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# THE CANONICAL CLASS AND THE $C^\infty$-PROPERTIES OF KÄHLER SURFACES.

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A compact complex surface $X$ with non negative Kodaira dimension, has a unique minimal model $X_{\min}$. The pullback of the canonical line bundle minimal model $\omega_{\min}$ is in some ways the most basic birational invariant of the surface, if only because it is the polarisation $\mathcal{O}(1)$ of the canonical model $\mathrm{Proj}(\oplus H^0(nK))$. Recently, Kronheimer, Mrowka and Tian, Yau proved that the cohomology class, $K_{\min} = c_1(\omega_{\min}) \in H^2(X, \mathbb{Z})$ is invariant under oriented diffeomorphism up to sign for minimal surfaces of general type with $p_g > 0$ [Ste]. While completing this manuscript, Friedman and Morgan posted a proof for the case $p_g = 0$ [FM3]. In the case of elliptic surfaces it was already known to be true by the joint effort of many people, as it is a direct consequence of the invariance of the multiplicities of the elliptic fibration.

The proof is based on fundamental work of Witten and Seiberg [Wit], who introduced a new set of non linear equations, the monopole equations, which allow to define new differentiable invariants, similar in spirit to the Donaldson invariants, but much easier to handle. During the Stillwater conference in November 1994, Stefan Bauer and I learned about these new invariants and the invariance of $K_{\min}$, and we decided to run a seminar on Seiberg Witten theory in Bielefeld. I started to think about a proof for the invariance of $K_{\min}$ using the SW invariants along the lines of [Bru]. A little to my surprise, it worked out beautifully. The monopole equations which define the SW classes, once specialised to the Kähler case, give all the necessary information, without the necessity to prove a full Thom conjecture type of result.

In fact it even turned out to be possible to deal with the case $p_g = 0$ and the elliptic surfaces simultaneously and almost uniformly. There is no need to use any result from “classical ” Donaldson theory either. From the point of view of classification of surfaces, it is very satisfactory that the various levels of nefness of $K_{\min}$ (nef and big, nef but not big, torsion) is what makes the proof work for Kodaira dimension $\kappa \geq 0$, what makes it fail for the rational and ruled case, and what makes for the difference in the different Kodaira dimensions. If $p_g = 0$ the higher plurigenera, and in particular the 2-canonical system plays a direct role.

While proving the invariance of the canonical class, we have to prove the invariance of $(-1)$-curves as well. This leads directly to the differentiable characterisation corollary 3 of rational and ruled surfaces which are characterised algebraically by the existence of a smooth rational curve $l$ with $l^2 \geq 0$ [BPV, Prop. V.4.3]. The

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invariance of the Kodaira dimension (Van de Ven conjecture) and the invariance of the plurigenera for surfaces of general type is then almost an afterthought. The Van de Ven Conjecture had already been solved with Donaldson theory ([FM2] for all surfaces but rational surfaces and surfaces of general type with $p_g = 0$, and Friedman Qin [FQ] and Pidstrigatch [P-T],[Pi2] for the remaining cases, see also [OT1] for an easy proof of the remaining case with Seiberg Witten theory).

**Theorem 1.** *If $X$ is a Kähler surface of non negative Kodaira dimension then*

1. *$K_{\min}$ is determined by the underlying oriented manifold up to sign,*
2. *every $(-1)$-sphere is $\mathbb{Z}$-homologous to a $(-1)$-curve up to sign.*

**Corollary 2.** *If a Kähler surface $X$ has non negative Kodaira dimension then every smooth sphere $S$ with $S^2 \geq 0$ is $\mathbb{Z}$-homologous to $0$,*

**Corollary 3.** *A Kähler surface is rational or ruled if and only if it contains a smooth sphere $S \neq 0 \in H^2(X, \mathbb{Z})$ with $S^2 \geq 0$.*

**Corollary 4.** *The Kodaira dimension of a Kähler surface is determined by the underlying differentiable manifold.*

To deal with the case $p_g = 0$, we encounter higher dimensional moduli spaces and more to the point, moduli spaces that have larger than virtual dimension. However, following Pidstrigatch and Tyurin, we will identify the multiplicity of a Seiberg Witten class as a localised Euler class of an infinite rank bundle with a Fredholm section, and the oversized moduli spaces will cause no problem at all. In addition without much extra work, the computation will give us the Seiberg Witten multiplicities of all Kähler elliptic surfaces. It demonstrates my belief that Seiberg Witten theory for surfaces is completely computable. An elegant argument of Stefan Bauer, then gives yet another proof that for elliptic surfaces with finite cyclic fundamental group the multiplicities of the elliptic fibration are determined by the underlying oriented manifold. The oriented homotopy type determines the multiplicities for other elliptic surfaces [FM2, Theorem S.7]. By the first two chapters of [FM2] (now probably the most difficult part of the story) this implies

**Theorem 5.** *Let $X \to C$ be an elliptic Kähler surface. Then the multiplicities of the elliptic fibration are determined by the underlying oriented smooth manifold. In particular for Kähler elliptic surfaces deformation equivalent, and oriented diffeomorphic are the same notions.*

This theorem has been well established with Donaldson theory by the work of Bauer, Kronheimer, Fintushel, Friedman, Morgan, Mrowka, O'Grady and Stern.

**Corollary 6.** *The plurigenera of a Kähler surface are determined by the underlying oriented manifold.*

Let me remark that it seems to be known that in the non Kähler case, with the exception of the equivalence of deformation and diffeomorphism equivalence of non Kähler elliptic surfaces, (where there can be a two to one discrepancy) all the previous statements are true as well, but seemingly for "classical" reasons like the homotopy type.

Inspired by results in the preprint of Friedman and Morgan I realised how the results in this article give an easy proof of

**Corollary 7.** *No Kähler surface of non negative Kodaira dimension admits a metric of positive scalar curvature*

While working on this article a flood of information on the Seiberg Witten classes came in. The holomorphic interpretation of the monopole equations is already in Wittens paper [Wit], and it seems that several people have remarked that his work implies that the canonical class is invariant for minimal surfaces of general type with $p_g > 0$ because of the numerical connectedness of the canonical divisor. Kronheimer informed me that he, Fintushel, Mrowka,Stern and Taubes are working on a note containing among many other things the mentioned proof of the invariance of $K_{\min}$. The results and methods of the before mentioned paper [FM3] of Friedman and Morgan are rather similar to the present one. The main difference seems to be that they deal mostly with the case $p_g = 0$, and that they rely on chamber changing formulas and a detailed analysis of the chamber structure. They also use a stronger version of the blow up formula which allows them to prove a stronger version of theorem 1.2: if a surface of non negative Kodaira dimension has a connected sum decomposition $X \cong X'\#N$, where $N$ is negative definite, then $H_2(N,\mathbb{Z}) \subset H_2(X,\mathbb{Z})$ is spanned by $(-1)$-curves. We will indicate how this result follow from the present methods. Finally Taubes shows that the results for Kähler surfaces are but the top of the iceberg. It seems that most results can be generalised to symplectic manifolds [Ta1],[Ta2].

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## 1. Preparation

We first prove the corollaries from the main theorems 1 and 5.

*Proof.* Corollary 2. Let $S$ be a positive sphere on a surface with $\kappa \geq 0$. Blow up $n = S^2 + 1$ times. Now $e = S + E_1 + \cdots + E_n$ is a $(-1)$-sphere. Hence there is a $(-1)$-curve $E_0$ such that $e = \pm E_0 \in H_2(X,\mathbb{Z})$. Then $S = \pm E_0$ , or $e = E_0 = E_1$ say. The first possibility leads to the contradiction $E_0^2 \geq 0$, the second to $S = 0 \in H_2(X,\mathbb{Z})$. (Reducing non negative spheres to $(-1)$-spheres is a well known trick, but I forgot where I read it precisely.)

Corollary 3 follows from corollary 2.

Corollary 4.By the above, a Kähler surface is of Kodaira dimension $-\infty$ if it contains a non trivial (0)-sphere. Clearly all ruled surfaces contain one. To deal with $\mathbb{P}^2$, note that there is no surface with $b_+ = b_1 = 0$. Thus diffeomorphisms between surfaces with $b_2 = 1$, $b_1 = 0$ are automatically orientation preserving.

Then a surface diffeomorphic to $\mathbb{P}^2$ must contain a $(+1)$-sphere, and is therefore of Kodaira dimension $-\infty$. Since $b_2 = 1$ it must in fact be equal to $\mathbb{P}^2$ (alternatively use Yau's result that $\mathbb{P}^2$ is the only surface with the homotopy type of $\mathbb{P}^2$ [BPV, Theorem 1.1], but this is a deep theorem). We conclude that Kodaira dimension $-\infty$ can be characterised by just diffeomorphism type. Without loss of generality we can therefore assume that $\kappa \geq 0$.

If $K_{\min}^2 > 0$, then $X$ is of general type. If $K_{\min}^2 = 0$ and $K_{\min}$ is not torsion, then $\kappa(X) = 1$, finally if $K_{\min}$ is torsion, $\kappa(X) = 0$. This proves that Kodaira dimension is determined by the oriented diffeomorphism type. If $X$ and $Y$ are orientation reversing diffeomorphic, both are minimal, otherwise one of them would contain a positive sphere. Then necessarily either $K_X^2 = K_Y^2 = 0$, or both have $K_X^2, K_Y^2 > 0$, i.e. $X$ and $Y$ are of general type. Now copy the argument of [FM2, lemma S.4]: for minimal surfaces with $\kappa = 0, 1$, the signature $\sigma = \frac{1}{3}(K^2 - 2e) \leq 0$. Thus $\sigma(X) = -\sigma(Y) = 0$, and $e(X) = e(Y) = 0$. In Kodaira dimension 0, this leaves only tori and hyperelliptic surfaces, which can fortunately be recognised by homotopy type [FM2, lemma 2.7].

Corollary 6. Since $P_1 = p_g$ is an oriented topological invariant we will whence assume that $n \geq 2$. We have to distinguish between the different Kodaira dimensions. For surfaces of general type (i.e $\kappa = 2$) we argue as follows. The plurigenera $P_n$ and $\chi(O_X)$ are birational invariants. Then by Ramanujan vanishing and Riemann Roch (cf. [BPV, corollary VII.5.6]) we have

$$P_n(X) = P_n(X_{\min}) = \tfrac{1}{2}n(n-1)K_{\min}^2 + \chi(\mathcal{O}_X) \tag{1}$$

Since $\chi(\mathcal{O}_X)$ is an oriented topological invariant the $P_n$ are oriented diffeomorphism invariants in this case. For surfaces with Kodaira dimension 0 or 1 with a fundamental group that is not finite cyclic, we simply quote [FM2, S.7]. For surfaces with finite cyclic fundamental group, it follows from the invariance of the multiplicities and the canonical bundle formula which gives an explicit formula for $P_n(X)$ in terms of the multiplicities and $\chi(O_X)$. (see [FM2, lemma I.3.18, prop. I.3.22]). □

Here is an other easy corollary

**Corollary 8.** *Every $(-2)$-sphere $\tau$ is orthogonal to $K_{\min}$ . If there is a $(-1)$-curve $E_1$ such that $\tau \cdot E_1 \neq 0$, then there is a $(-1)$-curve $E_2$ such that $\tau = \pm E_1 \pm E_2 \in H_2(X, \mathbb{Z})$.*

*Proof.* Let $R_\tau$ be the reflection in $\tau$. It is represented by a diffeomorphism with support in a neighborhood of $\tau$. By the invariance of $K_{\min}$ up to sign, $R_\tau K_{\min} = K_{\min} + (\tau \cdot K_{\min})\tau = \pm K_{\min}$. But if $K_{\min} \neq 0 \in H^2(X, \mathbb{Q})$, then $\tau$ and $K_{\min}$ are indepent, since $\tau^2 = -2$ and $K_{\min}^2 \geq 0$. Thus in either case $(\tau, K_{\min}) = 0$. Moreover if $E_1$ is a $(-1)$-curve then either $R_\tau E_1 = E_1$, $R_\tau E_1 = -E_1$, or there is a different $(-1)$-curve $E_2$ such that $R_\tau(E_1) = \pm E_2$. The first possibility gives $\tau \cdot E_1 = 0$, the second $(\tau \cdot E_1)^2 = 2$ i.e. is impossible, and the third $(\tau \cdot E_1) = \pm 1$. The statement follows. □

It will be convenient to first prove the main theorem 1 with (co)homology groups with $\mathbb{Q}$ coefficients, and later mop up to prove the theorem over $\mathbb{Z}$. Theorem 1 mod torsion is a formal consequence of the existence of a set of basic classes

$$\mathcal{K}(X) = \{K_1, K_2 \ldots\} \subset H^2(X, \mathbb{Z})$$

functorial under oriented diffeomorphism between 4-manifolds with $b_+ \geq 1$, and having the following properties:

**Properties** $(*)$**.** For every Kähler surface $X$ of non negative Kodaira dimension

1. the $K_i$ are of type $(1,1)$ i.e. represented by divisors,
2. if $X$ is minimal, then for every Kähler form $\Phi$, $\deg_\Phi(K_X) \geq |\deg_\Phi(K_i)|$,
3. if $\tilde{X} \xrightarrow{\sigma} X$ is the blow-up of a point $x \in X$, then $\sigma_* \mathcal{K}(\tilde{X}) = \mathcal{K}(X)$.
4. every $K_i$ is characteristic i.e. $K_i \equiv w_2(X) \pmod 2$,
5. $K_X \in \mathcal{K}$.

In the case that $X$ is an algebraic surface we could replace item 2 by weaker and more geometric requirement that $2g(H) - 2 \geq H^2 + |K_i \cdot H|$ for every very ample divisor $H$ without changing the results. We will see later that Seiberg Witten theory will give such an inequality for all surfaces minimal or not. This should not be confused with a Thom conjecture type of statement, since our methods do not give information about the minimal genus for arbitrary smooth real surfaces in a homology class. It is also clearly impossible to have a degree inequality like property 2 for all Kähler forms if $X$ is rational or ruled.

Recall that for algebraic surfaces, the Mori cone $\overline{\mathrm{NE}}(X) \subset H_2(X,\mathbb{R})$ is the closure of the cone generated by effective curves. It is dual to the nef (or Kähler) cone. In other words, the numerical equivalence class of a curve $D$ lies in $\overline{\mathrm{NE}}(X)$ if and only if $H \cdot D \geq 0$ for all $H$ ample. For a Kähler surface $(X,\Phi)$, it will be convenient to define the nef cone as closure of the positive cone in $H^{1,1}(X) \subset H^2(X,\mathbb{R})$ spanned by all Kähler forms, and containing $\Phi$. The Mori cone $\overline{\mathrm{NE}}$ is then just the dual cone in $H_2(X,\mathbb{R}) \cap {H^{1,1}}^\vee$ i.e.

$$\overline{\mathrm{NE}} = \{ C \in {H^{1,1}}^\vee \subset H_2(X,\mathbb{R}) \mid \int_C \omega \geq 0, \text{ for all Kähler forms } \omega \}.$$

(With this definition, a line bundle is nef iff for all $\epsilon > 0$, it admits a metric such that the curvature form $F$ has $\frac{\sqrt{-1}}{2\pi} F \geq -\epsilon\Phi$. A class $\omega \in \overline{\mathrm{NE}}$ if there exists a sequence of closed positive currents of type $(1,1)$ converging to the dual of $\omega$, i.e is $\overline{\mathrm{NE}}$ dual to $N_{\mathrm{psef}}$ in [Dem, proposition 6.6]. I am grateful to Demailly for explaining this to me). We will freely identify homology and cohomology by Poincaré duality.

**Lemma 9.** *If a class $L \in H^{1,1}(X)$ satisfies $\deg_\Phi(K_X) \geq |\deg_\Phi(L)|$ for all Kähler forms $\Phi$, then there is a unique decomposition of the canonical divisor $K_X = D_+ + D_-$ with $D_+$, $D_- \in \overline{\mathrm{NE}}(X)$ such that $L = D_+ - D_-$.*

*Proof.* Define $D_\pm = \frac{1}{2}(K_X \pm L)$. Then $K_X = D_+ + D_-$, $L = D_+ - D_-$, and $D_\pm \in \overline{\mathrm{NE}}$. □

The following simple lemma is a minor generalisation of the fact that the canonical divisor of a surface of general type is numerically connected [BPV, VII.6.1].

**Lemma 10.** *Let $X$ be a minimal Kähler surface of non negative Kodaira dimension. Suppose there is a decomposition $K_X = D_+ + D_-$ with $D_+$, $D_- \in \overline{\mathrm{NE}}(X) \subset H^{11}(X)$. Then $D_+ \cdot D_- \geq 0$, with equality if and only if say $K_X \cdot D_+ = D_+^2 = 0$. Thus if $X$ is of general type then $D_+ = 0$, if $\kappa(X) = 1$, then $D_+ = \lambda K_X$ with $0 \leq \lambda \leq 1$, and finally $D_+ = D_- = 0$ if $\kappa(X) = 0$.*

*Proof.* First assume that $D_+^2 \leq 0$. Since $K_X$ is nef, $D_+ \cdot D_- = (K_X - D_+) \cdot D_+ \geq -D_+^2 \geq 0$, with equality iff $K_X \cdot D_+ = D_+^2 = 0$. If $D_+^2 > 0$ and $D_-^2 > 0$, then using the Kähler form $\Phi$, we can write $D_+ = \alpha\Phi + C_+$ and $D_- = \beta\Phi + C_-$ with $\alpha, \beta > 0$ and $C_\pm \in \Phi^\perp$. By the Hodge index theorem,

$$D_+ \cdot D_- = \alpha\beta\Phi^2 + C_+ \cdot C_- \geq \alpha\beta\Phi^2 - \sqrt{-C_+^2}\sqrt{-C_-^2} > 0.$$

The statement for surfaces of general type follows directly from Hodge index and the fact that $K_X^2 > 0$. If $\kappa(X) = 1$, then $K_X$ is a generator of the unique isotropic subspace of $K_X^\perp$, so $D_+ = \lambda K_X$, and $D_- = (1-\lambda)K_X$. Since $K_X$, $D_+$ and $D_- \in \overline{\mathrm{NE}}(X)$, $\lambda$ is bounded by $0 \leq \lambda \leq 1$. Finally if $\kappa(X) = 0$, $K_X$ is numerically trivial and, $D_+$ and $D_-$ must be zero as well. □

**Lemma 11.** *Let $X$ be a surface of non negative Kodaira dimension with $(-1)$-curves $E_1, \dots E_m$. Assume that $\mathcal{K}$ has properties $(*)$. Then $K_i^2 \leq K_X^2$ for all $K_i \in \mathcal{K}$, with equality if and only if*

$$K_i = \lambda K_{\min} + \sum \pm E_i \in H^2(X, \mathbb{Q})$$

*where $\lambda = \pm 1$ if $X$ is of general type, $\lambda$ is a rational number with $|\lambda| \leq 1$ if $\kappa(X) = 1$, and where $\lambda = 0$ if $\kappa(X) = 0$.*

*Proof.* By property (3), and (4), $K_i = K_{i,\min} + \sum_j (2a_{ij} + 1)E_j$. Thus

$$K_i^2 \leq K_{i,\min}^2 - \#(-1)\text{-curves},$$

with equality if and only if $a_{ij} = 0$, or $-1$ for all $i, j$. Since $K_X^2 = K_{\min}^2 - \#(-1)$-curves, we can assume that $X$ is minimal. Using property (1) and (2) and lemma 9, write $K_X = D_+ + D_-$ and $K_i = D_+ - D_-$, with $D_\pm \in \overline{\mathrm{NE}}(X)$. Then by lemma 10 $K_i^2 = K_X^2 - 4D_+ \cdot D_- \leq K_X^2$ with equality under the stated condition. Note that this lemma does not use diffeomorphism invariance, nor that $K_X \in \mathcal{K}$. □

We are now in a position to formulate and prove half of the main theorem

**Proposition 12.** *Assume that for all 4-manifolds $X$ with $b_+ \geq 1$ there is a set of basic classes $\mathcal{K}(X) = \{K_1, K_2, \dots\} \subset H^2(X, \mathbb{Z})$ functorial under oriented diffeomorphism having properties $(*)$. Then $K_{\min}$ is an oriented $C^\infty$ invariant up to sign and torsion, and every $(-1)$-sphere is represented by a $(-1)$-curve up to sign and torsion.*

*Proof.* Using lemma 11 we can easily reduce the invariance of $K_{\min}$ up to sign and torsion to showing that $(-1)$-spheres are represented by $(-1)$-curves up to sign and torsion.

Since $K_X \in \mathcal{K}$, there is a nonempty subset $\mathcal{K}_0 = \{K_j\} \subset \mathcal{K}$ with $K_j^2 = K_X^2 = 2e(X) + 3\sigma(X)$. Consider the projection $K_{j,\min}$ of $K_j$ to the minimal model i.e. the projection to the orthogonal complement of the $(-1)$-spheres.

If $K_{j,\min}^2 > 0$, then by lemma 11, $X$ is of general type, and $K_{j,\min} = \pm K_{\min}$ up to torsion. If $K_{j,\min}^2 = 0$, there are two possibilities. If $K_{j,\min}$ is torsion for all $j$, then again by lemma 11, $X$ is of Kodaira dimension 0 i.e. $K_{\min}$ is also torsion. Otherwise we choose $j$ such that $K_{j,\min} \neq 0$ has maximal divisibility. Since $K_X \in \mathcal{K}_0$ our little lemma shows that, the Kodaira dimension is 1 and $K_{j,\min} = \pm K_{\min}$.

Now let $e$ be the class of a $(-1)$-sphere in $H^2(X, \mathbb{Q})$. Without loss of generality, we can assume that $K_X \cdot e < 0$. Consider $R_e$ the reflection generated by a $(-1)$-sphere $e$. It is represented by an orientation preserving diffeomorphism. Since $\mathcal{K}$ is invariant under oriented diffeomorphisms, the characterisation of basic classes with square $K_X^2$ tells us that

$$R_e K_X = K_{\min} + \sum E_i + 2(K_X \cdot e)e \tag{2}$$

$$= \lambda K_{\min} + \sum \pm E_i \tag{3}$$

with $|\lambda| \leq 1$. Taking intersection with $E_i$ we find that $(E_i \cdot e)(e \cdot K_X) = 0$ or 1. Since $K_X \cdot e \equiv e^2$ is odd, $e$ is either orthogonal to all $(-1)$ curves (i.e. $e \in H^2(X_{\min}, \mathbb{Q})$) or there is a $(-1)$-curve, say $E_1$, such that $K_X \cdot e = E_1 \cdot e = -1$. However, $e \in H^2(X_{\min})$ implies that $e = \frac{\lambda - 1}{2K_X \cdot e} K_{\min}$, which is impossible because $K_{\min}^2 \geq 0$. Thus, after renumbering the $(-1)$-curves, (2) and (3) can be rewritten to

$$e = \tfrac{1}{2}(1-\lambda)K_{\min} + \sum_{i=1}^{N} E_i \tag{4}$$

with $N = \frac{1}{4}(1-\lambda)^2 K_{\min}^2 + 1$.

Now reflect $e$ in $E_1^{\perp}$. $R_{E_1} e$ is yet another $(-1)$-sphere, so it has a representation as in equation (4), except possibly for an overall sign

$$\begin{aligned} R_{E_1} e &= \tfrac{1}{2}(1-\lambda)K_{\min} - E_1 + \sum_{i=2}^{N} E_i \\ &= \pm\big(\tfrac{1}{2}(1-\mu)K_{\min} + \sum_{j=1}^{M} E_{i_j}\big). \end{aligned}$$

Upon comparison, we see that the sign is minus, that $N = M = 1$, and that $0 \leq 1 - \lambda = \mu - 1 \leq 0$ unless $K_{\min} = 0$. In other words $e = E_1 \in H^2(X, \mathbb{Q})$. □

## 2. The localised Euler class of a Banach bundle.

We will use a construction pioneered by Pidstrigatch and Pidstrigatch Tjurin [Pi1], [P-T, §2], which is a convenient and general way to define fundamental cycles for moduli spaces arising from elliptic equations. Unfortunately their construction is not quite in the generality we will need it, and we will therefore set it up in fairly large generality here. The cycle is the localised homological Euler class of an infinite dimensional bundle. It can be used to give definitions that avoid transversality arguments needing small deformations, generic metrics etcetera, although transversality will be extremely useful for computations and proofs. The construction is modeled on Fultons intersection theory and in the complex case it makes the whole machinery of excess intersection theory available. However, although the construction is very simple in principle, the whole thing has turned a bit technical. On first reading it is best to ignore the difference between Čech and singular homology, and continue to proposition 14, the construction of the Euler class in the proof of this proposition and corollary 15. Some readers might even want to continue to the next section, since we will use rather little of the general machinery for the proofs of the theorems and corollaries in the introduction.

We first make some algebraic topological preparations. For any pair of topological spaces $A \subset X$, homology with closed support and with local coefficients $\xi$ is defined as

$$H_i^{\mathrm{cl}}(X, A; \xi) = \lim_{\leftarrow K} H_i(X, A \cup (X - K); \xi)$$

where we take the limit over all compacta $K \subset X - \overset{\circ}{A}$. $H^{\mathrm{cl}}_*$ is functorial under proper maps. Unfortunately this "homology theory" suffers the same tautness problems that singular homology has. To be able to work with well behaved cap products we will have to complete it. The following works well enough for our purposes but is a bit clumsy.

Suppose that $X$ is *locally modelable* i.e. is locally compact Hausdorff and has local models which are each subsets of some $\mathbb{R}^n$. Obviously locally compact subsets of locally modelable spaces are locally modelable, in particular a closed subset of a local modelable space is locally modelable. Then for every compact subset $K \subset X - \overset{\circ}{A}$ there is a neighborhood $U_K \supset K$ in $X$ which embeds in $\mathbb{R}^N$. We now define

$$\check{H}^{\mathrm{cl}}_i(X, A, \xi) = \lim_{\underset{K}{\leftarrow}} \check{H}_i(U_K, A \cap U_K \cup (U_K - K); \xi)$$

where for every pair $(Y, B)$ in a manifold $M$, Čech homology is defined as

$$\check{H}_i(Y, B) = \lim_{\leftarrow}\{H_i(V, W),\ (V, W) \text{ neighborhoods of } (Y, B) \text{ in } M\}$$

This definition depends neither on the choice of $U_K$, nor on the embedding $U_K \hookrightarrow \mathbb{R}^N$, since two embeddings are dominated by the diagonal embedding, and $\check{H}_*(Y, B)$ does not depend on $M$ but only on $(Y, B)$ (c.f [Dol, VIII.13.16]).

Fortunately we do not usually have to bother with Čech homology. Suppose in addition that $X$ is locally contractible e.g. locally a sub analytic set (c.f. [GM, §I.1.7], and the fact that Whitney stratified spaces admit a triangulation). Then $X$ is locally an Euclidean neighborhood retract (ENR) by [Dol, IV 8.12] and since in a Hausdorff space a finite union of ENR's is an ENR by [Dol, IV 8.10] we can assume that $U_K$ is an ENR. Now assume that $A$ is open. Then by [Dol, prop. VIII 13.17]

$$\check{H}_*(U_K, U_K \cap A \cup (U_K - K)) \cong H_*(U_K, U_K \cap A \cup (U_K - K)) \cong H_i(X, A \cup X - K).$$

Thus in this case $\check{H}^{\mathrm{cl}}_*(X, A) = H^{\mathrm{cl}}_*(X, A)$. If $A$ is closed and locally contractible then one should be able to organise things such that $U_K \cap A$ is an ENR and the same conclusion would hold.

**Lemma 13.** *Let $X$ be a locally modelable space, and $Z$ a locally compact (e.g. closed) subspace, then there are cap products*

$$\check{H}^i(X, X - Z, \xi) \otimes \check{H}^{\mathrm{cl}}_j(X, \xi') \xrightarrow{\cap} \check{H}^{\mathrm{cl}}_{j-i}(Z, \xi \otimes \xi')$$

*with the following properties.*

1. *If $Y$ is locally embeddable, $f\colon Y \to X$ is proper, and $\sigma' \in \check{H}^{\mathrm{cl}}_j(Y, Y - f^{-1}(Z))$, then the push-pull formula holds:*

$$f_*(f^* c \cap \sigma') = c \cap f_* \sigma'.$$

2. *If $Z \overset{i}{\hookrightarrow} W$ is proper and $W$ is locally compact, we can increase supports i.e.*

$$c|_{(X, X-W)} \cap \sigma = i_*(c \cap \sigma).$$

*Proof.* For every $c \in \check{H}^i(X, X - Z)$ and $\sigma \in \check{H}^{\mathrm{cl}}_j(X)$, we have to construct a class $c \cap \sigma \in \check{H}_{i-j}(Z, Z - K)$ for a cofinal family of compacta $\{K\}$. Since $Z$ is locally compact, every compactum $K$ is contained in a compactum $L \subset Z$ with $L \supset \overset{\circ}{L} \supset K$. Likewise there exists a compactum $L' \supset\supset L$. By excision it suffices to construct a class in $\check{H}_{i-j}(L, L-K)$. Let $U_{L'}$ be a neighborhood of $L'$ in $X$ which embeds in $\mathbb{R}^N$. Let $V_L$, $W_{L-K} \subset V_L$, and $V_K \subset V_K$ be neighborhoods of respectively $L$, $L_K$ and $K$ in $\mathbb{R}^N$. Define $U_L = V_L \cap U_{L'}$. We can assume that $U_L \cap Z = U_L \cap L'$, $V_K \cap Z = V_K \cap L$, and after replacing $V_{L-K}$ by $(V_{L-K} - (L' \cap W^c_{L-K}) \cup V_K$, that $V_L \cap (L' - K) = W_{L-K} \cap (L' - K)$. Then our task is to construct a class $c_L \cap \sigma_L \in H_{i-j}(V_L, W_{L-K})$ possibly after shrinking $V_L$ and $W_{L-K}$.

We have a restriction map $\check{H}^i(X, X-Z) \to \check{H}^i(U_L, U_L - L')$. After shrinking $V_L$ if necessary, $c|_{(U_L, U_L - L')}$ comes from a class $c_L \in H^i(V_L, V_L - L')$. By definition there is map

$$\check{H}^{\mathrm{cl}}_j(X) \to \check{H}_j(U_L, U_L - K) \to H_j(V_L, V_L - K).$$

Let $\sigma_L \in H_j(V_L, V_L - K)$ be the image of $\sigma$.

Now write $V_L - K = (V_L - L') \cup (W_{L-K} - K)$. Then the standard cap product [Dol, VII Def. 12.1] gives a map

$$H^i(V_L, V_L - L') \otimes H_j(V_L, V_L - K) \xrightarrow{\cap} H_{j-i}(V_L, W_{L-K} - K)$$

so we get a class $c_L \cap \sigma_L \in H_{j-i}(V_L, W_{L-K})$ as required. Since if $K' \supset K$, choices for $K'$ will work a fortiori for $K$, we can pass to the limit.

To prove the first property, note that since $f$ is proper, $f^{-1}Z$ is locally compact. Choose compacta $K \subset\subset L \subset\subset L' \subset Z$ giving compacta $f^{-1}K \subset\subset f^{-1}L \subset\subset f^{-1}L'$. Note that compacta of the form $f^{-1}K$ are a cofinal family of compacta in $f^{-1}(Z)$. Embed neighborhoods $U_{L'} \subset V_{L'} \subset \mathbb{R}^N$ and $U_{f^{-1}L'} \subset \mathbb{R}^M$. Now we carry out the construction above with the diagonal embedding of $U_{f^{-1}L'}$ in $\mathbb{R}^{N+M}$. Let $V_{f^{-1}L'}$ be a neighborhood of $U_{f^{-1}L'} \in \mathbb{R}^{N+M}$. We can assume that $V_{f^{-1}L'} \to V_{L'}$ under the projection $\pi$ to $\mathbb{R}^N$. We can also assume that $c|_{(U_L, U_L - L')}$ comes from a class $c_L \in H^i(V_L, V_L - L')$. Finally let $\sigma_{f^{-1}L'}$ be an image of $\sigma$ in $H_j(V_{f^{-1}L'}, \pi^{-1}W_{K-L})$. Then the first property follows from the identity

$$\pi_*(\pi^* c_L \cap \sigma'_{f^{-1}L'}) = c_L \cap \pi_* \sigma'_{f^{-1}L'}$$

in $H_j(V_l, W_{K-L})$. The second property is left to reader. □

A smooth manifold $X$ of dimension $n$, has an orientation system $or(X)$, the sheafification of the presheaf $U \to H^n(X, X-U)$. Equivalently, we can define $or(X)$ as the sheaf $R^d\pi_*(X \times X, X \times X - \Delta, \mathbb{Z})$, where $\Delta$ is the diagonal of $X \times X$, $\pi$ the projection on the first coordinate, and $R^d\pi_*$ the parametrised version of the $d$ th cohomology.

Likewise for a real vector bundle $E$ of rank $r$ there is an orientation system $or(E)$, the sheafification of $H_q(E|_U, E|_U - U)$. We have $or(X) = or(TX)^\vee$, as can be seen immediately from the alternative description of $or(X)$ and excision.

A manifold $X$ has a unique fundamental class $[X] \in H^{cl}_n(X, or(X))$ in singular or Čech homology such that for small $U$

$$[X]|_{X-U} \in H_d(X, X-U, H^d(X, X-U)) = \mathrm{Hom}(H^d(X, X-U), H^d(X, X-U))$$

is identified with the identity (cf [Spa, p. 357]).

Similarly, a bundle $E$ has a Thom class $\Phi_E \in \check{H}^r(E, E-X, or(E))$ [Spa, p. 283]. In turn for every section $s$ in $E$ with zero set $Z(s)$, the Thom class defines a localised cohomological Euler class $e(E,s) = s^*\Phi_E \in \check{H}^r(X, X - Z(s), or(E))$.

Let $M$ be a Banach manifold, $E$ a real Banach vector bundle on $M$ and $s$ a section of $E$ with zero set $Z(s)$. The section induces an exact sequence

$$0 \to E \to s^*TE \xrightarrow{\pi} TM \to 0, \tag{5}$$

which expresses that the vertical tangent bundle of the total space of $E$ is canonically isomorphic to the bundle $E$. On $Z(s)$ we have a canonical splitting of this sequence, given by the sequence

$$0 \to TM \xrightarrow{Ts_0} s_0^*TE \to E \to 0$$

defined by the zero section $s_0$, and the identification $s^*TE|_{Z(s)} = s_0^*TE|_{Z(s)}$ over $Z(s)$. This gives a canonical map

$$Ds: TM|_{Z(s)} \xrightarrow{Ts} s^*TE = s_0^*TE \to E|_{Z(s)}.$$

If $D$ is a connection on $E$ then $D(s)$ is a splitting that extends the canonical splitting over $Z(S)$ (hence the notation) but in general connections need not exist on Banach manifolds. We will avoid choosing non canonical splittings.

**Proposition 14.** *Let $M$ be a smooth Banach manifold, $E$ a banach bundle over $M$ and $s$ a section in $E$. Assume that*

1. *The map $Ds$ is a section in the bundle* $\mathrm{Fred}^{\mathrm{d}}(\mathrm{TM}|_{\mathrm{Z(s)}}, \mathrm{E}|_{\mathrm{Z(s)}})$ *of Fredholm maps of index $d$. We say that $Z(s)$ has virtual dimension $d$, and that $Ds$ is Fredholm of index $d$.*
2. *The real line bundle* $\det(\mathrm{Ind}(Ds))$ *is trivialised over $Z(s)$.*

*Then these data define a Čech homology class with closed support*

$$\mathbb{Z}(E, s) = \mathbb{Z}(s) \in \check{H}_d^{\mathrm{cl}}(Z(s), \mathbb{Z})$$

*with the following properties.*

1. *The class $\mathbb{Z}(s) = [Z(s)]$ if $Z(s)$ is smooth of dimension $d$ and carries the natural orientation defined by the trivialisation of* $\det(\mathrm{Ind}Ds)$,
2. *if $\{C\}$ is a family of closed subsets of $M$ such that $C \cap Z(s)$ is compact for all $C$, then there is a natural map $\check{H}_j(Z(s)) \to \lim\limits_{\leftarrow C} H_j(M, M - C, \mathbb{Z})$, and if $s_t$ is a one parameter family of sections with this property then $\mathbb{Z}(s_0) = \mathbb{Z}(s_1)) \in \lim\limits_{\leftarrow C} H_d(M, M - C, \mathbb{Z})$.*

*For every exact sequence*

$$0 \to E' \to E \to E'' \to 0,$$

*defined over a neighborhood of $Z(s)$, let $s''$ be the induced section in $E''$, and $s'$ the induced section of $E'|_{Z(s'')}$ with zero set $Z(s)$. Then*

3. *if $E'$ has finite rank*

$$\mathbb{Z}(s) = e(E'|_{Z(s'')}, s') \cap \mathbb{Z}(s''),$$

4. *if $Ds''|_{Z(s)}$ is surjective, then $Z(s'')$ is smooth in a neighborhood of $Z(s)$, $Ds': TZ(s'')|_{Z(s)} \to E'|_{Z(s)}$ is Fredholm, with* $\mathrm{Ind}Ds' \cong \mathrm{Ind}Ds$ *and*

$$\mathbb{Z}(E, s) = \mathbb{Z}(E'|_{Z(s'')}, s').$$

For property 2 there are two typical situations we have in mind. One is that we have a natural connected family of sections $s_t$ such that $Z(s_t)$ is compact for all $t \in T$. In this situation we get a homology class $\mathbb{Z}(s_{t_0}) \in H_d(M)$ independent of the choice of $t_0$ (take $\{C\} = \{M\}$). Such will be the case in Seiberg Witten theory. In the other case we again have a family of sections $s_t$ but there is "bubbling" which invariably means we lack some a priori estimate. For example in Donaldson theory, the moduli space of ASD connections with curvature bounded in the $L^4$ norm is compact. Therefore it is natural to define a family of subsets $\{\mathcal{B}^{\leq C}\}_{C \in \mathbb{R}^+}$ in the space $\mathcal{B}^*$ of all irreducible $L^2_2$ connections mod gauge, where $\mathcal{B}^{\leq C}$ the subset of connections with $L^4$ norm of the curvature bounded by $C$.

*Proof.* If $M$ (hence $E$) is a finite dimensional manifold of dimension $N+d$ then $E$ is a real vector bundle of rank $N$ with an isomorphism $\det(E) = \det(TM)$ over $Z(s)$. Let $[M] \in H^{\mathrm{cl}}_N(M, or(M))$ be the fundamental class, and $\Phi_E$ the twisted Thom class of $E$ in $H^{N-d}(E, E-M, or(E))$. Define

$$\mathbb{Z}(s) = e(E,s) \cap [M] \in \check{H}^{\mathrm{cl}}_d(Z, or(E) \otimes or(M)) = \check{H}^{\mathrm{cl}}_d(Z(s), \mathbb{Z})$$

i.e. $\mathbb{Z}(s)$ is the Poincaré dual of the localised cohomological Euler class. In the last step we used the chosen trivialisation of $or(E) \otimes or(M) = or(\det TM^\vee \otimes \det E) = or(\det(\mathrm{Ind}(Ds)))$ given by the trivialisation of the index.

In the infinite dimensional case we proceed similarly but we have to go through a limiting process and use that we know what to do when the section is regular. For each compactum $K \subset Z$ we have to construct a class $\mathbb{Z}_K \in \check{H}_d(Z, Z-K)$ such that for $K' \supset K$ the class $\mathbb{Z}_{K'}|_{Z-K} = \mathbb{Z}_K$ under the restriction map $H_d(Z, Z-K') \to H_d(Z, Z-K)$.

Over a neighborhood $U$ of $K$ in $M$ we can find a finite rank $N$ subbundle $F$ of $E$ such that $\mathrm{Im}(Ds)|_K + F|_K = E|_K$. Such a bundle certainly exists: we can choose a finite number of sections $s_1, \ldots s_N$ such that the $s_i$ span $\mathrm{Coker}(Ds_x)$ for every $x \in K$, and possibly after perturbing we can assume that the $s_i$ are linearly independent in a neighborhood. Let $\tilde{E}$ be the quotient bundle $E/F$ defined over $U$, and $\tilde{s}$ the induced section with zero set $M_f = Z(\tilde{s})$ ($f$ is for finite, $M$ is for, well, manifold).

Clearly the map $TM|_{Z(s)} \xrightarrow{Ds} E|_{Z(s)} \to \tilde{E}$ is surjective. Since the canonical map $D\tilde{s}$ on $M_f$ restricts to this composition on $Z(s)$, $D\tilde{s}$ is surjective on $M_f$ possibly after shrinking $U$. Hence $M_f$ is a smooth manifold. Let $T = \ker(TM|_{M_f} \to \tilde{E})$. There is a canonical identification $T \cong TM_f$. Now $T$ is a bundle of rank $N+d$ since

$$\mathrm{Ind}(Ds)|_K = T - F. \tag{6}$$

Thus $M_f$ has dimension $N+d$.

On $M_f$, the section $s$ in $E$ lifts to a section $s_f$ of the subbundle $F$. Clearly $Z(s_f) = Z(s) \cap U$. Define

$$\mathbb{Z}_K = e(F|_{M_f}, s_f) \cap [M_f] \in \check{H}_d(Z(s), Z(s) - K; \mathbb{Z})).$$

Here we have used the restriction map

$$\check{H}^{\mathrm{cl}}_d(Z(s) \cap U; or(F) \otimes or(M_f)) \to \check{H}_d(Z(s), Z(s) - K; or(F) \otimes or(M_f)),$$

the identification $or(\det(\mathrm{Ind}(Ds))) = or(F) \otimes or(M_f)$ and the chosen trivialisation of $\det(\mathrm{Ind}(Ds))$ as in the finite dimensional case.

This construction does not depend on the choices. If $F$ and $F'$ are two choices of subbundles of $E$ then there is third bundle $F''$ containing $F+F'$. We can therefore assume that $F$ is a subbundle of $F'$. Then using primes to denote objects we get out of the construction above using $F'$ instead of $F$, we have a section $s'_f$ in $F'$, a section $s''_f$ in $F'/F$ cutting out $M_f$ in $M'_f$ and the identity

$$\begin{aligned}\mathbb{Z}'_K &= e(F'|_{M'_f} s'_f) \cap [M'_f] \\ &= e(F|_{M_f}, s_f) \cap e(F'/F|_{M'_f}, s''_f) \cap [M'_f] \\ &= e(F|_{M_f}, s_f) \cap [M_f] = \mathbb{Z}_K.\end{aligned}$$

Note that in the third step we have used the identification $or(M_f) = or(M'_f) \otimes or(F'/F))|_{M_f}$. In particular, if $K' \supset K$ all choices on $K'$ work a fortiori for $K$, so we can pass to the limit.

The relation $\mathbb{Z}(s) = [Z(s)]$ for regular sections (property 1), and the compatibility with Euler classes of finite rank bundles (property 3) are now clear from the construction. The stability property 4 also follows from the construction. For every compactum $K$, we can choose the finite rank subbundle $F$ as a subbundle of $E'$. Then $\tilde{E} \twoheadrightarrow E''$. Now one checks that by a diagram chase that

$$Z(\tilde{E}, \tilde{s}) = Z(E'/F|_{Z(E'',s'')}, s' \bmod F)$$

and that

$$\begin{aligned}TZ(\tilde{E}, \tilde{s}) &= \mathrm{Ker}(TM \to \tilde{E}) \\ &= \mathrm{Ker}(\mathrm{Ker}(TM \to E'') \to E'/F) \\ &= \mathrm{Ker}(TZ(s'') \to E'/F) = TZ(E'/F).\end{aligned}$$

In particular, the orientations agree. Thus we see that

$$\begin{aligned}\mathbb{Z}_K(E, s) &= e(F, s_f) \cap [Z(\tilde{E}, \tilde{s})] \\ &= e(F, s_f) \cap [Z(E'/F|_{Z(E'',s'')}, s' \bmod F)] = \mathbb{Z}_K(E'|_{Z(s'')}, s').\end{aligned}$$

It only remains to pass to the limit over $K$.

To see that $\check{H}^{\mathrm{cl}}_j(Z(s))$ maps to $\lim_{\leftarrow C} H_j(M, M - C)$ note that for every compact subset $K = C \cap Z(s)$, we constructed a finite dimensional manifold $M_f \supset K$. Then we have maps

$$\begin{aligned}\check{H}^{\mathrm{cl}}_j(Z(s)) \to &\check{H}_j(Z(s), Z(s) - K) = \check{H}_j(Z(s) \cap M_f, Z(s) \cap M_f - K) \\ &\to H_j(M_f, M_f - C) \to H_j(M, M - C).\end{aligned}$$

Again this map is independent of choices, and we can pass to the limit.

The homotopy property of $\mathbb{Z}$ is a formal consequence of the compatibility with finite dimensional Euler classes. Consider the trivial bundle $\mathbb{R}$ over the interval $[-1, 2]$ with the one parameter family of sections $\theta - \tau$ where $\theta: [-1, 2] \to \mathbb{R}$ is the inclusion and $0 \leq \tau \leq 1$. Then clearly $e(\mathbb{R}, \theta) = e(\mathbb{R}, \theta - 1) \in H^1([-1, 2], \{-1, 2\})$ is the canonical generator. Consider $M \times [-1, 2]$. Let $\pi: M \times [-1, 2] \to M$ be the projection and $S: M \times [-1, 2] \to \pi^* E$ an extension of our one parameter family of sections e.g. $S_t = s_0$ for $t \leq 0$ and $S_t = s_1$ for $t \geq 1$. The bundle $\pi^* E \oplus \mathbb{R}$ has a one parameter family of sections $(S, \theta - \tau)$. Now

$$\begin{aligned}\mathbb{Z}(s_0) &\overset{4}{=} \pi_* \mathbb{Z}(\pi^* E \oplus \mathbb{R}; (S,\theta))\\ &\overset{3}{=} \pi_* e(\mathbb{R},\theta) \cap \mathbb{Z}(\pi^* E; S)\\ &= \pi_* e(\mathbb{R},\theta - 1) \cap \mathbb{Z}(\pi^* E; S)\\ &= \pi_* \mathbb{Z}(\pi^* E \oplus \mathbb{R}; (S,\theta - 1)) = \mathbb{Z}(s_1)\end{aligned}$$

□

**Corollary 15.** *(compare* [P-T, prop. III.2.4]*) Let $M$ be a complex Banach manifold, $E$ a holomorphic vector bundle and $s$ a holomorphic section with zero set $Z(s)$ Assume that $Ds$ is a section of* $\mathrm{Fred}^{\mathrm{d}}_{\mathbb{C}}(\mathrm{TM}|_{\mathrm{Z(s)}}, \mathrm{E}|_{\mathrm{Z(s)}})$. *We say that $Z(s)$ has complex virtual dimension $d$, and that $Ds$ is Fredholm of complex index $d$. Then the localised Euler class $\mathbb{Z}(s) = [Z(s)] \in H^{\mathrm{cl}}_{2d}(Z(s), \mathbb{Z})$, if $Z(s)$ is a local complete intersection of dimension $d$, and more generally*

$$\mathbb{Z}(s) = [c(\mathrm{Ind}(Ds))^{-1} c_*(Z(s))]_{2d} \tag{7}$$

*where $c_*(Z(s))$ is the total homological chern class of $Z(s)$ defined analogous to* [Ful, example 4.2.6] *by equation* (10) *and coincides with the Poincaré dual of the cohomological chern classes of the tangent bundle if $Z(s)$ is smooth.*

*Remark 16.* If $Z(s)$ is smooth we can even get away with an almost complex manifold $M$ and the assumption that $Ds$ is complex linear.

*Remark 17.* I have tacitly removed $M$ and $E$ from the notation of the homological Chern class $c_*(Z(s))$. I strongly believe that $c_*(Z(s))$ is independent of the embedding but I did not prove this. There is one case where independence of $c_*(Z(s))$ on the embedding can be proved completely analogous to [Ful, Example 4.2.6] by simply replacing algebraic arguments by complex analytic ones: if for every $K \subset Z(s)$ compact, there exists a *holomorphic* finite rank sub bundle $F \hookrightarrow E$ defined over a neighborhood of $K$ such that $F|_K + \mathrm{Im}(Ds)|_K = E|_K$. Then a neighborhood $U_K$ of $K$ in $Z(s)$ sits in a complex rather then almost complex finite dimensional manifold $M_f$. Such a bundle should typically exist if $Z(s)$ has the structure of a quasi projective variety, and $\mathrm{Coker}\, DS$ has the interpretation of a coherent sheaf as in [Pi1, §5, §6].

*Proof.* We will use Mac Phersons graph construction, that is we consider the limit $\lambda \to \infty$ of the map $(\lambda s{:}\, 1)$ in $\mathbb{P}(E \oplus \mathcal{O})$ or finite dimensional approximations thereof. We use the notations of the proof of proposition 14.

For a compactum $K \subset Z(s)$ we choose the finite rank bundle $F$ as follows. It is a complex bundle, and in every point of $Z(s)$ there are sections of $F$ which restricted to a neighborhood are holomorphic sections of $E$ and which span locally a subbundle $F^{\mathrm{hol}} \hookrightarrow F$, such that $Ds{:}\, TE|_{Z(s)} \twoheadrightarrow E/F^{\mathrm{hol}}|_{Z(s)}$ is a surjection. We do not assume that $F$ is a holomorphic subbundle, because I do not see a reason why such a bundle should exist. However since $F$ is a complex bundle, the quotient $\tilde{E} = E/F$ and

$$TM_f|_{Z(s)} = T|_{Z(s)} = \mathrm{Ker}(TM|_{Z(s)} \xrightarrow{Ds} E|_{Z(s)} \to \tilde{E}|_{Z(s)})$$

are complex bundles. We extend this complex structure on $TM_f$ over all of $M_f$, possibly after shrinking $M_f$, making it into an almost complex manifold of complex dimension $d + N$.

Consider the space $\mathbb{P}(F \oplus \mathcal{O}) \xrightarrow{\pi} M_f$. Then the total space of $F$ embeds in $\mathbb{P}(F \oplus \mathcal{O})$. The image of the zero section will also be called the zero section, and the complement of $F$ the divisor at infinity.

Let $Q$ be the universal quotient bundle. The bundle $Q$ has sections $\overline{(0,1)}$ , and $\overline{(\lambda s_f, 1)}$ cutting out the zero section and the graph of $\lambda s_f$ respectively. Equivalently we can cut out the graph of $\lambda s_f$ by $\overline{(s_f, 1/\lambda)}$. Then clearly as $\lambda \to \infty$ the graph degenerates to a set contained in the zero set of $\overline{(s_f, 0)}$.

Now $Z(\overline{(s_f,0)})$ has two "irreducible components". One component $\tilde{M}_f \hookrightarrow \mathbb{P}F$ is the closure of the image of $(s_f{:}\,0)\colon M_f - Z(s) \to \mathbb{P}F|_{M_f - Z(s)} \subset \mathbb{P}(F \oplus \mathcal{O})$ It will be called the strict transform. The other component is just $\mathbb{P}(F \oplus \mathcal{O})|_{Z(s)}$. Let $\mathcal{E}_f = \tilde{M}_f \cap \mathbb{P}(F \oplus \mathcal{O})|_{Z(s)}$. It will be called the exceptional divisor.

I claim that

$$\check{H}^{\mathrm{cl}}_{2d+2N-1+i}(\mathcal{E}_f) = 0 \text{ for } i \geq 0. \tag{8}$$

Accepting this claim we see that $\tilde{M}_f$ carries a unique fundamental class $[\tilde{M}_f]$ restricting to $[\tilde{M}_f - \mathcal{E}_f]$ by the exact sequence

$$\check{H}^{\mathrm{cl}}_{2d+2N}(\mathcal{E}_f) \to \check{H}^{\mathrm{cl}}_{2d+2N}(\tilde{M}_f) \to H^{\mathrm{cl}}_{2d+2N}(\tilde{M}_f - \mathcal{E}_f) \to \check{H}^{\mathrm{cl}}_{2d+2N-1}(\mathcal{E}_f)$$

Consider $C' = \mathbb{Z}(\overline{(s_f,0)}) - [\tilde{M}_f] \in \check{H}^{\mathrm{cl}}_{2d+2N}(Z(\overline{(s_f,0)}))$. Then $C'$ comes from a unique class $C \in \check{H}^{\mathrm{cl}}_{2d+2N}(\mathbb{P}(F \oplus \mathcal{O})|_{Z(s)})$ because of the sequence.

$$0 \to \check{H}^{\mathrm{cl}}_{2d+2N}(\mathbb{P}(F \oplus \mathcal{O})|_{Z(s)}) \to \check{H}^{\mathrm{cl}}_{2d+2N}(Z(\overline{(s_f,0)})) \to H^{\mathrm{cl}}_{2d+2N}(\tilde{M}_f - \mathcal{E}_f).$$

Now note that $Q$ restricted to the zero section is canonically isomorphic to $F$. We therefore have the following chain of equivalences

$$\begin{aligned}
\mathbb{Z}(s)_K &= e(F, s_f) \cap [M_f] \\
&= \pi_* e(\pi^* F, \lambda s_f) \cap e(Q, \overline{(0,1)}) \cap [\mathbb{P}(F \oplus \mathcal{O})] \\
&= \pi_* \left( e(Q, \overline{(\lambda s_f, 1)}) \cup e(Q, \overline{(0,1)}) \right) \cap [\mathbb{P}(F \oplus \mathcal{O})] \\
&= \pi_* e(Q, \overline{(0,1)}) \cap e(Q, \overline{(s_f, 1/\lambda)}) \cap [\mathbb{P}(F \oplus \mathcal{O})] \\
&= \pi_* e(Q, \overline{(0,1)}) \cap \left( e(Q, \overline{(s_f, 0)}) \cap [\mathbb{P}(F \oplus \mathcal{O})] \right). \\
&= \pi_* e(Q, \overline{(0,1)}) \cap \mathbb{Z}(\overline{(s_f, 0)})
\end{aligned}$$

If we accept the claim (8) for a moment and we note that the support of $\tilde{M}_f$ and $e(Q, \overline{(0,1)})$ are disjoint we see further that

$$\mathbb{Z}(s)_K = \pi_* e(Q, \overline{(0,1)}) \cap C' = \pi_* e(Q) \cap C$$

If we use that $e(Q) = c_{\mathrm{top}}(Q)$ this can be rewritten further to

$$\begin{aligned}
\mathbb{Z}(s)_K &= [\pi_* c(Q) \cap C)]_{2d} \\
&= [c(F) \pi_* \left( (1-h)^{-1} \cap C \right)]_{2d} \\
&= [c(F - T) \left( c(T) s_*(Z(s), M_f) \right)]_{2d}
\end{aligned}$$

where we used the notation $h = c_1(\mathcal{O}_{\mathbb{P}(F \oplus \mathcal{O})}(-1))$ and

$$s_*(Z(s), M_f) \stackrel{\mathrm{def}}{=} \pi_*(1-h)^{-1} C \tag{9}$$

for the total homological Segre class of the normal cone (this terminology will be justified in a minute). But $c(F-T) = c(\mathrm{Ind}Ds)^{-1}$ and since $T = TM_f$,

$$c_*(Z(s)) \stackrel{\text{def}}{=} c(T)s_*(Z(s), M_f) \tag{10}$$

is exactly the analogue of the homological chern classes of [Ful, example 4.2.6].

We show that $c_*(Z(s))$ does not depend on the choice of $F$. Again it suffices to treat the case that $F' \subset F$. We use primes whenever an object is associated to $F'$. The independence follows directly from a formula for the Segre classes which expresses how it behaves under the extension $M'_f \subset M_f$ in terms of the normal bundle $F/F'$ of $M'_f \subset M_f$.

$$s_*(Z(s), M_f) = c(F/F')^{-1} s_*(Z(s), M'_f). \tag{11}$$

Assuming (11), we see that

$$\begin{aligned} c_*(Z(s)) &= c(T)s_*(Z(s), M_f) \\ &= c(T)c(F/F')^{-1}s_*(Z(s), M'_f) = c(T')s_*(Z(s), M'_f). \end{aligned}$$

In particular we can take the limit over $K$.

Formula (11) is well known for integrable complex manifolds [Ful, example 4.1.5], and we will follow the proof closely. There are two terms in the class $C$ occurring in the definition (9) of the Segre class, which we treat separately.

Note that there is a regular section $\sigma$ of $F/F'(1)$ on $\mathbb{P}(F \oplus \mathcal{O})|_{M_f}$ cutting out $\mathbb{P}(F' \oplus \mathcal{O})|_{M_f}$. Therefore

$$\begin{aligned} [\mathbb{P}(F' \oplus \mathcal{O})|_{M'_f}] &= e(F/F', s_f \bmod F') \cap [\mathbb{P}(F' \oplus \mathcal{O})|_{M_f}] \\ &= e(F/F', s_f \bmod F') \cap e(F/F'(1), \sigma) \cap [\mathbb{P}(F \oplus \mathcal{O})|_{M_f}]. \end{aligned}$$

Since on $\mathbb{P}(F' \oplus \mathcal{O})|_{M_f}$ there is an exact sequence

$$0 \to Q' \to Q \to F/F' \to 0,$$

we have $e(Q', \overline{(s'_f, 0)}) \cup e(F/F', s_f \bmod F') = e(Q, \overline{(s_f, 0)})$. Then the above implies that

$$\begin{aligned} \mathbb{Z}(Q', \overline{(s'_f, 0)}) &= e(Q', \overline{(s'_f, 0)}) \cap [\mathbb{P}(F' \oplus \mathcal{O})|_{M'_f}] \\ &= e(Q, \overline{(s_f, 0)}) \cap e(F/F'(1), \sigma) \cap [\mathbb{P}(F \oplus \mathcal{O})|_{M_f}] \\ &= e(F/F'(1), \sigma) \cap \mathbb{Z}(Q, \overline{(s_f, 0)}) \end{aligned}$$

As for the other term, on $\tilde{M}_f$ there is a smooth section in $\mathcal{O}(-1)$ given by $(s_f, 0)$ which is an isomorphism $\mathcal{O} \cong \mathcal{O}(-1)$ on $\tilde{M}_f - \mathcal{E}$. It follows that

$$[\tilde{M}'_f - \mathcal{E}] = e(F/F', s_f \bmod F') \cap [\tilde{M}_f - \mathcal{E}] = e(F/F'(1), \sigma) \cap [\tilde{M}_f - \mathcal{E}].$$

Then we have the equality

$$[\tilde{M}'_f] = e(F/F'(1), \sigma) \cap [\tilde{M}_f].$$

because both left and right hand side are cycles supported on $\tilde{M}'_f - \mathcal{E} \cup \mathbb{P}(F' \oplus \mathcal{O})|_{Z(s)}$ restricting to $[\mathcal{M}'_f - \mathcal{E}]$.

For the computation of the Segre class we can forget about the support given by $\sigma$ and use

$$e(F/F'(1)) = c_{\mathrm{top}}(F/F'(1)) = \sum c_{\mathrm{top}-j}(F/F')h^j.$$

Thus we finally get the expression

$$\begin{aligned} s_*(Z(s), M'_f) &= \pi_*\Big(\sum h^{i+j} c_{\text{top}-j}(F'/F) \cap (\mathbb{Z}(Q, \overline{(s_f, 0)}) - [\tilde{M}_f])\Big) \\ &= c(F'/F) s_*(Z(s), M_f) \end{aligned}$$

which we set out to prove.

It remains to prove the claim (8). We first turn to the case that $Z(s)$ is smooth but possibly of the wrong dimension. This condition implies that $\operatorname{Im} Ds|_T \subset F$ has constant rank over $Z(s)$ because $\ker Ds|_T = \ker Ds = TZ(s)$. Then $\operatorname{Im} Ds|_T$ is just the normal bundle $\mathcal{N}$ of $Z(s)$ in $M_f$. Now let us identify the limit set $(s_f{:}\, 1/\lambda)(M_f)$ when $\lambda \to \infty$. If we have a smooth path $\gamma$ with $\gamma(0) = x_0 \in Z(s)$, then we see that $\lim_{t\to 0}(s_f{:}\, 0)(\gamma(t)) = (Ds_f(\frac{d}{dt}|_0\gamma){:}\, 0)$. Therefore $\tilde{M}_f$ is just the blowup $\hat{M}_f$ of $Z(s)$ in $M_f$. This makes sense even though $M_f$ is only an almost complex manifold since the normal bundle $\mathcal{N}$ has a complex structure. The blow up is obtained abstractly by identifying a tubular neighborhood $N_\epsilon$ of $Z(s)$ with the normal bundle, and replacing $N_\epsilon$ with $I = \{(l, x) \in \mathbb{P}\mathcal{N} \times N_\epsilon \mid l \ni x\}$. It is an almost complex manifold, so certainly carries a fundamental class $[\tilde{M}_f]$. It is also clear that $\mathcal{E}_f = \mathbb{P}\mathcal{N}$ is a submanifold of real codimension 2, and certainly satisfies the claim (8).

Let $\mathcal{O}(\mathcal{E}_f)$ be the smooth complex line bundle on the blow-up $\hat{M}_f$ defined by the exceptional divisor $\mathcal{E}_f$, and let $z \in A^0(\mathcal{O}(E))$ be a section cutting out $\mathcal{E}_f = \mathbb{P}\mathcal{N}$ with the proper orientation i.e. $\mathbb{Z}(\mathcal{O}(\mathcal{E}_f), z) = [\mathcal{E}_f]$. On $\hat{M}_f$ the pulled back section is of the form $s_f = z\hat{s}_f$ with $\hat{s}$ nowhere vanishing. Therefore the limit set of $(s_f{:}\, 1/\lambda)(\hat{M}_f)$ in $\mathbb{P}(F \oplus \mathcal{O})|_{\hat{M}_f}$ as $\lambda \to \infty$ is just $(\hat{s}{:}\, 0)(\hat{M}_f) \cup D$ where $D \subset \mathbb{P}(F \oplus \mathcal{O})|_{\mathcal{E}_f}$ is the $\mathbb{P}^1$ bundle joining the zero section $(0{:}\, 1)|_{\mathcal{E}_f}$ and the section $(\hat{s}_f{:}\, 0)$. Then down on $M_f$ the limit set of $(s_f{:}\, 1/\lambda)(M_f)$ is just $\tilde{M}_f \cup C\mathcal{E}_f$, where $C\mathcal{E}_f$ is cone bundle over $Z(s)$ joining $\mathcal{E}_f \subset \tilde{M}_f$ and the zero section.

Now $C\mathcal{E}_f$ represents the homology class $C$. Thus

$$s_*(Z(s), M_f) = \pi_*(1-h)^{-1} C\mathcal{E}_f = \pi_*(1-h)^{-1}\mathcal{E}_f = \pi_*(1-h)^{-1}\mathbb{P}\mathcal{N} = s(\mathcal{N}) \cap [Z(s)]$$

Therefore if $Z(s)$ is smooth we find the expected formula

$$c_*(Z(s)) = c(TM_f)s(\mathcal{N}) \cap [Z(s)] = c(TZ(s)) \cap [Z(s)].$$

Note that in deriving this formula we have not really used the holomorphicity of $s$. It was sufficient that $M$ has an almost complex structure and that $Ds$ is complex linear. Replacing manifolds by stratified spaces the proof carries over essentially verbatim if $Z(s)$ is a local complete intersection since this condition implies that $Ds|_T$ has constant rank, and that we have a well defined normal bundle.

In proving the claim (8) in the general case we use holomorphicity more strongly. We first blow up $Z(s)^{\text{red}}$ in $M$ to get a new infinite dimensional analytic space $\hat{M}$. That this is possible follows from the local analysis of the normal cone in [P-T, §III.1].

Locally on $M$, the exceptional divisor $\mathcal{E} \subset \hat{M}$ can be described as follows. Locally on $M$ we have an exact sequence of holomorphic bundles

$$0 \to F^{\text{hol}} \to E \to \tilde{E}^{\text{hol}} \to 0,$$

such that $TM|_{Z(s)} \twoheadrightarrow \tilde{E}|_{Z(s)}$ is surjective, i.e. locally $F^{\text{hol}}$ can take the role of $F$. Further, locally we can split the sequence since $F^{\text{hol}}$ has finite rank. Let the holomorphic subbundle $\tilde{\tilde{E}} \subset E$ be a lift of $\tilde{E}^{\text{hol}}$. We write $s = s_f^{\text{hol}} \oplus \tilde{\tilde{s}}$ corresponding to the decomposition $E = F^{\mathrm{hol}} \oplus \tilde{\tilde{E}}$. Then locally $\mathcal{E} \cong \mathcal{E}_f^{\mathrm{hol}} \times_{Z(s)} \mathbb{P}\tilde{\tilde{E}}$, where $\mathcal{E}^{\mathrm{hol}}$ is the exceptional divisor of the blow up of $Z(s)$ in $M_f^{\mathrm{hol}}$, and where $M_f^{\mathrm{hol}}$ is the integrable finite dimensional complex manifold $Z(\tilde{s}^{\mathrm{hol}})$. Moreover $\mathcal{E}_f^{\mathrm{hol}}$ is naturally embedded in $\mathbb{P}(F^{\mathrm{hol}} \oplus \mathcal{O})|_{Z(s)} \subset \mathbb{P}(E \oplus \mathcal{O})|_{Z(s)}$. If we are a little more careful and choose $\tilde{\tilde{E}}$ such that $\mathbb{P}\tilde{\tilde{E}}|_{Z(s)} \subset \mathcal{E}$ then $\mathcal{E} = \mathrm{Join}(\mathcal{E}_f^{\mathrm{hol}}, \mathbb{P}\tilde{\tilde{E}}|_{Z(s)}) \subset \mathbb{P}E|_{Z(s)}$.

Let $z \in H^0(\mathcal{O}(\mathcal{E}))$ be a section vanishing exactly along $\mathcal{E}$. On $\hat{M}$ we can decompose the section as $s = z^n \hat{s}$. Therefore, just as in the previous finite dimensional case, $(s{:}\, 1/\lambda)(\hat{M}) \to \mathbb{P}(E \oplus \mathcal{O})$ degenerates to $(\hat{s}{:}\, 0)(\hat{M}) \cup nD$ where $D$ is the $\mathbb{P}^1$ bundle over $\mathcal{E}$ joining the zero section $(0{:}\,1)|_{\mathcal{E}}$ and $(\hat{s}_f{:}\, 0)|_{\mathcal{E}}$. Down on $M$, this means that $(s{:}\, 1/\lambda)(M) \subset \mathbb{P}(E \oplus \mathcal{O})$ degenerates to $\tilde{M} \cup C\mathcal{E}$ where $\tilde{M} \subset \mathbb{P}E$ is isomorphic to $\hat{M}$ with $\hat{M} \cap \mathbb{P}(E \oplus \mathcal{O})|_{Z(s)} \cong \mathcal{E}$, and $C\mathcal{E}$ is the cone bundle over $Z(s)$ joining the zero section and $\mathcal{E}$.

Now we finally come to our claim (8). The set $\mathcal{E}_f = \mathbb{P}(F \oplus \mathcal{O}) \cap \mathcal{E}$. At the very beginning we chose $F$ such that $F \supset F^{\mathrm{hol}}$. Locally we define $\tilde{\tilde{F}} = F \cap \tilde{\tilde{E}}$, then locally $F = F^{\mathrm{hol}} \oplus \tilde{\tilde{F}}$ and locally $\mathcal{E}_f = \mathrm{Join}(\mathcal{E}_f^{\mathrm{hol}}, \mathbb{P}\tilde{\tilde{F}}|_{Z(s)})$. Thus $\mathcal{E}_f$ is a stratified space of real dimension $2d + 2N - 2$, and we are done. ☐

*Remark 18.* In the complex case we have obviously defined a class containing more information about the section. Let

$$\widehat{\mathbb{Z}}(s) = c(\mathrm{Ind}(Ds))^{-1} c_*(Z(s)).$$

## 3. Seiberg Witten classes

We will collect a few facts about Seiberg Witten basic classes in a formulation suitable for arbitrary Kähler surfaces. In the usual formulation, these classes are the support of a certain function on the set of $\mathrm{Spin}^c$-structures. However in the presence of 2-torsion, $\mathrm{Spin}^c$-structures cause endless confusion which is why I have chosen to base my exposition on SC-structures [Kar]. This notion catches the essence of $\mathrm{Spin}^c$-structures, the existence of spinors. It is well suited to the Kähler case and is equivalent to that of a $\mathrm{Spin}^c$-structure in dimension 4. For more details see [Kar].

Let $X$ be a closed oriented manifold of dimension $2n$. Choose a Riemannian metric $g$ with Levi-Civita connection $\nabla^g$, and Clifford algebra bundle $C(X, g) = C(T^\vee X, g)$. There is a natural isomorphism of bundles $c{:}\, \wedge^* T^\vee X \to C(X, g)$ given by anti-symmetrisation. It induces a connection and metric on $C(X, g)$ also denoted $\nabla^g$ and $g$.

An *SC-structure* is a smooth complex vector bundle $W$ of rank $2^n$ together with an algebra bundle isomorphism $\rho{:}\, C(X, g) \to \mathcal{E}nd(W)$. In other words an SC structure is a bundle with the irreducible Clifford algebra representation $\Delta$ in every fibre. A section $\phi \in A^0(W)$ is called a (smooth) spinor. An SC-structure exists if and only if $w_2(X)$ can be lifted to the integers [Kar, §3.4]. Existence will be clear in the case of Kähler surfaces.

SC-structures admit an invariant hermitian metric i.e. one such that Clifford multiplication by 1-forms is skew hermitian (sh). The chirality operator $\Gamma = (\sqrt{-1})^n c(\mathrm{Vol}_g)$ has square 1, and is hermitian. Thus $\Gamma$ has an orthogonal eigenbundle decomposition $W = W^+ \oplus W^-$ with eigenvalue $\pm 1$, the positive and negative spinors of the SC-structure. A one form $\omega \in A^1(X)$ defines an skew hermitian map $c(\omega){:}\, W^\pm \to W^\mp$ which is an isomorphism away from the zero set of $\omega$.

In this paragraph we assume $\dim(X) = 4$. Then $T_X^\vee \cong \mathcal{H}om(W^+, W^-)^{\mathrm{sh}}$. Let $L_W = \det W^+$. Then $L_W \cong \det W^-$, by the isomorphism induced from Clifford multiplication by a generic 1-form, which is an isomorphism outside codimension 4. Thus $W$ is a $\mathrm{Spin}^c(4)$-bundle if we identify

$$\mathrm{Spin}^c(4) = \{(U_1, U_2) \in U(2) \times U(2) \mid \det(U_1) = \det(U_2)\}.$$

We recover the usual definition $\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\mathbb{Z}/2/Z} U(1)$ from the isomorphism $\mathrm{Spin}(4) = SU(2) \times SU(2)$. In any case by chasing around the cohomology sequences of the diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Spin}^c(4) & \to & \mathrm{SO}(4) \times \mathrm{U}(1) & \to 1 \\ & & \| & & \uparrow & & \uparrow & \\ 0 & \to & \mathbb{Z}/2\mathbb{Z} & \to & \mathrm{Spin}(4) & \to & \mathrm{SO}(4) & \to 1 \end{array}$$

we see that $L_W + w_2(X) \equiv 0 \pmod 2$, and that this is the only obstruction to lifting the $SO(4) \times U(1)$ bundle to $\mathrm{Spin}^c(4)$. If $H^2(X, \mathbb{Z})$ has no 2-torsion, the line bundle $L \equiv w_2(X)$ determines such a lift completely, and it is common to speak of the $\mathrm{Spin}^c$-structure $L$.

An *SC-Clifford module* $(S, \langle , \rangle, \nabla)$, is an SC-structure with a non-degenerate invariant hermitian metric $\langle , \rangle$ and a unitary Clifford connection $\nabla$ i.e. a unitary connection such that for all vector fields $X$, spinors $\phi \in A^0(S)$, and 1-forms $\omega$ we have

$$\nabla_X(\omega \cdot \phi) = (\nabla^g_X \omega) \cdot \phi + \omega \cdot \nabla_X \phi.$$

The *Dirac operator* $\not{\partial}$ of a Clifford module is the composition

$$A^0(W) \xrightarrow{\nabla} A^1(W) \xrightarrow{\cdot} A^0(W).$$

It is an elliptic self adjoint first order differential operator, and it maps positive spinors to negative ones and vice versa (i.e. $\not{\partial}\colon A^0(W^\pm) \to A^0(W^\mp)$). Since $\rho$ is parallel, $\nabla$ respects the decomposition $W = W^+ \oplus W^-$. Thus $\nabla$ induces a connection on $L_W$ with curvature $F$.

Much of the usefulness of SC-structures is a consequence of the following easy lemma.

**Lemma 19.** *The set of isomorphism classes $\mathcal{SC}$ of SC-structures is an $H^2(X, \mathbb{Z})$ torsor i.e. if $\mathcal{SC} \neq \emptyset$ and we fix an SC-structure $W_0$, then for every SC-structures $W_1$, there exits a unique line bundle $\mathcal{L}$ such that $W_1 = W_0 \otimes \mathcal{L}$. Every SC-structure $S$ admits a Clifford module structure $(W, \langle , \rangle, \nabla)$. If we fix one SC-Clifford module $(W_0, \langle , \rangle_0, \nabla_0)$, there is a unique triple $(\mathcal{L}, h, d)$ of a smooth line bundle $\mathcal{L}$, with hermitian metric $h$ and unitary connection $d$, such that*

$$(W, \langle , \rangle, \nabla) \cong (W_0, \langle , \rangle_0, \nabla_0) \otimes (\mathcal{L}, h, d). \tag{12}$$

*Proof.* Clearly if $W_0$ is an SC structure, so is $W_0 \otimes \mathcal{L}$ for every line bundle $\mathcal{L}$. Conversely, the bundle of Clifford linear homomorphisms $\mathcal{L}(W_0, W) = \mathcal{H}om_C(W_0, W)$ has rank 1, and the natural map $W_0 \otimes \mathcal{L}(W_0, W) \to W$ is an isomorphism.

For existence of a Clifford module structure see [Kar, prop. 4.2.1, 4.5.1]. It will be clear for Kähler surfaces. It follows directly from the definition of a Clifford module that the natural connection and metric on $\mathcal{H}om(W_0, W)$ leaves $\mathcal{L}(W_0, W)$ invariant. Hence there is an induced metric and connection $(h, d)$ on $\mathcal{L}(W_0, W)$, which has property (12). Conversely if $(W, \langle , \rangle, \nabla)$ is defined by equation (12), then

$$(\mathcal{L}, h, d) = \mathcal{H}om_C\left((W_0, \langle , \rangle_0, \nabla_0), (W_0, \langle , \rangle_0, \nabla_0) \otimes (\mathcal{L}, h, d)\right)$$

which proves uniqueness. □

If a base SC-structure is chosen, the line bundle $\mathcal{L}$ will be called the twisting line bundle.

There is a natural gauge group $\mathcal{G}^{\mathbb{C}}$ acting on a Clifford module, the group of all smooth invertible Clifford linear endomorphisms. $\mathcal{G}^{\mathbb{C}}$ can be canonically identified with $C^\infty(X, \mathbb{C}^*)$. In the representation (12), $\mathcal{G}^{\mathbb{C}} = C^\infty(X, \mathbb{C}^*)$ acts in the usual way on the set of metrics and unitary connections on the twisting line bundle $\mathcal{L}$. Since every hermitian metric on a line bundle is gauge equivalent, so is every Clifford invariant metric on a Clifford module. Thus, up to gauge we can fix an invariant metric and we are left with a residual gauge group $\mathcal{G} = C^\infty(X, \mathrm{U}(1))$.

The set of Clifford connections $\mathcal{A}$ on a fixed hermitian SC structure $(W, \langle,\rangle)$ (i.e. Clifford module structures) is an affine space modeled on $\sqrt{-1}A^1_{\mathbb{R}}(X)$. Using the representation (12) and harmonic representatives, one shows that the set of connections mod gauge is

$$\mathcal{B} = \mathcal{A}/\mathcal{G} \cong \sqrt{-1}A^1_{\mathbb{R}}(X)/d\log C^\infty(X, \mathrm{U}(1)) \cong \mathrm{H}^1_{\mathrm{DR}}(\mathrm{X})/\mathrm{H}^1(\mathrm{X}, \mathbb{Z}) \oplus \ker \mathrm{d}^*$$

We set $\mathcal{P}^* = \mathcal{A} \times A^0(W^+)^*/\mathcal{G}$. It is a $\mathbb{CP}^\infty \times \mathbb{R}^+$ bundle over $\mathcal{B}$. Thus $\mathcal{P}^*$ has the homotopy type of $(S^1)^{b_1(X)} \times \mathbb{CP}^\infty$.

There is an alternative description of $\mathcal{B}$ and $\mathcal{P}^*$ that will be useful. Let $\mathcal{A}^{\mathbb{C}}$ be the set of all Clifford connections, and $\mathcal{H}$ the set of all hermitian metrics on $\mathcal{L}$. Let

$$\mathcal{A}^{\mathrm{mod}} = \{(\nabla, <,>),\ \nabla \text{ is } <,> \text{-unitary}\} \subset \mathcal{A} \times \mathcal{H}$$

be the set of Clifford module structures. Fix a metric $<,>_0$ and a $<,>_0$-unitary connection $\nabla_0$. The representation $\nabla = \nabla_0 + a$, models $\mathcal{A}^{\mathbb{C}}$ on $A^1_{\mathbb{C}}(X)$, and the representation $<,>= e^f <,>_0$ models $\mathcal{H}$ on $A^0_{\mathbb{R}}(X)$. A pair $(\nabla, <,>) \in \mathcal{A}^{\mathrm{mod}}$ if and only if $a + \bar{a} = df$. In particular $a$ is determined by $f$ and its imaginary part, so $\mathcal{A}^{\mathrm{mod}}$ is modeled on $A^0_{\mathbb{R}}(X) \times A^1_{\mathbb{R}}(X)$.

Now the diagonal action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{A}^{\mathbb{C}} \times \mathcal{H}$ leaves $\mathcal{A}^{\mathrm{mod}}$ invariant. Our alternative description of $\mathcal{B}$ and $\mathcal{P}^*$ is

$$\mathcal{P}^* = \mathcal{A}^{\mathrm{mod}} \times A^0(W^+)^*/\mathcal{G}^{\mathbb{C}} \to \mathcal{B} = \mathcal{A}^{\mathrm{mod}}/\mathcal{G}^{\mathbb{C}} \tag{13}$$

Finally, to do decent gauge theory we have to complete to Banach spaces and -manifolds. Seiberg Witten theory works fine with an $L^p_1$ completion of $\mathcal{A}$, $\mathcal{A}^{\mathbb{C}}$, and $A^0(W^+)$ and an $L^p_2$ completion of $\mathcal{G}$, $\mathcal{G}^{\mathbb{C}}$ and $\mathcal{H}$ if $p > \dim X$. In this range $L^p_1 \hookrightarrow C^0$, and therefore the two possible $L_p$ descriptions of $\mathcal{P}^*$ and $\mathcal{B}$ coincide. On the other hand, the Sobolev range does not seem optimal: with more care and work one can probably use all $p$-completions with $2 - \dim(X)/p > 0$. We will suppress completions from the notation, explicitly mentioning completions if necessary.

From now on we assume $\dim X = 4$. Fix an SC structure $W$ and choose an invariant hermitian metric $\langle,\rangle$. Choose a Riemannian metric $g$ and a real 2 form $\epsilon$, which are *admissible* in the following sense: $L_W$ admits no connection with $F^+ = -2\pi\sqrt{-1}\epsilon^+$, where as usual $+$ means taking the self dual part. Admissible metrics and forms exist if $b_+ \geq 1$, since the condition is certainly satisfied if $c_1(L_W) \notin \epsilon^{\mathrm{harm}} + H^-_g$ where $H^-_g$ is the space of $g$-anti-self-dual closed forms, and "harm" means projection to the harmonic part. Note that no use of Sard-Smale is made to define admissibility. Actually for most of our purposes it would be enough to let $\epsilon$ be a closed (hence harmonic) self-dual form.

By a transversality argument [Don], the admissible (metrics,forms) form a connected set if $b_+ \geq 2$. We say that a metric $g$ is admissible if $(g, 0)$ is. Even if $b_+ = 1$, all metrics are admissible when $L_W^2 \geq 0$, and $L_W$ is not torsion.

In dimension 4, the anti-symmetrisation map gives an isomorphism $c\colon \Lambda^+ \cong \mathrm{End}_0^{sh}(W^+)$ between the real self-dual forms and the traceless skew hermitian endomorphisms of $W^+$. This special phenomenon allows us (or rather Seiberg and Witten) to write down the monopole equations [Wit]

$$\not{\partial}\phi = 0 \qquad \phi \in A^0(W^+) \tag{14}$$

$$c(F^+) = 2\pi\phi\,\langle\phi, -\rangle - \pi|\phi|^2 - 2\pi\sqrt{-1}c(\varepsilon^+). \tag{15}$$

Let $\mathcal{M} = \mathcal{M}(W, g, \epsilon) \subset \mathcal{P}^*$ be the space of solutions modulo gauge.

As a technical remark, note that we use the conventions of [BGV], and that in their conventions the Weitzenböck (Lichnerowitz) formula restricted to $W^+$ reads

$$\not{\partial}^2 = \nabla^*\nabla + r/4 + c(F^+/2)$$

([BGV, th. 3.52] and the observation that the twisting curvature of an SC structure is $1/\operatorname{rank}(W^+)$ times the curvature on $\det(W^+)$.) The sign difference in the $c(F^+)$ term in [KM, lemma 2] explains the relative change of sign with respect to [KM, formula $(*)$] in the Seiberg Witten equations. It is chosen in such a way that the Weitzenböck formula gives $C^0$ control on the harmonic positive spinor $\phi$.

A basic property of the monopole equation noted by Witten, which follows from the Weitzenböck formula [KM, lemma 2] or a variational description [Wit, Section 3], is the following

**Proposition 20.** *The monopole equations have no solution with $\phi \neq 0$ if the metric has positive scalar curvature.*

Alternatively we can define $\mathcal{M}$ as the zero of a Fredholm section in an infinite dimensional vector bundle. Let $\mathcal{W}^\pm = (\mathcal{A} \times A^0(W^+)^* \times_{\mathcal{G}} A^0(W^\pm) \to \mathcal{P}^*$. Then $\mathcal{M}$ is the zero of the section in $\mathcal{W}^- \oplus A^+(X)$ given by the monopole equations (14), and (15).

To see that it is actually a Fredholm section we linearise the equations, assuming that $(\nabla, \phi)$ is a solution, and $(\nabla + \varepsilon a, \phi + \varepsilon\psi)$ with $a \in \sqrt{-1}A^1_{\mathbb{R}}(X)$ and $\psi \in A^0(W^+)$ is a solution up to order 1 in $\varepsilon$. We get (c.f [Wit, eq.2.4])

$$\not{\partial}\psi + a \cdot \phi = 0$$

$$c^{-1}(2\pi(\phi\,\langle\psi, -\rangle + \psi\,\langle\phi, -\rangle - \operatorname{Re}\langle\phi, \psi\rangle) - d^+a = 0.$$

The tangent space of the $\mathcal{G}$-orbit of $(\nabla, \phi)$ is $\{(a, \psi) = (-du, u\phi),\ u \in \sqrt{-1}A^0_{\mathbb{R}}(X)\}$. Thus the Zariski tangent space of $\mathcal{M}$ in $(\nabla, \phi)$ is the first cohomology of the Fredholm complex

$$\sqrt{-1}A^0_{\mathbb{R}}(X) \to \sqrt{-1}A^1_{\mathbb{R}}(X) \oplus A^0(W^+) \to \sqrt{-1}A^+_{\mathbb{R}}(X) \oplus A^0(W^-),$$

where the maps are given by the left hand side of the linearised equations. The virtual dimension is given by Atiyah Singer index formula and is

$$d(W) = \operatorname{vdim}_{\mathbb{R}}(\mathcal{M}) = \tfrac{1}{4}(L_W^2 - (2e(X) + 3\sigma(X))), \tag{16}$$

where $e(X)$ is the topological Euler characteristic, and $\sigma(X)$ the signature [Wit, eq. 2.5].

The crucial property that makes Seiberg Witten theory so much easier than Donaldson theory is

**Proposition 21.** [KM, Corollary 3],[Wit, §3] *The moduli space $\mathcal{M}$ is compact. For fixed $c > 0$ there are only finitely many SC-structure $W$ with $d(\mathcal{M}(W)) \geq -c$ and $\mathcal{M}(W, g, \epsilon) \neq \emptyset$.*

Note that for generic pairs $(g, \epsilon)$, moduli spaces of negative virtual dimension are empty, but I do not see an a priori reason why moduli spaces of arbitrary negative virtual dimension should not exist for special pairs. Likewise for generic pairs the moduli space is smooth of dimension $d(W)$ [KM]. However we have no need for this fact.

The index bundle $\mathrm{Ind}(Ds)$ of the deformation complex can be deformed by compact operators (over a compact space !) into the sum of the index of the signature complex and the index of the complex dirac operator. Thus the determinant line bundle $\det(\mathrm{Ind}(Ds)$ of the index is naturally oriented by choosing an orientation for $\det H^1(X, \mathbb{R})^\vee \otimes H^+(X, \mathbb{R})$. We will in fact assume that an orientation for both $H^+$ and $H^1$ is chosen. Suppose further that the pair $(g, \epsilon)$ is admissable (i.e. $\mathcal{M}((W, g, \epsilon) \subset \mathcal{P}^*)$, then proposition 14 in the previous section gives us a homology class $\mathbb{M} \in H_{d(W)}(\mathcal{P}^*)$, i.e. a homology class of the proper virtual dimension even if $\mathcal{M}$ is not smooth, not reduced and not of the proper dimension (note that in our case the moduli space $\mathcal{M} = Z(s)$ is compact, and homology with closed support is just ordinary homology). In case $\mathcal{M}$ is smooth and has the proper dimension it is just the fundamental class. The class $\mathbb{M}$ depends only on the connected component of $(g, \epsilon)$ in the space of admissable pairs, by the homotopy property of the localised Euler class proposition 14.2. In particular $\mathcal{M}$ is independent of the admissable pair if $b_+ \geq 2$.

If $b_+ = 1$ the choice of an orientation of $H^+$ is the choice of a connected component in $\{\omega^2 > 0\} \subset H^2(X, \mathbb{R})$. It will be called the forward timelike cone. For every metric $g$ let $\omega_g$ be the unique self dual form in the forward timelike cone with $\int \omega^2 = 1$. For a pair $(g, \epsilon)$ and an SC-structure $W$ define the *discriminant*

$$\Delta_W(g, \epsilon) = \int (c_1(L_W) - \epsilon)\omega_g \tag{17}$$

A pair $(g, \epsilon)$ is admissable if the discriminant $\Delta_W(g, \epsilon) \neq 0$, because it means precisely that $c_1(L_W) \notin \epsilon^{\mathrm{harm}} + H^-$. Clearly the discriminant depends only on the period $(\omega_g, \epsilon^{+\mathrm{harm}})$.

**Lemma 22.** *If $b_+ = 1$ a pair $(g, \epsilon)$ is admissable if and only if the discriminant $\Delta_W(g, \epsilon) \neq 0$. There are exactly two connected components of admissable pairs labeled by the sign of the discriminant.*

*Proof.* Suppose two pairs $(g_i, \epsilon_i)$, $i = 0, 1$, have discriminants $\Delta_i$ of equal sign. Connect them by a path $(g_t, \epsilon_t)$ in the space of all pairs. Let $(\omega_t, \epsilon_t^{+,\mathrm{harm}})$ be the corresponding path of periods. Then the discriminant

$$\Delta_t = \int (c_1(L_W) - \epsilon_t^{+,\mathrm{harm}})\omega_t$$

is continuous in $t$ but may change sign. However if we modify the path by setting

$$\epsilon'_t = \epsilon_t + (\Delta_t - (1-t)\Delta_0 - t\Delta_1)\omega_t$$

then using $\Delta_W(g, \epsilon + \delta) = \Delta_W(g, \epsilon) - \int \delta \wedge \omega_g$ and $\int \omega^2 = 1$ we see that

$$\Delta'_t = \Delta_W(g_t, \epsilon'_t) = (1-t)\Delta_0 + t\Delta_1.$$

In particular $\Delta'_t$ does not change sign, so that $(g_t, \epsilon'_t)$ is a path of admissable pairs.

Conversely if $c_1(L_W) \in \epsilon^{\rm harm} + H^-$, then any connection $\nabla$ with induced Chern form $\epsilon^{\rm harm}$ determines a "reducible" solution $(\nabla, 0) \in \mathcal{P} - \mathcal{P}^*$ of the monopole equations. $\square$

**Definition 23.** If $b_+ \geq 2$, the *SW-multiplicity* is the map

$$\begin{aligned} n\colon \mathcal{SC} &\to (\Lambda^* H^1(X,\mathbb{Z})[h])^\vee \cong H^*(\mathcal{P}^*,\mathbb{Z})^\vee \\ W &\mapsto \mathbb{M}(W,g,\epsilon) \end{aligned}$$

where $(g,\epsilon)$ is any $W$-admissable pair. If $b_+ = 1$ the *SW-multiplicities* $n_+$ and $n_-$ are defined similarly but with pairs $(g_\pm, \epsilon_\pm)$ having positive respectively negative discriminant.

It should be remarked that the SW-multiplicity (ies) depend(s) implicitly on the orientation of $H^+$ and $H^1$. For $b_+ > 1$ this is only a matter of sign, but for $b_+ = 1$ the orientation of $H^+$ determines in addition which invariant is $n_+$ and which is $n_-$.

All known examples with $b_+ \geq 2$ have non trivial multiplicities only when the virtual dimension $d(W) = 0$. However for surfaces with $p_g = 0$ it is easy to give examples with one of $n_\pm$ is non trivial for $d(W) > 0$ we will in fact use such an invariant. If $b_1 \neq 0$, the $H^1$ part of the multiplicity becomes essential.

*Remark 24.* Since $H_i(\mathcal{P}^*) = 0$ for $i < 0$, a moduli space of negative virtual dimension never defines a nontrivial class. Thus if for a class $L \in H^2(X,\mathbb{Z})$ there exists an SC-structure $W$ with $L = c_1(L_W)$ and the multiplicity $n(W) \neq 0$ (respectively one of $n_\pm(W) \neq 0$ then $L^2 \geq 3e(X) + 2\sigma(X)$ (c.f. equation (16)).

*Remark 25.* In the case $b_+ = 1$ we can alternatively consider the multiplicity as depending in addition on a chamber structure in

$$\Gamma = \{(\omega, \epsilon) \in H^2(X,\mathbb{R})^2 \mid \omega^2 = 1,\ \omega_0 > 0\}$$

where a chamber is defined by walls which are in turn defined by all classes $L \equiv w_2(X)$ through equation (17). This is particularly useful when we consider structures with $L_W^2 \geq 0$, $L_W$ is not torsion. Then all pairs $(g,0)$ are admissable and have discriminant of equal sign, because the forward timelike cone is strictly on one side of the hyperplane $L_W^\perp \subset H^2(X,\mathbb{R})$. Thus for this subset we have a preferred chamber.

We will say that $L \in H^2(X,\mathbb{Z})$ with $L \equiv w_2(X)$ has non trivial multiplicity if there is an SC-structure $W$ such that $L = c_1(L_W)$ and $W$ has non trivial multiplicity. If $b_+ = 1$ we will further qualify which multiplicity is non trivial (i.e. $n_+$ or $n_-$) or which chamber is chosen. We will simply write $n(L) \neq 0$ or $n_+(L) \neq 0$ etc.

A final and important piece of general theory is the following blow-up formula [Ste],[FS, §8]. We will give a proof valid for Kähler surfaces in section 5.

**Theorem 26.** *Let $X$ be a closed oriented 4-manifold with $b_+ \geq 1$. An SC-structure $\tilde W$ on $X\#\bar{\mathbb{P}}^2$ can be decomposed as $\tilde W = W\#W_k^{\bar{\mathbb{P}}^2}$, with determinant lines $L_{\tilde W} = L_W + (2k+1)E$. If the multiplicity $n_{(\pm)}(\tilde W) \neq 0$ then $d(\tilde W) = d(W) - k(k+1) \geq 0$, and the multiplicity $n_{(\pm)}(W) \neq 0$. Moreover if $L_{W_{\bar{\mathbb{P}}^2}} = \pm E$ (i.e. $E \cdot L_{\tilde W} = \pm 1$) then $n_{(\pm)}(\tilde W) = n_{(\pm)}(W)$ under the identification $H^1(X,\mathbb{Z}) \cong H^1(\tilde X,\mathbb{Z})$.*

Here, $n_{(\pm)} = n$ if $b_+ > 1$, and if $b_+ = 1$, it is understood that we compare say $n_+(W\#W_k^{\bar{\mathbb{P}}^2})$ with $n_+(W)$.

## 4. Seiberg Witten classes of Kähler surfaces

From now on, $(X, \Phi)$ denotes a Kähler surface. Then $X$ has a natural base SC-structure
$$W_0 = \Lambda^{0,*}X$$
with Clifford multiplication given by
$$c(\omega^{10} + \omega^{01}) = \sqrt{2}\left(-i(\omega^{10}) + \varepsilon(\omega^{01})\right),$$
where $i$ is contraction and $\varepsilon$ is exterior multiplication. The metric and connection induced by the Kähler structure on $\Lambda^{0*}X$ define a Clifford module structure on $W_0$. For an arbitrary SC structure $W = W(\mathcal{L})$ the spinor bundles are of form
$$W^+ = (\Lambda^{00} \oplus \Lambda^{02}) \otimes \mathcal{L}, \qquad W^- = \Lambda^{01}(\mathcal{L}).$$
and $L_W = \det(W^+) = -K \otimes \mathcal{L}^2$ (c.f. lemma 12). We call $\mathcal{L}$ the twisting line bundle.

We now turn to the monopole equations (see also [Wit, Section 4]). In the decomposition of $W^+$, a positive spinor will be written $\phi = (\alpha, \beta)$. The Dirac equation is then [BGV, Propos. 3.67].
$$\partial\!\!\!/\phi = \sqrt{2}(\bar\partial\alpha + \bar\partial^*\beta) = 0.$$

Since $X$ is Kähler, we can locally choose holomorphic geodesic coordinates $(z_1, z_2)$. A basis of the self dual forms is then the Kähler form $\Phi = \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar z_1 + dz_2 \wedge d\bar z_2)$, $dz_1 \wedge dz_2$ and $d\bar z_1 \wedge d\bar z_2$. Let $h$ be an hermitian metric on $\mathcal{L}$. Choose a unit generator $e$ for $\mathcal{L}$, then an orthonormal basis for $W^+$ is $e$ and $\frac{1}{2}e d\bar z_1 \wedge d\bar z_2$.

Using the definition of Clifford Multiplication we compute:
$$\begin{aligned} c(\Phi)e &= \tfrac{\sqrt{-1}}{2}(-i(dz_1)\varepsilon(d\bar z_1) + \varepsilon(d\bar z_1)i(dz_1) - i(dz_2)\varepsilon(d\bar z_2) + \varepsilon(d\bar z_2)i(dz_2))e \\ &= -2\sqrt{-1}e. \end{aligned}$$

In exactly the same way we compute $c(\Phi)$, $\frac{1}{2}e d\bar z_1 \wedge d\bar z_2$, and the action of $c(dz_1 \wedge dz_2)$ and $c(d\bar z_1 \wedge d\bar z_2)$ on $e$ and $\frac{1}{2}e d\bar z_1 \wedge d\bar z_2$. The result in matrix form is given by
$$c(\Phi) = \begin{pmatrix} -2\sqrt{-1} & 0 \\ 0 & 2\sqrt{-1} \end{pmatrix} \quad c(dz_1 \wedge dz_2) = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \quad c(d\bar z_1 \wedge d\bar z_2) = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}.$$

On the other hand, writing $\alpha = \alpha_e e$, and $\beta = \frac{1}{2}\beta_{\dot 1\dot 2} e d\bar z_1 \wedge d\bar z_2$,
$$(\alpha + \beta)\langle \alpha + \beta, - \rangle = \begin{pmatrix} |\alpha_e|^2 & \alpha_e \bar\beta_{\dot 1\dot 2} \\ \bar\alpha_e \beta_{\dot 1\dot 2} & |\beta_{\dot 1\dot 2}|^2 \end{pmatrix}.$$
Thus if we define $\alpha^* = h(\alpha, -)$, $\beta^* = h(\beta, -)$ and take the trace free part, we get the healthy global expression
$$(2\pi(\alpha + \beta)\langle \alpha + \beta, - \rangle)_0 = -2\pi\sqrt{-1}c\left(\tfrac{1}{2}(|\beta|_h^2 - |\alpha|_h^2)\Phi + \sqrt{-1}(-\alpha\beta^* + \beta\alpha^*)\right)$$

Plug all this in the monopole equations (14),(15). Writing $c_1(F) = \frac{-1}{2\pi i}F$, and using that $\Lambda\Phi = 2$ the monopole equation for a Kähler metric and perturbation $\epsilon = \lambda\Phi$ can be rewritten to
$$\bar\partial\alpha + \bar\partial^*\beta = 0 \tag{18}$$
$$F^{02} = 2\pi\beta\alpha^* \tag{19}$$
$$F^{20} = -2\pi\alpha\beta^* \tag{20}$$
$$\Lambda c_1(F)^{11} = (|\beta|^2 - |\alpha|^2) + 2\lambda. \tag{21}$$

Note that $F$ is the curvature on $L_W$, but that these are equations for a unitary connection $d = \partial + \bar\partial$ on $\mathcal{L}$ and sections $\alpha \in A^{00}(\mathcal{L})$, and $\beta \in A^{02}(\mathcal{L})$ through the identity $F = -F(K) + 2F(\mathcal{L}, d)$. Here $F(K)$ is the curvature of the canonical line bundle i.e. minus the Ricci form.

In terms of the twisting bundle the virtual (real) dimension of the moduli space reads

$$d(\mathcal{L}) = d(\Lambda^{0*}(\mathcal{L})) = \tfrac{1}{4}(L^2 - K^2) = \mathcal{L}\cdot(\mathcal{L} - K). \tag{22}$$

A more precise description is given by

**Proposition 27.** *A necessary condition for the existence of solutions to the monopole equations* (18) *to* (21)*, is that* $(\mathcal{L}, \bar\partial)$ *is a holomorphic line bundle, and that*

$$-\deg_\Phi(K) \le \deg_\Phi(L) < \int(\lambda\Phi^2), \text{ or} \tag{23}$$

$$\int \lambda\Phi^2 < \deg_\Phi(L) \le \deg_\Phi(K), \text{ or} \tag{24}$$

$$\int \lambda\Phi^2 = \deg_\Phi(L) \tag{25}$$

*In particular* $L_W = -K \otimes \mathcal{L}^2$ *has a natural holomorphic structure. In case* (23) *the moduli space* $\mathcal{M} = \mathcal{M}(\mathcal{L}, \Phi, \lambda)$ *of solutions can be identified as a real analytic space with the moduli space of pairs of a holomorphic structures* $\bar\partial$ *on* $\mathcal{L}$*, and a divisor* $\alpha \in |(\mathcal{L}, \bar\partial)|$*, in particular the Zariski tangent space in* $(\bar\partial, \alpha)$ *is canonically identified with* $H^0(\mathcal{L}|_{Z(\alpha)})$*. In case* (24) *the moduli space* $\mathcal{M}$ *of solutions can be identified with the moduli space of pairs of a holomorphic structure* $\bar\partial$ *on* $\mathcal{L}$*, and an element* $\beta \in \mathbb{P}H^2(\mathcal{L}) = |K \otimes \mathcal{L}^\vee|^\vee$*, in particular the Zariski tangent space at* $(\bar\partial, \beta)$ *is isomorphic to* $\overline{H^0(K \otimes \mathcal{L}|_{Z(\bar\beta)})}$*. In case* (25) *the "moduli space"* $\mathcal{M} \subset \mathcal{P} - \mathcal{P}^*$ *(i.e.* $\alpha = \beta = 0$*) can be identified with the space of holomorphic structures* $\bar\partial$ *on* $\mathcal{L}$*.*

*Proof.* Combining (18) and (19) yields

$$\bar\partial\bar\partial^*\beta = -\bar\partial^2\alpha = -F^{02}\alpha = -2\pi|\alpha|^2\beta. \tag{26}$$

Integrating both sides against $\langle\beta, -\rangle$, immediately gives that $\alpha\beta = 0$ and $\bar\partial\beta = \bar\partial\alpha = 0$. Thus $F^{02} = F^{20} = 0$, Since $F^{02} = 2F^{02}(\mathcal{L}, d)$, $\bar\partial$ is a holomorphic structure on $\mathcal{L}$, and either $0 \ne \alpha \in H^0(\mathcal{L})$ and $\beta = 0$ or $0 \ne \beta \in H^2(\mathcal{L})$ and $\alpha = 0$, or $\alpha = \beta = 0$. Note that if for example $\alpha \ne 0$, then $\beta = 0$ is cut out transversely by equation (26). The last monopole equation (21) gives the condition

$$\deg(L) = -\deg(K) + 2\deg(\mathcal{L}) = \tfrac{1}{2}\int \Lambda c_1(F)\Phi^2 = \tfrac{1}{2}\int(|\beta|^2 - |\alpha|^2 + 2\lambda)\Phi^2$$

which fixes the global $L_2$ norm of $\alpha$ and $\beta$, and determines whether $\alpha \ne 0$ or $\beta \ne 0$ or $\alpha = \beta = 0$.

Finally we deal with equation (21). If $\alpha \ne 0$ and $\beta = 0$ then we are dealing essentially with the abelian vortex equation studied by Steve Bradlow [Br1, §4] Oscar Garcia-Prada and earlier in a different guise by Kazdan Warner [KW]. See also [Br2] and [OT2]. I thank Steve Bradlow for pointing out that almost all of the work had already been done by him and Oscar Garcia-Prada. To identify the moduli space as a real analytic space we just jazz up Bradlow's results a bit. This is necessary because we have to to understand how the moduli space is cut out in order to apply the localised Euler class machinery in the next section.

It is slightly more convenient to use our alternative description (13) of $\mathcal{P}^*$, and solve for a pair $(d_L, h)$ where $h = e^f h_0$ is a hermitian metric on $L$ and $d_L = \partial + \bar\partial = d_0 + a$ is $h$ unitary, and mod out the full gauge group $\mathcal{G}^{\mathbb{C}}$ of all complex nowhere vanishing functions. To be precise we take $d_L$ in $L^p_1$, and $\mathcal{G}^{\mathbb{C}}$ and $f$ in $L^p_2$ with $p > 4$. The sections $\alpha$ and $\beta$, being disguised spinors, are as before in $L^p_1$.

For an $h$-unitary connection we have , $\partial h(s,t) = h(\partial s, t) + h(s, \bar\partial t)$ for all sections $s, t \in A^0(\mathcal{L})$. Thus $d_L$ is determined by $\bar\partial$ and $h$, or equivalently, $a^{01}$ and $f$. Expressed in $a^{01}$ and $f$, equation (21) becomes

$$\Delta f = 2\pi(|\beta|^2_{h_0} - |\alpha|^2_{h_0})e^f - 2\sqrt{-1}\Lambda(\partial_0 a^{01} - \bar\partial_0 a^{\bar 01}) + \mu \tag{27}$$

where $\mu = 2\pi(2\lambda + (\Lambda c_1(F(K)) - 2\Lambda c_1(\mathcal{L}, \nabla_0)$ (compare [Br1, lemma 4.1]).

If $\beta$ is small in $L^p_1$ hence in $C^0$, we can solve for $f$ in equation (27) with the solution depending real analytically on $(a^{01}, \alpha)$ by the analytical lemma 28. Moreover, variation of (27) with respect to $f$ when $\beta = 0$ gives

$$\delta\text{“eqn (27)”} = (\Delta + 2\pi|\alpha|^2 e^f)\delta f. \tag{28}$$

Thus, equation (27) cuts out this solution transversely. More invariantly, if $\beta$ is small, there is a unique metric $h(\bar\partial, \alpha, \beta) = h_0 e^{f(\bar\partial - \bar\partial_0, \alpha, \beta)}$ solving the last monopole equation (21).

**Lemma 28.** *Let $X$ be a compact Riemannian manifold, and* $\dim(X) < p < \infty$ *a Sobolev weight. Then for every real non negative function $0 \le w_0 \in L^p$, with $\int w_0 > 0$ and real function $\mu_0 \in L^p$, with $\int \mu_0 > 0$, there exists a neighborhood $U_{(w_0,\mu_0)} \subset L^p \times L^p$ such that for all $(w, \mu) \in U_{(w_0,\mu_0)}$ the equation*

$$\Delta f = -we^f + \mu \tag{29}$$

*has a unique $L^p_2$ solution depending analytically on $w$ and $\mu$. The solution is smooth if $w$ and $\mu$ are smooth.*

*Proof.* As in [Br1, lemma 4] make the substitution $f = \tilde f - g$ where $g$ is the unique solution of $\Delta g = \int \mu - \mu$ to reduce to the case where $\mu$ is constant. Then apply [KW, theorem 10.5(a)] to solve the equation for $w_0, \mu_0$ (note that Kazdan Warners Laplacian is negative definite and that the proof works fine with $w \in L^p$ instead of $C^\infty$). Since at a solution $f_0$ for $(w_0, \mu_0)$ we have

$$\delta\text{“eqn (29)”} = (\Delta + w_0 e^{f_0})\delta f$$

and $(\Delta + w_0 e^{f_0})$ is invertible, we conclude with the implicit function theorem that there continues to exist a solution for $(w, \mu)$ in a small neighborhood of $(w_0, \mu_0)$, and that this solution depends real analytically on $(w, \mu)$. Regularity follows from standard bootstrapping techniques. Uniqueness follows from the weak maximum principle ([GT, theorem 8.1], c.f. [KW, remark 10.12]). ☐

In geometric terms, this has the following consequence. let $\mathcal{A}^{01}$ be (the $L^p_1$-completion) of the space of $\bar\partial$-operators on $\mathcal{L}$ modeled on $A^{01}(X)$ through $\bar\partial = \bar\partial_0 + a^{01}$. The complex gauge group $\mathcal{G}^{\mathbb{C}}$ acts naturally by conjugation. Let

$$\mathcal{P}^{01*} = \mathcal{A}^{01} \times (A^{00}(\mathcal{L}) \oplus A^{02}(\mathcal{L}))^*/\mathcal{G}^{\mathbb{C}}$$

Clearly there is a projection $\mathcal{P}^* \to \mathcal{P}^{01*}$ forgetting $h$. What we have done is showing that there is section

$$\mathcal{P}^{01*} \to \mathcal{P}^*$$
$$(\bar\partial, \alpha, \beta) \to (\bar\partial, \alpha, \beta, h(\bar\partial, \alpha, \beta))$$

in a neighborhood of $\beta = 0$, whose image is cut out as a real analytic space by the last monopole equation (21).

So far we have not used the other equations. Suppose we are in case (23), i.e. where a solution corresponds to sections. Then $\mathcal{M}$ is cut out by $\bar\partial^2 = 0$, $\bar\partial\alpha = 0$, $\beta = 0$ and, by the preceding argument, $h = h(\bar\partial, \alpha, \beta)$. Thus projection identifies $\mathcal{M}$ with

$$\mathcal{M}^{\mathrm{BN}} = \{(\bar\partial, \alpha, \beta) \in \mathcal{P}^{01},\ \bar\partial^2 = 0,\ \bar\partial\alpha = 0, \beta = 0\}$$

For the Zariski tangent space it gives

$$\begin{aligned} T_{(\nabla,\alpha,0,h)}\mathcal{M} &= T_{(\bar\partial,\alpha,0)}\mathcal{M}^{\mathrm{BN}} \\ &= \mathrm{Ker}\begin{pmatrix} \bar\partial & \alpha \\ & \bar\partial \end{pmatrix} \Big/ \mathrm{Im}\begin{pmatrix} \alpha \\ -\bar\partial \end{pmatrix} \\ &= \mathbb{H}^1(0 \to \mathcal{O} \xrightarrow{\alpha} \mathcal{L} \to 0) \\ &= H^0(\mathcal{L}|_{Z(\alpha)}). \end{aligned}$$

It is easy to check that the linearised versions of equations (18), (19), (20), and (27) give the same result (as it should).

Case (24) is reduced to the previous case by Serre duality. In case (25) the metric $h$ we look for is an (almost) Hermite-Einstein metric. $\square$

**Corollary 29.** *Let $X$ be Kähler surface. and $L \equiv w_2(X)$ be a class in $H^2(X, \mathbb{Z})$ with $n(L) \neq 0$. Then $L$ is of type $(1,1)$. Moreover if $p_g > 0$, then for all Kähler forms $\Phi$ on $X$, the class $L$ satisfies*

$$\deg_\Phi(K_X) \geq \deg_\Phi(L) \geq -\deg_\Phi K_X$$

*If $p_g = 0$, and $n_-(L) \neq 0$ (resp. $n_+(L) \neq 0$), then*

$$\deg_\Phi(L) \geq -\deg_\Phi(K_X) \quad (\textit{resp. } \deg_\Phi(L) \leq \deg_\Phi(K_X))$$

*Proof.* First we consider the case $p_g > 0$. Under the conditions of the corollary, there is an SC-structure $W$ with $L_W = L$ which admits at least one solution to the monopole equation for *every* admissable pair $(g, \epsilon)$. In particular $W$ admits a solution for every Kähler metric and $\epsilon = \lambda\Phi$. Thus $L = L_W$ is of type $(1,1)$. Moreover the necessary condition for the existence of a solution of section *or* cosection type (i.e. equation 23 *or* 24 in proposition 27) gives precisely the required inequality in the limit $\lambda \to 0$.

If $p_g = 0$, then $L$ is automatically of type $(1,1)$ and say the condition $n_-(L) \neq 0$ means that there is an SC structure $W$ with $L_W = L$ such that for any Kähler metric, $W$ admits solutions of section type (i.e. equation 23) if $\lambda$ is sufficiently large. This gives a lower bound but no upper bound on $\deg_\Phi(L)$. $\square$

*Remark 30.* If $p_g = 0$ and we restrict to perturbation $\epsilon = 0$ (or small), then the same argument as in the $p_g > 0$ case gives the stronger degree inequality if $L^2 \geq 0$, $L$ is not torsion, since in this case all metrics are admissable and have discriminant of equal sign. In particular on a Del Pezzo surface such classes do not exist.

**Corollary 31.** *Let $X$ be a Kähler surface with base SC structure $W_0 = \Lambda^{0*}X$. Then $n(W_0) = 1$ if $p_g > 0$ and $n_-(W_0) = 1$ if $p_g = 0$, in particular $n(-K_X) \neq 0$ resp. $n_-(-K_X) \neq 0$. Likewise, $n(W_0(K_X) = \pm 1$ if $p_g > 0$ and $n_+(W_0(K_X) = \pm 1$, in particular $n(K_X) \neq 0$ resp. $n_+(K_X) \neq 0$. Moreover $W_0$ is the only SC-structure $W$ with $L_W = -K_X$ mod torsion and non trivial multiplicity $n$ respectively $n_-$. In* *particular if $L \in H^2(X,\mathbb{Z})$, such that $L = -K \in H^2(X,\mathbb{Q})$ and $n(L) \neq 0$ resp. $n_-(L) \neq 0$ then $L = -K \in H^2(X,\mathbb{Z})$.*

*Proof.* We will prove the statement for $-K_X$. Then we have to consider SC-structures $W = \Lambda^{0*}(\mathcal{L})$ with $c_1(\mathcal{L})$ torsion. Choose a Kähler metric and $\lambda \gg 0$. Then $\mathcal{M}(W) \cong \mathcal{M}^{\mathrm{BN}}(\mathcal{L})$ the moduli space of line bundles with a section. But $\mathcal{M}^{BN}(\mathcal{L})$ is just a reduced point if $\mathcal{L}$ is trivial, and empty if $c_1(\mathcal{L})$ is non trivial torsion. Thus $W_0 = \Lambda^{0*}X$ is unique among the SC-structures $W$ with $L_W = -K_X$ mod torsion with $n(W) \neq 0$ (resp. $n_-(W) \neq 0$). In fact its multiplicity is 1. The case $+K_X$ can be dealt similarly with Serre duality. Its multiplicity is $\pm 1$ because of the unpleasant orientation switches. □

**Corollary 32.** *Let $D$ be an effective divisor with $D \cdot (D-K) = 0$, $h^0(\mathcal{O}(D)) = 1$, $h^0(\mathcal{O}_D(D)) = 0$, and $h^0(\mathcal{L}(D)) = 0$ for every line bundle $\mathcal{L} \in \mathrm{Pic}^0(X)$. Then $n(-K_X + 2D) \neq 0$ if $p_g > 0$ and $n_-(-K_X + 2D) \neq 0$ if $p_g = 0$. Likewise, $n(K_X - 2D) \neq 0$ if $p_g > 0$ and $n_+(K_X - 2D) \neq 0$ if $p_g = 0$.*

*Proof.* This corollary is proved just as the previous one, and reduces to it if $D = 0$. The conditions of the corollary ensure precisely that $\mathcal{M}^{\mathrm{BN}}(\mathcal{O}(D))$ consists of one smooth point and that $\mathrm{vdim}(\Lambda^{0*}(D)) = 0$. □

We are finally in the position to prove the main theorem 1 and corollary 7.

Our first task is to define a set $\mathcal{K}$ of basic classes.

**Definition 33.** If $b_+ \geq 2$ then the basic classes are defined by

$$\mathcal{K} = \{K \in H^2(X,\mathbb{Z}) \mid n(K) \neq 0\}$$

If $b_+ = 1$ then $\mathcal{K} = \mathcal{K}_- \cup \mathcal{K}_+$ where

$$\begin{aligned}\mathcal{K}_- = \{K \in H^2(X,\mathbb{Z}) \mid n_-(K) \neq 0, \text{ and } \exists L \text{ with } n_-(L) \neq 0 \\ \text{such that } n_-(L - m(K+L)) \neq 0 \text{ for some } m \geq 1 \}.\end{aligned}$$

The set $\mathcal{K}_+$ is defined similarly in terms of $n_+$. Here we are allowed to take $m \geq 1$ rational as long as $m(K+L)$ is two divisible.

These basic classes are rightfully *the* Seiberg-Witten basic classes when $b_+ \geq 2$, but for $b_+ = 1$ the definition is geared towards the specific application we have in mind. We will show that $\mathcal{K}$ has all properties $(*)$.

It is clear that $\mathcal{K}$ is an oriented diffeomorphism invariant, and that the basic classes are characteristic. The pushforward property $(*)$.3 follows immediately from the blow up formula theorem 26 or 40. For Kähler surfaces the classes are of type $(1,1)$ by corollary 29. The degree property $((*).2)$ (for all surfaces minimal or not) follows also from corollary 29. This is immediate for $p_g > 0$. If $p_g = 0$ assume that $K \in \mathcal{K}_+$ say, the case $K \in \mathcal{K}_-$ being essentially the same. Then the corollary gives the three inequalities

$$\deg K \leq \deg K_X, \tag{30}$$

$$\deg L \leq \deg K_X, \tag{31}$$

$$-m \deg K \leq \deg K_X + (m-1)\deg L \leq m \deg K_X. \tag{32}$$

If $p_g > 0$ then $K_X \in \mathcal{K}$ by corollary 31. Thus it remains to check that $K_X \in \mathcal{K}$ if $p_g = 0$. In fact we will check that $-K_X \in \mathcal{K}$.

We have already seen in corollary 31 that $n_-(-K_X) \neq 0$. Either directly from corollary 32, or using the invariance under the reflection in the exceptional curves $E_1, \dots, E_n$ we see that $n_-(-K_X + 2\sum E_i) \neq 0$. Then denoting

$$\mathcal{L}_m = mK_{\min} + \sum E_i,$$

we have to check that $n_-(-K_X + 2\mathcal{L}_m) \neq 0$. We will distinguish four cases.

If $\kappa(X) = 0$, then $K_{\min}$ is torsion and we can take $m = \operatorname{ord}(K_{\min})$, since $n_-(-K_X + 2\sum E_i) \neq 0$.

If $\kappa(X) = 1$, then $X_{\min}$ has a unique elliptic fibration $X_{\min} \xrightarrow{\pi} C$. By the canonical bundle formula, $K_{\min} = \pi^*\mathcal{L}_C(\pi^*K_C + \sum(p_i - 1)F_i)$, where $\mathcal{L}_C$ is a line bundle on $C$ of degree $\chi$. Since $p_g = 0$ and $\chi \geq 0$, we have $0 \leq g \leq q \leq 1$, and we distinguish further between $g = 0$ and $g = 1$.

If $g = 0$, then $c_1(\pi^*\mathcal{L}_C(K_C)) = (\chi - 2)F$, where $F$ is a general fibre, and there are at least $3 - \chi$ multiple fibers because $K_{\min} > 0$. Now the class $K_{\min} + \sum_{i=1}^{2-\chi} F_i = \sum_{j=3-\chi}^{n} (p_j - 1)F_j$ is of the form $mK_{\min}$ with rational $m > 1$. Again by corollary 32, we have

$$n_-(-K_X + 2\mathcal{L}_m) = n_-(-K_X + 2(\sum_{j=3-\chi}^{n} (p_j - 1)F_j + \sum E_i)) \neq 0$$

If $g = 1$, then $\chi = 0$, and $K_C = 0$. In this case we can take $m = 1$ since $c_1(\mathcal{L}_C) = 0 \in H^2(X, \mathbb{Z})$ and by corollary 32

$$n_-(-K_X + 2\mathcal{L}_1) = n_-(-K_X + 2(\sum(p_i - 1)F_i + \sum E_i)) \neq 0.$$

The most instructive case is when $X$ is of general type. Then the irregularity $q = 0$ since $p_g = 0$ and $\chi(\mathcal{O}_X) > 0$. Take $m = 2$, then $\mathcal{M}^{\mathrm{BN}}(\mathcal{L}_2) = |2K_{\min} + \sum E_i|$. By formula (1) (or directly by Ramanujan vanishing)

$$\dim_{\mathbb{C}} \mathcal{M}^{\mathrm{BN}}(\mathcal{L}_2) = P_2 - 1 = K_{\min}^2 = \tfrac{1}{2}\operatorname{vdim}_{\mathbb{R}}(W_2).$$

Thus the moduli space is again smooth of the proper dimension and we conclude that $n_-(-K_X + 2\mathcal{L}_2) \neq 0$. In fact $n_-(\Lambda^{0*}(\mathcal{L}_2)) = t^{K_{\min}^2}$ since the $\mathcal{O}(1)$ on $\mathcal{P}^*$ corresponds to the $\mathcal{O}(1)$ on $\mathcal{M}^{\mathrm{BN}}$. This is because both measure the weight of the action of the constant gauge transformations on the spinors respectively sections.

It now follows from lemma 11 that if $\kappa(X) \geq 0$, all SW-structure have a moduli space of virtual dimension $d = 0$, and up to torsion, the basic classes are of type

$$K = \lambda K_{\min} + \sum \pm E_i \bmod \text{Torsion}, \qquad |\lambda| \leq 1. \tag{33}$$

Moreover by proposition 12, $K_{\min}$ is invariant up to sign and torsion and every $(-1)$-sphere is represented by a $(-1)$-curve up to sign and torsion.

We first get rid of torsion in the $(-1)$-curve conjecture i.e. theorem 1 part 2. Let $e$ be a $(-1)$-sphere, giving a connected sum decomposition $X = X' \# \bar{\mathbb{P}}^2$. As we have used before, there is a diffeomorphism $R_e = \mathrm{id}\,\#\mathbb{C}$-conjugation representing the reflection in $e$.

I claim that for any SC-structure $W$ on a 4-manifold

$$R_e^*(W) = W \otimes \mathcal{O}((c_1(L_W), e)e),$$

where $\mathcal{O}(e)$ is the line bundle corresponding to the Poincaré dual of $e$. In fact if we write $R_e^*W = W \otimes \mathcal{L}$, then $\mathcal{L} = \mathcal{H}om_C(W, R_e^*W)$ (c.f. the proof of 12). Now we can just identify $W$ and $R_e^*W$ on $X'$, i.e. $\mathcal{L}$ is trivialised on $X'$. Thus

$$c_1(\mathcal{L}) \in \operatorname{Im} H^2(X, X - X', \mathbb{Z}) \cong H^2(\bar{\mathbb{P}}^2) \subset H^2(X, \mathbb{Z}).$$

Write $\mathcal{L} = \mathcal{O}(ae)$ for some integer $a$. Since

$$L_W + 2ae = L_{R_e^*W} = R_e^* L_W = L + 2(e, L_W)e$$

the claim is proved.

Going back to the Kähler case, we can assume that $e$ is homologous to a $(-1)$-curve $E$ up to torsion. Consider $W = R_e^* R_E^*(\Lambda^{0*}X) = \Lambda^{0*}(E - e)$. By oriented diffeomorphism invariance $n_{(-)}(W) \neq 0$ (in case $p_g = 0$ we have tacitly used the fact that $R_e^* R_E^*$ induces the identity on rational cohomology so in particular does not change the orientation of $H^+$). Moreover $c_1(L_W) = -K_X$ up to torsion. By corollary 31, we conclude that $W = \Lambda^{0*}X$, so $e = E \in H^2(X, \mathbb{Z})$.

Finally for the invariance $\pm K_{\min}$, consider any basic class of the form $L = \pm K_{\min} + \sum \pm E_i$ up to torsion. After reflections in the $(-1)$-curves, we get a class $L'$ equal to $\pm K_X$ up to torsion. By corollary 31 $L' = \pm K_X \in H^2(X, \mathbb{Z})$. Now for any basis $E_1', \dots E_n'$ of the lattice in $H^2(X, \mathbb{Z})$ spanned by the $(-1)$-spheres (e.g. the $(-1)$-curves) we have the identity

$$\pm K_{\min} = L' + \sum (E_i', L')E_i' = L + \sum (E_i', L)E_i' \in H^2(X, \mathbb{Z}).$$

This finally proves theorem 1.

*Remark 34.* It is easy to give a definition of basic classes for $b_+ = 1$ that satisfies all properties $(*)$ except the invariance under blow down (i.e. property $(*)$.3). A class $K$ is then basic if there exists a metric $g$ such for all $\delta > 0$ there exists an admissable pair $(g, \epsilon)$ with $\|\epsilon^{+,\mathrm{harm}}\| < \delta$ such that $n(g, \epsilon, K) \neq 0$. The degree inequality for minimal surfaces then follows from remark 25. But alas, if $K^2 < 0$ one can not avoid the possibility that a chamber on the blow up realisable with small $\epsilon$ can only be realised for large $\epsilon$ on the blow down. In my original treatment I used this definition. I am grateful to Robert Friedman whose insistent questions about my definitions made me realise this mistake.

*Remark 35.* An easy application of the techniques of the next section gives the following. If $\mathcal{L}$ is a holomorphic line bundle on a surface with $p_g = q = 0$ with $h^0(\mathcal{L}) \geq \chi(\mathcal{L}) \geq 1$, then $n_-(\Lambda^{0*}(\mathcal{L})) = t^{\frac{\mathcal{L}(\mathcal{L} - K_X)}{2}}$. If $p_g = q = 0$ and $\kappa(X) \geq 0$ we can apply this to $\mathcal{L}_2 = 2K_{\min} + \sum E_i$. Then by the Castelnuovo criterion and the above we conclude $n_-(-K_X + 2L_2) \neq 0$. This gives an alternative way to prove that $-K_X \in \mathcal{K}$ in this case. Conversely the degree inequality $(*)$.2 cannot hold true for rational and ruled surfaces for Kähler forms $\Phi$ such that $\deg_\Phi(K_X) < 0$. Since in deriving the degree inequality we did not use that $\kappa(X) \geq 0$, we conclude that for $\kappa(X) = -\infty$ the set of the above defined basic classes $\mathcal{K} = \emptyset$. In particular we see that the following proposition is a rather direct analog of to the classical Castelnuovo criterion.

**Proposition 36.** *A Kähler surface is rational if and only if $b_1 = 0$, and $\mathcal{K} = \emptyset$.*

*Remark 37.* After reading [FM3] I realised the following. The blow up formula 26 can be generalised to connected sum decompositions $X = X' \# N$ with $N$ negative definite and $H_1(N, \mathbb{Z}) = 0$. The latter condition is automatic for Kähler surfaces of non negative Kodaira dimension by a beautiful observation of Kotschick (an unramified covering $\tilde{N} \to N$ of degree $d$ gives an unramified covering $\tilde{X} = dX' \# \tilde{N} \to$

$X'\#N$ which is an algebraic surface of non negative Kodaira dimension with a connected sum decomposition with a factor with $b_+ > 0$). Such smooth negative definite manifolds $N$ have $H_2(N) = \oplus_{i=1}^n \mathbb{Z} n_i$. SC structures $W_N$ on $N$ are determined by $L_N = \sum(2a_i + 1)n_i$. Thus the reflections $R_{n_i}$ in $n_i^\perp$, act on the SC structures on $N$. SC -structures on $X'\#N$ are of the form $W = W_{X'}\#W_N$. Now the blow up formula is as if $N = n\bar{\mathbb{P}}^2$: $W = W_{X'}\#W_N$ is an SW-structure on $X'\#N$ if and only if $W_{X'}$ is a SW-structure on $X'$ and $d(W) \geq 0$. In particular the Seiberg Witten structures are invariant under the operation $R_{n_i}\colon W_{X'}\#W_N \to W_{X'}\#R_{n_i}W_N$, and $\mathcal{H}om_C(W, R_{n_i}W)$ has a trivialisation over $X'$. With these remarks the arguments for $(-1)$-spheres carry over directly to prove that for Kähler surfaces $X$ with $\kappa(X) \geq 0$, with a connected sum decomposition $X = X'\#N$, $H_2(N) \subset H_2(X)$ is spanned by $(-1)$-curves.

Stefan Bauer showed me how to use the Seiberg Witten multiplicities and the basic classes to determine the multiplicities of the elliptic surface. If the surface does not have finite cyclic fundamental group, the multiplicities can be read off from the topology. Thus we consider a minimal elliptic surface $X_{pq}$ fibred over $\mathbb{P}^1$ with 2 multiple fibers of multiplicity $p$ and $q$ We will assume that $p \leq q$.

**Corollary 38.** *(Bauer) The multiplicities $p$ and $q$ are determined by the underlying oriented differentiable manifold, unless $p_g = 0$, $p = 1$ and $q$ arbitrary. The surfaces $X_{1q}$ are all rational and diffeomorphic.*

*Proof.* If the canonical class $K_X$ is not torsion , we can write $K_X$ in terms of the primitive vector $\kappa$ in the ray spanned by $K_X$, normalised so that $\kappa\Phi > 0$

$$K_X = (p_g-1)F + (p-1)F_p + (q-1)F_q = \frac{(p_g+1)pq - p - q}{\gcd(p,q)}\kappa \in H^2(X,\mathbb{Z})/\text{Torsion}.$$

Let $d(p,q) = ((p_g+1)pq - p - q)\,/\gcd(p,q)$ be the oriented divisibility of $K_X$. If $K_X$ is torsion we simply set $d(p,q) = 0$.

The divisibility $d(p,q) < 0$ if and only if $p_g = 0$, $p = 1$ and $q$ is arbitrary. But this implies that $K_X$ is rational. We have already seen that we can recognise rationality as Kodaira dimension $-\infty$ and $b_1 = 0$ (corollary 4 or proposition 36). Thus we can assume that $X_{pq}$ has non negative Kodaira dimension. Then $\pm K_X \in \mathcal{K}$ are the basic classes with the highest divisibility (or torsion) and the oriented divisibility $d(p,q) \geq 0$ is just the unoriented divisibility of $\pm K_X$. The number $\gcd(p,q)$ is also determined by the oriented manifold, being the order of the fundamental group. Choose one of these classes, say $-K_X$.

First consider the case $p_g > 0$. Suppose that $K = -K_X + 2F_q \leq 0$, (i.e. on the same side of 0 as $-K_X$), then it is the basic class with second largest divisibility since $F_q$ is the smallest effective vertical divisor, and $n(-K_X + 2F_q)) \neq 0$ by lemma 32 above. Thus if there exist basic classes other then $\pm K_X$, we can reconstruct $p$ from $(2p/\gcd(p,q))\kappa = K - (-K_X)$. Since $d(p,q)$, $p_g$ and $\gcd(p,q)$ are known, this determines $q$ as well. Obviously if we have chosen $+K_X$ the same arguments works with $K = K_X - 2F_q$, there is nothing that prefers $K_X$ over $-K_X$.

In the case $p_g = 0$ we make a small modification. We choose an orientation of $H^+$, which for a moment we assume is the standard one. Consider the classes $K \in H^2(X,\mathbb{Z})$ mod torsion in the half ray spanned by 0 and $-K_X$ with unoriented divisibility at most $d(p,q)$ (i.e. in between 0 and $-K_X$) such that $n_-(K) \neq 0$. Note that $-K_X$ is just the basic class with largest divisibility in $\mathcal{K}_-$. Then if $K = -K_X + 2F_q \leq 0$ we can use exactly the same argument as in the case $p_g > 0$.

If we choose a different orientation of $H^+$, we replace $-K_X$ by $+K_X$ but just as above the conclusion is the same.

If $\mathcal{K} = \pm K_X$ or for $p_g = 0$ if $\{K \in [-K_X, 0] \mid n_-(K) \neq 0\} = -K_X$ then $d(p,q)\gcd(p,q) < 2p$. The few possibilities are listed in the following table

| | $(p,q)$ | $\gcd(p,q)$ | $d(p,q)$ | Type |
|---|---|---|---|---|
| $p_g = 0$ | $(2,2)$ | 2 | 0 | Enriques |
| | $(2,3)$ | 1 | 1 | |
| | $(2,4)$ | 2 | 1 | |
| | $(2,5)$ | 1 | 3 | |
| | $(3,3)$ | 3 | 1 | |
| | $(3,4)$ | 1 | 5 | |
| $p_g = 1$ | $(1,1)$ | 1 | 0 | K3 |
| | $(1,2)$ | 1 | 1 | |

Clearly, in this case the pair $(p,q)$ is determined by the oriented differentiable manifold as well. $\square$

To prove that no surface with $\kappa \geq 0$ admits a metric with positive scalar curvature (corollary 7), first consider the case $p_g > 0$. Then the statement is clear, and one of Witten's basic observations. By proposition 20, for 4-manifolds with positive scalar curvature $n(K) = 0$ for all $K \in H^2(X,\mathbb{Z})$, since for our metric with positive scalar curvature $g$ and small perturbations $\epsilon$, we have $\mathcal{M}(W,g,\epsilon) = \emptyset$ for all SC-structures $W$. On the other hand we just showed that $n(-K_X) \neq 0$ using a Kähler metric.

The same argument works if $p_g = 0$ and $K_X^2 \geq 0$: $n(-K_X, g, \epsilon)$ is independent of the metric $g$ and of $\epsilon$ as long as $\epsilon$ is small, with the exception of the case $-K_X$ torsion in which case we have to choose $\epsilon$ in the forward light cone. But we can do better.

For the general case $p_g = 0$, we choose a perturbation $\epsilon = \lambda\Phi$ with $0 < \lambda \ll 1$ say. Now suppose that the metric with positive scalar curvature $g$ has period $\omega_g = \omega_{\min} + \sum \eta_i E_i$ where $\omega_{\min}$ is the projection to the cohomology of minimal model. Then since $\omega_g$ is in the interior of the forward light cone, and $K_{\min}$ is in the closure of the forward light cone, $\omega \cdot K_{\min} = \omega_{\min} \cdot K_{\min} \geq 0$ with equality iff $K_{\min}$ is torsion. Then for *some* choice of signs in $-K_{\min} - \sum \pm E_i$ we have

$$\omega_g \cdot (-K_{\min} - \sum \pm E_i) \leq 0 < \lambda \int \omega_g \Phi$$

Thus for *some* choice of signs we compute $n_-$ (rather than $n_+$) with our metric of positive scalar curvature and small perturbation. Hence $n_-(-K_{\min} - \sum \pm E_i) = 0$. On the other hand $n_-(-K_{\min} - \sum \pm E_i) = n_-(-K_X) \neq 0$, a contradiction just like before.

## 5. Some computations of Seiberg-Witten multiplicities

In this section we will go beyond determining potential basic classes and compute the Seiberg Witten multiplicity of elliptic surfaces. We also prove an algebraic version of the blow up formula. It is here that our excess intersection formulas pay off. We first show how to go over to a fully complex point of view. Then we use the special geometry of elliptic surfaces to compute the multiplicities and finally we prove a blow up formula.

From now on we identify an SC-structure with the corresponding twisting line bundle $\mathcal{L}$. We will consider the solutions of the monopole equations of section type, i.e. corresponding to equation (23).

We have already seen that the variation of the last monopole equation (21) with respect to the hermitian metric is $h$ is given by $(\Delta + |\alpha|_h^2)h^{-1}\delta h$ (c.f. equation 27). Therefore the solutions to the fourth monopole equation (21) is a smooth submanifold of $\mathcal{P}^*$ in a neighborhood of the moduli space $\mathcal{M}(\mathcal{L})$. In the proof of proposition 27 we have seen that we can identify this submanifold with the "vortex locus" $\{h = h(\bar\partial, \alpha, \beta)\}$ i.e. the image of the section $\mathcal{P}^{01*} \to \mathcal{P}^*$. The vortex locus is well defined in a neighborhood of the moduli space $\mathcal{M}(\mathcal{L})$ only, but this will not affect our arguments, as the construction of the localised Euler class in section 2 depends only on a neighborhood of $\mathcal{M}(\mathcal{L})$. Since the vortex locus is given by a function, we can identify it with its domain $\mathcal{P}^{01*}$ which carries a natural complex structure.

By property 4 of proposition 14 we are allowed to compute the localised Euler class $\mathbb{M}(\mathcal{L})$ of the moduli space by considering $\mathcal{M}(\mathcal{L})$ as a zero set of a section $S$ over the vortex locus cut out by the remaining equations, which define the same ideal as

$$\bar\partial^2 = 0, \qquad \bar\partial\alpha = 0, \quad \beta = 0$$

i.e. complex equations ! Moreover the deformation complex of these equations on $\mathcal{P}^{01*}$ in a point $(\bar\partial, \alpha, 0)$ is

$$A^{00}(X) \to A^{01}(X) \oplus A^{00}(\mathcal{L}) \oplus A^{02}(\mathcal{L}) \to A^{02}(X) \oplus A^{01}(\mathcal{L})$$

where the map is complex linear. We trivialise the determinant of the index using the complex structure. This has brought us safely in complex waters, and allows us to use proposition 15 and in particular formula 7.

From now on we identify $\mathcal{M}(\mathcal{L})$ with $\mathcal{M}^{\mathrm{BN}}(\mathcal{L})$. Define the vector bundles

$$\mathsf{A}^{pq}(\mathcal{L}) = \left(\mathcal{A}^{01} \times (A^{00}(\mathcal{L} \oplus A^{02}(\mathcal{L}))^*\right) \times_{\mathcal{G}^{\mathbb{C}}} A^{pq}(\mathcal{L})$$

over $\mathcal{P}^{01*}$. Then $\mathcal{M}^{\mathrm{BN}}$ is given by a section $s$ in $E = A^{02}(X) \oplus \mathsf{A}^{01}(\mathcal{L})$, and the tangent space is given by

$$T\mathcal{P}^{01*} \cong \left(A^{01}(X) \oplus \mathsf{A}^{00}(\mathcal{L}) \oplus \mathsf{A}^{02}(\mathcal{L})\right)/A^{00}(X).$$

The deformation complex can be considered as a map $T\mathcal{P}^* \to E$ and is exactly what we called $Ds$ in section 2.

To identify the index $\mathrm{Ind}Ds$ we first make a compact perturbation, keeping only the differential operator part of the deformation complex. Then it splits naturally in the $\bar\partial$ complex on $X$ with index $\mathbb{C}^{\chi(\mathcal{O}_X)}$ and the index of complex

$$0 \to \mathsf{A}^{00}(\mathcal{L}) \xrightarrow{\bar\partial} \mathsf{A}^{01}(\mathcal{L}) \xrightarrow{\bar\partial} \mathsf{A}^{02}(\mathcal{L}) \to 0$$

where $\bar\partial$ is the universal $\bar\partial$ operator descended to $\mathcal{P}^{01*}$.

To rewrite this index in holomorphic terms, consider the universal divisor

$$\Delta = \{(\bar\partial, \alpha, x) \mid \alpha(x) = 0\}$$

on the pull back of $X \times \mathcal{M}^{\mathrm{BN}}$. Now if $\Omega^{pq}$ is the sheaf of $C^\infty$ $(p,q)$-forms on $X$ considered as an $\mathcal{O}(X)$ module, then I claim that on the pull back of $X \times \mathcal{M}^{\mathrm{BN}}$ to $\mathcal{A}^{01} \times (A^{00}(\mathcal{L} \oplus A^{02}(\mathcal{L}))^*$, there is a $\mathcal{G}^{\mathbb{C}}$ equivariant exact sequence

$$0 \to \mathcal{O}(\Delta) \xrightarrow{\bar\partial} p_1^*\Omega^{00}(\mathcal{L}) \xrightarrow{\bar\partial} p_1^*\Omega^{01}(\mathcal{L}) \xrightarrow{\bar\partial} p_1^*\Omega^{02}(\mathcal{L}) \to 0.$$

In fact this only says that $(\bar\partial, \alpha, x) \to \alpha(x)$ is a $\mathcal{G}^{\mathbb{C}}$ invariant section vanishing along $\Delta$ with multiplicity 1 lying in the kernel of $\bar\partial$, which is obvious. Now descend this whole complex to $X \times \mathcal{M}^{\mathrm{BN}}$ and take push forward to $\mathcal{M}^{\mathrm{BN}}$. Then we get an exact sequence of complexes

$$0 \to Rp_*\mathcal{O}(\Delta) \xrightarrow{\bar\partial} \mathsf{A}^{00}(\mathcal{L}) \xrightarrow{\bar\partial} A^{01}(\mathcal{L}) \xrightarrow{\bar\partial} \mathsf{A}^{02}(\mathcal{L}) \to 0$$

where we are considering $Rp_*\mathcal{O}(\Delta))$ as a complex with zero boundary operator and $\mathsf{A}^{pq}(\mathcal{L})$ as a complex concentrated in degree 0. Thus for the index we find

$$\operatorname{Ind}(Ds) = \operatorname{Ind}\left(Rp_*\mathcal{O}(\Delta)\right) + \mathbb{C}^\chi \tag{34}$$

A more precise description of $\mathcal{M}^{\mathrm{BN}}$ depends on the surface. Here we will do the case of elliptic surfaces.

**Proposition 39.** *Let $X \xrightarrow{\pi} C$ be a Kählerian elliptic surface over a curve $C$ of genus $g$, with multiple fibers $F_1, \dots F_r$ of multiplicity $p_1, \dots p_r$ of holomorphic Euler characteristic $\chi$. Consider the line bundle $\mathcal{L} = \mathcal{O}(\pi^* D + \sum a_i F_i)$ where $D$ is a divisor on $C$ of degree $d$, and $0 \le a_i < p_i$. Then the Seiberg Witten multiplicity is zero if $d < 0$, and if $d \ge 0$ it is given by*

$$n_{(-)}(\Lambda^{0*}(\mathcal{L})) = \begin{cases} (-1)^d \binom{\chi+2g-2}{d} & \text{if } \chi + g - 2 \ge 0 \\ \sum (-1)^j \binom{1-g-\chi+d-j}{d-j}\binom{g}{j} & \text{if } \chi + g - 2 < 0 \end{cases}$$

Note that if the topological Euler characteristic $e > 0$ (or equivalently $\chi > 0$) then $g = q = \frac{1}{2} b_1(X)$ [FM2, corollary II.2.4], so in this case $\chi + g - 2 = p_g - 1$. Note that the second formula is just 1 if $p_g = q = 0$ (i.e. $e > 0$). This illustrates remark 35.

If $p_g > 0$ and $q = g = 0$, so in particular $e = 12\chi > 0$, Witten proves this formula by choosing a general $\omega \in H^0(K_X)$ and using the perturbation $\epsilon = \omega + \bar\omega$. He then argues that the multiplicity $n(\mathcal{L})$ is the number of ways we can decompose a fixed canonical divisor $K_0$ as $K_0 = D_+ + D_-$ with $D_+ \in |(\mathcal{L}, \bar\partial_0)|$, and $D_- \in |K \otimes (\mathcal{L}, \bar\partial_0)^\vee|$, where $\bar\partial_0$ is the unique holomorphic structure that $\mathcal{L}$ admits [Wit, eq. (4.23) e.v.].

To be honest, this is what I read out of it. Note for example that his sheaf $R$ is just $\mathcal{L}|_Z(\alpha)$, and that

$$h^0(R) = h^0(\mathcal{L}|_{Z(\alpha)}) = \dim T_{(\bar\partial, \alpha)} = d$$

(the last equality we will see in a minute). Actually I think that the computations below are the mathematical version of (I paraphrase) "integrating over the bosonic and fermionic collective coordinates in the path integral" and "computing the Euler class of the bundle of the cokernel of the operator describing the linearised monopole equations over the moduli space (the bundle of antighost zero modes)" [Wit, above (4.11)]. In fact with hindsight, the latter seems a dual description of the localised Euler class in the case that the cokernel has constant rank.

*Proof.* We choose a Kähler metric and $\lambda$ such that $\deg_\Phi(\mathcal{L}^{\otimes 2}(-K)) < \lambda \operatorname{Vol}(X)$. This means that if $\mathcal{L}$ has non zero multiplicity, it must carry a holomorphic structure with a section. In case $p_g = 0$ it also means we are looking at $n_-$. But $(\mathcal{L}, \bar\partial)$ has a section if and only if $D$ is an effective divisor on $C$. In fact a family of vertical line bundles with a section gives a family of effective divisors on $C$ by pushforward of the line bundle, and conversely a family of effective divisors on $C$

gives a family of vertical line bundles with a section by pull back and multiplication with a fixed section in $\mathcal{O}(B) = \mathcal{O}(\sum a_i F_i)$ ($B$ for base locus). Thus there is a natural isomorphism

$$\mathcal{M}^{\mathrm{BN}} \cong \mathcal{M}_C^{\mathrm{BN}} = C^d$$

where $C^d$ is the $d^{\text{th}}$ symmetric power of $C$. The functorial isomorphism comes with an isomorphism $\mathcal{O}(\Delta_X) = \mathcal{O}(\pi^*\Delta_C + B)$.

Next we use Grothendieck Riemann Roch (an alias of the family index theorem). Let $q: C \times C^d \to C^d$ be the projection map. Then the projection $p: X \times \mathcal{M}^{\mathrm{BN}} \to \mathcal{M}^{\mathrm{BN}}$ can be factored as $p = q \circ \pi \times \mathrm{id}$. Thus writing $\pi \times \mathrm{id}$ as $\pi$,

$$\begin{aligned}
\mathrm{ch}(Rp_*\mathcal{O}(\Delta)) &= \mathrm{ch}\left(Rq_*\left(\mathcal{O}(\Delta_C) \otimes R\pi_*\mathcal{O}(B)\right)\right)\\
&= q_*\left(ch(\mathcal{O}(\Delta_C)\,\mathrm{ch}\,R\pi_*\mathcal{O}(B)\,\mathrm{td}(C)\right)\\
&= q_*\left(ch(\mathcal{O}(\Delta_C))\pi_*\left(e^B(1-K/2+\chi(\mathcal{O}_X)(pt\times C^d)\right)\right)\\
&= \chi(\mathcal{O}_X)q_*\left(ch(\mathcal{O}(\Delta_C))(pt\times C^d)\right)\\
&= \mathrm{ch}(\mathcal{O}(1)^\chi),
\end{aligned}$$

where we have abbreviated the holomorphic Euler characteristic by $\chi$. If we denote by $x$ the chern class of $\mathcal{O}(1)$, then our computation shows that

$$c_t(\mathrm{Ind}Ds) = (1+tx)^\chi,$$

at least over the rationals.

The chern classes of the tangent bundle of $C^d$ are computed in [ACGH, eq. VII.5.4]. Denoting the pullback of the $\theta$ divisor on $\mathrm{Pic}^d$ to $C^d$ by $\theta$ the result is

$$c_t(T_{C^d}) = (1+tx)^{d+1-g}e^{-t\theta/(1+tx)}$$

Combining these two expressions, our multiplicity drops out

$$\begin{aligned}
n(\Lambda^{0*}(\mathcal{L})) &= c(\mathrm{Ind}Ds)^{-1}c(T_C^d)\cap[C^d]\\
&= [(1+tx)^{d+1-g-\chi}e^{-t\theta/1+tx}]_{t^d}
\end{aligned}$$

With the following identity of formal power series [ACGH, eq. VIII.3.1]

$$[(1+xt)^a f(-t/(1+xt))]_{t^b} = [(1-xt)^{b-a-1}f(-t)]_{t^b}.$$

the expression becomes

$$n(\Lambda^{0*}(\mathcal{L})) = [(1-tx)^{\chi+g-2}e^{-t\theta}]_{t^d} = \begin{cases} (-1)^d\sum_{j=0}^d \binom{\chi+g-2}{d-j}\frac{\theta^j x^{d-j}}{j!} & \text{if } \chi+g-2\geq 0\\ \sum_{j=0}^d(-1)^j\frac{1-g-\chi+d-j}{d-j}\frac{\theta^j x^{d-j}}{j!} & \text{if } \chi+g-2<0\end{cases}$$

Now $\theta^j x^{d-j}\cap[C^d] = j!\binom{g}{j}$ [ACGH, below eq. VIII.3.3]. The elementary identity $\sum_j\binom{a}{j}\binom{b}{c-j} = \binom{a+b}{c}$ then gives the answer as stated. □

As a second application of the methods developed we give a complex analytic version of the blow up formula.

**Proposition 40.** *Let $(X,\Phi)$ be a Kähler surface, and $\mathcal{L}$ a line bundle on $X$. Suppose that $\deg_\Phi(\mathcal{L}^{\otimes 2}(-K) < \lambda\,\mathrm{Vol}(X)$. Let $\sigma: \tilde{X} \to X$ be the blow up of $X$ in a point, with Kähler form $\tilde{\Phi}$, and let $\tilde{\mathcal{L}} = \mathcal{L}(aE)$ be a line bundle on $\tilde{X}$ with $a \geq 0$. Suppose that the cohomology class of $\tilde{\Phi}$ is close to $\Phi$. Then there is a natural identification $\mathcal{M}(\tilde{\mathcal{L}}) = \mathcal{M}(\mathcal{L})$, and*

$$\mathbb{M}(\tilde{\mathcal{L}}) = [(1+x)^{a(a-1)/2}\widehat{\mathbb{M}}(\mathcal{L})]_{\dim_{\mathbb{R}}=\mathcal{L}\cdot(\mathcal{L}-K)-a(a-1)}.$$

*Here* $\widehat{\mathbb{M}}$ *is the class defined in remark 18, and* $x$ *the class of the natural bundle* $\mathcal{O}(1)$ *over* $\mathcal{M}$*. In particular if* $a = 0, 1$ *then* $n(\tilde{\mathcal{L}}) = n(\mathcal{L})$.

Of course this proposition determines the multiplicity

$$n_{(-)}(\Lambda^{0*}(\mathcal{L}(aE))) = n_{(-)}(\Lambda^{0*}(\mathcal{L}(-aE)))).$$

Since quite in general $n_+(\Lambda^{0*}(\mathcal{L})) = \pm n_-(\Lambda^{0*}(K \otimes \mathcal{L}^\vee)$ it determines the corresponding relation for $n_+$ up to sign which is really all we need here.

*Proof.* The conditions on the degree imply that a solution of the monopole equations correspond to a holomorphic structure on $\mathcal{L}$ with a section. Since $\tilde{\Phi}$ is close to $\Phi$ we have (by definition of close) $\deg_{\tilde{\Phi}}(\tilde{L}) < \lambda \operatorname{Vol}(\tilde{X})$, hence solutions on the blowup also correspond to holomorphic structures on $\tilde{\mathcal{L}}$ with a section.

Now $aE$ is contained in the base locus of the sections. Thus similar to what we did for elliptic surfaces, we get an identification of $\mathcal{M}(\mathcal{L})$ with $\mathcal{M}(\tilde{\mathcal{L}})$ by multiplication of the section with a section in $\mathcal{O}(aE)$, and the universal divisor on $\tilde{X} \times \mathcal{M}(\tilde{\mathcal{L}})$ is $\tilde{\Delta} = \Delta + aE$.

Again, identify the chern class of the index of the deformation complex with formula (34). Let $\tilde{p}$ be the projection $\tilde{X} \times \mathcal{M}(\mathcal{L}) \to \mathcal{M}(\mathcal{L})$, and $p$ the projection $X \times \mathcal{M}(\mathcal{L}) \to \mathcal{M}(\mathcal{L})$. Then the total chern class of the index is

$$c(R\tilde{p}_*(\tilde{\Delta})) = c\left(Rp_*\left(\mathcal{O}(\Delta) \otimes R\sigma_*\mathcal{O}(aE)\right)\right).$$

By induction on $a$, one shows that

$$R\sigma_*\mathcal{O}(aE) = \mathcal{O} - \mathcal{O}_{pt}^{a(a-1)/2}.$$

Since $\mathcal{O}(\Delta|_{pt \times \mathcal{M}(\mathcal{L})}) = \mathcal{O}(1)$ it gives

$$c(R\tilde{p}_*(\tilde{\Delta})) = c(Rp_*\mathcal{O}(\Delta))/c(\mathcal{O}(1))^{a(a-1)/2}.$$

Formula (7) gives us

$$\mathbb{M}(\tilde{L}) = \left[(1+x)^{a(a-1)/2}\left(c(Rp_*(\Delta))^{-1}c_*(\mathcal{M}(\mathcal{L})\right)\right]_{d(\tilde{L})}$$

Since the real virtual dimension of $\mathcal{M}(\tilde{\mathcal{L}})$ is $d(\tilde{\mathcal{L}}) = \mathcal{L} \cdot (\mathcal{L} - K) - a(a-1)$ and the term in brackets is exactly $\widehat{\mathbb{M}}(\mathcal{L})$, we have proved the formula. ☐

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