On the convergence of asynchronous iteration methods for nonlinear paracontractions and consistent linear systems

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Abstract

In [6] the authors introduced the concept of paracontracting operators for fixed point problems and their solution with asynchronous iteration methods. We will extend their results essentially with respect to the criterion of contraction for the *pool* of operators, from which a common fixed point is searched and a more general kind of asynchronous iterations, which we will call confluent. As an application of this theory, we will consider asynchronous iteration methods for consistent linear systems $Ax = b$, where A is a singular M-matrix. In [7] Lubachevski and Mitra investigated asynchronous iteration methods for the Perron-vector problem, e.g. determining a positive solution of $Tx = \rho(T)\,x$, where T is an irreducible nonnegative matrix and its spectral radius is known. We will show that we can extend their results by using the developed theory of paracontractions and confluence to the reducible-affine case which arises, if we have those kind of linear systems.

Key words: Asynchronous iterations, paracontracting operators, nonnegative matrices, singular M-systems, Frobenius vectors.

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1 Introduction

In [6] Elsner, Koltracht and Neumann studied the convergence of the sequential iteration

$$x(j+1) := G^{k(j)}(x(j)) \qquad j = 0, 1, \ldots \tag{1}$$

and the asynchronous iteration

$$x(j+1) := \alpha_j\, x(j) \;+\; (1-\alpha_j)\, G^{k(j)}(x(s(j))) \qquad j = 0, 1, \ldots, \tag{2}$$

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where in both cases the $G^{k(j)}$'s were chosen from a finite *pool* of real operators $\mathcal{G} = \{G^k \mid k \in \mathbb{K}\}$, here $\mathbb{K}$ is any set. This pool was assumed to be paracontracting, that is: All the operators $G : \mathbb{R}^n \to \mathbb{R}^n$ in $\mathcal{G}$ are continuous, and for any fixed point ξ of an operator G of this pool, and for any $x \in \mathbb{R}^n$ either

$$\| G(x) - \xi \| < \| x - \xi \| \tag{3}$$

or x itself is a fixed point of G. The α_j's in (2) are chosen from a finite set of numbers from $[0,1)$, with the restriction that $\alpha_j = 0$ is only *permitted,* if $s(j) = j$ ($s(j)_{\, j=0,1,\ldots}$ is a sequence from $\mathbb{N}_0$, with $s(j) \le j \;\forall\, j \in \mathbb{N}_0$, while $k(j)_{\, j=0,1,\ldots}$ is a sequence from $\mathbb{K}$). The underlying model of parallelism in (2) is as follows: Assume we have K processors $\pi_1, \ldots, \pi_K$. At iteration step $s(j)$ processor $\pi_{k(j)}$ retrieves the global approximation $x(s(j))$, which resides in some shared memory, and computes a local iteration $G^{k(j)}(x(s(j)))$. At the *time* j, when this computation is done, the global iteration in the shared memory is updated by a convex combination as in (2). The classical asynchronous iteration scheme was introduced for linear systems by Chazan and Miranker in [4] and in a variant as an asynchronous *multisplitting*-method in [3]. In [1] this concept was extended to the nonlinear case. This scheme was introduced for only one operator $G : \mathbb{R}^n \to \mathbb{R}^n$:

$$x_i(j+1) := \begin{cases} x_i(j) & \text{, if } \; i = p_j \\ G_i\left(\begin{pmatrix} x_1(s^1(j)) \\ \vdots \\ x_n(s^n(j)) \end{pmatrix} \right) & \text{, if } \; i \ne p_j \end{cases} \qquad j = 0, 1, \ldots, \tag{4}$$

where again $s_i(j)_{\, j=0,1,\ldots}$, $i = 1, \ldots, n$ are sequences from $\mathbb{N}_0$, with $s_i(j) \le j \;\; \forall\, i, j$, and $p_{j\; j=0,1,\ldots}$ is a sequence from $\{1, \ldots, n\}$. The concept of parallelism here is substantially the same as above, but here the different processors update single or (as in the original, but after renumbering equivalent definition in [1]) groups of components of the system. Criteria of contraction were introduced in [1] and [10]. If the sequences $s^i(j)_{\, j=0,1,\ldots}$ and $p_{j\; j=0,1,\ldots}$ fulfil expectable conditions, then these criteria are necessary and sufficient for general convergence in the linear case, if the spectral radius of the Jacobian of G at its unique fixed point ξ satisfies $\rho(|G'(\xi)|) \ne 1$.
In the second section we will define the asynchronous iteration scheme, which we shall study. The convex combination in (2) is to guarantee something like a coupling in the iteration scheme. For example, let G^1 be an operator with fixed point ξ, and G^2 be one with fixed point $\eta (\ne \xi)$, and let both vectors arise as iterations in (2), then, if the α_j's were zero, without the restriction described above, it would be easy to construct divergent sequences, even if there is a common fixed point of the pool $\{G^1, G^2\}$, namely some, which alternate between η and ξ. If there is only one operator, which has two fixed points, the same phenomenon could arise, and the iteration process would not converge. We will formulate another concept of coupling, to avoid the convex combination in (2). Asynchronous iterations, which fulfil these coupling-postulates, will be called confluent.
In the third section we will collect some criteria of contraction for the pool $\mathcal{G}$, which

is, in our definition, an infinite pool of *multiple point* operators. That is: All the operators are defined on (different) products of $\mathbb{R}^n$. A new criterion of contraction will perhaps be interesting for future research, because the class of operators, which satisfies this criterion, contains operators, which have non-convex sets of fixed points: The modification of the postulate in (3) is as follows: In (3) the norm is independent from both, the operator and the fixed point, and we do not use the latter. In this criterion of contraction we will also not use, that the operators of the pool are continuous. Therefore the new criterion will be called (n)-paracontracting, and that one described above (c)-paracontracting. We will also compare them with two other criteria of contraction, and give some examples.
The results of convergence shall comprise the fourth section. We will prove a Theorem, which now extends the results of [6] into different points: confluent iterations, multiple point operators, infinite pools, nonconvex sets of fixed points, noncontinuous operators, and non strictly convex vector norms (as in [6]). We will add results for *approximate pools*. These are sequences of operators, where subsequences converge to the operators of another pool.
As an application of this theory, we will analyze in section five the iteration scheme (4) for affin-linear systems, e.g. $G(x) = Tx + f$, where T is a nonnegative matrix of spectral radius one. We will extend the result of [7] to the case that $f \neq 0$ and that T is reducible, but semiconvergent and its Frobenius normal form has at least one positive diagonal element in each of the diagonal blocks, which are of unit spectral radius.

2 A class of asynchronous iterative methods

Below we give a definition of asynchronous iterations where, in difference to the original ([4]), the whole vector is updated in every iteration step, and where all components of a vector, which is retrieved from the shared memory, as described above, have the same delay.

Definition 2.1 Let $\mathbb{K}$ be some set, $m \in \mathbb{N}$ a fixed number, and $\mathcal{G} = \{G^k \mid k \in \mathbb{K}\}$ be a *pool* of operators $G^k : D^{m_k} \subset \mathbb{R}^{nm_k} \to D$, where $m_k \in \{1, ..., m\}\ \forall\, k \in \mathbb{K}$ and $D \subset \mathbb{R}^n$ is closed. Furthermore let $\mathcal{X}_{\mathcal{O}} = \{x(0), ..., x(-M)\} \subset D$ be a set of given vectors. Then, if there are sequences $\mathcal{K} = k(j)_{\ j=0,1,...}$ of elements in $\mathbb{K}$, $\mathcal{S} = \{s^1(j), ..., s^{m_{k(j)}}(j) \mid j = 0, 1, ...\}$ of m_k-tuple from $\mathbb{N}_0 \cup \{-1, ..., -M\}$ with $s^l(j) \leq j$ for each $j \in \mathbb{N}_0$, $l = 1, ..., m_{k(j)}$, we call the sequence $x(j)_{\ j=0,1,...}$, given by

$$x(j+1) := G^{k(j)}(x(s^1(j)), ..., x(s^{m_{k(j)}}(j))) \qquad j = 0, 1, ... \tag{5}$$

an *asynchronous iteration*.

Remark, that we intentionally did not define all the G^k's on $\mathbb{R}^{nm}$, because we need this type of definition for later studies. An asynchronous iteration corresponding to $\mathcal{G}$, starting with $\mathcal{X}_{\mathcal{O}}$ and defined by $\mathcal{K}$ and $\mathcal{S}$ can be denoted now by $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$.

Of course, a fixed point ξ of a multiple point operator $G : \mathbb{R}^{nm} \to \mathbb{R}^n$ is now a vector $\xi \in \mathbb{R}^n$, which satisfies

$$G(\xi, ..., \xi) = \xi \,, \tag{6}$$

and a common fixed point of a pool is one, which is a fixed point of all its operators in this sense.

Remark 1: Obviously one can embed (2) in the iteration scheme described above: Let $\{\beta_1, ..., \beta_K\}$ be the set from which the α_j's come, except zero. Then define operators

$$G^{(k,\beta_l)} : \mathbb{R}^{n\,2} \to \mathbb{R}^n$$

by

$$G^{(k,\beta_l)}(x^1, x^2) := \beta_l \, x^1 + (1 - \beta_l) \, G^k(x^2) \quad \forall \; l = 1, ..., K, k \in \mathbb{K} \tag{7}$$

and add them to the pool $\mathcal{G}$, reset $\mathbb{K}$ by $\mathbb{K} = \mathbb{K} \cup \mathbb{K} \times \{\beta_1, ..., \beta_K\}$ and redefine $\mathcal{K} = k(j)_{\, j=0,1,...}$ by

$$k(j) = \begin{cases} k(j) & \text{, if } \alpha_j = 0 \\ (k(j), \alpha_j) & \text{, if } \alpha_j \neq 0 \end{cases} .$$

Finally define $\mathcal{S}$ by $s^1(j) = j \;\; \forall \, j \in \mathbb{N}_0$ and $s^2(j) = s(j) \;\; \forall \, j \in \{ l \in \mathbb{N}_0 \, | \, \alpha_l \neq 0 \, \}$.

In order to develop Theorems of convergence, we now have to formulate some conditions for the elements of an asynchronous iteration. First we observe $\mathcal{S}$ and $\mathcal{K}$. In [3] the terms *admissible* and *regulated* were introduced. We now add the word *indexwise-regulated*:

Definition 2.2 Let $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$ be an asynchronous iteration. Then

(i) $\mathcal{S}$ is called *admissible*, if

$$s^l(j) \to \infty, \quad \text{whenever} \quad j \to \infty \qquad \forall \; l = 1, ..., m_{k(j)} \,. \tag{8}$$

(ii) $\mathcal{S}$ is said to be *regulated*, if

$$s := \max_{j,l} \; j - s^l(j) \tag{9}$$

exists.

(iii) $\mathcal{K}$ is *admissible*, if for all $j \in \mathbb{N}_0$

$$\{k(j)\} \cup \{k(j+1)\} \cup ... = \mathbb{K} \tag{10}$$

holds.

(iv) $\mathcal{K}$ is an *indexwise-regulated* sequence, if for all $k \in \mathbb{K}$ there is a number $c_k \in \mathbb{N}_0$, such that for all $j \in \mathbb{N}_0$

$$k \in \{k(j)\} \cup \{k(j+1)\} \cup ... \cup \{k(j + c_k)\} \,. \tag{11}$$

(v) $\mathcal{K}$ is called *regulated*, if there is a number $c \in \mathbb{N}_0$, such that for all $j \in \mathbb{N}_0$

$$\{k(j)\} \cup \{k(j+1)\} \cup ... \cup \{k(j + c)\} = \mathbb{K} \,. \tag{12}$$

It is clear, that $\mathbb{K}$ has to be finite, if $\mathcal{K}$ is regulated. If $\mathbb{K}$ is $\mathbb{N}$, the implicitly defined sequence

$$k(j) = k \Longleftrightarrow j \bmod 2^k = 2^{k-1}$$

is indexwise-regulated, because every $k \in \mathbb{N}$ appears in a period of $c_k = 2^k$ in $\mathcal{K}$, and so $\mathbb{K}$ can be infinite, if $\mathcal{K}$ is indexwise-regulated or admissible.

The next step in our analysis is to formulate that coupling of an iteration process, which avoids the divergence-phenomenons, described in section one. For this, we associate a directed graph with $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$. Every iteration including the initial vectors gets a vertex, so the set of vertices is $\mathcal{V} = \mathbb{N}_0 \cup \{-1, .., -M\}$. An edge $e = (j, k)$ is now an element of the set of edges $\mathcal{E}$ of the directed graph $(\mathcal{V}, \mathcal{E})$, if and only if the j-th iteration vector was used for the computation of the k-th iteration vector. As a demonstration for the concept of coupling we have in mind, we should observe again method (2) in the modification as described in Remark 1. Here all edges of the form $(j, j+1)$ would lie in $\mathcal{E}$. This means, that there is a directed path in $(\mathcal{V}, \mathcal{E})$ from every vertex $j \in \mathcal{V}$ to every vertex $k \in \mathcal{V}$, if $k \geq j+1$. Our modified postulate will be that there are constants $b, n_0 \in \mathbb{N}$ and a sequence $b_{j\ n_0, n_0+1, \ldots}$ in $\mathbb{N}$, such that $j - b_j$ is bounded by b, and such that there is a directed path in $(\mathcal{V}, \mathcal{E})$ from every vertex b_j to every vertex $k \in \mathcal{V}$ with $k \geq j$. Finally we also need some criterion, which guarantees that every operator is involved in the iteration process. In [6] this was, that $\mathcal{K}$ is regulated.

Definition 2.3 Let $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$ be an asynchronous iteration. Then we call the directed graph $(\mathcal{V}, \mathcal{E})$, whose vertices $\mathcal{V}$ are given by $\mathcal{V} = \mathbb{N}_0 \cup \{-1, .., -M\}$, and whose edges $\mathcal{E}$ are given by

$(j, j_0) \in \mathcal{E}$, iff there is a $1 \leq l \leq m_{k(j_0-1)}$, such that $s^l(j_0 - 1) = j\ \forall\, j \geq -M$, $j_0 \geq 1$,

the graph of $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$. $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$ is called *confluent*, if there are numbers $n_0 \in \mathbb{N}$, $b \in \mathbb{N}$ and a sequence $b_{j\ n_0, n_0+1, \ldots}$ in $\mathbb{N}$, such that for all $j \geq n_0$ the following is true:

(i) For every vertex $j_0 \geq j$ there is a directed path from b_j to j_0 in $(\mathcal{V}, \mathcal{E})$,

(ii) $j - b_j \leq b$,

(iii) $\mathcal{S}$ is regulated,

(iv) for every $k \in \mathbb{K}$ there are numbers $c_k \in \mathbb{N}$, so that for all $j \geq n_0$ there is a vertex w_j^k in $\mathcal{V}$, which is a successor of b_j and a predecessor of b_{j+c_k}, and for which is $k(w_j^k - 1) = k$.

Confluent asynchronous iterations are perhaps useful in analysing a variation of method (2), if there is no shared, but only local memory: The numbering of the iterations then could be determined through the real time, when there is an updating of an iteration vector on one of the local memories. Then maybe every processor retrieves its local approximation $x(s^2(j))$ at time $s^2(j)$ from its own local memory, computes $G^{k(j)}(x(s^2(j)))$ and, to guarantee some communication, computes a convex combination with a local approximation $x(s^1(j))$, lying in the local memory of one of its neighbour processors, computed at time $s^1(j)$. We do not want to analyze exactly when such a method is confluent, but at least the flow of data has to reach all processors in a certain fixed time.

3 Criteria of contraction

Now we have to investigate the pool $\mathcal{G}$, from which a common fixed point is searched. In introducing a new criterion of contraction, we need the following generalization of *equicontinuity* for multiple point operators. To describe this, we shall now use the notation $X = (x^1, ..., x^m)$ for an element of $\mathbb{R}^{nm}$, where the x^j, $j = 1, ..., m$ are elements of $\mathbb{R}^n$.

Definition 3.1 Let $\mathbb{K}$ be any set, $\|\cdot\|$ a norm on $\mathbb{R}^n$ and m_k, $k \in \mathbb{K}$ be natural numbers, such that $m_k \leq m \; \forall \, k \in \mathbb{K}$ for a $m \in \mathbb{N}$. A set of functionals $\{\omega^k : D^{m_k} \subset \mathbb{R}^{nm_k} \to \mathbb{R} \,|\, k \in \mathbb{K}\}$ is called *equicontinuous* at $X = (x^1, ..., x^m) \in D^m$, if for all $\varepsilon > 0$ there is a $\delta > 0$, such that

$$\max_{1 \leq l \leq m} \|y^l - x^l\| < \delta$$

implies that

$$|\, \omega^k(y^1, ..., y^{m_k}) - \omega^k(x^1, ..., x^{m_k}) \,| < \varepsilon \qquad \forall \, k \in \mathbb{K} .$$

If $\{\omega^k \,|\, k \in \mathbb{K}\}$ is equicontinuous at all $X \in D^m$, then it is said to be *equicontinuous* on D.

Definition 3.2 Let $\mathcal{G}$ be a pool of operators as in definition 2.1.

(i) If for all $k \in \mathbb{K}$, $X, Y \in D^{m_k}$, an $0 \leq \omega < 1$ and a norm $\|\cdot\|$

$$\|G^k(X) - G^k(Y)\| \leq \omega \, \max_j \|x^j - y^j\| ,$$

then $\mathcal{G}$ is called *contractive* on D.

(ii) If for all $k \in \mathbb{K}$, $X, Y \in D^{m_k}$ and a norm $\|\cdot\|$

$$\|G^k(X) - G^k(Y)\| < \max_j \|x^j - y^j\|$$

$$\text{or} \qquad G^k(X) - G^k(Y) = x^j - y^j \quad \forall \, j \in \{1, ..., m_k\} ,$$

then $\mathcal{G}$ is called *strictly nonexpansive* on D.

(iii) If for all $k \in \mathbb{K}$, $X \in D^{m_k}$ and a norm $\|\cdot\|$, G^k is continuous on D^{m_k} , then $\mathcal{G}$ is *(c)-paracontracting* on D , if for any fixed point $\xi \in \mathbb{R}^n$ of G^k ,

$$\| G^k(X) - \xi \| < \max_j \| x^j - \xi \|$$

or $X = (x, ..., x)$ and x is a fixed point of G^k .

(iv) If for every vector $x \in D$ there is a norm $\|\cdot\|_x$ on $\mathbb{R}^n$, such that there is a constant $c \geq 1$, so that $\| z \|_x \leq c \| z \|_y \ \forall\, x, y \in D$, $z \in \mathbb{R}^n$, and if $\xi \in \mathbb{R}^n$ is a fixed point of exactly the nonempty subset $\{ G^k \,|\, k \in \mathbb{K}_\xi \}$ of $\mathcal{G}$, and if this entails for all $k \in \mathbb{K}_\xi$ and $X \in D^{m_k}$

$$\| G^k(X, h) - \xi \|_\xi \leq \omega_\xi^k(X, h) \ \max_j \ \| x^j - \xi \|_\xi \, ,$$

where the set of functionals

$$\{ \omega_\xi^k(\cdot) : D \to [0, 1] \mid k \in \mathbb{K}_\xi \ \}$$

is equicontinuous on D , and for all $X \in D^{m_k}$, for which

$$\mathbb{K}_{X,\xi} := \ \{ k \in \mathbb{K}_\xi \mid \exists\, j \in \{1, ..., m_k\} \ : \ G^k(x^1, ..., x^{m_k}) \neq x^j \ \}$$

is not empty,

$$\sup_{k \in \mathbb{K}_{X,\xi}} \ \omega_\xi^k(x^1, ..., x^{m_k}) < 1 \tag{13}$$

valids, then $\mathcal{G}$ is called *(n)-paracontracting* on D .

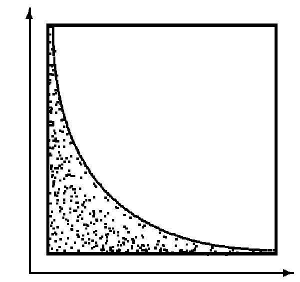
The criteria (i) and (ii) are variations of standard definitions (see for example [9]) extended to pools and multiple data operators. (iii) corresponds to paracontracting pools as defined in [6] (compare (3)). If $\mathcal{G}$ is a single operator $\{ G : \mathbb{R}^{nm} \to \mathbb{R}^n \}$, (i) is that criterion, where a unique fixed point has to exist, this is a consequence of Banach's fixed point Theorem, applied to the operator $\tilde{G} : \mathbb{R}^n \to \mathbb{R}^n$ with $\tilde{G}(x) := G(x, ..., x)$. In the case, that D is convex, we have as a consequence of the Theorem of Schauder, that operators, which fulfill the criteria (ii) and (iii), have fixed points, because they are continuous. It is easy to see, that the set of fixed points in these cases are convex. Let us consider some examples, the first one will show that this is not true anymore in the (n)-paracontracting case.

Example 1: Observe for given $C > 1$ the function $G : [\frac{1}{C}, C]^2 \to [\frac{1}{C}, C]^2$ defined by

$$G(\begin{pmatrix} x \\ y \end{pmatrix}) := \begin{pmatrix} \min\{ \frac{1}{y} \, , \ x \ \} \\ \min\{ \frac{1}{x} \, , \ y \ \} \end{pmatrix} \ . \tag{14}$$

Now $G(\xi) = \xi$, iff $\xi_1 \leq \frac{1}{\xi_2}$, and the set of fixed points are not convex. Define for every $x \in [\frac{1}{C}, C]^2$ a weighted maximum-norm $\| \cdot \|_x$ by

$$\|v\|_x := \max\left\{ \frac{v_1}{x_1}, \frac{v_2}{x_2} \right\} \qquad \forall\, v \in \mathbb{R}^2 , \qquad (15)$$


Set of nonconvex fixed points

and we can show that, if ξ is a fixed point of G,

$$\omega_\xi(x) := \begin{cases} \dfrac{\| G(x) - \xi \|_\xi}{\| x - \xi \|_\xi} & , \text{if } x \neq \xi \\ \\ 1 & , \text{if } x = \xi \end{cases} \qquad (16)$$

is less than one, if $G(x) \neq x$, and for that reason continuous. Hence $\{G\}$ is (norm=)(n)-paracontracting, because C^2 is a common equivalent constant for the defined norms. It suffices to show that

$$\max\left\{ \frac{|\frac{1}{x_2} - \xi_1|}{\xi_1}, \frac{|\frac{1}{x_1} - \xi_2|}{\xi_1} \right\} < \max\left\{ \frac{|x_1 - \xi_1|}{\xi_1}, \frac{|x_2 - \xi_2|}{\xi_2} \right\}, \qquad (17)$$

because it is clear, that $\omega_\xi(x)$ then is continuous. Since $G(x) \neq x$, we have $x_1 > \frac{1}{x_2}$, thus

$$|\tfrac{1}{x_2} - \xi_1| = \begin{cases} \frac{1}{x_2} - \xi_1 & < & x_1 - \xi_1 & , \text{if } \frac{1}{x_2} \geq \xi_1 \\ \\ \xi_1 - \frac{1}{x_2} & \leq & \frac{1}{\xi_2\, x_2}(x_2 - \xi_2) & , \text{if } \frac{1}{x_2} < \xi_1 \end{cases} \qquad (18)$$

and likewise

$$|\tfrac{1}{x_1} - \xi_2| = \begin{cases} \frac{1}{x_1} - \xi_2 & < & x_2 - \xi_2 & , \text{if } \frac{1}{x_1} \geq \xi_2 \\ \\ \xi_2 - \frac{1}{x_1} & \leq & \frac{1}{\xi_1\, x_1}(x_1 - \xi_1) & , \text{if } \frac{1}{x_1} < \xi_2 \end{cases} . \qquad (19)$$

In the case $\frac{1}{x_2} < \xi_1$ and $\frac{1}{x_1} \geq \xi_2$, we have $\frac{|\frac{1}{x_1} - \xi_2|}{\xi_2} < \frac{|x_2 - \xi_2|}{\xi_2}$ and

$$\frac{|\frac{1}{x_2} - \xi_1|}{\xi_1} \leq \frac{|x_2 - \xi_2|}{\xi_2\, x_2\, \xi_1} < \frac{|x_2 - \xi_2|}{\xi_2},$$

since $\xi_1\, x_2 > 1$. The case $\frac{1}{x_1} < \xi_2$ and $\frac{1}{x_2} \geq \xi_1$ is appropriate. The case $\frac{1}{x_1} \geq \xi_2$, $\frac{1}{x_2} \geq \xi_1$ follows directly from (18) and (19) and in the case $\frac{1}{x_1} < \xi_2$, $\frac{1}{x_2} < \xi_1$ we would have

$$\frac{1}{x_2} < \xi_1 \leq \frac{1}{\xi_2} < x_1 ,$$

and therefore x is a fixed point.

Example 2: One example of a (continuous=)(c)-paracontracting pool, which was given in [6], is as follows: Let $\{q_k : \mathbb{R}^n \to \mathbb{R} \,|\, k \in \mathbb{K}\}$ be a finite set of differentiable convex functionals, then the operators

$$G^k(x) := \begin{cases} x - \gamma_k \dfrac{q^k(x) - f^k}{\|\, q_k'(x) \,\|_2^2}\, q_k'(x) & \text{,if } f^k > 0 \\ \\ x & \text{,if } f^k \leq 0 \end{cases} \qquad \forall\, k \in \mathbb{K}\,, \tag{20}$$

where $\gamma_k \in (0,2)\ \forall\, k \in \mathbb{K}$, form a (c)-paracontracting pool with respect to the $\|\cdot\|_2$-norm. The fixed points η of the single operators here are that one with $q^k(\eta) \leq f^k$. In a former paper ([5]) the authors used affin-linear q_k's to develop an asynchronous algorithm as in (2) for *tomographic reconstruction from incomplete data.* In general pools of such operators can be useful to solve constrained optimization problems asynchronously, if the constrains are convex and the functional, which is to optimate, is also convex and its optimum is known.

Example 3: Consider, that the function $g : [-1+\varepsilon, 1-\varepsilon] \to [-1+\varepsilon, 1-\varepsilon]$ with $g(x) := x^2$ is for every $\frac{1}{2} > \varepsilon > 0$ (c)-paracontracting, since $|g(x)| \leq |x|\,|x|$, but it is not strictly nonexpansive, because we have for $1 > x > y > \frac{1}{2}$, that $|g(x) - g(y)| = |x+y||x-y| > |x-y|$. On the other hand it is easy to see, that every strictly nonexpansive pool is (c)-paracontracting.

Example 4: Defining functions like in (16)(with $\|\cdot\|_\xi = \|\cdot\|\ \forall\, \xi$), it is also easy to see, that every finite (c)-paracontracting pool is (n)-paracontracting. This statement does not hold in the infinite case, as the example $\{\, g^n(x) := x^n \,|\, n \in \mathbb{N}\backslash\{1\}\,\}$ with $D = [-1+\varepsilon, 1-\varepsilon]$, $\frac{1}{2} > \varepsilon > 0$ shows. Here the supremum-condition in (13) is disturbed. Indeed, in [6] only finite (c)-paracontracting pools were investigated, except asymptotical pools (see below).

Example 5: A further difference between (c)- and (n)-paracontracting pools is that the operators of a (n)-paracontracting pool do not have to be continuous. If in an asychronous method, which corresponds to the pool described in Example 2, we wish that the *descendence*, this means the difference between the values $q_k(y) - f^k$ of a new iteration y and $q_k(x) - f^k$ of the old iteration x, fulfills a given criterion, then we could apply the operator G^k onto the iteration x as often, until this criterion is reached. The iterations computed in this manner can be considered as an evaluation of another, in general discontinuous, but (n)-paracontracting operator: The iteration operator for the convex function $q(x) = x^2$, with $\gamma = 1$ would be $g(x) = \frac{1}{2}\,x$. Now define another iteration function $\tilde{g}$ by the following algorithm:

$y := x$
while $y > \min\{x^2, \frac{1}{2}\,x\}$ do $y := g(y)$
$\tilde{g}(x) := y$.

It is easy to see, that $\tilde{g}$ now is (n)-paracontracting, with $\omega_0(x) = \frac{1}{2}\ \forall\, x \in \mathbb{R}$, but $\tilde{g}$

is discontinuous at $\frac{1}{2}$.

Example 6: For linear pools $\mathcal{G} = \{T^k \in L(\mathbb{R}^{nm_k}, \mathbb{R}^n) \,|\, k \in \mathbb{K}\}$ the criteria strictly nonexpansive and (c)-paracontracting are equivalent. If $D = \mathbb{R}^n$, it is obvious that such a pool can only be strictly nonexpansive, if

$$\sigma\left(\sum_{i=1}^{m_k} T_i^k\right) \subset (-1,1] \qquad \forall\, k \in \mathbb{K}\,, \tag{21}$$

where $\sigma(T)$ is the spectra of a matrix T and T_i^k is the i-th $n \times n$ quadratic part of T^k. If all operators are of spectral radius less than unit, a finite pool is contractive, and the null-vector is the only fixed point of all the operators. If $m = 1$, all pools of positive definite, symmetric operators, which satisfy (21), are strictly nonexpansive with respect to the $\|\cdot\|_2-$norm. Just so, a pool of irreducible nonnegative square matrices, which have a common Perron vector v, is strictly nonexpansive with respect to the weighted maximum norm

$$\| x \|_v := \max_i \left| \frac{x_i}{v_i} \right|$$

(compare [2]). As an example for a strictly nonexpansive pool of linear multiple point operators, we remark that, if T is a nonnegative substochastic $K \times m-$matrix, and $t_{k,m_k(j)}$ is the j-th of m_k nonzero entries in its k-th row, the operators $\{T_k : \mathbb{R}_{+0}^{m_k} \to \mathbb{R}_{+0} \,|\, k = 1, ..., K\}$ defined by

$$T_k(x^1, ..., x^{m_k}) := \sum_{j=1}^{m_k} t_{k,m_k(j)}\, x^j \tag{22}$$

form a strictly nonexpansive pool of operators. If T is stochastic, all nonnegative numbers are common fixed points, if T is really substochastic, then zero is the one and only common fixed point.

Assume that we want to solve asynchronously a convex optimization problem with operators as described in Example 2, but the optimum f^0 of its cost functional q^0 is not known. A way out is perhaps, to adapt successively and approximately estimates for this optimum. This is one example, for which the following definition, which we will use later in a different context, could be helpful.

Definition 3.3 An infinite pool of operators $\mathcal{F} = \{F^j \mid j \in \mathbb{N}_0\}$

$$\text{with } F^j : D^{\tilde{m}_j} \subset \mathbb{R}^{n\tilde{m}_j} \to D \text{ and } \tilde{m}_j \in \{1, ..., m\} \quad \forall\, j \in \mathbb{N}_0$$

is said to be *approximate* to the pool

$$\mathcal{G} = \{G^k \mid k \in \mathbb{K}\} \text{ with } G^k : D^{m_k} \subset \mathbb{R}^{nm_k} \to D \quad \forall\, k \in \mathbb{K}\,,$$

if for all $j \in \mathbb{N}_0$ there are $k_j \in \mathbb{K}$ and a norm $\|\cdot\|$, such that $\tilde{m}_j = m_{k_j} \ \forall\, j \in \mathbb{N}_0$ and

$$\lim_{j\to\infty} \|F^j(X) - G^{k_j}(X)\| = 0 \tag{23}$$

uniformly for all $Y = (X,Z) \in D^{m_k} \times D^{m-m_k}$. If even

$$\sum_{j=0}^{\infty} \|F^j(X) - G^{k_j}(X)\| < \infty \tag{24}$$

uniformly for all $(X,Z) \in D^{m_k} \times D^{m-m_k}$, then $\mathcal{F}$ is called *asymptotical* to $\mathcal{G}$.

4 Convergence theorems

In this section we give the first of our main results. The convergence Theorem in [6] was as follows:

Theorem 4.1 *Let $\|\cdot\|_{sc}$ be a strictly convex vector norm on $\mathbb{R}^n$, $M = 0$, e.g. $\mathcal{X}_{\mathcal{O}} = \{x(0)\} \subset \mathbb{R}^n$, and $\mathcal{G}$ be a finite (c)-paracontracting pool on $\mathbb{R}^n$ with $m = 1$ and with respect to $\|\cdot\|_{sc}$. Finally let $\mathcal{K}$ be regulated. Then the asynchronous iteration $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$ converges if and only if a common fixed point of $\mathcal{G}$ exist. Moreover, if $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$ converges, it converges to a common fixed point of $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$.*

If we return to the pool defined by a *one point-operator*-pool in remark 1, and if we assume that the latter one is (c)-paracontracting with respect to a strict convex vector norm, then we have for

$$\omega_{\xi}^{(k,\beta_l)}(x^1,x^2) := \frac{\|\,\beta_l\, x^1 \;+\; (1-\beta_l)G^k(x^2) - \xi\,\|_{sc}}{\max\limits_{i=1,2} \|\, x^i - \xi\,\|_{sc}}$$

by the strict convexity of $\|\cdot\|_{sc}$ and the triangle inequality, that

$$\omega_{\xi}^{(k,\beta_l)}(x^1,x^2) \;<\; 1\ , \text{ if } \ G^k(x^2) \neq x^2 \text{ or if } \ x^1 \neq x^2\ .$$

Hence, the new pool is also (c)- and (n)-paracontracting, as defined in definition 3.2 (iii) and (iv).

The Theorem motivated by the above considerations is now as follows:

Theorem 4.2 : *Let $\mathcal{G}$ be a (n)-paracontracting pool on $D \subset \mathbb{R}^n$, and assume that $\mathcal{G}$ has a common fixed point $\xi \in D$, then a confluent asynchronous iteration $(\mathcal{G}, \mathcal{X}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$ converges to a common fixed point of $\mathcal{G}$.*

Proof: The proof is divided into six parts:

I.: Notations and statements.

(a) All terms are as defined above. Furthermore we need the set

$$D_0 \;:=\; \{x \in D \;\mid\; \|x-\xi\|_\xi \leq C\,\}\;, \tag{25}$$

where C was given by

$$C := \max_{-M\leq j\leq 0} \;\|x(j)-\xi\|_\xi\;.$$

Since $\mathcal{G}$ is (n)-paracontracting and ξ is a common fixed point of $\mathcal{G}$, all iterations $x(j)_{\,j=0,1,...}$ stay in the compact set D_0, thus

$$\lambda := \limsup_{j\to\infty} \;\|x(j)-\xi\|_\xi \tag{26}$$

exists.

(b) On D_0 there is for all $\delta > 0$ some $\varepsilon > 0$, which is independent of k, such that

$$\omega^k_\xi(x^1,...,x^{m_k}) \;>\; 1-\varepsilon$$

implies, that there is a fixed point η of G^k, which satisfies

$$\|x^l \;-\; \eta\|_\xi < \frac{1}{c^2}\,\delta \qquad \forall\;\; l = 1,...,m_k\;.$$

Prf.: W.l.o.g. we can assume, that $m_k = m\;\forall\;k\in\mathbb{K}$, otherwise we could prove this statement for all of the subsets $\mathbb{K}_i := \{k\in\mathbb{K}\mid m_k = i\,\}$, $i = 1,...,m$. Assume now there would be sequences $X_{j\;j=0,1,...}$ in D_0^m, $k_{j\;j=0,1,...}$ in $\mathbb{K}_0\subset\mathbb{K}$ for which

$$\lim_{j\to\infty}\;\omega^{k_j}_\xi(X_j) = 1$$

and since D_0 is compact

$$\lim_{j\to\infty} X_j = X = (x^1,...,x^m)\;,$$

but for all $k\in\mathbb{K}_0$ there would be some $l(k)\in\{1,...,m\}$, such that

$$G^k(X) \neq x^{l(k)}.$$

This would contradict

$$\sup_{k\in\mathbb{K}_0}\omega^k_\xi(X) < 1$$

and

$$\lim_{j\to\infty}\left(\omega^{k_j}_\xi(X_j) - \omega^{k_j}_\xi(X)\right) \;=\; 0\;. \qquad \text{comp. def. 3.2(iv)}$$

(c) Vice versa: Let $\eta \in D_0$ be no fixed point of exactly all G^k with $k \in \mathbb{K}_0 \subset \mathbb{K}$. Then for all $\tilde{\varepsilon} \in (0,1)$ there is a $\tilde{\delta} > 0$, such that for all $k \in \mathbb{K}_0$

$$\|x^l - \eta\|_\xi < \tilde{\delta} \qquad \text{for some } \ l \in \{1, ..., m_k\}$$

implies, that

$$\omega_\xi^k(x^1, ..., x^{m_k}) \ < \ 1 - \tilde{\varepsilon} \ .$$

Prf.: Again assume $m_k = m \ \forall \ k \in \mathbb{K}$. Say there would be sequences $X_j = (x_j^1, ..., x_j^m)_{\ j=0,1,...}$ in D_0^m, $k_{j \ j=0,1,...}$ in $\mathbb{K}_0$ and $l_{j \ j=0,1,...}$ in $\{1, ..., m\}$, such that

$$\lim_{j \to \infty} \ \omega_\xi^{k_j}(X_j) = 1$$

and

$$\lim_{j \to \infty} \ x_j^{l_j} \ = \ \eta \ .$$

Since D_0^m is compact, $X_{j \ j=0,1,...}$ would have a point of accumulation $X = (x^1, ..., x^m) \in D_0^m$, with $x^l = \eta$ for some $l \in \{1, ..., m\}$. Due to this point of accumulation, we would have by (13), that

$$\sup_{k \in \mathbb{K}} \omega_\xi^k(X) \ \leq \ \omega \ \ ,$$

for some $\omega < 1$. Hence, there would be a sequence $j_{\kappa \ \kappa=0,1,...}$ in $\mathbb{N}_0$ for which

$$\begin{aligned} 1 \ &= \ \lim_{\kappa \to \infty} \ \omega_\xi^{k_{j_\kappa}}(X_{j_\kappa}) \\ &\leq \ \lim_{\kappa \to \infty} \ |\omega_\xi^{k_{j_\kappa}}(X_{j_\kappa}) - \omega_\xi^{k_{j_\kappa}}(X)| \ + \ \omega \\ &= \ \omega \ . \end{aligned}$$

II.: Let $\tilde{\varepsilon}, \tilde{\delta}, \alpha \in \mathbb{R}_+$, $\alpha < \lambda$. A sequence of positive numbers $\alpha_{L \ L=0,1,...}$ and a sequence of pairs $(\delta_L, \ \varepsilon_L)_{\ L=0,1,...}$, which all satisfy some (δ, ε)-criterion as in I.(b), can be defined as follows:

$$\begin{aligned} \alpha_0 \ &:= \ \frac{\lambda - \alpha}{2} \\ \delta_0 \ &:= \ \min\{\frac{\lambda - \alpha_0}{2.1}, \frac{\tilde{\delta}}{4}\} \\ \varepsilon_0 \ & \quad \text{belonging to } \delta_0 \text{ as described in I.(b) and satisfying } 0 < \varepsilon_0 < \tilde{\varepsilon} \end{aligned}$$

$$\begin{aligned} \alpha_{L+1} \ &:= \ \max\{\frac{1 - \varepsilon_L}{1 - \varepsilon_L \cdot \ 0.9} \ \lambda \ , \ 2 \cdot \delta_L + \alpha_L\} \qquad (27) \\ \delta_{L+1} \ &:= \ \min\{\frac{\lambda - \alpha_{L+1}}{2.1}, \frac{\delta_L}{4}\} \\ \varepsilon_{L+1} \ & \quad \text{belonging to } \delta_{L+1} \text{ and satisfying } \ 0 < \varepsilon_{L+1} < \varepsilon_L < 1 \qquad \forall \ L \geq 0 \ . \end{aligned}$$

Then we have

$$0 < \alpha_L < \alpha_{L+1} \quad \text{and} \quad \alpha_L < \lambda \qquad \forall\, L \geq 0\,.$$

Thus (27) is well-defined. Furthermore it is

$$\sum_{r=0}^{\infty} 2\,\delta_r \leq \tilde{\delta}.$$

III.: By induction we can show, that for arbitrary $\alpha, \tilde{\varepsilon}, \tilde{\delta}$ the sequences defined in II. satisfy the following assertion: Assume $J \in \mathbb{N}_0$ is a number, such there is a $L \in \mathbb{N}_0$ for which

$$\|x(s^l(j)) - \xi\|_\xi < \frac{\lambda}{1 - \varepsilon_L \cdot 0.9} \qquad \forall\, j \geq J,\ l = 1, ..., m_{k(j)}\,, \tag{28}$$

then

$$\|x(J) - \xi\|_\xi \;<\; \alpha_0$$

implies, that for all j_0, for which there is a directed path in $(\mathcal{V}, \mathcal{E})$ of length less than $L+1$ from J to j_0,

$$\|x(j_0) - \xi\|_\xi \;<\; \alpha_L \tag{29}$$

is validated.

Prf.: This is no question for $L = 0$. Now assume the assertion is proved for all $0 \leq L_0 < L$. Then we have for all j_0, for which there is a path from J to j_0 of length L in $(\mathcal{V}, \mathcal{E})$, that there is a $l_0 \in \{1, ..., m_{k(j_0-1)}\}$, such that $\|x(s^{l_0}(j_0 - 1)) - \xi\|_\xi \;<\; \alpha_L$. Since

$$\frac{\lambda}{1 - \varepsilon_L \cdot 0.9} < \frac{\lambda}{1 - \varepsilon_{L-1} \cdot 0.9}\,,$$

this is a consequence of the assumption we made. Furthermore, due to (28), we have

$$\begin{aligned}\|x(j_0) - \xi\|_\xi \;&=\; \|G^{k(j_0-1)}(X_{j_0-1}) - \xi\|_\xi \\ &\leq \omega_\xi^{k(j_0-1)}(X_{j_0-1}) \cdot \frac{\lambda}{1 - \varepsilon_L \cdot 0.9}\,.\end{aligned}$$

Using I.(d), the latter term is less than

$$\frac{1 - \varepsilon_L}{1 - \varepsilon_L \cdot 0.9}\,\lambda \;\leq\; \alpha_{L+1}$$

or there is a fixed point η of $G^{k(j_0-1)}$, which satisfies

$$\|x^l(s^l(j_0 - 1)) - \eta\|_\xi < \frac{1}{c^2}\,\delta_L \qquad \forall\; l = 1, ..., m_{k(j_0-1)}\,.$$

But in the latter case, we have

$$\begin{aligned}\|\; x(j_0) - \eta \;\|_\xi \quad &\leq\quad c\,\|G^{k(j_0-1)}(X_{j_0-1}) - \eta\|_\eta \\ &\leq\quad c^2\,\omega_\eta^{k(j_0-1)}(X_{j_0-1})\;\max_l\;\|x^l(\;s^l(j_0 - 1)\;) - \eta\|_\xi \\ &<\quad \delta_L\,.\end{aligned} \tag{30}$$

Thus

$$\begin{aligned}\|x(j_0)-\xi\|_\xi &\leq \|x(j_0)-\eta\|_\xi + \|\eta - x(s^{l_0}(j_0-1))\|_\xi + \|x(s^{l_0}(j_0-1))-\xi\|_\xi \\ &< 2\,\delta_L + \alpha_L \qquad \text{assumption} \\ &\leq \alpha_{L+1}\,.\end{aligned}$$

IV.: It has to be

$$\lim_{j\to\infty} \|x(b_j)-\xi\|_\xi = \lambda\,. \tag{31}$$

Prf.: Assume there is a subsequence $x(b_{j_\kappa})_{\kappa=0,1,\dots}$ of $x(b_j)_{j=n_0,n_0+1,\dots}$ for which

$$\lim_{\kappa\to\infty} \|x(b_{j_\kappa})-\xi\|_\xi = \alpha \tag{32}$$

$$\text{and} \qquad \alpha < \lambda\,.$$

Now define sequences as in II. by this α. The assumption (32) then implies the existence of a $\kappa_0 \in \mathbb{N}$, for which at the same time

$$\|x(b_{j_{\kappa_0}})-\xi\|_\xi < \alpha_0$$

and

$$\|x(s^l(j))-\xi\|_\xi < \frac{\lambda}{1-\varepsilon_{b+s}\cdot 0.9} \qquad \forall\, j \geq b_{j_{\kappa_0}},\ l=1,\dots,m_{k(j)} \tag{33}$$

holds. Since in $(\mathcal{V},\mathcal{E})$ there is for all $j \geq j_{\kappa_0}$ a directed path of length less or equal than $b+j-j_{\kappa_0}$ from $b_{j_{\kappa_0}}$ to j, we have by III.

$$\|x(j)-\xi\|_\xi < \alpha_{b+\hat{s}} \qquad \forall\, j \in \{\, j_{\kappa_0},\dots,j_{\kappa_0}+s\,\}$$

But this means $\|x(j)-\xi\|_\xi < \alpha_{b+s}$ for all $j > j_{\kappa_0}+s$, since s is an upper bound for the *delay*. Hence, λ cannot be the limes superior defined in I.(a), and (31) valids.

V.: W.l.o.g. we can assume: There is at least one $k \in \mathbb{K}$, such that there is a fixed point η of G^k and such that

$$\lim_{\kappa\to\infty} x(b^0_{j_\kappa}) = \eta \tag{34}$$

for an infinite subsequence $b^0_{j_\kappa}\ {}_{\kappa=0,1,\dots}$ of $b_j\ {}_{j=0,1,\dots}$, which satisfies $k(b^0_{j_\kappa}-1) = k\ \forall\, \kappa \geq 0$.

Prf.: For any fixed $k \in \mathbb{K}$ the sequence $w^k_j\ {}_{j=n_0+c_k,n_0+c_k+1,\dots}$ satisfies the criteria claimed in definition 2.3 for the sequence $b_j\ {}_{j=n_0,n_0+1,\dots}$ (comp. (i),(ii),(iv)). Thus, assume that there is a k, such that

$$k(b_{j_\kappa}-1) = k \qquad \forall\, \kappa \geq 0$$

for an infinite subsequence b_{j_κ} $_{\kappa=0,1,...}$ of b_j $_{j=0,1,...}$. Because of IV., this sequence holds

$$\lambda = \lim_{\kappa\to\infty} \|x(b_{j_\kappa}) - \xi\|_\xi \le \limsup_{\kappa\to\infty} \omega_\xi^k(X_{b_{j_\kappa}-1}) \max_l \|x^l_{b_{j_\kappa}-1} - \xi\|_\xi \le \lambda \,.$$

Hence, there is a subsequence $X_{b^0_{j_\kappa}-1}$ $_{\kappa=0,1,...}$ of $X_{b_{j_\kappa}-1}$ $_{\kappa=0,1,...}$, which converges to the m_k-time product of a fixed point η of G^k, because if $\lambda \neq 0$, there has to be a subsequence of

$$\omega_\xi^{k(b_{j_\kappa}-1)}(X_{b_{j_\kappa}-1}) \quad {}_{\kappa=0,1,...},$$

which converges to 1, and thus

$$\limsup_{\kappa\to\infty} \|x(b^0_{j_\kappa}) - \eta\|_\eta \le \limsup_{\kappa\to\infty} \omega_\eta^k(\, X_{b^0_{j_\kappa}-1}) \max_l \|\, x^l_{b_{j_\kappa}-1} - \eta\|_\eta \;=\; 0$$

VI.: Conclusion.

If we can show, that $\eta \in D_0$, observed in V., is a common fixed point of $\mathcal{G}$, the Theorem is proved, because after repeating the whole argumentation, substituting ξ by η, the assertion is a consequence of IV. Let now $\mathbb{K}_0$ be the set of all indices of that operators of which η is not a fixed point, and let $k_0 \in \mathbb{K}_0$. Using I.(c), we know, that we can choose an $\tilde\varepsilon > 0$, such that there is a $\tilde\delta > 0$, which satisfies for all $X \in D_0^m$, $k \in \mathbb{K}_0$, that

$$\|x^l - \eta\|_\xi < \tilde\delta$$

for an $l \in \{1, ..., m_k\}$ implies, that

$$\omega_\xi^k(x^1, ..., x^{m_k}) \;<\; 1 - \tilde\varepsilon \,.$$

Now, for an an arbitrary $0 < \alpha < \lambda$, let α_L $_{L=0,1,...}$, δ_L $_{L=0,1,...}$ and ε_L $_{L=0,1,...}$ be the sequences defined in II. by $\tilde\delta$, $\tilde\varepsilon$, α, then we have $\alpha_{c_{k_0}+b} < \lambda \;\forall\, k \in \mathbb{K}$, because of II., and there is a j_0, which satisfies at the same time

$$\begin{aligned} \|x(b_{j_0}) - \eta\|_\xi &< 2\,\delta_0 && \text{since V.}\\ \|x(b_j) - \xi\|_\xi &> \alpha_{c_{k_0}+b} \quad \forall\, j \ge j_0 && \text{since IV.}\\ \text{and}\qquad \|x(j) - \xi\|_\xi &< \frac{\lambda}{1 - \varepsilon_{c_{k_0}+b}\, 0.9} \quad \forall\; j \ge b_{j_0} - s\,. && (35)\end{aligned}$$

Further, there is a directed path $b_{j_0} = w^0, ..., w^{(j_0,k_0)} = b_{j_0+c_{k_0}}$ in $(\mathcal{V},\mathcal{E})$, which leads across a vertex $w^{k_0}_{j_0}$, for which is $k(w^{k_0}_{j_0} - 1) = k_0$ and which length is less or equal to $c_{k_0} + b$. By induction on the vertices w^L $L = 0, ..., (j_0, k_0)$ we will show now, that for every $k(w^L - 1)$, $L = 0, 1, ..., (j_0, k_0)$ η is a fixed point of $G^{k(w^L-1)}$ and we will conclude in that manner, that $\mathbb{K}_0$ is empty. In the case $L = 0$ the assertion can be assumed, because of V.. Now let the assertion and also

$$\|x(w^q) - \eta\|_\xi < \sum_{r=0}^{q} 2\,\delta_r$$

hold for all $0 \le q < L \le (j_0, k_0)$. Assume now $k(w^L - 1) \in \mathbb{K}_0$, then by

$$\|x(w^{L-1}) - \eta\|_\xi < \tilde{\delta} \,,$$

(compare I.(c)) we would have

$$\|x(w^L) - \xi\|_\xi = \|G^{k(w^L-1)}(X_{w^L-1}) - \xi\|_\xi < \frac{(1-\tilde{\varepsilon})\,\lambda}{1 - \varepsilon_{c_{k_0}+b}\, 0.9}$$

$$< \frac{(1-\varepsilon_0)\,\lambda}{1-\varepsilon_0\, 0.9} \le \alpha_1 \,,$$

but, by III., this would mean, that

$$\|x(b_{j_0+c_{k_0}}) - \xi\|_\xi < \alpha_{c_{k_0}+b} \,.$$

It has even to be

$$\|x(w^L) - \xi\|_\xi \;\ge\; \alpha_L \qquad \forall\, L \ge 0 \,,$$

since otherwise by III. $\|x(b_{j_0+c_{k_0}}) - \xi\|_\xi < \alpha_{c_{k_0}+b}$. Thus

$$\begin{aligned} \|x(w^L) - \xi\|_\xi &= \|G^{k(w^L-1)}(X_{w^L-1}) - \xi\|_\xi \\ &\le \omega_\xi^{k(w^L-1)}(X_{w^L-1})\,\frac{\lambda}{1-\varepsilon_{c_{k_0}+b}\cdot 0.9} \qquad \text{comp. (35)} \\ &< \omega_\xi^{k(w^L-1)}(X_{w^L-1})\,\frac{\lambda}{1-\varepsilon_L\cdot 0.9}\,, \end{aligned}$$

implies, by (27), that

$$\omega_\xi^{k(w^L-1)}(X_{w^L-1}) \;>\; 1 - \varepsilon_L.$$

Hence, I.(d) guarantees the existence of a fixed point η^0 of $G^{k(w^L-1)}$ with

$$\|x^l_{w^L-1} - \eta^0\|_\xi \;<\; \delta_L\,\frac{1}{c^2} \qquad \forall\, l = 1, ..., m_{k(w^L-1)}$$

This means (compare (30))

$$\|x(w^L) - \eta^0\|_\xi \le \delta_L \,,$$

and therefore

$$\begin{aligned} \|x(w^L) - \eta\|_\xi &\le \|x(w^L) - \eta^0\|_\xi \;+\; \|x(w^{L-1}) - \eta^0\|_\xi \;+\; \|x(w^{L-1}) - \eta\|_\xi \\ &< \delta_L \;+\; \delta_L\,\frac{1}{c^2} \;+\; \sum_{r=0}^{L-1} 2\,\delta_r \\ &\le \sum_{r=0}^{L} 2\,\delta_r \,. \end{aligned}$$

This completes the proof. □

Remark: In the case of (n)-paracontracting pools it is not possible to prove the sufficient condition of Theorem 4.1, e.g. that an asynchronous iteration converges only, if a common fixed point exists, since the operators of such a pool are generally not continuous. But it is easy to see, that the continuity of the operators and the admissibility of $\mathcal{K}$ are in any case the only assumptions, that we need to prove such an assertion.

Now we turn to approximate pools. We will begin with an appropriate result for pools, which are asymptotical to strictly nonexpansive pools and in the infinite case also (n)-paracontracting.

Theorem 4.3 *Let $\mathcal{G} = \{G^k \mid k \in \mathbb{K}\}$ be an (n)-paracontracting and at the same time strictly nonexpansive pool on D with respect to $\|\cdot\|$, and let there be a common fixed point $\xi \in D$ of $\mathcal{G}$. Further let $\mathcal{F}$ be asymptotical to $\mathcal{G}$, therefore let*

$$\| F^j(X) - G^{k_j}(X) \| \leq \varepsilon_j \quad \forall\, X \in D^{m_{k_j}}\,,\ j \in \mathbb{N}_0$$

for a sequence $\mathcal{K} = k_j\ {}_{j=0,1,...}$ in $\mathbb{K}$ and a null sequence $\varepsilon_j\ {}_{j=0,1,...}$ in $\mathbb{R}_{+0}$, which satisfies

$$\varepsilon_\Sigma := \sum_{j=0}^{\infty} \varepsilon_j \ \in\ \mathbb{R}\,.$$

Furthermore let

$$D_0 := \{x \in D \mid y \in D\,,\ \text{if}\ \| x - \xi \| \leq \| y - \xi \| + \varepsilon_\Sigma \}\,, \tag{36}$$

$\mathcal{X}_\mathcal{O} \subset D_0$ and denote the sequence $0, 1, \ldots$ by $\mathcal{N}_\mathcal{O}$. Then, if the asynchronous iteration $(\mathcal{G}, \mathcal{X}_\mathcal{O}, \mathcal{K}, \mathcal{S})$ is confluent, the asynchronous iteration $(\mathcal{F}, \mathcal{X}_\mathcal{O}, \mathcal{N}_\mathcal{O}, \mathcal{S})$ is well-defined and converges to a common fixed point of $\mathcal{G}$.

Proof: We will show by induction, that, if $y^0(j)\ {}_{j=0,1,...}$ is the asynchronous iteration $(\mathcal{G}, \mathcal{X}_\mathcal{O}, \mathcal{K}, \mathcal{S})$, then

$$\|x(j) - y^0(j)\| \leq \sum_{q=0}^{j-1} \varepsilon_q \quad \forall\ j \geq 0\,, \tag{37}$$

and thus

$$\|x(j) - \xi\| \ \leq\ \max_{0 \leq l \leq M} \| x(-l) - \xi \|_\xi + \varepsilon_\Sigma \qquad \forall\, j \in \mathbb{N}_0\,,$$

which means that $(\mathcal{F}, \mathcal{X}_\mathcal{O}, \mathcal{N}_\mathcal{O}, \mathcal{S})$ is well-defined. (37) holds for $j = 0$ and if (37) is shown for all $0 \leq j_0 < j$, then

$$\|x(j+1) - y^0(j+1)\| \ \leq\ \varepsilon_j \ +\ \max_{1 \leq l \leq m_{k(j)}} \|x(s^l(j)) - y(s^l(j))\| \leq \sum_{q=0}^{j} \varepsilon_q\,.$$

Now let w.l.o.g. $M = s$ and define for all $j_0 \in \mathbb{N}_0$ sequences $\mathcal{K}^{j_0} = k_j^{j_0}\ {}_{j=0,1,\ldots}$ in $\mathbb{K}$, $\mathcal{S}^{j_0} = \{s^1(j)^{j_0}, \ldots, s^{m_{k_{j-j_0}}}(j)^{j_0}\}_{j=0,1,\ldots}$ of $m_{k_{j-j_0}}-$tuple in $\mathbb{N} \cup \{0, \ldots, -M\}$ and subsets $\mathcal{X_O}^{j_0} = \{x^{j_0}(-s), \ldots, x^{j_0}(0)\}$ in $\mathbb{R}^n$ by

$$\begin{aligned} k^{j_0}(j) &:= k_{j-j_0} \ , \\ s^l(j)^{j_0} &:= s^l(j-j_0) \qquad \forall\ l = 1, \ldots, m_{k(j-j_0)} \ , \\ x^{j_0}(-l) &:= x(j_0 - l) \qquad \forall\ l = 0, \ldots, s \qquad\qquad \forall\ j \geq 0 \ . \end{aligned} \tag{38}$$

Because of Theorem 4.2, the confluent asynchronous iterations $(\mathcal{G}, \mathcal{X_O}^{j_0}, \mathcal{K}^{j_0}, \mathcal{S}^{j_0})$, which are now denoted by $y^{j_0}(j)\ {}_{j=0,1,\ldots}$, converge for all $j_0 \in \mathbb{N}_0$ to common fixed points $\eta^j\ {}_{j=0,1,\ldots}$ of $\mathcal{G}$. Moreover, all of the common fixed points lie in the bounded set D_0 (compare the induction at the beginning of the proof). Therefore $\eta^j\ {}_{j=0,1,\ldots}$ has an accumulation point in D_0 , say η . Now assume the sequence $\eta^{\kappa_j}\ {}_{j=0,1,\ldots}$ converges to η , then for all $\varepsilon > 0$ there are some $j_0, j_1 \in \mathbb{N}$, such that

$$\begin{aligned} \| \eta^{\kappa_{j_0}} - \eta \| &\leq \frac{\varepsilon}{3} \ , \\ \sum_{q=\kappa_{j_0}}^{\infty} \varepsilon_q &\leq \frac{\varepsilon}{3} \ , \\ \| y^{\kappa_{j_0}}(j) - \eta^{\kappa_{j_0}} \| &\leq \frac{\varepsilon}{3} \qquad \forall\ j \geq j_1 \ . \end{aligned} \tag{39}$$

Therefore, for all $j \geq j_1$, we have by (37)

$$\| x(j+\kappa_{j_0}) - \eta \| \ \leq \ \| x(j+\kappa_{j_0}) - y^{\kappa_{j_0}}(j) \| + \| y^{\kappa_{j_0}}(j) - \eta^{\kappa_{j_0}} \| + \| \eta^{\kappa_{j_0}} - \eta \| \leq \varepsilon \ .$$

Since for all $k \in \mathbb{K}$ in the (n)-paracontracting formulation of criterion of contraction

$$1 = \lim_{j \to \infty} \omega_\xi^k(\eta^{\kappa_j}, \ldots, \eta^{\kappa_j}) = \omega_\xi^k(\eta, \ldots, \eta) \ ,$$

if $\eta \neq \xi$, η is a common fixed point of $\mathcal{G}$, since ($\mathcal{K}^{j_0}$ is at least indexwise-regulated for all $j_0 \in \mathbb{N}_0$. $\Box$

A simple example shows that this Theorem is not true anymore, if $\mathcal{F}$ is only approximate to $\mathcal{G}$:

Example 7: To the (n)-paracontracing *pool* $\{g : \mathbb{R}_{+0} \to \mathbb{R}_{+0}\}$ with

$$g(x) := x\ \frac{x^4}{x^4+1} \ ,$$

which has the fixed point 0 , the pool

$$f^j(x) := g(x) + \frac{1}{h_j} \qquad j = 0, 1, \ldots \tag{40}$$

is approximate, if $h_j \to \infty$, whenever $j \to \infty$. Setting $h = \frac{x^5+x}{x^4+x^2+1}$ $(= O(x))$, we have

$$g(x) + \tfrac{1}{h} \;=\; x + \tfrac{1}{x}.$$

Hence, for $x(0) > 0$ the sequential iteration

$$\begin{aligned} h_j &:= \frac{x(j)^5 + x(j)}{x(j)^4 + x(j)^2 + 1} \\ x(j+1) &:= f^j(x(j)) \qquad\qquad j = 0, 1, \ldots \end{aligned} \tag{41}$$

is diverging as

$$x(j) \;\geq\; \sum_{n=1}^{c_j} \frac{1}{n} \qquad \forall\, j \in \mathbb{N}_0 \tag{42}$$

for an ascending sequence $c_j\ {}_{j=0,1,\ldots}$ from $\mathbb{N}_0$.

To prove a Theorem of convergence for approximate pools we will assume that $\mathcal{G}$ is contractive:

Theorem 4.4 *Let $\mathcal{G}$ be a contractive pool on $D \subset \mathbb{R}^n$ and assume, that $\mathcal{G}$ has a common fixed point $\xi \in D$, and that the pool $\mathcal{F}$ is approximate to $\mathcal{G}$, e.g. there is a null sequence $\varepsilon_j\ {}_{j=0,1,\ldots}$ in $\mathbb{R}_{+0}$ and a sequence $\mathcal{K} = k_j\ {}_{j=0,1,\ldots}$ in $\mathbb{K}$, such that w.l.o.g. with respect to same norm $\|\cdot\|$, for which $\mathcal{G}$ is contractive*

$$\| F^j(X) - G^{k_j}(X) \| \leq \varepsilon_j \quad \forall\, X \in D^{m_{k_j}},\ j \in \mathbb{N}_0\,.$$

Further let respectively

$$\begin{aligned} p_0 &:= -M - 1\,, \\ p_t &:= \min\{p \in \mathbb{N}_0 \,|\, s^l(j) > p_{t-1}\ \forall\, j \geq p,\ l = 1, \ldots, m_{k_j}\} \quad \forall\, t \in \mathbb{N}\,, \\ \overline{\varepsilon}_{-1} &:= \max_{-M \leq l \leq 0} \| x(l) - \xi \|\,, \\ \overline{\varepsilon}_j &:= \sum_{l=p_j}^{p_{j+1}-1} \varepsilon_j \\ \text{and} \qquad \overline{\varepsilon}_{\max} &:= \max_j \overline{\varepsilon}_j\,. \end{aligned} \tag{43}$$

Now, if $0 \leq \omega < 1$ is the constant belonging to the contractive pool $\mathcal{G}$, as required in definition 3.2, and

$$D_0 := \{x \in D \mid y \in D\ ,\ \mathit{if}\ \| x - \xi \| \leq \frac{1 - \omega}{\overline{\varepsilon}_{\max}} \| y - \xi \| \}\,, \tag{44}$$

then , if $\mathcal{X}_{\mathcal{O}} \subset D_0$, the asynchronous iteration $(\mathcal{F}, \mathcal{X}_{\mathcal{O}}, \mathcal{N}_{\mathcal{O}}, \mathcal{S})$ is well-defined and converges to ξ.

Moreover, if $\mathcal{F}$ is asymptotical to $\mathcal{G}$ and $\mathcal{S}$ is regulated, then $(\mathcal{F}, \mathcal{X}_{\mathcal{O}}, \mathcal{N}_{\mathcal{O}}, \mathcal{S})$ converges to ξ, and

$$\sum_{j=-M}^{\infty} \| x(j) - \xi \| \tag{45}$$

exists.

Proof: We will prove by induction, that

$$\max_{p_j < l \leq p_{j+1}} \| x(p_j + l) - \xi \| \leq \sum_{q=0}^{j} \overline{\varepsilon}_{q-1}\, \omega^{j-q} \quad \forall\, j \in \mathbb{N}_0\,. \tag{46}$$

$(\mathcal{F}, \mathcal{X}_{\mathcal{O}}, \mathcal{N}_{\mathcal{O}}, \mathcal{S})$ is well-defined, because the iterations stay by (44) in D_0.

The definition of ε_{-1} is just the statement for $j = 0$.

Now let (46) be shown for all $j_0 = 0, ..., j$, then

$$\begin{aligned} \| x(p_j + 1) - \xi \| &= \| F^j(X_{p_j}) - \xi \| \leq \| G^{k_j}(X_{p_j}) - \xi \| + \varepsilon_{p_j} \\ &\leq \omega \max_l \| x^l_{p_j} - \xi \| + \varepsilon_{p_j} \\ \text{since } s^l(p_j) > p_{j-1}\ \forall\, l \qquad &\leq \omega \sum_{q=0}^{j-1} \overline{\varepsilon}_{q-1}\, \omega^{j-1-q} + \varepsilon_{p_j}. \end{aligned} \tag{47}$$

Now, continuing with this argumentation for $l = p_j + 2, ..., p_{j+1}$, we get that

$$\| x(p_{j+1}) - \xi \| \leq \omega \sum_{q=0}^{j-1} \overline{\varepsilon}_{q-1}\, \omega^{j-1-q} + \sum_{l=p_j}^{p_{j+1}-1} \varepsilon_l = \sum_{q=0}^{j} \overline{\varepsilon}_{q-1}\, \omega^{j-q}.$$

In the approximate case, the Theorem now follows from the so called Toeplitz lemma (see [11]): Let $\omega_{j,l}\ j = 0, 1, ...;\ l = 0, 1, ..., j$ and C be real numbers, such that

$$\lim_{j \to \infty} \omega_{j,l} = 0 \qquad \forall\, l \in \mathbb{N}_0\,,$$

$$\text{and} \qquad \sum_{l=0}^{j} |\omega_{j,l}| \leq C \quad \forall\, j \in \mathbb{N}\,.$$

Further let $\varepsilon_j\ {}_{j=0,1,...}$ be a null sequence, then

$$\alpha_j := \sum_{l=0}^{j} \varepsilon_l\, \omega_{j,l} \quad j = 0, 1, ...$$

is a null sequence.

Replace $\omega_{j,l}$ by ω^{j-l} in inequality (46) and the first assertion is proved. If $\mathcal{S}$ is regulated, then $p_{j+1} - p_j \;_{j=0,1,\ldots}$ is bounded by s, and we have

$$\begin{aligned}\sum_{j=0}^{\infty} \| x(j) - \xi \| &\leq s \sum_{j=0}^{\infty} \sum_{q=0}^{j} \overline{\varepsilon}_{q-1} \, \omega^{j-q} \\ &= s \, \frac{\overline{\varepsilon}_{-1} + \sum_{j=0}^{\infty} \varepsilon_j}{1 - \omega} ,\end{aligned}$$

which exists, if $\mathcal{F}$ is asymptotical to $\mathcal{G}$, and this states the proof. $\square$

5 Asynchronous iterations for singular systems and the Frobenius vector problem

As a first application, the developed theory in the previous section will now be used for solving asychronously singular linear systems and the Frobenius vector problem, e.g. determining nonnegative eigenvectors of nonnegative matrices, whose spectral radii are known. In the case of irreducible matrices, these are the Perron vectors and they are positive and unique after normization. To compute them asynchronously, we can use the following result of Lubachevsky and Mitra (see [7], the notations are usual or as introduced in the previous sections):

Theorem 5.1 *Let $T \in L(\mathbb{R}^n)$ be a nonnegative, irreducible matrix of unit spectral radius. Let there be $i_0 \in \{1, ..., n\}$, such that*

$$t_{i_0 i_0} > 0 , \tag{48}$$

then the iteration

$$x_i(j+1) := \begin{cases} \sum_{l=1}^{n} x_l(s^l(j)) \, t_{li} & , \text{ if } \quad i = p_j \\ \\ x_i(j) & , \text{ otherwise } \quad j = 0, 1, \ldots \end{cases} \tag{49}$$

converges to a positive left-hand eigenvector of T, if

$$\begin{array}{ll} (i) & \mathcal{X}_{\mathcal{O}} = \{ x(0) \} \\ (ii) & x(0) \in \{ x \in \mathbb{R}^n \mid x \geq 0 \ \ \wedge \ \ x_{i_0} > 0 \} \\ (iii) & j - s^l(j) \leq s \quad \text{for a} \quad s \in \mathbb{N}_0 \quad \forall \ l, j \\ (iv) & \{p_j\} \cup \{p_{j+1}\} ... \cup \{p_{j+c}\} = \{1, ..., n\} \quad \forall \ j \geq 0 \quad \text{for a} \quad c \in \mathbb{N}_0 \\ (v) & s^{i_0}(j) = j \quad , \text{ if } \ p_j = i_0 \ \ \forall \ j \geq 0 \ . \end{array} \tag{50}$$

Remarks: The algorithm is at the same time a method for computing Perron vectors of any nonnegative irreducible matrix T, for which the spectral radius $\rho(T)$ is known, by observing $\frac{1}{\rho(T)}Tx = x$, $x > 0$.

If there is no $1 \leq i_0 \leq n$, which has the required property (48), we can use the matrix $(1-\alpha)\, I + \alpha\, T$, where $\alpha \in (0,1)$, which has the same eigenvectors as T.

(50)(v) is not really restrictive. For example, if only one processor is computes the $i_0 - th$ component, then it has only to save this component in its local memory, in order to use it for the next updating.

While in [7] the authors preferred for their proof a somewhat more constructive approach from the theory of Markov chains, we will embed this iteration method into the theory developed in the previous sections. We will consider an equivalent problem, which allows us to analyze the affin linear case: If we interpret the different rows of a stochastic matrix as multiple data operators, as described in example 6, we get a strongly nonexpansive pool of operators.

Theorem 5.2 **(i)** *Let $T \in L(\mathbb{R}^n)$ be a matrix, $b \in \mathbb{R}^n$, let $s^l(j)_{\, j=0,1,\dots}$, $l = 1,\dots,n$ be sequences from $\mathbb{N}_0$, $z(0) \in \mathbb{R}^n$ and $p_{j\; j=0,1,\dots}$ be a sequence from $\{1,\dots,n\}$. Then the iteration*

$$z_i(j+1) := \begin{cases} \sum\limits_{l=1}^{n} t_{il}\, z_l(s^l(j)) + b_i & , \textit{ if } \ i = p_j \\[2ex] z_i(j) & , \textit{ otherwise} \qquad j = 0,1,\dots, \end{cases} \tag{51}$$

converges, if and only if for any permutation or diagonal matrix $P \in L(\mathbb{R}^n)$, with $\det(P) \neq 0$, *the iteration*

$$x_i(j+1) := \begin{cases} \sum\limits_{l=1}^{n} (PTP^{-1})_{il}\, x_l(s^l(j)) + (P^{-1}b)_i & , \textit{ if } \ i = p_j \\[2ex] x_i(j) & , \textit{ otherwise} \qquad j = 0,1,\dots, \end{cases} \tag{52}$$

where $x(0) := P^{-1}z(0)$, converges.

(ii) *Moreover, is there a $\xi \in \mathbb{R}^n$, such that $(I - T)\xi = b$, the iteration (51) converges, if and only if*

$$x_i(j+1) := \begin{cases} \sum\limits_{l=1}^{n} t_{il}\, x_l(s^l(j)) & , \textit{ if } \ i = p_j \\[2ex] x_i(j) & , \textit{ otherwise} \qquad j = 0,1,\dots, \end{cases} \tag{53}$$

where $x(0) := z(0) - \xi$, *converges.*

(iii) *Let* $t_{i,m_i(j)}$ *be for all* $i \in \{1,...,n\}$ *the* $j-$*th of* m_i *nonzero entries in* T*'s i-th row, or let* $m_i(j) = j \ \forall \ j = 1,...,n$ *and* $m_i = n$, *if this row is zero. The pool of operators* $\mathcal{T} = \{T^i \mid i = 1,...,n\}$, *defined by*

$$T^i : \mathbb{R}^{m_i} \to \mathbb{R} ,$$

$$T^i(y^1,...,y^{m_i}) := \sum_{j=1}^{m_i} t_{i\,m_i(j)}\, y^j \qquad\qquad i = 1,...,n , \tag{54}$$

is strongly nonexpansive on all closed intervals $D \subset \mathbb{R}$, *if* $T\,e = e$. $\mathcal{T}$ *is contractive on* $\mathbb{R}$ *and on* $\mathbb{R}_{+0}$, *if* $T\,e < e$.

(iv) *Assume now w.l.o.g., that all components* $x_l(s^l(j))$, *which are used to update a new iteration component in (53), themselfs are updated at time step* $s^l(j)$, *e.g.*

$$p_{s^i(j)-1} = i , \ \text{for all} \ j \in \mathbb{N} , i \in \{1,...,n\} \ \text{with} \ s^i(j) \geq 1 , \tag{55}$$

and that all initial vectors are multiples of e , *therefore define*

$$x(-j) \ := \ x_j(0)\, e \quad \forall \ j = 1,...,n$$

and renumber likewise that elements of the sequences of $s^l(j)_{\,j=0,1,...}$, $l = 1,...,n$, *for which* $s^l(j) = 0$.

Then the asynchronous iteration $(\mathcal{T}, \mathcal{Y}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$, *given by*

$$y(j+1) := T^{p_j}\big(y(\tilde{s}^1(j)), \ \ldots \ , y(\tilde{s}^{m_{p_j}}(j))\big) \qquad\qquad j = 0, 1, \ldots , \tag{56}$$

where $\mathcal{T}$ *is as in (iii),* $\mathcal{K} = p_{j\ j=0,1,...}$, $\mathcal{S}$ *is given by*

$$\tilde{s}^i(j) := s^{m_{p_j}(i)}(j) \quad \forall \ j \in \mathbb{N}_0 , \ i = 1,...,m_{p_j} \tag{57}$$

and $\mathcal{Y}_{\mathcal{O}}$ *by* $y(-l) := x_1(-l)$, $l = 1,...,n$, *produces*

$$y(j+1) = x_{p_j}(j+1) \qquad \forall \ j \in \mathbb{N}_0 \ . \tag{58}$$

(v) *Let now* T *be irreducible and* i_0 *be an index, such that* $t_{i_0 i_0} \neq 0$, *and assume*

$$\begin{aligned} &(a) \quad \mathcal{K} \text{ is regulated },\\ &(b) \quad s^{i_0}(j) = \max\{\, j_0 \leq j \mid p_{j_0 - 1} = i_0 \ \}\\ &\qquad\ \ \text{for all} \ \ j > \min\{\, j_0 \in \mathbb{N}_0 \mid p_{j_0} = i_0 \,\} \ \ \text{with} \ \ p_j = i_0 ,\\ &(c) \quad j - s^l(j) \leq s \quad \forall \ j \in \mathbb{N}_0 , \ l = 1,...,n , \quad \text{for a} \ \ s \in \mathbb{N}_0. \end{aligned} \tag{59}$$

then $(\mathcal{T}, \mathcal{Y}_{\mathcal{O}}, \mathcal{S}, \mathcal{K})$ *from (56) is confluent.*

Proof: (i) and (ii) are clear. It is easy to show by a short induction on j, that $x(j) = P^{-1}z(j)$, respectively that $x(j) = z(j) - \xi \quad \forall\, j \in \mathbb{N}_0$.

(iii) is also clear, since we have for all $i \in \{1, ..., n\}$, x, $y \in \mathbb{R}^{m_i}$, that

$$|\,T^i(x) \,-\, T^i(y)\,| \,\le\, \sum_{i=1}^{n} t_{il}\,|\,x^l - y^l\,| \,\le\, \max_{1\le l\le m_i}\, |x^l - y^l| \tag{60}$$

and equality holds only in the case $x^l - y^l = r = T^i(x-y)\ \forall\, l = 1, ..., m_i$ for some $r \in \mathbb{R}$. In the case $T_i\, e = 1$, we have also

$$|\,T^i(x) \,-\, T^i(y)\,| \,\ge\, \min_{1\le l\le m_i}\, |x^l - y^l| \ \forall\, x, y \in \mathbb{R}^{m_i}\,,$$

so T^i is *selfmapping* on every closed interval of $\mathbb{R}$.

(iv): (58) follows by simple induction on j.

(v): Since (59) (b), in the graph of $(\mathcal{T}, \mathcal{Y}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})$ all the vertices $j > 1$ with $p_{j-1} = i_0$ are connected by a directed path. Since T is irreducible, there is a $n_1 \in \mathbb{N}$, such that all vertices $j \ge n_1$ are successors of vertices a_j, for which $p_{a_j - 1} = i_0$. Because of the regularity of $\mathcal{K}$ and (59)(c) $j - a_j\,{}_{j=0,1,...}$ is bounded by some $a \in \mathbb{N}$. Hence, there is a $n_0 \in \mathbb{N}$, such that for all $j \ge n_0$

$$b_j := \min\{\, j_0 \in \mathbb{N} \,|\, j_0 \le a_j - a \,\wedge\, p_{j_0 - 1} = i_0\,\}$$

exists and such, that for all $j_0 \ge j$, j_0 is a successor of b_j. The sequence $j - b_j\,{}_{j=n_0,n_0+1,...}$ is also bounded by some $b \in \mathbb{N}$.

All vertices b_j, if j is large enough, are by the irreducibility of T successors of vertices v^i_j , such that $p_{v^i_j - 1} = i_0$ and, by the regularity of $\mathcal{K}$ and (59)(c), such that $b_j - v^i_j\,{}_{j=...}$ is bounded by some $c \in \mathbb{N}$. Thus, since $b_{j+c+b} - c \ge j \quad \forall\, j \ge n_0$, there is a vertex w^i_j for every $j \ge n_0$, such that w^i_j is a successor of b_j and a predecessor of b_{j+b+c} . That is the assertion (v). □

Remark 2: Theorem 5.2 confirms Theorem 5.1 by Theorem 4.2. If T is a nonnegative irreducible matrix of unit spectral radius, and $\mathrm{diag}(v) \in L(\mathbb{R}^n)$ is a diagonal matrix, whose diagonal entries are the components of a Perron vector v of T, in the same order, then we have

$$\mathrm{diag}(v)^{-1}\, T\, \mathrm{diag}(v)\, e \,=\, e\,.$$

But by (i), the convergence of the iteration (53) is independent of scaling, and since T is irreducible, the corresponding method in (v) is convergent, therefore $x(j) \to \lambda\, e$, for some $\lambda \in \mathbb{R}$. Because of condition (ii) of (50), the corresponding $y(j)$'s will be positive after some first iteration steps, which means by the selfmapping of the operators in (iii) of the previous Theorem, that the limit of $x(j)\,{}_{j=0,1,...}$ cannot vanish.

With help of Theorem 5.2, we can state a result on asynchronous iterations for linear systems of equations $Ax = b$. Asynchronous iterations were investigated first by Chazan and Miranker in [4] (as *chaotic iterations*). They considered the iteration

$$x_i(j+1) := \begin{cases} \sum\limits_{l=1}^{n} (B^{-1}C)_{il}\, x_l(s^l(j)) + (B^{-1}b)_i & , \text{if } i = p_j \\ \\ x_i(j) & , \text{otherwise} \quad j = 0, 1, \ldots \end{cases} , \tag{61}$$

where the pair of matrices (B, C) is a *splitting* of A, e.g. $A = B - C$ and $\det(B) \neq 0$. The other notations are as usual.

A typical condition for convergence Theorems for such methods is, to assume, that $A \in L(\mathbb{R}^n)$ is an M-matrix. That is a matrix, which has a decomposition $A = sI - T$, such that T is nonnegative, and it is spectral radius $\rho(T)$ is less or equal to s. Further, to assume that the splitting is *weak regular*, that is: B^{-1} and $B^{-1}C$ are nonnegative. In this case, using Perron and Frobenius theory, we have a vector $v \in \mathbb{R}^n$, such that,

$$\begin{array}{lll} (a) & A\,v = (sI - T)\,v \;>\; 0\,,\; v > 0 & , \text{if } A \text{ is regular} \\ (b) & A\,v = (sI - T)\,v \;=\; 0\,,\; v > 0 & , \text{if } A \text{ is singular and irreducible} \\ (c) & A\,v = (sI - T)\,v \;\geq\; 0\,,\; v \geq 0 & , \text{if } A \text{ is singular} \end{array} \tag{62}$$

Therefore, if the splitting is weak regular, we have in case (b), that

$$B^{-1}C\,v \;=\; (\,I \;-\; B^{-1}A\,)\,v \;=\; v\,.$$

Now, due to (i) of Theorem 5.2, we can assume w.l.o.g., that T is stochastic. Because of (ii), we can assume, that the linear system is homogeneous, and by (iv) we can observe the equivalent one dimensional asynchronous iteration (56), produced by a strictly nonexpansive pool, due to (iii). Thus, from Theorem 4.2, it suffices to carry over the conditions of (v) to the origin iteration method (61).

It was already shown by [4], that (61) converges, if A is a regular M-matrix, but we have a new result for singular M-matrices. In [4] was given an often cited necessary and sufficient condition for general convergence of method (61), which was that $\rho(B^{-1}C)$ has to be less than one. But this criterion based on regular matrices A and, on the whole, unsteered sequences $s^l(j)_{\,j=0,1,\ldots}$. Under these considerations, we can give the following Theorem without a proof.

Theorem 5.3 *Let $A \in L(\mathbb{R}^n)$ be an irreducible M-matrix and $b \in \mathbb{R}^n$. Further let (B, C) be a weak regular splitting of A. Assume that $B^{-1}C$ is irreducible, there is an i_0, such that the i_0-th diagonal entry of $B^{-1}C$ is positive and, that*

$$\begin{array}{ll} (i) & j - s^l(j) \leq s \quad \forall\, l, j\,, \;\; \textit{for a} \;\; s \in \mathbb{N} \\ (ii) & \mathcal{K} \;\; \textit{is regulated,} \\ (iii) & s^{i_0}(j) \geq \max\{\, j_0 \leq j \mid p_{j_0 - 1} = i_0 \,\} \\ & \textit{for all} \;\; j > \min\{\, j_0 \in \mathbb{N}_0 \mid p_{j_0} = i_0 \,\} \;\; \textit{with} \;\; p_j = i_0\,. \end{array} \tag{63}$$

Then, if $Ax = b$ is consistent, the iteration (61) converges to a solution of it.

Our last topic is the reducible case. It is well known (see for example [2]), that in this case the corresponding sequential methods need not converge, even if the system is consistent. The condition here is, that the matrix T be semiconvergent, that means that

$$\lim_{j\to\infty} T^j$$

exists. A matrix is semiconvergent, if and only if every similar matrix is semiconvergent. Therefore, if we transform T first into its Frobenius normal form, that is

$$P\,T\,P^{-1} = \begin{pmatrix} T^{11} & 0 & \cdot & \cdot & \cdot & 0 \\ T^{21} & T^{22} & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & 0 \\ T^{p1} & \cdot & \cdot & \cdot & T^{pp-1} & T^{pp} \end{pmatrix}, \tag{64}$$

where P is a permutation matrix, such that all $T^{ii} \in L(\mathbb{R}^{n_i})$, $n_i \in \mathbb{N}$, $i = 1, ..., p$ are square and irreducible or 1×1, and then scaling it by $\mathrm{diag}(v_i) \in \mathrm{L}(\mathbb{R}^n)$, which be the matrix, whose diagonal elements are the components of the Perron vectors of the diagonal blocks of $P\,T\,P^{-1}$ or 1, if this is zero, then

$$T_0 := \mathrm{diag}(v_i)^{-1}\,P\,T\,P^{-1}\,\mathrm{diag}(v_i)$$

is semiconvergent, if and only if T is semiconvergent. For that reason and because of Theorem 5.2(i), we can assume with loss of generality, that $B^{-1}C$ is in Frobenius normal form, and its diagonal blocks are either stochastic or all row sums of them are less than one.

Theorem 5.4 *Let $A \in L(\mathbb{R}^n)$ be an M-matrix, $b \in \mathbb{R}^n$ and (B, C) be a weak regular splitting of A. Assume, that $T := B^{-1}C$ is semiconvergent, and in Frobenius normal form, given by (64). Further let there be a subset I_O of $\{1, ..., n\}$, such that for each $i \in \{1, ..., p\}$, for which T^{ii} is of unit spectral radius, there is some $i_0 \in I_O$, such that $t^{ii}_{i_0 i_0}$ is positive.*
In addition let

$$\begin{array}{ll} (i) & j - s^l(j) \le s \quad \forall\ l = 1, ..., n\,,\ j \in \mathbb{N}_0\,,\ \text{for a } s \in \mathbb{N}\,, \\ (ii) & \mathcal{K} \text{ be regulated,} \\ (iii) & s^{i_0}(j) = \max\{\, j_0 \le j \mid p_{j_0 - 1} = i_0 \ \wedge\ k(j_0 - 1) \in \mathbb{K}_{I_O} \,\} \end{array} \tag{65}$$

for all $j > \min\{\, j_0 \in \mathbb{N}_0 \mid p_{j_0} = i_0 \,\}$ with $p_j = i_0$ and this for all $i_0 \in I_O$.

Then the iteration method (61) converges to a solution of $Ax = b$, *if the system is consistent.*
If further

$$(iv) \qquad b = 0\,,$$
$$x(0) \in \{x \in \mathbb{R}^n_{+0} \mid x_{i_0} > 0 \;\forall\, i_0 \in I_O\,\}$$
$$\textit{and} \qquad \rho(T) = 1\,,$$

then (61) converges to a nonnegative eigenvector of T .

Proof: Using Theorem 5.2 (i) and (ii), we can assume w.l.o.g., that $b = 0$ and $T^{ii}\,e$ is for all $i \in \{1, ..., p\}$ either less or equal to e .
We will show by induction, that, if $x^i(j)_{\,j=0,1,...}$, $i = 1, ..., p$ are the sequences, which partition the iterations $x(j)_{\,j=0,1,...}$ in the same manner as T's Frobenius normal form, that

$$(a) \quad \lim_{j\to\infty} x^i(j) \;=\; z^i \quad , \qquad \text{where} \quad T^{ii}\,z^i \;=\; z^i - \sum_{l=1}^{i-1} T^{il}\,z^l \;,$$

$$(b) \qquad \text{If (iv) holds and } \; z^i = 0\;, \text{ then}$$
$$\sum_{j=0}^{\infty} \parallel x^i(j) - z^i \parallel \;<\; \infty \;, \tag{66}$$
$$\tag{67}$$

whether (iv) holds or not.

For $i = 1$ and $T^{11}\,e \;=\; e$, this was stated by Theorem 5.1, or the previous Theorem. If $T^{11}\,e \;<\; e$ this is an old result of the theory of asynchronous iterations (compare for example the nonlinear case in [1]). But this result can also be derived from Theorem 4.4: The pool $\mathcal{T}$, which belongs by Theorem 5.2 (iii) to T^{11}, is contractive, its unique fixed point is zero and it is asymptotical to itself.
Now let (66) be shown for all $1 \le l < i$, and define recursively: $l_{-1} := -1$

$$l_{j+1} := \min\left\{\, l > l_j \,|\, p_l \in \left[1 + \sum_{k=1}^{i-1} n_k\,,\, \sum_{k=1}^{i} n_k\right]\right\} \;\forall\, j \in \mathbb{N}_0\,. \tag{68}$$

If we denote the $l-$th entry of T's $p-$th row, which lies in block ik by t^{ik}_{pl}, then

$$\begin{aligned}
& x_{p_{l_j}}(l_j+1) - \sum_{k=1}^{i-1}\sum_{l=1}^{n_l} t^{ik}_{p_{l_j}\,l}\left(\, x^k_l(s^l(l_j)) \;-\; z^k_l\right) \\
=\;& y(l_j+1) - \sum_{k=1}^{i-1}\sum_{l=1}^{n_l} t^{ik}_{p_{l_j}\,l}\left(\, x^k_l(s^l(l_j)) \;-\; z^k_l\right) \qquad (69) \\
=\;& T^{p_{l_j}}(y(\tilde{s}^1(l_j)), ..., y(\tilde{s}^{m_{p_{l_j}}}(l_j))) - \sum_{k=1}^{i-1}\sum_{l=1}^{n_l} t^{ik}_{p_{l_j}\,l}\left(\, x^k_l(s^l(l_j)) \;-\; z^k_l\right) \qquad j = 0, 1, ... \,,
\end{aligned}$$

where $y(j)_{j=0,1,...}$ is the sequence produced by the asynchronous iteration

$$(\mathcal{T}, \mathcal{Y}_{\mathcal{O}}, \mathcal{K}, \mathcal{S})\ ,$$

belonging to the iteration (61) as in Theorem 5.2, converges, if and only if the sequence $x^i(j)_{j=0,1,...}$ converges to some z^i, for which (a) holds. This holds also for the speed of convergence (in the sense of 66) , since $\mathcal{K}$ is regulated.

Now define a sequence $\varepsilon_j\,_{j=0,1,...}$ by

$$\varepsilon_j := \sum_{k=1}^{i-1} \sum_{l=1}^{n_l} t^{ik}_{p_{l_j}\,l} \left(x^k_l(s^l(l_j)) - z^k_l \right) \ \forall\, j \in \mathbb{N}_0\ , \tag{70}$$

define a strongly nonexpansive pool of operators $\mathcal{T}_{\mathcal{I}}$ as that one, which is belonging to $T^{ii} := \{T^l_i \,|\, l = 1, ..., n_i\}$ by Theorem 5.2(iii),

assume, that for all $j \in \mathbb{N}_0$, if $T^{ii} \neq (0)$, the $p_{l_j}-$th row of T has $m^{p_{l_j}}$ positive entries in T's $i-$th diagonal block, and define a sequences $\mathcal{S}_{\mathcal{I}}$ by

$$\begin{aligned} s^l_i(j) &:= \tilde{s}^{m_{p_{l_j}} - m^{p_{l_j}} + l}(l_j) \quad \forall\ j \in \mathbb{N}_0\,,\ l = 1, ..., m^{p_{l_j}} \\ \text{or} \quad s^1_i(j) &:= s^l(l_j) \quad \forall\ j \in \mathbb{N}_0\ \text{, if } T^{ii} = (0) \end{aligned} \tag{71}$$

Then the asynchronous iteration $(\mathcal{F}, \mathcal{Y}_{\mathcal{O}}, \mathcal{N}_{\mathcal{O}}, \mathcal{S}_{\mathcal{I}})$, where the pool $\mathcal{F} = \{\, F^j : \mathbb{R}^{m_{p_{l_j}}} \to \mathbb{R} \,|\, j = 0, 1, ...\,\}$, respectively $\mathcal{F} = \{\, F^j : \mathbb{R} \to \mathbb{R} \,|\, j = 0, 1, ...\,\}$, if $T^{ii} = (0)$ is given by

$$F^j(z^1, ..., z^{m^{p_{l_j}}}) := T_i^{p_{l_j} - \sum_{k=1}^{i-1} n_k} (z^1, ..., z^{m^{p_{l_j}}}) + \varepsilon_j \quad \forall\ j \in \mathbb{N}_0\,, \tag{72}$$

respectively $F^j(z^1) := \varepsilon_j \ \forall\ j \in \mathbb{N}_0\,,$

generates the sequence $z(j)_{j=0,1,...}$, which we have to analyze, e.g.

$$z(j) = y(l_j + 1) - \sum_{k=1}^{i-1} \sum_{l=1}^{n_l} t^{ik}_{p_{l_j}\,l}\, z^k_l \quad \forall\ j \in \mathbb{N}_0\ . \tag{73}$$

First assume now $T^{ii}\, e < e$, then $\mathcal{T}_{\mathcal{I}}$ is contractive, and since $\mathcal{F}$ is approximate to $\mathcal{T}_{\mathcal{I}}$ the assertion follows by Theorem 4.4. If (iv) holds and $x^i(j)_{j=0,1,...}$ converges to zero, then either $T^{li} = 0$ or $z^l = 0$ for all $l < i$, and this means, by the assumption of our induction, that the sum of $\varepsilon_j\,_{j=0,1,...}$ converges. Therefore, in this case, again by Theorem 4.4, the sum of $\| x^i(j) - z^i \|_{j=0,1,...}$ converges, even if (iv) not holds.

Now assume, that $T^{ii}\,e = e$, and that there are numbers $1 \le l < i$, $1 + \sum_{q=1}^{i-1} n_q \le p \le \sum_{q=1}^{i} n_q$, $1 \le k \le n_l$, such that $t_{pk}^{il} \ne 0$, but $z_k^l > 0$, then

$$\begin{pmatrix} T^{11} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & & T^{ll} & \cdot & \cdot \\ \cdot & & & \cdot & 0 \\ T^{p1} & \cdot & T^{il} & \cdot & T^{ii} \end{pmatrix} \begin{pmatrix} \vdots \\ z^l \\ \vdots \\ e \end{pmatrix} \ge \begin{pmatrix} \vdots \\ z^l \\ \vdots \\ T^{il}\,z^l \; + \; e \end{pmatrix}$$

and T would not be semiconvergent. Therefore, either the sum of $\| x^l(j) - z^l \|_{\,j=0,1,\dots}$ converges, or $T^{il} = 0$ for all $l < i$.

Again, $\mathcal{F}$ is asymptotical to $\mathcal{T}_{\mathcal{I}}$ and the first part of the assertion follows by Theorem 4.3. Since in the case that (iv) holds, $x^i(j) \ge v(j)\ \forall\ j \in \mathbb{N}_0$, where $v(j)_{\,j=0,1,\dots}$ denote the sequence, which would be generated by the corresponding iteration to compute a Perron vector of T^{ii}, using the initial vector $x^i(0)$, z^i has to be positive, and we need not prove (b). □

The theory developed so far, can be extended in lot of ways, for example:

- Using multisplittings (see [8],[3]) for computing the solution of singular linear (M-)systems. Then we have to guarantee confluence of the different iteration matrices T^k, $k \in \mathbb{K}$.

- Other singular systems than M, for example weak monotone matrices (see [2]), or such that T is no longer nonnegative.

- A corresponding nonlinear problem, for which perhaps (n)-paracontracting pools are useful, is to determine the eigenfunctions of differential equations, if the associated eigenvalue is known.

- Using (n)-paracontracting pools for fixed point problems, where the set of common fixed points is not convex.

- Infinite pools, by using indexwise-regulated sequences. Observe, that we have used a finite induction in Theorem 5.4, and it would also hold in the case, where T has infinite many rows, if $p_j\,{}_{j=0,1,\dots}$ is indexwise-regulated.

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