On asynchronous Newton-iterative methods

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Abstract

In [9] A.H. Sherman investigated Newton-iterative methods for nonlinear systems $F(x) = 0$. These are methods, where instead of the next Newton step, the linear system $F'(x(j))\, y = F'(x(j))\, x(j) - F(x(j))$ was not solved directly, but approximated by finite many steps of a linear iterative method. He showed, that the principle of many numerical methods, namely to start with rough and to end with fine approximations, which means here to increase the number of *inner* or *linear iterations*, leads to a significant speed up of convergence, assuming, that the evaluations of the function F and its partial derivatives are expensive. R.E. White [11] investigated the same problem by using a synchronized parallel multisplitting method for the inner iterations. We will introduce an asynchronous Newton-iterative method, which has the same properties in speed of convergence, but in addition, inner and outer iterations are totally decoupled, and moreover, the partial derivatives of F can be evaluated or approximated totally asynchronous. Interpreting the evaluation of component functions and partial derivatives as the most complex task, our method will converge superlinearly, and under suitable conditions at least quadratically.

Key words: Asynchronous iterations, Newton-iterative methods.

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1 Introduction

Baudet [1] extended asynchronous iterations, introduced by Chazan and Miranker [4] for linear systems, to nonlinear systems. Further he introduced *asynchronous iterations with memory,* where the operator of iteration depends on more than one of each components of already computed iterations. In the definition below, we will add the idea of a *pool* of iteration operators to this concept (compare [5]) and also that the real operators depend on parameters.

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Definition 1.1 Let $\mathbb{K}$ be some set and

$$G^k : D_0^m \times D_h \subset \mathbb{R}^{nm} \times \mathbb{R}^p \to \mathbb{R}^n \,, \qquad \text{for all} \quad k \in \mathbb{K} \tag{1}$$

be a *pool* of operators, $\mathcal{K}$ be a sequence from $\mathbb{K}$, given by

$$k : \mathbb{N}_0 \to \mathbb{K}\,, \tag{2}$$

$\mathcal{S}$ be a sequence from $\mathbb{N}_0^{nm}$, denoted by

$$\begin{aligned} s_i^l : \mathbb{N}_0 \to \mathbb{N}_0 \quad i = 1, ..., n \,,\; l = 1, ..., m \,, \\ \text{with} \quad s_i^l(j) \le j \quad \forall\, j \in \mathbb{N}_0 \,,\, l = 1, ..., m \,,\, i = 1, ..., n \end{aligned} \tag{3}$$

and $\mathcal{J}$ a sequence of subsets of $\{1, ..., n\}$:

$$J_j \subset \{1, ..., n\} \quad j = 0, 1, \ldots \,. \tag{4}$$

Finally, let $\mathcal{H}$ be a sequence from D_h,

$$h_j \in D_h \quad j = 0, 1, ... \tag{5}$$

and $x(0) \in D_0$ be an initial vector, then the iteration

$$x_i(j+1) := \begin{cases} x_i(j) & , \text{if } \; i \notin J_j \\ G_i^{k(j)}\left(\begin{pmatrix} x_1(s_1^1(j)) \\ \vdots \\ x_n(s_n^1(j)) \end{pmatrix}, ..., \begin{pmatrix} x_1(s_1^m(j)) \\ \vdots \\ x_n(s_n^m(j)) \end{pmatrix}, h_j \right) & , \text{if } \; i \in J_j \end{cases} \tag{6}$$
$$j = 0, 1, \ldots \,.$$

is called an *asynchronous iteration* .

Now we can denote an asynchronous iteration by $(\mathcal{G}, \mathcal{K}, \mathcal{S}, \mathcal{J}, \mathcal{H}, x(0))$. Of course, the problem to be solved, is determining a *common fixed point* ξ of the pool $\mathcal{G}$, e.g.

$$G^k(\xi, ..., \xi, h) = \xi \quad \forall\, k \in \mathbb{K}\,,\, h \in D_h \,,$$

where throughout this paper we will always assume that such a fixed point exists and is locally unique.

The concept of asynchronous parallelism we have in mind can be interpreted as a pure or a combination of two strategies: The idea of [4] was as follows: Each of the single processors of a multiprocessor, denoted by $\pi_1, ..., \pi_K$ retrieves some components of an iteration it belongs to from some shared memory, or the local memory of processors to which it is linked, computed at *time* steps $s_i^l(j)$, respectively.

Then it computes a set J_j of components of the system, it is ordered to, e.g. for example $J_j \in \{J_{\pi_1}, ..., J_{\pi_K}\} \quad \forall\ j \in \mathbb{N}_0$. This computation is assumed to be done at time step j, e.g. the processor is saving it at this time on the shared or its own memory. A different strategy of asynchronous iterations is as follows: Each processor is ordered to compute a whole iteration vector using one of the operators. For example the multisplitting-method for linear systems $Ax = b$, introduced by [6](synchronized) and further investigated in [3](asynchronous), can be embedded into this concept: A *multisplitting* of a matrix A is, see [6], a finite set of triples of matrices $\{\,(E^k, B^k, C^k)\,|\, k \in \mathbb{K}\,\}$, such that

$$\begin{aligned} &\text{(i)} && \sum_{k\in\mathbb{K}} E^k = I\,, \\ &\text{(ii)} && \det(B^k) \neq 0 \qquad \forall\ k \in \mathbb{K}\,, \\ &\text{(iii)} && B^k - C^k = A \qquad \forall\ k \in \mathbb{K}\,. \end{aligned} \tag{7}$$

Now defining operators

$$G^k(x^1, x^2) := (I - E^k)\,x^1 + E^k\Big(B^{k^{-1}}\,C^k\,x\ -\ B^{k^{-1}}\,b\Big)\ \ \forall\ k \in \mathbb{K}\,,$$

we get in (6) exactly the asynchronous method introduced in [3], if we set $J_j = \{1, ..., n\}$, $s_i^1(j) = j$ and $s_i^2(j) = s(j)\ \forall\ i \in \{1, ..., n\}$, $j \in \mathbb{N}_0$ for some sequence $s(j)_{\ j=0,1,...}$ from $\mathbb{N}_0$. If $E_{ii}^k \in \{0,1\}\ \forall\ i \in \{1, ..., n\}$, $k \in \mathbb{K}$, we are back again in the previous strategy.

For further investigations we need some criteria and notations, which we will use throughout the rest of the paper.

Definition 1.2 Let $(\mathcal{G}, \mathcal{K}, \mathcal{S}, \mathcal{J}, \mathcal{H}, , x(0))$ be an asynchronous iteration, then

$$X_j := (x_j^1, ..., x_j^m) := \left(\begin{pmatrix} x_1(s_1^1(j)) \\ \vdots \\ x_n(s_n^1(j)) \end{pmatrix}, ..., \begin{pmatrix} x_1(s_1^m(j)) \\ \vdots \\ x_n(s_n^m(j)) \end{pmatrix} \right) \qquad \forall\ j \in \mathbb{N}_0 \tag{8}$$

are the approximations of the solution used in the $j+1$-st step of iteration.
If the sequence $s_j\ _{j=0,1,...}$ with

$$s_j\ :=\ \min\{s \in \mathbb{N}_0\ |\ \ s_i^l(j_0 + s) \geq j \quad \forall\ j_0 \geq j,\ i = 1, ..., n\ ,\ l = 1, ..., m\} \qquad \forall\ j \in \mathbb{N}_0\,, \tag{9}$$

is well-defined, then $\mathcal{S}$ is called *admissible*. If

$$s = \limsup_{j\to\infty}\ s_j\ < \infty\,, \tag{10}$$

then $\mathcal{S}$ is called *regulated.* If there is a sequence $t_{j\ j=0,1,\ldots}$ from $\mathbb{N}_0$, such that

$$J_j \cup \ldots \cup J_{j+t_j} = \{1, \ldots, n\} \quad \forall\, j \in \mathbb{N}_0 \,, \tag{11}$$

then $\mathcal{J}$ is said to be *admissible.* If

$$t = \limsup_{j\to\infty}\ t_j \tag{12}$$

exists, then $\mathcal{J}$ *is regulated.*

In addition to criteria on $\mathcal{S}$ and $\mathcal{J}$ we need a criterion for the pool $\mathcal{G}$. The following one is in the case $|\,\mathbb{K}\,| = 1$, $D_h = \{h_0\}$ essentially equivalent to that one, which was given by El Tarazi in [10].

Definition 1.3 Let $\mathbb{K}$ be a set and $\mathcal{G} = \{G^k \mid k \in \mathbb{K}\}$ be a pool of operators with

$$G^k : D_0^m \times D_h \subset \mathbb{R}^{nm} \times \mathbb{R}^p \to D_0 \quad \forall\, k \in \mathbb{K} \,. \tag{13}$$

Furthermore let D_0 be closed, and such that for all diagonal matrices $P \in L(\mathbb{R}^n)$ with $P_{ii} \in \{0,1\}$ $i = 1, \ldots, n$ and all $x, y \in D_0$

$$P\,x \,+\, (I - P)\,y \in D_0 \tag{14}$$

holds. If then for all $h \in D_h$, $k \in \mathbb{K}$, $X \in D_0^m$, some $0 \le \omega < 1$ and a positive vector $v \in \mathbb{R}^n$ valids: $\mathcal{G}$ has a common fixed point $\xi \in D_0$ and

$$|G^k(X,h) - \xi| \le (\,(A_1^k(X,h), A_2^k(X,h), \ldots, A_m^k(X,h)\,)(|x^1 - \xi|^t, \ldots, |x^m - \xi|^t)^t \tag{15}$$

for matrices $A_i^k(X,h) \in L(\mathbb{R}^n)$ $i = 1, \ldots, m$ with

$$\sum_{j=1}^{m} |A_j^k(X,h)|\ v \le \omega^k(X,h)\ v \,, \tag{16}$$

where the image and the definition area of the set of functionals $\{\omega^k \mid k \in \mathbb{K}\}$ is given by

$$\omega^k : D_0^m \times D_h \to [0, \omega] \,,$$

then $\mathcal{G}$ is called *weakly contracting* on D_0.

In [1] Baudet introduced a somewhat stronger criterion, which he called *contracting*. El Tarazi assumed in [10], that a *one-point* operator, independent on parameters, satisfies

$$\|G(x)-\xi\|^v \leq \omega\, \|x-\xi\|^v \ , \tag{17}$$

where $\|\cdot\|^v$ is a weighted maximum-norm, that is

$$\| z \|^v := \max_i \left| \frac{z_i}{v_i} \right| \quad \forall\ z \in \mathbb{R}^n \ , \tag{18}$$

here $v \in \mathbb{R}^n$ is a positive vector. If we have (17) on some set D_0, then

$$\begin{aligned} |G(x)-\xi| &\leq \|G(x)-\xi\|^v\ v \\ &\leq \omega\ \|x-\xi\|^v \\ &\leq \omega\ v\ \frac{1}{v_{i_0}}\, e_{i_0}^t\ |x-\xi| \\ &\qquad \text{for some } 1 \leq\ i_0\ \leq n. \end{aligned} \tag{19}$$

Now $A(x) \in L(\mathbb{R}^n)$, given by

$$A(x)_{ij} := \omega\ \frac{v_i}{v_{i_0}} \cdot \delta(i_0, j) \quad \text{for all} \quad i,j=1,...,n, \tag{20}$$

where $\delta(\cdot,\cdot)$ is the Kronecker-symbol, satisfies (15) and (16). If on the other hand (15) and (16) hold, we get

$$\begin{aligned} \|G(X)-\xi\|^v \ &\leq\ \| \sum_j |A_j(X)|\ \ | x^j - \xi \mid \|^v \\ \leq\ \| \sum_j |A_j(X)|\ \ &\max_j \| x^j - \xi \|^v\ \ v\ \ \|^v \\ \leq\ \omega\ &\max_j \| x^j - \xi \|^v . \end{aligned} \tag{21}$$

Therefore, our definition is equivalent. Remark, that we added the conditions (13), (14) and that D_0 is closed. Therefore, if $x(0) \in D_0$, the iterations will stay in D_0, because combinations of components of different vectors of D_0 are again in D_0. At last we need to have some measure for the speed of convergence:

Definition 1.4 Let $x(j)\ _{j=0,1,...}$ be a sequence from $\mathbb{R}^n$, which converges to $\xi \in \mathbb{R}^n$ and $\|\cdot\|$ some vector norm, then

$$\mathcal{R}\ :=\ \limsup_{j\to\infty}\ \sqrt[j]{\|x(j)-\xi\|} \tag{22}$$

is called the *rate of convergence* of $x(j)\ _{j=0,1,...}$ and

$$\mathcal{O}\ :=\ \liminf_{j\to\infty}\ \sqrt[j]{-\log(\|x(j)-\xi\|)} \tag{23}$$

the *order of convergence* of $x(j)\ _{j=0,1,...}$.

Remark, that these values are independent of the norm and the logarithm used. El Tarazi's Theorem of convergence in [10] was as follows:

Theorem 1.5 *Let* $\mathcal{G} = \{G\}$ *be a weakly contracting pool on* D_0 , $D_h = \{h_0\}$ *and* $\mathcal{S}$ *and* $\mathcal{J}$ *be admissible. Then the asynchronous iteration* $(\{G\}, \mathcal{S}, \mathcal{J}, x(0))$ *is well-defined and converges to* ξ . *Moreover, if* $\mathcal{S}$ *and* $\mathcal{J}$ *are regulated, then it converges at least linear with*

$$\mathcal{R} \leq \sqrt[s+t+1]{\omega}. \tag{24}$$

Just to give a slight extension, namely to carry over this result to pools of operators, which depend on parameters, we give below a short proof of a Theorem of convergence for asynchronous iteration methods.

Lemma 1.6 *Let* $\mathcal{G}$ *be a weakly contracting pool of operators on* $D_0 \subset \mathbb{R}^n$ *and* $x(0) \in D_0$. *Then the asynchronous iteration* $(\mathcal{G}, \mathcal{K}, \mathcal{S}, \mathcal{J}, \mathcal{H}, x(0))$ *is well-defined and converges to the common fixed point* $\xi \in D_0$ *of* $\mathcal{G}$, *if* $\mathcal{S}$ *and* $\mathcal{J}$ *are admissible. The sequence*

$$\begin{aligned} \lambda_0 &:= \min\{\lambda \geq 0 \mid |x(0) - \xi| \leq \lambda\, v\ \} \\ \lambda_j &:= \min\{\lambda \geq 0 \mid |x(j_0) - \xi| \leq \lambda\, v \quad \forall\, j_0 \geq j\ \} \qquad j = 1, 2, \ldots \end{aligned} \tag{25}$$

is also well-defined, and suffices

$$\lambda_{j+s_j+t_{j+s_j}+1} \leq \max_{j+s_j \leq j_0 \leq j+s_j+t_{j+s_j}+s_{j+s_j+t_{j+s_j}+1}} \omega^{k(j_0)}(X_{j_0}, h_{j_0})\ \ \lambda_j \tag{26}$$

$$\forall\, j \in \mathbb{N}_0 .$$

Proof: λ_j $_{j=0,1,\ldots}$ is well-defined, if we can show, that $|x(j) - \xi| \leq \lambda_0\, v \ \ \forall\, j \geq 0$. $|\,x(0) - \xi\,| \leq \lambda_0\, v$ is an assumption. If now $|x(j_0) - \xi| \ \leq \lambda_0\, v$ is shown for all $j_0 \leq j$, then, because of

$$|\,x_i(s_i^l(j)) - \xi_i\,| \leq \lambda_0\, v_i \qquad i = 1, \ldots, n,\ \ l = 1, \ldots, m\ ,$$

we have

$$|\,x_i(j+1) - \xi_i\,| \ = \ |\,G_i^{k(j)}(X_j, h_j) - \xi_i\,|$$

$$\leq \ \sum_{l=1}^{m} A_l^{k(j)}(X_j, h_j)_i\ \ |\,x_j^l - \xi\,| \ \leq \ \sum_{l=1}^{m} A_l^{k(j)}(X_j, h_j)_i\ \ \lambda_0\, v$$

$$\leq\ \omega^{k(j)}(X_j, h_j)\, \lambda_0\, v_i\ , \qquad \text{if} \ \ i \in J_j\ ,$$

and if $i \notin J_j$ anyhow

$$| x_i(j+1) - \xi_i | \; = \; | x_i(j) - \xi_i | \; \le \; \lambda_0 \, v_i \; .$$

Therefore $|x(j+1) - \xi| \le \lambda_0 \, v$. Using the same argumentation, we can point out (26). Because of

$$| x_i(s_i^l(j_0)) - \xi_i | \le \lambda_j \, v_i \qquad \forall \; j_0 \ge j + s_j, \; i = 1, ..., n, \; l = 1, ..., m \; ,$$

it is

$$| x_i(j + s_j + 1) - \xi_i | \le \; \omega^{k(j+s_j)}(X_{j+s_j}, h_{j+s_j}) \, \lambda_j \, v_i \quad , \text{if } \; i \in J_{j+s_j} \; ,$$

and

$$| x_i(j + s_j + 1) - \xi_i | \; = \; | x_i(j + s_j) - \xi_i | \le \; \lambda_j \, v_i \quad \text{otherwise} \; .$$

Now using the fact, that $J_{j+s_j} \cup \; ... \; \cup J_{j+s_j+t_{j+s_j}} \; = \{1, ..., n\}$, we get

$$| x(j + s_j + t_{j+s_j} + 1) - \xi | \; \le \max_{j + s_j \le j_0 \le j + s_j + t_{j+s_j}} \omega^{k(j_0)}(X_{j_0}, h_{j_0}) \, \lambda_j \, v \tag{27}$$

After $s_{j+s_j+t_{j+s_j}+1}$ further steps of iteration, only by the factor

$$\max_{j + s_j \le j_0 \le j + s_j + t_{j+s_j} + s_{j+s_j+t_{j+s_j}+1}} \omega^{k(j_0)}(X_{j_0}, h_{j_0})$$

better iterations were used to update new iterations, therefore (26) is valid. □

Remark, if $\mathcal{S}$ and $\mathcal{J}$ are regulated, then (26) implies (24).

2 Regular roots, consistent approximations and nonlinear multisplittings

Now we are ready to analyze asynchronous iterations for nonlinear systems. Therefore, we introduce an assumption, which guarantees the local uniqueness of a solution ξ of the system $F(x) = 0$.

Definition 2.1 Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function, such that $F(\xi) = 0$ for some $\xi \in D$. If there is an open neighborhood $D_0 \subset D$ of ξ , such that

(i) F' exists and is continuous on D_0 ,

(ii) $\| F'(x) - F'(\xi) \| \; \le \; L \, \| x - \xi \|$ for all $x \in D_0$ and some $L \ge 0$,

(iii) $\det(F'(\xi)) \ne 0$,

then ξ is called a *regular root* of F .

If ξ is a regular root of F, then we have by [7],§3.2.12, that for any convex subset D_1 of D_0

$$\| F'(\xi)(x-\xi) + F(x) - F(\xi) \| \leq \frac{L}{2} \| x - \xi \|^2 \qquad \forall\, x \in D_1 \,. \tag{28}$$

For that reason, Newton's method converges locally quadratically to ξ. Newton's method is somehow elementary for nonlinear systems, but according to the given system the costs for the *single parts* of Newton's method, like evaluating the component functions of F, evaluating the entries of F's Jacobian and the solution of the linear systems to which it belongs, can be very different. Therefore, in literature, we find many variants of Newton's method, which are helpful in handling with the special difficulties of a given system. We will assume now, that the computation of F and its partial derivatives are hard tasks, or impossible. In [7],§11.2.1 is a hint for the theory of sequential iterative methods, if this is our problem. Below, we will extend this theory to the $n \cdot m-$dimensional case.

Definition 2.2 Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function, differentiable on $D_0 \subset D$ and $J : D_J^m \times D_h \subset \mathbb{R}^{n^m} \times \mathbb{R}^p \to L(\mathbb{R}^n)$. Then J is called a *consistent approximation* at F' on D_0, if $D_0 \subset D_J$ and

(i) $$0 \in \overline{D}_h,$$

(ii) $$\lim_{\substack{h \to 0;\, h \in D_h \\ \max_l \|x^l - x\| \to 0}} J(x^1, ..., x^m, h) = F'(x)$$

uniformly for all $x \in D_0$.

If there are constants $C_0, C_1 \geq 0$, such that

(iii) $$\|F'(x) - J(x^1, ..., x^m, h)\| \leq C_0 \,\|h\| + C_1 \max_l \|x^l - x\| \,,$$

$$\forall\, x, x^1, ..., x^m \in D_0 \,,\, h \in D_h,$$

then J is said to be a *strongly consistent approximation* at F' on D_0.

Example 1: The idea of consistent approximations is obviously: Replacing the system matrices $F'(x)$ in Newton's method by $J(x^1, ..., x^m, h)$. In the sequential case, or if the partial derivatives of F are incalculable, $J(x^1, ..., x^m, h)$ is, on the whole, only assumed to be a *good enough* approximation for F's Jacobian, just to avoid problems. In the asynchronous case, there is a new meaning of consistent approximations. Let us assume, that we have so many processors, that perhaps a whole block of them can be ordered to compute only partial derivatives, using thereby approximations of ξ, which itself are computed at different time steps. So observe, that

$$J(x^1, ..., x^{n^2}, 0)_{ij} := F'(x^{(i-1)n+j})_{ij} \qquad \forall\, i, j = 1, ..., n \qquad D_h = \{0\}\,, \tag{29}$$

is a strongly consistent approximation at F' in some neighborhood of a regular root.

Example 2: Even if F's partial derivatives are incalculable, or so expensive, that it is senseless to compute them exactly,

$$J(x^1,...,x^{n^2},h)_{ij} :=$$
$$\frac{F_i(x^{(i-1)n+j} + h_{(i-1)n+j}\ e_j) - F_i(x_x^{(i-1)n+j})}{h_{(i-1)n+j}} \quad , \quad \forall\, i,j = 1,...,n \qquad D_h = \mathbb{R}_+^{n^2} \tag{30}$$

and

$$J(h_x^1,...,h_x^{n^2},0)_{ij} :=$$
$$\begin{cases} \dfrac{F_i(x^{(i-1)n+j} + F_i(x^{(i-1)n+j})\ e_j) - F_i(x^{(i-1)n+j})}{F_i(x^{(i-1)n+j})} & \text{, if } F_i(x^{(i-1)n+j}) \neq 0 \\ F'(x^{(i-1)n+j})_{ij} & \text{, otherwise} \end{cases} \tag{31}$$
$$D_h = \{0\}\ ,\ \forall\, i,j = 1,...,n$$

are in neighborhoods of regular roots strongly consistent approximations, as we can point out by the one-dimensional mean value Theorem.

The next step is to extend the term of multisplittings (7) to nonlinear functions:

Definition 2.3 Let $F : D_0 \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function, differentiable on D_0. A pair $(\mathcal{B},\mathcal{C})$ of sets of operators $\mathcal{B} = \{B^k \mid k \in \mathbb{K}\}$

$$B^k : D_0^m \times D_{\tilde{h}^1} \subset \mathbb{R}^{n^m} \times \mathbb{R}^p \to L(\mathbb{R}^n)$$

and $\mathcal{C} = \{C^k \mid k \in \mathbb{K}\}$

$$C^k : D_0^m \times D_{\tilde{h}^1} \subset \mathbb{R}^{n^m} \times \mathbb{R}^p \to L(\mathbb{R}^n) \qquad \forall\, k \in \mathbb{K}$$

is called a *multisplitting* of F' on D_0, if

$$\text{(i)} \qquad 0 \in \overline{D_{\tilde{h}^1}}$$
$$\text{(ii)} \qquad \det\Big(B^k(X,h)\Big) \neq 0$$
$$\text{(iii)} \qquad \lim_{\substack{\max_l \|x^l - x\| \to 0 \\ h \to 0 \\ h \in D_{\tilde{h}^1}}} B^k(x^1,...,x^m,h) - C^k(x^1,...,x^m,h) = F'(x)$$
$$\forall\, X \in D_k^m\ ,\ k \in \mathbb{K}\,.$$

Example 3: The most known condition for Theorems of convergence for asynchronous iterations is to assume that the system is an M-system (or related to such a system): Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function and $\xi \in D$ be a regular root of it. $F'(\xi)$ be a regular M-matrix and $J : D^m \times D_{\tilde{h}} \subset \mathbb{R}^{nm} \times \mathbb{R}^p \to L(\mathbb{R}^n)$ a consistent approximation at F' on D. It is well known, that an M-matrix (s. for example [2]) has many decompositions or splittings

$$F'(\xi) = B^k - C^k \qquad k \in \mathbb{K}\,,$$

such that there is a positive vector $v \in \mathbb{R}^n$ and a $\omega \in [0,1)$ with

$$B^{k^{-1}}\, C^k\, v \;\leq\; \omega\, v \qquad\qquad \forall\; k \in \mathbb{K}\,, \tag{32}$$

and where in addition the B^k's are *easier to invert* than $F'(\xi)$. For example, they are lower triangular matrices or something similar. Since the set of matrices with property (32) is open in $L(\mathbb{R}^n)$, and by assuming, that a likewise decomposition of consistent approximations at F' depends continuously on $x^1, ..., x^m$ and h, it is reasonable to expect by continuity, that there are open neighborhoods D_0 of ξ and a number $c \in \mathbb{R}_+$, such that for all $h \in \{\, h_0 \in D_{\tilde{h}^1} \mid \|\, h_0 \,\| \leq c \,\}$ and $x^i \in D_0$, $i = 1, ..., m$

$$B^k(x^1, ..., x^m, h)^{-1}\, C^k(x^1, ..., x^m, h)\, v \;\leq\; \omega_0\, v \qquad\qquad \forall\; k \in \mathbb{K}$$

holds for some $\omega_0 \in [0,1)$ and the same v as above.

3 Main results

Sequential Newton-iterative methods are methods, where the problem of computing the next Newton step is avoided and replaced by the following strategy: Given an approximation (or iteration) $x(j) = x(j_0)$ and a splitting $F'(x(j)) = B(x(j)) - C(x(j))$. Now (*inner or linear*) approximations for the next Newton step are computed by

$$x(j_i) := B(x(j))^{-1} C(x(j))\; x(j_{i-1}) \;+\; B(x(j))^{-1}\Big(F'(x(j))\, x(j) - F(x(j))\Big) \tag{33}$$
$$i = 1, ..., k\,,$$

and thereafter a new (*outer or nonlinear*) iteration is obtained by setting $x(j+1) := x(j_k)$. In [7],§11.1.5 it was shown, that now

$$x(j+1) \;=\; x(j) \;-\; \sum_{q=0}^{k-1} \Big(B(x(j))^{-1} C(x(j))\Big)^q\; B(x(j))^{-1}\; F(x(j)) \qquad j = 0, 1, \ldots\,. \tag{34}$$

and, if $\rho(B(\xi)^{-1} C(\xi)) < 1$, that such methods converge locally and at least linear with

$$\mathcal{R} \;\leq\; \rho\Big(B(\xi)^{-1}\, C(\xi)\Big)^k\,.$$

This means, on the whole, k-times as fast as the so called Newton-SOR method, where $k = 1$. Thereby in a k-step Newton-iterative method, we have to evaluate F and its Jacobian one time for each outer iteration, and in the Newton-SOR method, we have to evaluate F and its Jacobian also one time for each iteration step. A.H. Sherman investigated in [9] in a variant with $k(j) = 2^j$ and showed the optimal quadratic order of convergence. R.E. White extended this result in [11] to the synchronized multisplitting method, introduced by [6]. The following algorithm now makes it possible to compute asynchronously and thereby to decouple outer and inner iterations totally.

Algorithm: Asynchronous Newton-iterative methods

Compute $(\mathcal{G}, \mathcal{K}, \mathcal{S}, \mathcal{J}, \mathcal{T}, \mathcal{H}, x(0))$, where the pool $\mathcal{G}$ is given by

$$G^k(x^1, x^2, h_x^1, h_x^2, h^1, h^2) := x^1 - B^k(h_x^1, h^1)^{-1}\Big(F(x^2) - J(h_x^2, h^2)\,(x^2 - x^1)\Big) \;\; \forall\, k \in \mathbb{K} \tag{35}$$

with help of a multisplitting $(\mathcal{B}, \mathcal{C})$ of F' and some consistent approximation J at F' , where the sequence $\mathcal{H}$ is given by:

$$(h_x^1, h_x^2, h^1, h^2)_j := (h_x^{11}, ..., h_x^{2m}, h^1, h^2)_j := (\begin{pmatrix} x_1(h_1^{11}(j)) \\ \vdots \\ x_n(h_n^{11}(j)) \end{pmatrix}, ..., \begin{pmatrix} x_1(h_1^{2m}(j)) \\ \vdots \\ x_n(h_n^{2m}(j)) \end{pmatrix}, h_j^1, h_j^2),$$

where $h_i^{1l}(j)$ $_{j=0,1,...}$, $h_i^{2l}(j)$ $_{j=0,1,...}$ are sequences from $\mathbb{N}_0$

with $h_i^{rl}(j) \le j \quad \forall\, i = 1, ..., n\,,\; l = 1, ..., m\,,\; r = 1, 2\,,\; j = 0, 1, ...$.

Remark, that we have denoted the approximations of ξ , J and $\mathcal{B}$ depending on, by a parameter. The reason is, that $G^k(\xi, \xi, h_x^1, h_x^2, h^1, h^2) = \xi$ is valid, independent of the values of h_x^1, h_x^2, h^1 and h^2 . Remark also, that there are a lot of equivalent or essentially equivalent possibilities for computing the G^k's practical, for example as in (33).

To prove our main result in Theorem 3.2, we need the following Lemma of perturbation from [7],§2.3.2.:

Lemma 3.1 *Let $A \in L(\mathbb{R}^n)$ be a regular matrix, $\|\cdot\|$ some operator norm. Then any matrix $B \in L(\mathbb{R}^n)$ with*

$$\|A - B\| \cdot \|A^{-1}\| \, < \, 1 \tag{36}$$

is also regular, and

$$\|B^{-1}\| \,\le\, \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\,\|A - B\|} \,. \tag{37}$$

Theorem 3.2 *Let* $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ *be a function,* $\xi \in D$ *its regular root and* $J : D^m \times D_{\tilde{h}^2} \to L(\mathbb{R}^n)$ *be a strongly consistent approximation at* F' *on* D. $\|\cdot\|^v$ *be some weighted maximum norm as in (18). Then:*

(i) *Let* $(\mathcal{B},\mathcal{C})$ *be a multisplitting of* F' *on* D *like in definition 2.3, such that for constants* $C_2, C_3, C_4 \geq 0$

$$\| B^k(h_x, h) - B^k(\xi, ..., \xi, h) \| \leq C_2 \max_l \|h_x^l - \xi\| + C_3 \| h^1 \|,$$
$$\|B^k(\xi, ..., \xi, h)^{-1} \| \leq C_4 \quad \forall\ k \in \mathbb{K}\ ,\ h_x \in D^m\ ,\ h \in D_{h^1}\ . \tag{38}$$

If there is a postive $v \in \mathbb{R}^n$, *such that*

$$| B^k(\xi, ..., \xi, h)^{-1} C^k(\xi, ..., \xi, h) |\ v \leq \omega\ v \tag{39}$$
$$\text{for some } \omega < 1 \quad \forall\ h \in D_{h^1}\ ,\ k \in \mathbb{K}\ ,$$

then there is an open neighborhood D_0 *of* ξ, *such that the pool* $\mathcal{G} = \{G^k \mid k \in \mathbb{K}\}$, *given by*

$$G^k : D_0^2 \times D_0^{m^2} \times D_{h^1} \times D_{h^2} \to \mathbb{R}^n$$
$$G^k(x^1, x^2, h_x^1, h_x^2, h^1, h^2) := x^1 - B^k(h_x^1, h^1)^{-1} \Big(F(x^2) - J(h_x^2, h^2)\,(x^2 - x^1) \Big)$$
$$\forall\ k \in \mathbb{K}$$

is weakly contracting on D_0*'s closing, if for suitable* $c_1, c_2 > 0$:

$$\begin{aligned} D_{h^1} &= \{ h^1 \in D_{\tilde{h}^1} \mid \|h^1\| \leq c_1 \}\ , \\ D_{h^2} &= \{ h^2 \in D_{\tilde{h}^2} \mid \|h^2\| \leq c_2 \}\ . \end{aligned} \tag{40}$$

(ii) *Furthermore, the asynchronous Newton-iterative method* $(\mathcal{G}, \mathcal{K}, \mathcal{S}, \mathcal{J}, \mathcal{H}, x(0))$ *converges to* ξ, *if* $x(0) \in D_0$ *and if* $\mathcal{S}$ *and* $\mathcal{J}$ *are admissible. If they are regulated and*

$$h_i^{rl}(j) \to \infty, \ \text{ if } \ j \to \infty \qquad \forall\ r = 1, 2,\ l = 1, ..., m\ ,\ i = 1, ..., n\ , \tag{41}$$

and

$$h_j^1 \to 0\ ,\ h_j^2 \to 0\ , \quad \text{if} \quad j \to \infty\ , \tag{42}$$

then it converges at least linear to ξ *with*

$$\mathcal{R} \leq \sqrt[s+t+1]{\omega} \tag{43}$$

(iii) *Under the conditions of (i), there is an open neighborhood* D_1 *of* ξ *, such that there are constants* K_0 , K_1 , K_2 , $K_3 \geq 0$ *and nonnegative matrices*

$$A_1^k(h_x^1, h_x^2, h^1) \qquad \forall\, h_x^1, h_x^2 \in D_1^m \,,\; h^1 \in D_{h^1} \,,\; k \in \mathbb{K}$$

and

$$A_2^k(x^2, h_x^1, h_x^2, h^2) \qquad \forall\, x^2 \in D_1 \,,\; h_x^1, h_x^2 \in D_1^m \,,\; h^2 \in D_{h^2} \,,\; k \in \mathbb{K} \,,$$

such that

$$\begin{aligned} &|\, G^k(x^1, x^2, h_x^1, h_x^2, h^1, h^2) - \xi \,| \\ \leq\; & A_1^k(h_x^1, h_x^2, h^1)\,|\, x^1 - \xi \,| \;+\; A_2^k(x^2, h_x^1, h_x^2, h^2)\,|\, x^2 - \xi \,| \end{aligned} \tag{44}$$

and

$$\Big(\, A_1^k(h_x^1, h_x^2, h^1) \;+\; A_2^k(x^2, h_x^1, h_x^2, h^2) \Big)\, v \tag{45}$$

$$\leq \Big(\omega \;+\; K_0 \max_l \|\, h_x^{2l} - \xi \,\|^v \;+\; K_1 \,\|\, h^1 \,\|^v \;+\; K_2 \,\|\, h^2 \,\|^v \;+\; K_3 \|\, x^2 - \xi \,\|^v \Big)\, v$$

$$\forall\, k \in \mathbb{K} \,,\; x^1, x^2 \in D_1 \,,\; h_x^1, h_x^2 \in D_1^m \,,\; h^1 \in D_{h^1} \,,\; h^2 \in D_{h^2} \;.$$

Proof: After multiplying with constants, we can assume that all introduced norms are $\|\cdot\|_\infty$. Then the following inequality holds in a suitable open neighborhood of ξ , if, in addition, c_1 of (40) is suitable.

$$\begin{aligned} &|\, G^k(x^1, x^2, h_x^1, h_x^2, h^1, h^2) - \xi \,| \\ =\; & |\, x^1 - B^k(h_x^1, h^1)^{-1} \Big(F(x^2) - J(h_x^2, h^2)\,(x^2 - x^1) \Big) - \xi \,| \\ \leq\; & |\, x^1 - B^k(\xi, ..., \xi, h^1)^{-1} \Big(F(x^2) - F'(\xi)\,(x^2 - x^1) \Big) - \xi \,| \\ & \quad + \Big| B^k(h_x^1, h^1)^{-1} \Big| \; \Big| J(h_x^2, h^2) - F'(\xi) \Big| \Big(|x^1 - \xi| \;+\; |x^2 - \xi| \Big) \\ & \quad + \Big|\, B^k(h_x^1, h^1)^{-1} - B^k(\xi, ..., \xi, h^1)^{-1} \Big| \;|\, F(x^2) - F'(\xi)\,(x^2 - x^1)\,| \\ \leq\; & \Big| B^k(\xi, ..., \xi, h^1)^{-1}\, C^k(\xi, ..., \xi, h^1) \Big| \;|\, x^1 - \xi \,| \\ & \quad + |\, B^k(\xi, ..., \xi, h^1)^{-1} \,| \;|\, F(x^2) - F(\xi) - F'(\xi)\,(x^2 - \xi)\,| \\ & \quad + \|\, B^k(h_x^1, h^1)^{-1} \,\|_\infty \, e\, e^t \;(\, C_1 \max_l \|\, h_x^{2l} - \xi \,\|_\infty + C_0 \,\|h^2\|_\infty \,)\, e\, e^t \\ & \qquad \cdot \Big(\,|x^2 - \xi| \;+\; |x^1 - \xi| \Big) \end{aligned}$$

$$+ \| B^k(h_x^1, h^1)^{-1} \|_\infty \, e\, e^t \, \| B^k(h_x^1, h^1) - B^k(\xi, ..., \xi, h^1) \|_\infty \, e\, e^t \, C_4 \, e\, e^t$$

$$\cdot \Big(\, | F(x^2) - F(\xi) - F'(\xi)\,(x^2 - \xi) \,| \; + \; | F'(\xi) \,|\, | x^1 - \xi \,| \Big)$$

$$\leq \Bigg(\Big| B^k(\xi, ..., \xi, h^1)^{-1} \, C^k(\xi, ..., \xi, h^1) \Big| + \frac{n \; C_4}{1 - C_4\,(C_2 \, \max_l \|h_x^{1l} - \xi\|_\infty \; + \; C_3 \, \|h^1\|_\infty)}$$

$$\cdot \Bigg(C_1 \, \max_l \| \, h_x^{2l} - \xi \,\|_\infty + C_0 \,\| \, h^2 \,\|_\infty$$

$$+ \, n^2 \,(\, C_2 \;\; \max_l \| \, h_x^{1l} - \xi \,\|_\infty \; + \; C_3 \, \|h^1\|_\infty) \, C_4 \, | \, F'(\xi) \,| \Big) \Bigg) \, e\, e^t$$

$$\cdot \, | \, x^1 - \xi \, |$$

$$+ \Bigg(n \, C_4 \; \frac{L}{2} \| \, x^2 - \xi \,\|_\infty \, e\, e^t \; + \; \frac{n \quad C_4}{1 - C_4\,(\, C_2 \, \max_l \|h_x^{1l} - \xi\|_\infty \; + \; C_3 \, \|h^1\|_\infty)}$$

$$\cdot \Bigg(C_1 \, \max_l \| \, h_x^{2l} - \xi \,\|_\infty + C_0 \, \|h^2\|_\infty$$

$$+ \, n^2 \,(\, C_2 \, \max_l \|h_x^{1l} - \xi\|_\infty \; + \; C_3 \, \|h^1\|_\infty) \, C_4 \, \frac{L}{2} \, \|x^2 - \xi\|_\infty \Bigg) \, e\, e^t \Bigg)$$

$$\cdot \, | \, x^2 - \xi \, |$$

$$=: \;\; A_1^k(h_x^1, h_x^2, h^1) \, | \, x^1 - \xi \, | \; + \; A_2^k(x^2, h_x^1, h_x^2, h^2) \, | \, x^2 - \xi \, | \tag{46}$$

Here, there are implicitly defined matrices, which satisfy (iii). Using lemma 1.6, we also get (i) and (ii). □

Using the chain of inequalities (46), it is possible to obtain some D_0 explicitly, but observe, that we have used rough estimates. (46) shows also, that we do not need *strongly* consistent approximations and the first of the inequalities of (38) for a Theorem of convergence, but then the method generally does not converge linearly anymore.

We have asserted, that asynchronous Newton-iterative methods converge in some sense superlinearly, even if there are fewer and fewer evaluations of component functions of F and the entries of its Jacobian during the iteration process, than inner iterations, but in Theorem 3.2(ii) we have ordered $\mathcal{S}$ and $\mathcal{J}$ to be regulated and showed linear convergence. Therefore, we have to observe the influence of $s_i^2(j)_{\;j=0,1,...}$ and $h_i^{2l}(j)_{\;j=0,1,...} \, \forall \, i, l$ on the behaviour of $x(j)_{\,j=0,1,...}$ more precisely.

Theorem 3.3 *Let the conditions of Theorem 3.2 hold and let additionial* $(\mathcal{G}, \mathcal{K}, \mathcal{S}, \mathcal{J}, \mathcal{H}, x(0))$ *fulfilling, that*

(i) $\quad x(0) \in D_0$

$$
\begin{array}{lll}
\text{(ii)} & t_j \le t & \forall\, j \ge 0 \\
\text{(iii)} & j - s_i^1(j) \le \hat{s}^1 & \forall\, j \ge 0\,,\; i = 1, \dots, n \;\; \textit{for some} \;\; \hat{s}^1 \in \mathbb{N}_0 \\
\text{(iv)} & 2\, s_i^2(j) - j \ge -\hat{s}^1 & \forall\, j \ge 0\,,\; i = 1, \dots, n \\
\text{(v)} & 2\, h_i^{2l}(j) - j \ge -\hat{s}^1 & \forall\, j \ge 0\,,\; i = 1, \dots, n\,,\; l = 1, \dots, m \\
\text{(vi)} & \| \, h_j^2 \, \|^v \le K_3 \, \| \, x_j^2 - \xi \, \|^v & \forall\, j \ge 0\,, \;\; \textit{for some} \;\; K_3 \ge 0
\end{array} \tag{47}
$$

Then there are constants $\omega < 1$ *and* $\lambda_0 \ge 0$ *, such that*

$$
| \, x(j) - \xi \, | \; \le \; \omega^{\,j} \, \lambda_0 \, v \qquad \forall\, j \ge 0 \,. \tag{48}
$$

Proof: It suffices to prove this for some λ_0 , such that

$$
0 < \lambda_0 \; < \; \frac{1 - \omega_0}{K_0 + K_1 \, K_3 \, + K_2}
$$

and

$$
\omega \; := \; \sqrt[\hat{s}^1 + t + 1]{\omega_0 \; + \; (\, K_0 + K_1 \, K_3 \; + K_2 \,) \, \lambda_0}
$$

in the case of

$$
\| \, x(0) - \xi \, \|^v \; \le \; \lambda_0 \, \omega_0^t \,.
$$

By an induction we will show, that

$$
| \, x_i(j) - \xi_i \, | \; \le \; \omega^{\,j + t} \, \lambda_0 \, v_i \qquad \forall\, i \in J_{j-1} \;\; \forall\, j \ge 1 \,, \tag{49}
$$

which completes the proof, since the case $i \notin J_j$ can arise by (ii) only t-times in a series. If $i \in J_0$, then

$$
\begin{aligned}
| \, x_i(1) - \xi_i \, | \; &= \; | \, G_i^{k(0)}(x_0^1, x_0^2, (h_x^1, h_x^2, h^1, h^2)_0) - \xi_i \, | \\
&\le \; A_1^{k(0)}((h_x^1, h_x^2, h^1)_0)_i \, | \, x(0) - \xi \, | \; + \; A_2^{k(0)}(x(0), (h_x^1, h_x^2, h^2)_0)_i \, | \, x(0) - \xi \, | \\
&\le \; \omega_0 \, \lambda_0 \, \omega^t \, v_i \; + \; \left(K_0 \, \max_l \, \| \, (h_x^{2l})_0 - \xi \, \|^v \; + \; K_1 \, \| \, h_0^2 \, \|^v \; + \; K_2 \, \| \, x(0) - \xi \, \|^v \right) \omega^t \, \lambda_0 \, v_i \\
&\le \; \omega^t \, \lambda_0 \, v_i \left(\omega_0 \; + \; \omega^t \, \lambda_0 \, (\, K_0 \; + \; K_1 \, K_3 \; + \; K_2 \,) \right) \\
&\le \; \omega^{t+1} \, \lambda_0 \, v_i \qquad .
\end{aligned}
$$

If the assertion is shown for all $q = 1, \dots, j$, then, if $i \in J_j$:

$$
\begin{aligned}
| \, x_i(j+1) \, - \xi_i \, | \; &\le \; \omega_0 \, \lambda_0 \, \omega^{\,j - \hat{s}^1} \, v_i \\
&\quad + \left(K_0 \, \max_l \, \| \, (h_x^{2l})_j - \xi \, \|^v \; + \; K_1 \, \| \, h_j^2 \, \|^v \; + \; K_2 \, \| \, x_j^2 - \xi \, \|^v \right) \omega^{\min_i \{ s_i^2(j) \}} \, \lambda_0 \, v_i \\
&\le \; \omega^{\,j} \, \lambda_0 \, v_i \left(\omega_0 \, \omega^{\,-\hat{s}^1} \; + \; \omega^{-j} \, \lambda_0 \, (\, K_0 \; + \; K_1 \, K_3 \; + \; K_2 \,) \, \omega^{\frac{j - \hat{s}^1}{2}} \omega^{\frac{j - \hat{s}^1}{2}} \right) \qquad (50) \\
&\le \; \omega^{\,j+t} \, \lambda_0 \, v_i \, \omega^{\,-\hat{s}^1 - t} \, (\omega_0 \; + \; (\, K_0 \; + \; K_1 \, K_3 \; + \; K_2 \,) \, \lambda_0 \,) \; = \omega^{\,j+1+t} \, \lambda_0 \, v_i \;\; \square
\end{aligned}
$$

Now we have obtained, that $x(j)_{\, j=0,1,...}$ converges linearly to ξ, even if all $s_i^2(j)_{\, j=0,1,...}$ and $h_i^{2l}(j)_{\, j=0,1,...}$ are going only half as fast to infinity, than all $s_i^1(j)_{\, j=0,1,...}$. If now the evaluation of F's component functions and the entries of its Jacobian, are such a hard task, that we can ignore the number of inner iterations, then we can attain quadratical convergence:

Corollary 3.4 *Let $p : \mathbb{N}_0 \to \{0, ..., \lfloor \frac{\hat{s}^1}{2} \rfloor\}$ be some map. If then for some $i_0 \in \{1, ..., n\}$*

$$s_{i_0}^2(j) \;=\; 2^{\lfloor \lg(j) \rfloor} - p(\lfloor \lg(j) \rfloor) \qquad \forall\, j \geq 1 \,, \tag{51}$$

condition (iv) in Theorem 3.3 is not violated. The sequence $2^j{}_{j=0,1,...}$ denotes for large j that steps of iteration, where a new evaluation of F_{i_0} is involved. $x(2^j)_{\, j=0,1,...}$ converges, if the other conditions in Theorem 3.3 also hold, with

$$\mathcal{O} \geq 2 \tag{52}$$

to ξ.

Remark: The results of this article are part of [8].

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