# Tame Minimal Non-Polynomial Growth Strongly Simply Connected Algebras

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**Abstract**

A finite dimensional basic connected algebra $A$ over an algebraically closed field $K$ is called strongly simply connected if every full convex subcategory of $A$ is simply connected. We give a complete classification of tame minimal non-polynomial growth strongly simply connected algebras by quivers and relations.

Throughout this article, $K$ will denote a fixed algebraically closed field. By an algebra we mean a finite dimensional $K$–algebra, which we shall assume to be basic. An algebra $A$ can be written as a bound quiver algebra $A \simeq KQ/I$ where $Q = Q_A$ is the quiver of $A$ and $I$ is an admissible ideal in the path algebra $KQ$ of $Q$. Equivalently, every algebra $A = KQ/I$ will equally be considered as a $K$–category of which the object class is the set of vertices of $Q$, and the set of morphisms from $x$ to $y$ is the vector space $KQ(x,y)$ of all linear combinations of paths in $Q$ from $x$ to $y$ modulo the subspace $I(x,y) = I \cap KQ(x,y)$. We shall call an algebra $A$ triangular whenever its quiver $Q_A$ has no oriented cycles. Following [AS] a triangular algebra $A$ is called simply connected if, for any presentation $A \simeq KQ/I$ of $A$ as a bound quiver algebra, the fundamental group $\Pi_1(Q,I)$ of $(Q,I)$ is trivial. Further, $A$ is called strongly simply connected if every full convex subcategory of $A$ is simply connected [S1]. It was shown in [S1] that a triangular algebra $A$ is strongly simply connected if and only if the first Hochschild cohomology $H^1(C,C)$ of any full convex subcategory $C$ of $A$ vanishes. This is also equivalent to the fact that every full convex subcategory $C$ of $A$ has the separation property (in the sense of [BLS].) Recall also that an algebra $A$ is called tame, if for any dimension $d$, there is a finite number of $K[X]$–$A$–bimodules $M_i$ which are finitely generated and free as left $K[X]$–modules and satisfy the following condition: all but a finite number of isomorphism classes of indecomposable right $A$–modules of dimension $d$ are of the form $K[X]/(X-\lambda) \otimes_{K[X]} M_i$ for some $\lambda \in K$ and some $i$. Let $\mu_A(d)$ be the least number of bimodules $M_i$ satisfying the above condition. Then $A$ is called of polynomial growth if there is a natural number $m$ such that $\mu_A(d) \leq d^m$ for all $d \geq 1$.

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It is shown in [S2],[S3], that a strongly simply connected algebra $A$ is of polynomial growth if and only if $A$ does not contain a convex subcategory which is hypercritical or pg-critical. The hypercritical algebras (being the preprojective tilts of minimal wild hereditary tree algebras) have been classified by quivers and relations in [U] (see also [L],[W]). The definition of pg-critical algebras presented in [S2] is in some general terms (see section 1), and does not allow to apply the above criterion in practice. Note that it follows from the above criterion that the class of strongly simply connected pg-critical algebras coincides with the class of tame strongly simply connected algebras which are minimal not of polynomial growth (they themselves are not of polynomial growth but every proper convex subcategory is of polynomial growth).

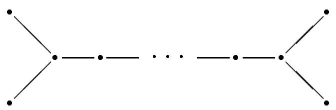
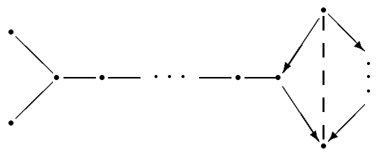
The aim of this paper is to establish a handy description of the tame minimal non-polynomial growth strongly simply connected algebras by quivers and relations. In the course of our proofs we also obtain a description of tame minimal non-polynomial growth incidence algebras of finite partially ordered sets.

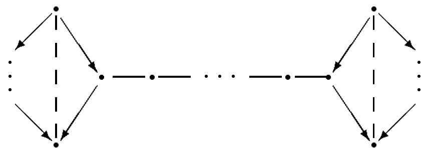
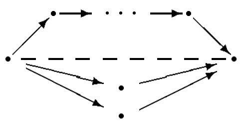
Throughout the paper, by an $A$–module always is meant a finitely generated right $A$–module, and we shall denote their module category by mod$A$. For each vertex $i$ of $Q_A$ we denote by $S_A(i)$ the corresponding simple $A$–module, by $P_A(i)$ the projective cover of $S_A(i)$, and by $I_A(i)$ the injective envelope of $S_A(i)$. We shall denote by $\Gamma_A$ the Auslander-Reiten quiver of $A$ and by $\tau_A = \mathrm{DTr}$ and $\tau_A^- = \mathrm{TrD}$ the Auslander-Reiten translations. Finally, we shall call an $A$-module $M$ thin, if the entries of its dimension vector $\underline{\dim} M = (\dim_K \mathrm{Hom}_A(P_A(i), M))_{i \text{ vertex in } Q_A}$ all are less or equal to 1. For a basic background from representation theory we refer to [Ri].

# 1 The main results

We shall present a complete description of tame minimal non-polynomial growth strongly simply connected algebras by quivers and relations. From [S2] we know, that this class of algebras coincides with the class of strongly simply connected pg-critical algebras. We first recall the definition of pg-critical algebras. For an algebra $A = KQ/I$, generators of $I$ are usually called relations. In our quivers, a dashed line indicates a relation being the sum of all paths from the starting point to the end point.

We know from [B], [HV] that there are only four families of critical algebras of type $\tilde{\mathbf{D}}_n$, $n \geq 4$, given by the following quivers and relations:

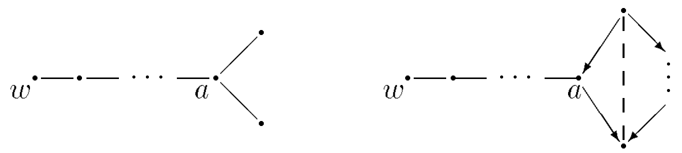
(1)  (2) 

(3)  (4) 

where the number of vertices is equal to $n+1$ and $\bullet\!-\!\bullet$ means $\bullet\!\rightarrow\!\bullet$ or $\bullet\!\leftarrow\!\bullet$. Note that all of the above algebras are strongly simply connected, and those of types (1) – (3) are completely separating [Dr]. It is known (see [Ri]) that if $C$ is a critical algebra of type $\tilde{\mathbf{D}}_n, n \geq 4$, then the Auslander-Reiten quiver $\Gamma_C$ consists of a preprojective component $\mathcal{P}$, a preinjective component $\mathcal{I}$ and a $\mathbb{P}_1(K)$–family $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes. Moreover, two tubes in $\mathcal{T}$ have rank 2, one tube has rank $n-2$, and the remaining ones have rank 1. The modules lying in $\mathcal{T}$ are called regular and they form a serial category whose simple objects are just the modules lying on the mouth of the tubes $(\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}_1(K)}$. Following [S2], by a pg-critical algebra we mean an algebra of one of the following forms or their opposites:

$$D(M) = \begin{bmatrix} K & M \\ 0 & C \end{bmatrix}, \quad E(N) = \begin{bmatrix} K & 0 & N \\ 0 & K & N \\ 0 & 0 & C \end{bmatrix}, \quad F(N) = \begin{bmatrix} \Lambda & N^t \\ 0 & C \end{bmatrix}$$

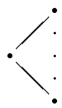
where $C$ is a critical algebra of type $\tilde{\mathbf{D}}_n, n \geq 4$, $M$ is an indecomposable $C$–module of regular length 2 lying in a tube of rank $n-2$, $N$ is a simple regular $C$–module lying in a tube of rank $n-2$, $\Lambda$ is given by one of the following quivers



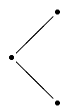

(possibly $w = a$, but no loop) and such that $F(N)(x,y) = N(y) \otimes_K \Lambda(x,w)$ for any object $x$ in $\Lambda$ and any object $y$ in $C$. Observe that the structure of the above modules $M$ and $N$ depends on the orientations of arrows in the quiver of $C$.

In order to save space in the theorem below and to make the list below more accessible we write down only the possible frames. Given such a frame, we allow the following admissible operations:

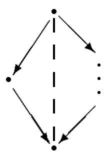
(i) Replacing each subgraph



by



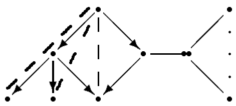
or



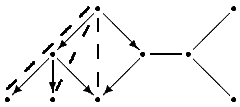
(ii) Choice of arbitrary orientations in non-oriented edges.

(iii) Constructing the opposite algebra.

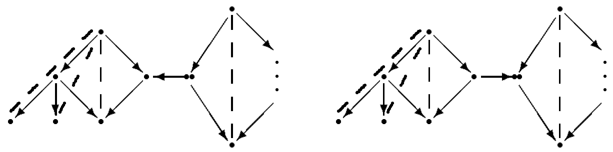
We want to illustrate the admissible operations. For example, consider the following frame:



We obtain 6 different algebras by choosing orientations for the non-oriented edges in

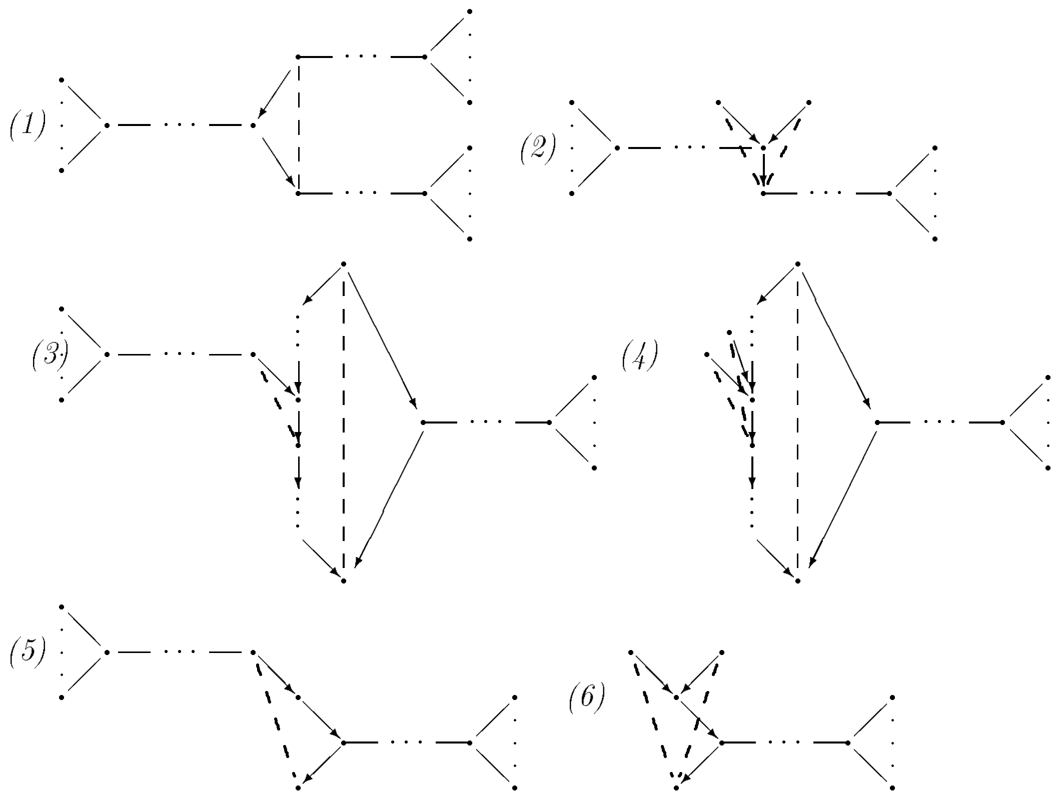


and the following 2 families of algebras



Finally, we also have to take the opposite algebras of the ones above. Hence this frame yields 12 non-isomorphic algebras with 9 vertices and 4 families of algebras.

**THEOREM 1** *Let $A$ be a strongly simply connected algebra. Then $A$ is tame minimal non-polynomial growth if and only if $A$ is obtained from a frame in the following list by admissible operations:*



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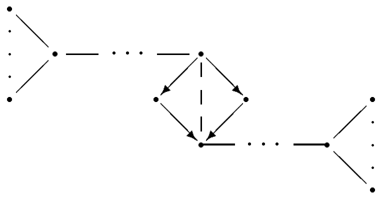

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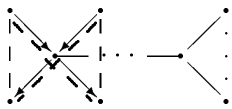

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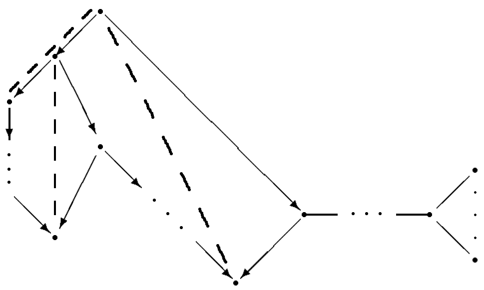

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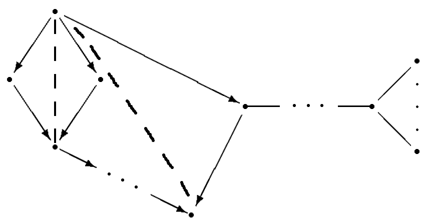
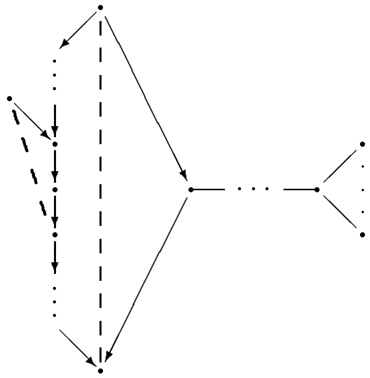


*(11)*



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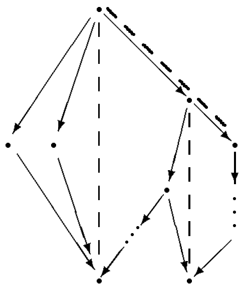

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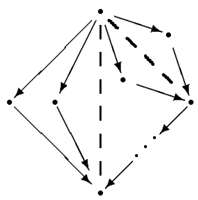

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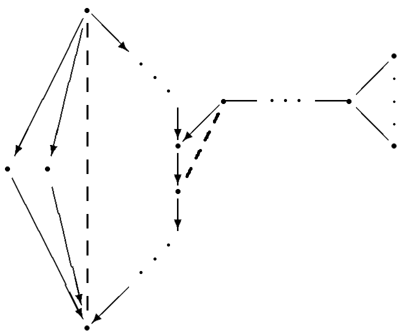

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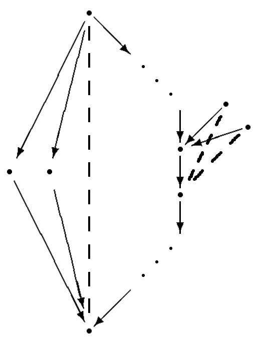

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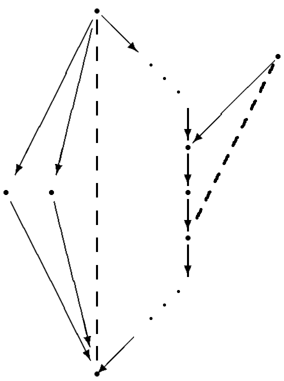


*In all cases, directed paths of non-fixed length denoted by* $\bullet\!\!\longrightarrow \cdots \longrightarrow\!\!\bullet$ *may consist of only one point, but in the cases (3) and (4) this may not occur simultaneously for both of these paths.*

Following [Dr] a triangular algebra $A$ is called completely separating if it is schurian ($\dim_K(\mathrm{Hom}_A(P_A(x), P_A(y)) \leq 1$ for all vertices $x$ and $y$ of $Q_A$) and every full convex subcategory of A has the separation property.

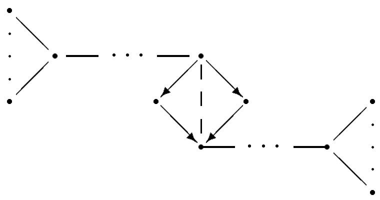
We then get the following consequence of the above theorem

**COROLLARY 1** *Let $A$ be a completely separating algebra. Then $A$ is tame of minimal non-polynomial growth if and only if $A$ is obtained from one of the frames (1) – (11) by applying admissible operations.*

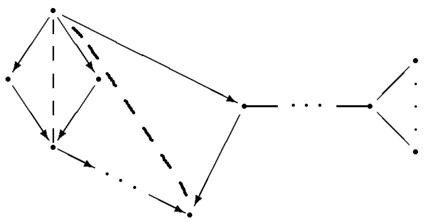
In the course of the proof we also obtain the following description of tame incidence algebras which are minimal of non-polynomial growth.

**COROLLARY 2** *Let $A$ be the incidence algebra of a finite partially ordered set. Then $A$ is tame minimal of non-polynomial growth if and only if $A$ is obtained by admissible operations from one of the following frames:*

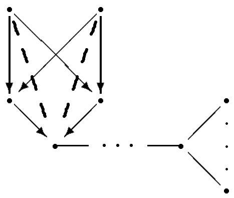
*A)*



*B)*



*C)*



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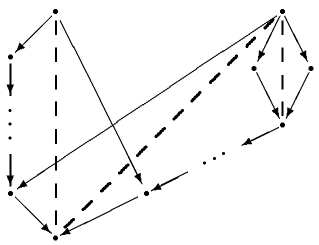


# 2 Constructions

By definition, a complete list of pg-critical algebras can be obtained by computing the modules of regular lengths one and two lying in the tubes of rank n-2 for the critical algebras of type $\tilde{D}_n$. We will do this separately for the cases (1) – (3) of completely separating and for the case (4) of canonical algebras, and will start with the canonical algebras, as they are easier to handle because of the fixed orientations of their quivers.

During the construction, we shall discuss, whether a given class of algebras consists of incidence algebras of partially ordered sets only for such classes, as do not occur in the list of the main theorem, since for those classes occurring there, this should be quite obvious from their given quivers.

## 2.1 pg-critical algebras arising from simple regular modules of canonical algebras

Let us start by fixing some notations. For $n \geq 4$, let $Q$ be the quiver:

$$\begin{tikzcd}
 & x_0 \arrow[ddl] \arrow[dd] \arrow[dr] & \\
 & & x_1 \arrow[d] \\
x_{n-1} \arrow[ddr] & x_n \arrow[dd] & \vdots \arrow[d] \\
 & & x_{n-3} \arrow[dl] \\
 & x_{n-2} &
\end{tikzcd}$$

Let $C = kQ/I$ where $I$ is the ideal generated by the sum of all pathes from $x_0$ to $x_{n-2}$ in $kQ$.

We observe, that for $i \in \{0, \ldots, n-3\}$ the module $P_C(x_{i+1})$ is a submodule of $P_C(x_i)$ and set

$$X_i := P_C(x_i)/P_C(x_{i+1})$$

This gives a family of pairwise non-isomorphic indecomposable thin modules. In fact, for $i = 1, \ldots, n-3$ the module $X_i$ is just the simple $A$-module $S_C(x_i)$, while $X_0$ is isomorphic to the module

$$\begin{tikzcd}
 & k \arrow[ddl, "1"'] \arrow[dd, "1"'] \arrow[dr] & \\
 & & 0 \arrow[d] \\
k \arrow[ddr, "-1"'] & k \arrow[dd, "1"'] & \vdots \arrow[d] \\
 & & 0 \arrow[dl] \\
 & k &
\end{tikzcd}$$

Moreover these modules have the following property:

**LEMMA 1** *The $\tau$-translate of each module in the sequence*

$$(X_0, X_{n-3}, X_{n-2}, \ldots, X_2, X_1, X_0)$$

*is given by its unique predecessor in the sequence.*

*Proof* We observe, that the modules $X_i$ indeed are given in terms of their projective resolutions

$$0 \longrightarrow P_C(x_{i+1}) \longrightarrow P_C(x_i) \longrightarrow X_i \longrightarrow 0.$$

This enables us to compute the $\tau$-translates $\mathrm{DTr}X_i$ of the modules $X_i$ explicitly. As $\mathrm{DHom}(P_C(x_j), C) = I_C(x_j)$, $\tau X_i$ is just the kernel of the map $I_C(x_{i+1}) \longrightarrow I_C(x_i)$, which is easily identified as $X_{i+1}$ for $i = 0, \ldots, n-4$ or $X_0$ in case $i = n-3$ respectively.

**COROLLARY 3** *The modules $(X_0, \ldots, X_{n-3})$ are simple regular and form the mouth of a tube of rank $n-2$ in $\Gamma_C$.*

*Proof* By the above lemma, the modules $X_0, \ldots, X_{n-3}$ form a cyclic $\tau$-orbit. As they are pairwise non-isomorphic, the length of this cycle is $n-2$. Thus these modules lie in a tube $T$ of rank $n-2$. (For the structure of $\Gamma_C$ we refer to [Ri].) As furthermore $X_1$ is a simple module, it also is simple regular. Therefore $X_i$ lies in the mouth of the tube $T$ and its $\tau$-orbit $(X_0, \ldots, X_{n-3})$ consits of simple regular modules and forms the mouth of this tube.

Let us point out, that by the above construction, we obtain all simple regular modules of $C$ lying in a tube of rank $n-2$. In particular, for $n > 4$ there is exactly one such tube, and by the corollary, the simple regular objects are just the constructed modules $(X_0, \ldots, X_{n-3})$.
For $n = 4$, we have three equivalent choices for the point $X_1$, because of the internal symmetry of the quiver $Q$. Each of these choices yields a family $(X_0^{(\kappa)}, X_1^{(\kappa)})$ forming the mouth of a tube of rank 2. As by construction these families are pairwise disjoint and as in $\Gamma_C$ there exist exactly 3 tubes of rank 2, the modules $(X_0^{(\kappa)}, X_1^{(\kappa)})_{\kappa=1,2,3}$ are just the simple regular modules of these tubes.

It is now easy to construct the pg-critical algebras obtained from $C$ by means of the simple regular modules $(X_i)_{i=0,\ldots,n-3}$ as defined in section 1. In fact, the algebras of type

$$E(X_i) = \begin{bmatrix} K & 0 & X_i \\ 0 & K & X_i \\ 0 & 0 & C \end{bmatrix}$$

are just the algebras of type (15) in the list of the main theorem. Those of type

$$F(X_i) = \begin{bmatrix} \Lambda & X_i^t \\ 0 & C \end{bmatrix}$$

are just those of type (14) in the list.

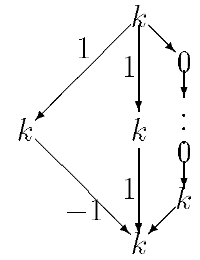
## 2.2 pg-critical algebras arising from modules of canonical algebras of regular length 2

Let $Q$ and $C$ be as above. Knowing the simple regular modules $(X_0, \ldots, X_{n-3})$ in a tube of rank $n-2$ together with their $\tau$-translates allows easily to compute the modules of regular length 2 in such a tube, as these modules are given by the middle terms $Y_i$ of almost split sequences of the form

$$0 \longrightarrow \tau X_i \longrightarrow Y_i \longrightarrow X_i \longrightarrow 0 \quad .$$

As $\dim_K \operatorname{Ext}^1(X_i, \tau X_i) = \dim_K \operatorname{Hom}(\tau X_i, \tau X_i) = 1$, it is sufficient to find an indecomposable module $Y_i$ that allows for such an exact sequence. Obviously, for $i = 0, \ldots, n-4$

such a module $Y_i$ is given by $Y_i = P_C(x_i)/P_C(x_{i+2})$. For $i = n-3$ the following module will do:




Again it is easy to identify the corresponding pg-critical algebras. In fact, for $i = 0, \ldots, n-4$, the algebras of type $D(Y_i)$ are just those of type (16) in the list of theorem 1.

Let us finally consider the algebra

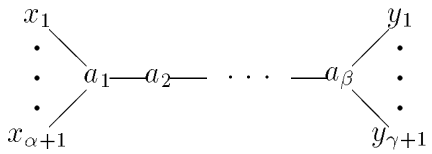
$$D = D(Y_{n-3}) = \begin{bmatrix} K & Y_{n-3} \\ 0 & C \end{bmatrix}.$$

As this is a one-point extension of $C$, the quiver $Q_D$ of $D$ is obtained from the quiver $Q_C$ by adding a source $w$, the additional arrows and relations being of course defined by $Y_{n-3}$.

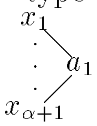
Deleting the vertex $x_{n-2}$ from $Q_D$ we will obtain a corresponding convex subcategory $D'$ of $D$, for which the projective module $P_{D'}(w)$ does not have separating radical. (The radical of $P_{D'}(w)$ being just the restriction of $Y_{n-3}$ to $D'$.) Therefore $D$ is not strictly simply connected.

## 2.3 pg-critical algebras arising from simple regular modules of completely separating algebras

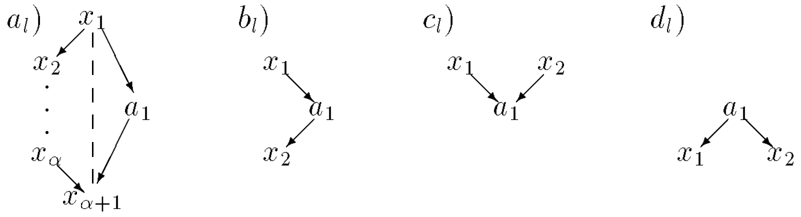
For $n \geq 4$, let $Q$ be a bound quiver with $n+1$ points of the following form:




Let $C = kQ/I$ be the corresponding algebra. With the notations introduced in section 1, this decribes all of the three families of completely separating critical algebras of type $\tilde{D}_n$. In order to control the effect of orientation of edges, we will, for the 'left end' (




), distinguish the following four cases:



By $a_r$), $b_r$), $c_r$) and $d_r$) we will denote the corresponding cases for the 'right end'. Note that only the cases $a_r$) ans $a_l$) contribute any relations to the ideal $I$.

In order to control the orientation of the 'center line' $a_1$—$a_2$—$\cdots$—$a_\beta$ we introduce some further notations:

We will call the point $a_i$ a *left slope point* of $C$ if $i > 1$ and there is an arrow from $a_i$ to $a_{i-1}$ in $Q$. We will call $a_i$ a *right slope point* if $i < \beta$ and there is an arrow from $a_i$ to $a_{i+1}$ in $Q$.

Let $\lambda$ be the number of left slope points and let $i_1 < i_2 < \ldots < i_\lambda$ indices such that the set $S_l := \{a_{i_k} | k = 1, \ldots, \lambda\}$ is the set of all left slope points of $C$.

Let $\rho$ be the number of right slope points and let $j_1 < j_2 < \ldots < j_\rho$ be indices such that the set $S_r := \{a_{j_k} | k = 1, \ldots, \rho\}$ is the set of all right slope points of $C$.

The sum $\rho + \lambda$ is invariant under change of orientation, as turning an arrow $a_i \to a_{i+1}$ into an arrow $a_{i+1} \to a_i$ will remove the point $a_i$ from the set $S_r$ and add $a_{i+1}$ to the set $S_l$ and vice versa. Therefore $\rho + \lambda = \beta - 1$, (which is the number of right slope points of the oriented chain $a_1 \to a_2 \to \ldots \to a_\beta$, there being no left slope points.)

For each left slope point $a_{i_k}$, ($k \in \{1, \ldots, \lambda\}$) we observe, that $P_C(a_{i_k - 1})$ is a submodule of $P_C(a_{i_k})$ and set

$$U_k := P_C(a_{i_k})/P_C(a_{i_k - 1}).$$

We shall call $U_k$ the left slope module corresponding to $a_{i_k}$.

In the same way we define for each right slope point $a_{j_k}$ a right slope module $V_k$ by

$$V_k := P_C(a_{j_k})/P_C(a_{j_k + 1}).$$

Depending on the shape of the left end, we define modules $X_1, \ldots, X_\alpha$ in the following way:

In case $a_l$) set $X_i = P_C(x_i)/P_C(x_{i+1})$ for $i = 1, \ldots \alpha$.

In case $b_l$) set $X_1 = P_C(x_1)/P_C(x_2)$.

In case $c_l$) let $X_1$ be the cokernel of the map

$$P_C(a_1) \xrightarrow{\binom{\iota_1}{-\iota_2}} P_C(x_1) \amalg P_C(x_2)$$

Here $\iota_k$ denotes the inclusion map $P_C(a_1) \longrightarrow P_C(x_k)$ (k=1,2).

In case $d_l$) set $X_i = P_C(a_1)/(P_C(x_1) \amalg P_C(x_2))$.

In essentially the same way, substituting $y_i$ for $x_i$, $\gamma$ for $\alpha$, and $a_\beta$ for $a_1$, we define modules $Y_1, \ldots, Y_\gamma$, depending on the shape of the right end.

Thus we obtain a family

$$(V_\rho, V_{\rho-1}, \ldots, V_1, X_\alpha, X_{\alpha-1}, \ldots, X_1, U_1, \ldots, U_\lambda, Y_\gamma, \ldots, Y_1)$$

of $\lambda + \rho + \alpha + \gamma = \beta - 1 + \alpha + \gamma = n - 2$ pairwise non-isomorphic thin modules. As in section 2.1 we get:

**LEMMA 2** *The $\tau$-translate of each module of the sequence*

$$(Y_1, V_\rho, V_{\rho-1}, \dots, V_1, X_\alpha, X_{\alpha-1}, \dots, X_1, U_1, \dots, U_\lambda, Y_\gamma, \dots, Y_1)$$

*is given by its unique predecessor.*

*Proof* Observe, that again, as in lemma 1 the modules to be considered are given in terms of projective resolutions. Thus determining their $\tau$-translates is a matter of elementary computation, which can indeed be carried out, since the definition of the left and right slope modules in terms of left and right slope points gives sufficient control of the possible orientations. Also due to left-right symmetry it is sufficient to compute the translates of the left slope modules $U_k$ and the left end modules $X_i$. Still due to the effects of orientations the need to distinguish some cases remains.

As we will have to discuss all these cases arising from orientation in somewhat more detail when computing the modules of regular length 2 in a tube on rank $n-2$ in section 2.4, we do not list them right now but rather refer to section 2.4 for a complete account.

Let us note in addition, that neither left nor right slope modules need to exist, whereas always there is at least one left end module $X_1$ and one right end module $Y_1$. Thus the sequence given above is to be read in the following way: The $\tau$-translate of the module $X_\alpha$ is the module $V_1$ if there is a right slope point, or else it is $Y_1$.

**COROLLARY 4** *The modules*

$$(Y_1, V_\rho, V_{\rho-1}, \dots, V_1, X_\alpha, X_{\alpha-1}, \dots, X_1, U_1, \dots, U_\lambda, Y_\gamma, \dots, Y_1)$$

*are simple regular and form the mouth of a tube of rank $n-2$ in $\Gamma_C$.*

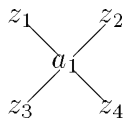
*Proof* By the above lemma, the $n-2$ modules in question form a cyclic $\tau$-orbit und thus lie in a tube of rank $n-2$.

In case $a_l$) the module $X_1$ is a simple module, thus simple regular. Therefore its whole $\tau$-orbit consists of simple regular modules.

By [Ri] the sum $s$ of the dimension vectors of all modules in a given $\tau$-orbit inside a tube is a multiple $\nu c$ of the critical vector $c \in \mathbb{Z}^{n+1}$ of the quadratic form $q_C$ associated with $C$, and $\nu$ is the regular length of every module in this orbit.

For the orbit in question, for any feasible shape of the left end and any orientation of arrows, $x_1$ always lies in the support of exactly one of the above modules. In case $b_l$) and $c_l$) this is $X_1$. In case $d_l$) this is $V_1$ if there is a right slope point or else it is $Y_1$. Therefore the component $s_{x_1}$ of $s$ in $x_1$ is $(\nu c)_{x_1} = s_{x_1} = 1$. Therefore $\nu = 1$ and thus all modules in question are simple regular.

As we have seen, the construction described above will yield the modules in the mouth of one tube of rank $n-2$. For $n > 4$ there is a unique tube of that rank, so that by this uniqueness we have constructed all such modules. For $n = 4$ there are three tubes of rank $n - 2 = 2$. Let us recall, that in this case the quiver $Q_C$ of $C$ up to orientation is




As there is no 'middle line', we are free to divide the set $\{z_1, z_2, z_3, z_4\}$ into two disjoint subsets containing 2 elements each, one forming the right end, the other the left end. To do this, there are up to left-right symmetry $\binom{4}{2} = 3$ possibilities. Each of these choices leading to one tube of rank 2, the three together will then yield all three different tubes of length 2.

Knowing the modules in the mouth of the tubes of rank $n - 2$ it again is easy to determine the corresponding pg-critical algebras.

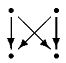
Because of left-right symmetry the families of algebras obtained from the right slope modules $V_k$ are the same as those obtained from the left slope modules $U_k$, and those obtained from the right end modules $Y_i$ are the same as those obtained from the left end modules $X_i$.

In particular the algebras of type $F(U_k)$ (or $F(V_k)$ respectively) are just those of type (1) in the list of the main theorem. The algebras of type $E(U_k)$ (resp. $E(V_k)$) are just those of type (2).

In case $a$) for the left (resp. right) end, the algebras of type $F(X_i)$ (resp. $F(Y_i)$) are just those of type (3) and those of type $E(X_i)$ (resp. $E(Y_i)$) are just those of type (4).

In case $b$), the algebras of type $F(X_1)$ (resp. $F(Y_1)$) are just those of type (5) and those of type $E(X_1)$ (resp. $E(Y_1)$) are just those of type (6).

In case $c$) the algebras of type $F(X_1)$ (resp. $F(Y_1)$) are just those of type (7) in the list of the main theorem. Let us point out, that these are as yet the first algebras obtained during our construction, that are algebras of partially ordered sets, and thus are listed in corollary 2 as item A). The algebras of type $E(X_1)$ (resp. $E(Y_1)$) also are algebras of partially ordered sets, namely just those of type C) in corollary 2. As however they contain the following algebra



of type $\tilde{A}_3$(as a convex subcategory), which is not simply connected, they themselves are not strongly simply connected.

In case $d$), the algebras of type $F(X_1)$ (resp. $F(Y_1)$) are just the duals of those of type (2). The algebras of type $E(X_1)$ (resp. $E(Y_1)$) are just those of type (8).

## 2.4 pg-critical algebras arising from modules of completely separating algebras of regular length 2

Let $Q = Q_C$ and $C$ be as above. The knowledge of the simple regular modules in the tubes of rank $n - 2$ in $\Gamma_C$ together with their $\tau$ translates again enables us to compute the modules of regular length 2 in these tubes, as of course these again are given by the middle term $Z$ of almost split sequences

$$0 \longrightarrow \tau X \longrightarrow Z \longrightarrow X \longrightarrow 0$$

where $X$ is a simple regular module. We will in this situation call the module $Z$, which is an extension of $X$ by $\tau X$, the corresponding module of regular length 2 to $X$. As in section 2.2 because of $\dim_K \operatorname{Ext}^1(X, \tau X) = 1$ it is sufficient to find a short exact sequence of the above form, which is non-split, in order to determine $Z$. As a consequence of the left-right symmetry in our description of the quiver $Q_C$ and of the simple regular modules it furthermore is sufficient to consider only such modules $Z$ which are an extension of a left end module $X_i$ or of a left slope module $U_k$.

Depending on orientation, the shape of the left end and on the specific module itself, the $\tau$-translate $\tau X_i$ of a left end module $X_i$ may be the left end module $X_{i+1}$, the right slope module $V_1$ or the right end module $Y_1$. Also the $\tau$-translate $\tau U_k$ of a left slope module $U_k$ may be the left end module $U_{k-1}$ or the left end module $X_1$ (cf. lemma 2). We will consider these situations separately below.

In general, every pg-critical algebra of type $D(Z)$ for an appropriate $C$-module $Z$ is by definition a one-point extension of $C$ by $Z$. Therefore the quiver $Q_{D(Z)}$ of $D(Z)$ is obtained from the quiver $Q_C$ of $C$ by adding one additional source $\omega$, the additional arrows and relations being such, that the radical of the new projective module $P_{D(C)}(\omega)$ is $Z$.

In order to abbreviate our notation, for two vertices $x$ and $y$ of $Q_{D(C)}$, we will write $x \preceq y$ if there is a directed path in $Q_{D(C)}$ from $y$ to $x$ or if $x = y$. As $\omega$ is a source in $Q_{D(C)}$, for $x \neq \omega \neq y \neq x$ this is equivalent to the existence of a directed path for $y$ to $x$ in $Q_C$.

Let us also recall from [Dr] that an indecomposable thin module over a completely separating algebra such as $C$ is up to isomorphism uniquely determined by its support. This property will in many cases enable us to identify the modules of regular length 2 just by calculation of their dimension vectors.

### 2.4.1 Extensions of left end modules by left end modules

This situation only occurs in case $a_l$) for the left end. Here for $i = 1, \ldots, \alpha - 1$ there is a non-split short exact sequence

$$0 \longrightarrow X_{i+1} \longrightarrow P_C(x_i)/P_C(x_{i+2}) \longrightarrow X_i \longrightarrow 0$$

As $X_{i+1} = \tau X_i$ by lemma 2, the corresponding module of regular length 2 to $X_i$ is $Z = P_C(x_i)/P_C(x_{i+2})$. The algebras of type $D(Z)$ obtained from these modules then are just those of type (11) in the list of theorem 1.

### 2.4.2 Extensions of left slope modules by left slope modules

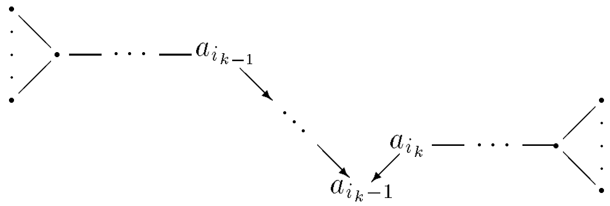
In this situation, which occurs if and only if there are at least two left slope points, we have to consider non-split extensions of the form

$$0 \longrightarrow U_{k-1} \longrightarrow Z \longrightarrow U_k \longrightarrow 0$$

for $k \in \{2, \dots \lambda\}$.

If $a_{i_k-1}$ is a right slope point, this implies $i_k-1 = i_{k-1}$. Therefore $U_k = P_C(a_{i_k})/P_C(a_{i_k-1})$, $U_{k-1} = P_C(a_{i_{k-1}})/P_C(a_{i_{k-1}-1}) = P_C(a_{i_k-1})/P_C(a_{i_k-2})$, and $Z = P_C(a_{i_k})/P_C(a_{i_k-2})$. The extensions of type $D(Z)$ then are the duals of those of type (5) in the list of theorem 1.

If $a_{i_k-1}$ is not a right slope point, then $a_{i_k-1} \preceq a_{i_k-2}$ and $Q_C$ is of the form




From the dimension vector $\underline{\dim}(Z) = \underline{\dim}(U_{k-1}+\underline{\dim}(U_k)$ we find, that $Z$ is a thin module with support $\{a_{i_{k-1}}, \dots, a_{i_k-1}\} \cup \{x | x \preceq a_{i_k}\}$. Therefore $Z$ has to be the unique (up to isomorphism) thin module with this support. The algebras of type $D(Z)$ obtained this way are then special cases of the duals of algebras of type (3) in the list of theorem 1.

### 2.4.3 Extensions of left slope modules by left end modules

Here we have to consider extensions of the form

$$0 \longrightarrow X_1 \longrightarrow Z \longrightarrow U_1 \longrightarrow 0$$

From the dimension vector $\underline{\dim}(Z) = \underline{\dim}(U_1) + \underline{\dim}(X_1)$ we find, that in any case for the left end, and thus for $X_1$, $Z$ is a thin module, the support of which then is of course $\mathrm{supp}(Z) = \mathrm{supp}(U_1) \cup \mathrm{supp}(X_1)$.

By this, $Z$ as an indecomposable thin module again is uniquely determined up to isomorphism, and thus readily attainable from the knowledge of the modules $X_1$ and $U_1$.

In particular, in case $a_l$) for the left end, the algebras of type $D(Z)$ are just those of type (9) in the list of theorem 1, in case $c_l$) they are those of type (10), which also occur as item $B$) in corollary 2. In case $b_l$) we obtain special cases of duals of algebras of type (11) and for case $d_l$) we obtain special cases of duals of algebras of type (4), both classes having in total already been accounted for.

Note, that in the cases $a_l$), $b_l$) and $d_l$) it is essentially the structure of $Z$ inherited from the module $X_1$, that induces a zero relation that prevents the algebra $D(Z)$ from being an algebra of a partially ordered set.

### 2.4.4 Extensions of left end modules by right slope modules

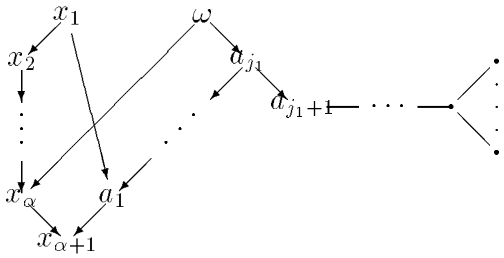
Here we have to consider extensions of the following form:

$$0 \longrightarrow V_1 \longrightarrow Z \longrightarrow X_\alpha \longrightarrow 0$$

This especially means, there is at least one right slope point.

In case $a_l$) for the left end, we again observe from the dimension vector, that $Z$ is a thin module with support $\mathrm{supp}(Z) = \mathrm{supp}(V_1) \cup \mathrm{supp}(X_\alpha) = \{x_{\alpha+1}, a_1, \ldots, a_{j_1}\} \cup \{x_\alpha\}$.

The quiver $Q_{D(Z)}$ of $D(Z)$ then is



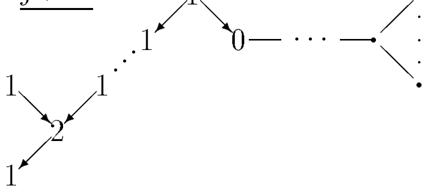

However $D(Z)$ is neither strongly simply connected, as it contais a convex subalgebra of type $\tilde{A}_\nu$, nor is it the algebra of a poset.

In the remaining cases for the left end, i.e. in the cases $b_l$), $c_l$) ans $d_l$) computing $\underline{\dim}(Z) = \underline{\dim}(V_1) + \underline{\dim}(X_\alpha)$, we obtain $(\underline{\dim}(Z))_x = 1$ for all $x \in \mathrm{supp}(Z) \setminus \{a_1\}$ and $(\underline{\dim}(Z))_{a_1} = 2$.

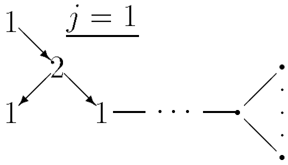
For a closer analysis of the situation, we take a look at the quiver $Q_C$ of $C$, writing at a vertex $x$ the corresponding entry $(\underline{\dim}(Z))_x$ of the dimension vector of $Z$. The resulting picture does not only depend on the shape of the left end, but also on whether $j_i = 1$ or $j_i > 1$. That way we get:

In case $b_l$):

$\underline{j > 1}$

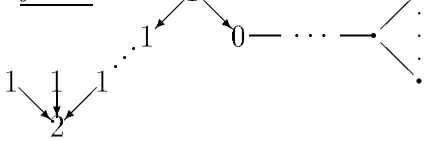


$\underline{j = 1}$

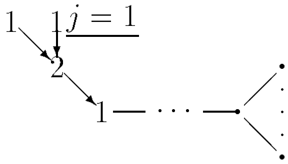


In case $c_l$):

$\underline{j > 1}$

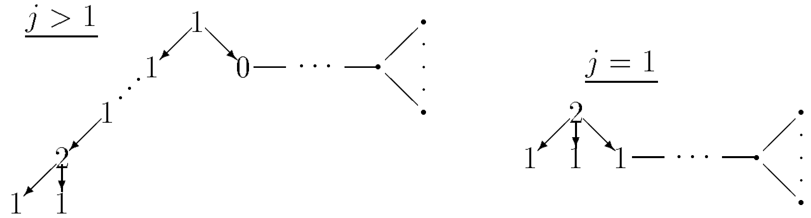


$\underline{j = 1}$



In case $d_l$):

$\underline{j > 1}$ $\underline{j = 1}$



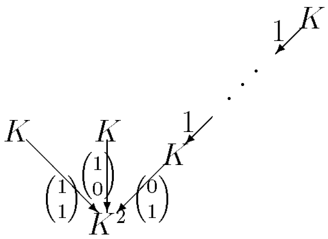
(Note, that in the cases with $j_i = 1$ the support of $Z$ is given by $\text{supp}(Z) = \{x_1, x_2\} \cap \{x | x| \preceq a_1\}$.)

We now observe, that an algebra of type $D(Z)$ cannot be strongly simply connected, if the full subquiver $Q'$ of $Q_C$, whose set of vertices is given by $R = \{x \in \text{supp}(Z) | x \preceq a_1 \text{and} x \neq a_1\}$ has at most two connected components.

In order to see this, we recall, that the quiver $Q_{D(Z)}$ of $D(Z)$ is obtained from $Q_C$ by adding a single source $\omega$ such that $\text{rad}P_{D(Z)}(\omega) = Z$. We now consider the full convex subcategory $D'$ of $D(Z)$ induced by the set of all vertices $x$ of $Q_{D(Z)}$ with the property $a_1 \preceq x \preceq \omega$. The radical $\text{rad}P_{D'}(\omega)$ of $P_{D'}(\omega)$ then is just the restriction $Z'$ of $Z$ to $D'$. By assumption we may write the set $R$ as disjoint union $R = R_1 \cup R_2$, $R_1 \cap R_2 = \emptyset$, where $R_i$ is either a connected component of $Q'$ or the empty set. Then $Z'$ decomposes over $D'$ into two indecomposable thin modules $Z'_1$ and $Z'_2$ with supports $\text{supp}(Z'_i) = R_i \cup \{a_1\}$, $i = 1, 2$. As the intersection $\text{supp}(Z'_1) \cap \text{supp}(Z'_2) = (R_1 \cup \{a_1\}) \cap (R_2 \cup \{a_1\}) = \{a_1\}$ is non-empty, the module $P_{D'}(\omega)$ does not have separating radical. Therefore $D(Z)$ is not strongly simply connected.

Thus in all the cases stated above, the algebras $D(Z)$ obtained, are not strongly simply connected, except for the case $c_l$) and $j_1 > 1$.

Also it is easy in this remaining case to find an indecomposable extension $Z$ of $X_\alpha$ by $V_1$. Such an extension is given over its support by:



The algebras $D(Z)$ obtained this way are special cases of duals of the algebras of type (14) in the list of theorem 1.

Let us finally mention, that in all these cases, no further algebras which belong to partially ordered sets occur, as for such an algebra, the indecomposable projective modules and thus their radicals are thin moduls, whence $Z$ would have to be thin.

### 2.4.5 Extensions of left end modules by right end modules

Here we have to consider extensions of the form:

$$0 \longrightarrow Y_1 \longrightarrow Z \longrightarrow X_\alpha \longrightarrow 0$$

This situation is very similar to that of the previous section. We therefore will omit most of the rather technical details. As in section 2.4.4 it turns out, that $Z$ is a thin module if the left end is of shape $a_l$) whereas otherwise $(\underline{\dim}(Z))_x = 1$ for all $x \in \operatorname{supp}(Z) \setminus \{a_1\}$ and $(\underline{\dim}(Z))_{a_1} = 2$. Also like in the previous section, the algebras of type $D(Z)$ are strongly simply connected if the left end is of shape $c_l$). In this case, for a right end of shape $a_r$) the algebras $D(Z)$ are just those of type (12) in the list of theorem 1, in case $c_r$) for the right end they are those of type (13). Thus all families in the list of theorem 1 now have been accounted for.

Furthermore, in case $b_r$) for the right end, the algebras of type $D(Z)$ are special cases of duals of those of type (16) and in case $d_r$) they are special cases of duals of those of type (15).

As was already mentioned, for an algebra $D(Z)$ to belong to a poset, the module $Z$ has to be thin. Thus such algebras can only occur in case $a_l$) for the left end. Similarly to the considerations made in section 2.4.3, in the cases $a_r$),$b_r$) and $d_r$) the structure of $Z$ inherited from the module $Y_1$ will impose a zero relation upon $D(Z)$ that prevents it from being the algebra of a poset. Therefore, algebras of posets can only occur in the case of $c_l$) for the left and $c_r$) for the right end. Indeed in this case, the algebras $D(Z)$ are algebras of posets, namely those of type $D$) in the list of corollary 2, which also completes that list.

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