Bicombings of triangle buildings

Gennady A. Noskov

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Abstract

For an arbitrary triangle building Δ we define some "rational" bicombing, which satisfies the "k-fellow traveller property" and which is invariant under "type rotating" automorphisms of Δ. In particular we obtain a purely geometric proof of biautomacity of a finitely generated group,acting simply transitively on the vertices of Δ.

Introduction

The idea of bicombing, introduced by W.Thurston, became quite popular in combinatorial group theory, see, for example [AB], [ECHLPT], [GS], [NS]. A bicombing of a space is a choice of a family of paths in X between each (or almost each) pair of points of X. The idea is that bicombings with rather nice dependence of the paths on varying end points generalize the geometry of geodesics in a simply connected manifold of nonpositive curvature. This approach permits [AB] *"... to transfer as much information as possible about the asymptotic geometry of nice spaces, such as nonpositively curved manifolds, to related spaces whose lack of local structure precludes local-to-global analysis"*.

Every affine building, being a CAT(0)- space admits a geodesic bicombing. In this paper we are interested in bicombings which are "sufficiently rational", having in mind to apply this to constructing biautomatic structures on groups. The following definition of rationality seems to be good enough: a bicombing $\mathcal{C}$ on an affine building Δ is *rational* if

1)For every pair of vertices of Δ there is a path $\gamma \in \mathcal{C}$ joining these points,

2) Every $\gamma \in \mathcal{C}$ lies in the 1-dimensional skeleton of Δ,

3) There exists a natural number n, such that every $\gamma \in \mathcal{C}$ consists of at most n geodesics,

4) $\mathcal{C}$ satisfies the "fellow traveller property" (see section 3 for definition).

We believe that a rational bicombing exists on every affine building. In this paper we construct such a bicombing on every triangle building ($\tilde{A}_2$–building in Tits' classification). This permits us to obtain a purely geometric proof of the biautomacity of a finitely generated group acting simply transitively on the vertices of a triangle building, see [CS].

I am grateful to the SFB 343, University of Bielefeld for their hospitality while I carried out much of this work. I would like to thank Herbert Abels for his kind invitation, interest and support.

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I would like to thank Herbert Abels for his kind invitation, interest and support. I gratefully acknowledge the suggestions of Herbert Abels and John Klein, improving the text.

1 Triangle buildings

1.1 Definitions

We adopt the following direct definition of a triangle building which can be easily extracted from [Bro]. Let Σ_0 be the abstract simplicial complex associated with the Euclidean plane, tiled by equalaterial triangles. We call a simplicial complex Δ a triangle building if it can expressed as the union of a family of subcomplexes Σ, called apartments,satisfying the following conditions:

B0) Each apartment Σ is isomorphic to Σ_0,

B1) Any two simplices of Δ are contained in an apartment,

B2) Given two apartments Σ, Σ' with a common 2-simplex (=chamber), there is an isomorphism $\Sigma \overset{\approx}{\to} \Sigma'$ fixing $\Sigma \cap \Sigma'$ pointwise.

We prefer to consider the geometric realization $|\Delta|$ of Δ rather than Δ itself. One can easily show that there is a canonical bijection between $|\Sigma_0|$ and the Euclidean plane, which Σ_0 triangulates. Thus Σ_0 carries a metric, which we normalize, agreeing that all 1-simplices(edges) have length 1. If we do the same for each apartment Σ, then we see that the simplicial isomorphism $\Sigma \overset{\approx}{\to} \Sigma'$ in the definition of Δ extends uniquely to an isometry $\Sigma \overset{\approx}{\to} \Sigma'$.

It will be convenient to apply to $|\Delta|$ the same terminology as to Δ itself.In particular, we will refer to $|\Delta|$ as a building and to the subsets $E = |\Sigma|$ as apartments. Moreover, we will use the notation Δ for $|\Delta|$.

So each apartment E of Δ is a Euclidean space with a metric $|x-y|_E, x, y \in E$. Moreover, the isomorphisms between apartments given by the building axiom (B2) are isometries. The metrics $|x-y|_E$ can be pieced together to make the entire building Δ a metric space. The resulting metric will be denoted by $x, y \mapsto |x-y|$. It is known (see [Bro],Thm. VI.3) that the metric space Δ is complete. Besides, for any $x, y \in \Delta$ the line segment $[x, y]$ joining them in any apartment E, the line segment $[x, y]$ is independent of the choice of E and can be characterized by

$[x, y] = \{z \in X; |x-y| = |x-z| + |z-y|\}$

Besides, $[x, y]$ is *geodesic*, that is the shortest path joining x and y and there is no other geodesic joining x and y.

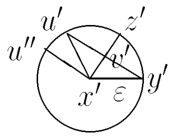
Let us call a subset $M \subset \Delta$ *Euclidean* if it is contained in some apartment. The subset of Δ is called *convex* if it contains the geodesic connecting any two of its points. The *convex hull* of a subset of Δ is defined as the intersection of all convex subsets containing the given one. Obviously, the convex hull of a Euclidean subset M is Euclidean and it coincides with the convex hull of M in every apartment containing M.

1.2 Triangle inequality for angles

We need the notion of *angle* between segments (see [Bro],p.169,exercises). Given $x, y, z \in \Delta$ with $x \neq y$ and $x \neq z$ there is a well-defined angle θ between $[x, y]$ and $[x, z]$ in x. To define it take $y' \in [x, y]$ and $z' \in [x, z]$ close enough to x such that $[x, y']$ and $[x, z']$ are each contained in a closed simplex. Then by axiom (B0) both $[x, y']$ and $[x, z']$ are contained in an apartment E, so the convex hull of the set x, y', z' is a Euclidean triangle and the angle θ between $[x, y']$ and $[x, z']$ is defined. In view of the properties of the convex hull, the value of θ does not depend on the choice of y' and z'. We will write $\theta = \angle yxz$. Sometimes, when there is no danger of confusion we will use the notation $\angle x$. The following lemma is a "triangle inequality" for angles.

Lemma. *Let* $[x, y], [x, z], [x, u]$ *be non-degenerate segments in* Δ*, then* $\angle yxz + \angle zxu \geq \angle yxu$

Proof. We may assume that all the segments are of the same length $\epsilon > 0$ and that each of them is contained in a closed simplex and thereby each pair of the segments is contained in an apartment. Thus each of the triangles yxz, xzu, yxu is Euclidean. The inequality desired is obvious if the sum $\angle yxz + \angle zxu$ is not smaller than π, so we may assume that this sum is $\leq \pi$. Suppose, that the inequality is not true, then draw in the Euclidean plane the following triangles isometric to ours:



All the letters have an obvious meaning, except, maybe, of u'',where we mean that the triangle $y'x'u''$ is isometric to the triangle yxu. By assumption $\angle y'x'z' + \angle z'x'u' < \angle y'x'u''$, so $|u' - y'| < |u'' - y'|$. Let v' be an intersection of $[y', u']$ and $[x', z']$ and let v be the corresponding point in $|\Delta|$. Then

$|u - y| \leq |u - v| + |v - y| = |u' - v'| + |v' - y'| = |u' - y'| < |u'' - y'| = |u - y|$.

This contradiction proves the lemma.

1.3 The sum of angles in polygons

Lemma. *The sum of angles of a geodesic triangle* xyz *in* Δ *is not bigger than* π.

Proof. Let us call a triangle fat if the sum above is strictly bigger than π. Suppose that xyz is fat. Let u be any internal point of $[x, y]$. We assert that one of the triangles xuz, yuz is fat. Indeed,

$(\angle x + \angle xuz + \angle uzx) + (\angle y + \angle yuz + \angle uzx) \geq$

(applying the previous lemma to the angles in z and u)
$\angle x + \angle yux + \angle y + \angle z =$
one should explain $\angle x$ etc.
$\geq \pi + \angle x + \angle y + \angle z > 2\pi$ (since xyz is fat).

Now, divide the segment $[x, y]$ by points $u_1, u_2, \dots, u_n$ in such a way, that the segments $[x, u_1]$, $[u_1, u_2], \dots [u_n, y]$ are each contained in a closed simplex. It follows from the remark above that one of the triangles x_1uz, $u_1u_2z \dots u_nyz$ is fat. Thus we have a fat triangle with one side contained in a closed simplex. This simplex and the opposite vertex of the side are contained in an apartment and consequently our fat triangle is Euclidean. But every Euclidean triangle has π as sum of its angles. This contradiction proves the lemma.

Remarks. The lemma is actually a manifestation of "nonpositive curvature" of Δ but we prefer to give this short and direct proof. See [Bri], where a survey is given of non positive curvature theory in piecewise Euclidean complexes.

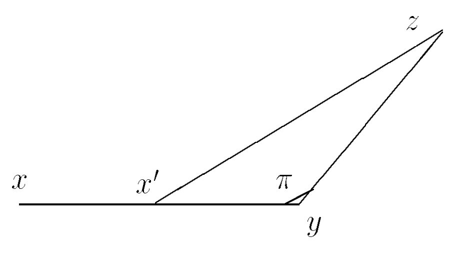
Lemma. *The sum of angles of a geodesic quadrangle in Δ is not greater than 2π.*

Proof. If $xyzu$ is a geodesic quadrangle, then,considering the triangles xyz and xuz, we have the inequalities
$\angle zxy + \angle y + \angle xzy \leq \pi$,
$\angle uxz + \angle xzu + \angle u \leq \pi$.
and in view of triangle inequality for angles we have that
$\angle x \leq \angle zxy + \angle zxu$,
$\angle z \leq \angle yzx + \angle xzu$.
It follows from these four inequalities that
$\angle x + \angle y + \angle z + \angle u \leq 2\pi$.

1.4 Local geodesics are geodesic

Lemma. *If $\angle xyz = \pi$,then the union $[x, y] \cup [y, z]$ is a geodesic and coincides with $[x, z]$.*

Proof. Consider the triangle xyz. Let $[x, x'] = [x, y] \cap [x, z]$.



If $x' = y$, then y lies on the segment $[x, z]$ and there is nothing to prove. If $x' \neq y$, then $\angle zx'y \neq 0$, so in the triangle $x'uz$ the sum of angles is strictly bigger then π. This contradiction proves the lemma.

Lemma *(Local geodesics are geodesic). If $[x_0, x_1], [x_1, x_2], \dots, [x_n, x_{n+1}]$ is a sequence of segments, such that the angle between any two subsequent edges is π,then the union of all the segments coincides with $[x_0, x_n]$.*

Proof by induction on n applying the previous lemma.

1.5 The types of edges

Let Δ be a triangle building and Δ^0, Δ^1 be its sets of vertices (0–simplices) and edges (1–simplices), respectively. It is known that Δ (as any building) is *labellable* ([Bro],p.78). This means that there is a function $\tau : \Delta^0 \to \mathbf{Z/3} = \{0,1,2\}$, which takes the vertices of every chamber bijectively onto $\mathbf{Z/3}$. We call τx the *type* of x and we call $(\tau x, \tau y) \in \mathbf{Z/3} \times \mathbf{Z/3}$ the type of the oriented edge $[x,y]$ of Δ^1.

The diagonal subgroup $D = \{(x,x) : x \in \mathbf{Z/3}\}$ acts on $\mathbf{Z/3} \times \mathbf{Z/3}$ by translations and thus defines the orbit equivalence relation on $\mathbf{Z/3}\times\mathbf{Z/3}$. There are two equivalence classes: the first one, called "positive", consists of (0,1), (1,2), (2,0) and the second one, called "negative", consists of (1,0), (2,1), (0,2). The terminology comes from an orientation of the triangle with vertices in $\mathbf{Z/3}$. The edges with corresponding types will be called *positive* and *negative*, respectively. The *sign* of the edge is defined to be "+" oder "-" iff the edge is positive or negative, respectively.

The following properties can be easily seen:

If $[x,y],[y,z]$ are directed edges with mutual angle π in y, then they have the same sign. On the other hand, if this angle is $2\pi/3$, then the edges have different signs.Further, if e_1, e_2 are directed distinct edges with the same starting point and of the same sign, then the angle between them in the common starting point is equal to $2\pi/3$.

2 Combing triangle buildings

2.1 Global definition

Let Δ be a triangle building, labelled by the symbols 0, 1, 2. We want to "comb" Δ, that is to find a family of paths $\mathcal{C}$ in Δ^1, which is well behaved under varying the end points of paths. We will consider piecewise linear paths in Δ^1,parametrized by the length and joining the points of Δ^0.Such a path γ is uniquely determined by the sequence of the vertices $\gamma(0), \gamma(1), \ldots \gamma(|\gamma|)$, where $|\gamma|$ is the length of γ. Sometimes we will identify the path γ with its image, when there will be no danger of confusion. It is convenient to extend the definition of γ onto the whole halfline $[0,\infty)$, making $\gamma(t)$ stop after the moment $t = |\gamma|$, i.e. setting $\gamma(t) = \gamma(|\gamma|)$ for $t \geq (|\gamma|)$. Let us denote $\gamma(0)$ by γ_0 and $\gamma(|\gamma|)$ by γ_∞.

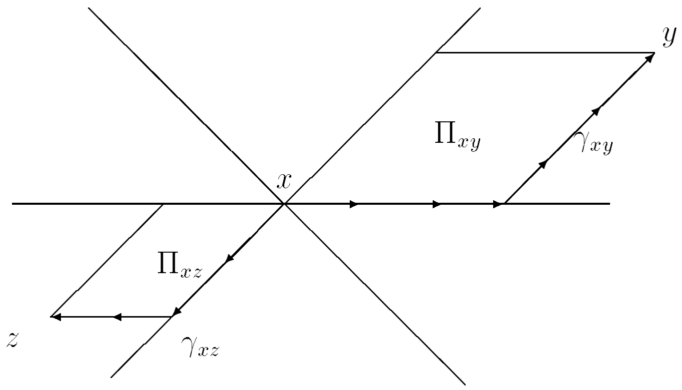
In order to define the combing take an arbitrary pair of vertices x, y and let Σ be an arbitrary apartment containing x and y. Define Π_{xy} as the smallest convex subcomplex of Σ containing x and y. Thus either Π_{xy} is a segment $[x,y]$ or a parallelogramm such that

a)the boundary of Π_{xy} lies in Δ^1,

b)both x and y are the vertices of Π_{xy},

c)the angles of Π_{xy} at both x and y are equal to $\pi/3$.

It is easy to see that Π_{xy} is uniquely defined by the properties a), b), c).



Note, that Π_{xy} doesn't depend on the choice of Σ, since if Σ' is another apartment, containing both x and y, then $\Sigma \cap \Sigma'$ is a convex subcomplex in both of Σ and Σ' and thus $\Sigma \cap \Sigma'$ contains Π_{xy} .

Now consider two cases:

1) $\Pi_{xy} = [x, y]$ *lies in* Δ^1. Then we take the segment $[x, y]$ as the path γ_{xy} of our combing $\mathcal{C}$, joining the vertexes x and y.

2) Π_{xy} *is a parallelogramm.* In this case we take γ_{xy} to be the path joining x and y and consisting of two sides of the boundary of Π_{xy}, such that the initial edge of γ_{xy} is positive, see 1.5.

2.2 Local definition of $\mathcal{C}$

The definition of $\mathcal{C}$ given above is "global" in the sense that a path from $\mathcal{C}$ "knows" where it goes. In this section we show that a path from $\mathcal{C}$ can be defined by a simple local "direction set"– either go in any direction with zero turning angle in every vertice or at most one turning in $\pi/3$-angle is allowed but in the last case the initial directed edge has to be positive. Then, passing a finite number of edges, stop. This description will be important when we will construct a biautomatic structure.

By definition the *turning angle* of the path γ at the point $\gamma(n) \in \Delta^1$ is the angle $\pi - \angle\gamma(n-1)\gamma(n)\gamma(n+1)$.

Now define the family $\mathcal{C}'$ of all path γ in Δ^1,such that

1) The turning angle of γ in every internal vertex is zero, except in possibly at most one vertex, where it equals $\pi/3$,

2) If some of the turning angles is equal $\pi/3$, then the initial edge of γ is positive.

It follows from the definition and from the geodesicity of local geodesics (see 1.4) that for any $\gamma \in \mathcal{C}'$ there exist an integer $n \in [0, |\gamma|]$, such that both subpaths $\gamma_+ = \gamma|_{[0,n]}$ and $\gamma_- = \gamma|_{[n,|\gamma|]}$ are geodesic. Moreover, it follows from the properties of signs , see 1.6, that if γ_+ is trivial then it is positive (that is all of its directed edges are positive) and if γ_- is different from γ then it is negative.

2.3 $\mathcal{C} = \mathcal{C}'$

We have defined two bicombings, namely $\mathcal{C}$ in 2.1 and $\mathcal{C}'$ in 2.2.

Theorem. *The bicombings $\mathcal{C}$ and $\mathcal{C}'$ coincide.*

Proof. It follows immediately from the global definition of $\mathcal{C}$ that it is contained in $\mathcal{C}'$.

Now let $\gamma' \in \mathcal{C}'$. Take $\gamma \in \mathcal{C}$, having the same beginning and the same end as γ', that is $\gamma_0 = \gamma'_0$ and $\gamma_\infty = \gamma'_\infty$. We will show that $\gamma = \gamma'$. Consider the following cases for γ and γ'.

Case 1. *Both γ and γ' are geodesic.*

In this case the assertion of the theorem immediately follows from the uniqueness of geodesics in Δ.

Case 2. *γ' is geodesic and γ is not geodesic.*

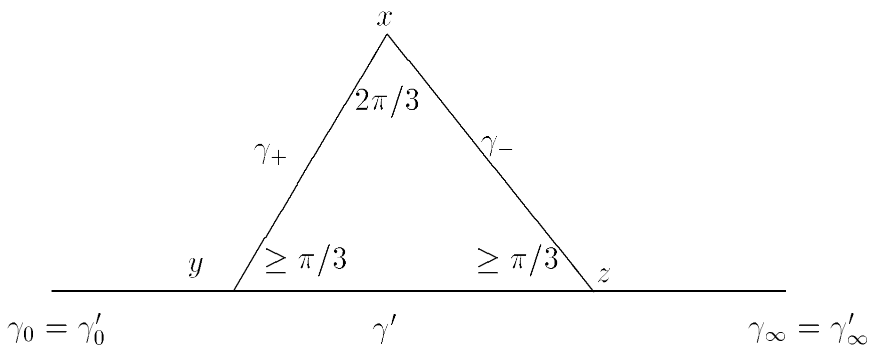
Then $\gamma = \gamma_+ \cup \gamma_-$ and both γ_+ and γ_- are nontrivial. Let x be the turning point of γ.

Case 2a. *The vertex x belongs to γ'.*

Then, as follows from the uniqueness of geodesics, both γ_+ and γ_- lie in a geodesic γ' and, consequently, the turning angle of γ in x is equal 0 or π. But this angle equals $\pi/3$ (see property 1) of $\mathcal{C}$) and this contradiction shows that the case 2a) is impossible.

Case 2b. *The vertex x does not belong to γ'.*

The intersection of two geodesics is geodesic again, so $\gamma_+ \cap \gamma' = [\gamma_0, y]$ for some vertex y and similarly $\gamma_- \cap \gamma' = [z, \gamma_\infty]$ for some vertex z.



Now consider the geodesic triangle xyz. Since the edges which meet in y and z are distinct, the angles of the triangle in x and y are not smaller than $\pi/3$ and $\angle yxz$ is equal to $2\pi/3$. Thus the sum of the angles in our triangle is not smaller that $4\pi/3$, a contradiction.

Case 3. *γ is geodesic and γ' is not geodesic.*

The proof is the same as for case 2. Here we rely on the the fact that the only nonzero turning angle of $\gamma \in \mathcal{C}$ is equal to $\pi/3$ and this is also true for $\mathcal{C}'$.

Case 4. *Both γ and γ' are geodesic.*

Denote x and x' the turning vertexes of γ and γ' respectively.

Case 4a. *The vertex x belongs to γ'.*

Then, as it follows from the uniqueness of geodesics, γ_+ lies in a geodesic γ' and, applying case 2a) to the subpath of γ and γ', beginning in x and ending in γ_∞ we get that $\gamma = \gamma'$.

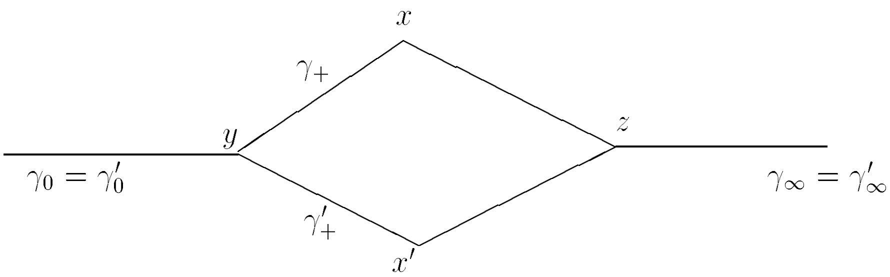
Case 4b. *The vertex x' belongs to γ.*

This is completely symmetrical to 4a), taking into account the remark to case 3).

Case 4c. *The vertex x' does not belong to γ and the vertex x does not belong to γ.*

Let y and z have the same meaning as in case 2b), that is

$[\gamma_0, y] = [\gamma_0, x] \cap [\gamma_0, x']$ and $[z, \gamma_\infty] = [x, \gamma_\infty] \cap [x', \gamma_\infty]$



Remark that the first (directed) edges of the subpath $[y, x]$ and $[y, x']$ are positive, so, in view of the properties of signs, the angle $\angle xyx'$ is equal to $2\pi/3$. Analogously, the angle $\angle xzx'$ is also equal to $2\pi/3$. By the definition of $\mathcal{C}'$ the angles of the geodesic quadrangle $xyx'z$ at x and x' are both equal to $2\pi/3$. Thus the sum of the angles in $xyx'z$ is equal to $8\pi/3 > 2\pi$ contradicting 1.3.

We conclude from the considerations above that $\gamma = \gamma'$ and this proves the theorem.

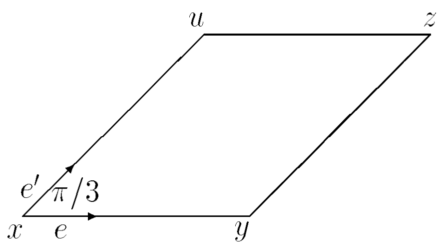
3 Fellow traveller property

Theorem. *The bicombing $\mathcal{C}$ constructed above satisfies the "2-traveller property", that is if $\gamma, \gamma' \in \mathcal{C}$ begin and end at a distance at most one apart, then $|\gamma(t) - \gamma'(t)| \leq 2$ for all $t \geq 0$.*

Proof. Consider all possible configurations, which the end points of γ and γ' can have.

Case 1. *γ and γ' have a common origin and common end.*

Let $\gamma_0 = \gamma'_0 = x$ and $\gamma_\infty = \gamma'_\infty = z$. If $[x, z] \subset \Delta^1$, then $\gamma = \gamma' = [x, z]$ and there is nothing to prove. Otherwise the parallelogramm $\Pi = \Pi_{xz}$ is defined. Let y, u be the remaining two vertices of Π. If both γ and γ' consist of $[x, y] \cup [y, z]$ or that of $[x, u] \cup [u, z]$, then γ and γ' just coincide. We show that the opposite case, when, say, $\gamma = [x, y] \cup [y, z]$ and $\gamma' = [x, u] \cup [u, z]$, is impossible. Indeed, let e, e' are the initial (directed) edges of γ

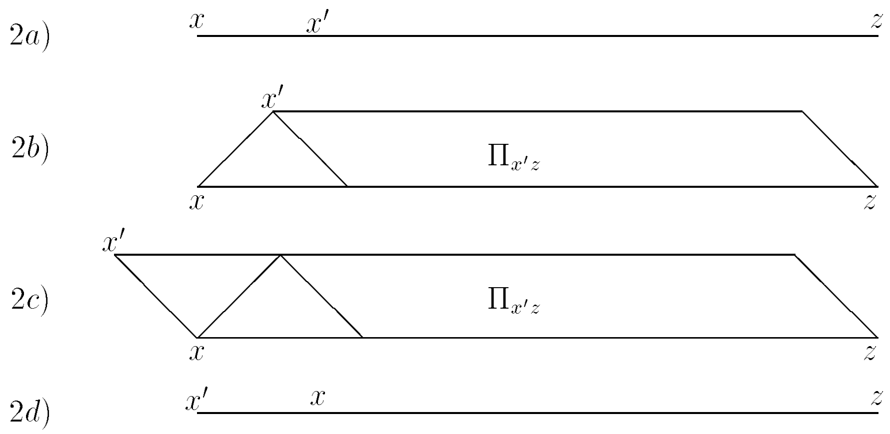


and γ' respectively.

By definition of $\mathcal{C}$ they are both positive. On the other hand, since the angle between e and e' is equal to $\pi/3$, they both cannot be positive, as follows from the properties of signs, see 1.5.

Case 2. *γ and γ' have a common end and γ is geodesic.*

Let $\gamma_0 = x, \gamma'_0 = x'$ and $\gamma_\infty = \gamma'_\infty = z$ and $|x - x'| \leq 1$. The case $x = x'$ is considered already,so we may assume that $|x - x'| = 1$ and we can put $[x, x']$ and z into some apartment. The following configurations are possible apriori:

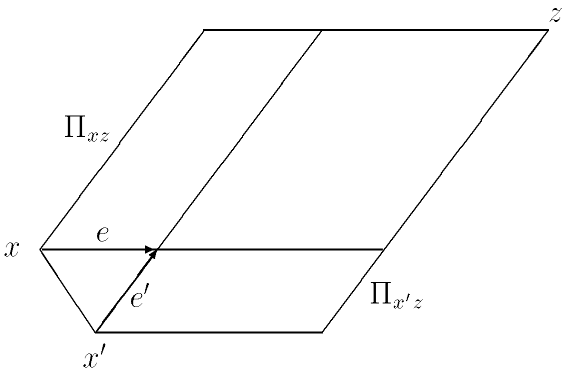


In the case 2a) the path γ and γ' obviously 1-travel along the path $[x, z]$. In the cases 2b) and 2c) the path γ' either comes down and travels along $[x, z]$ or , the other way around, firstly travels "horizontally" and on the last step, comes down to z. In both cases γ and γ' 1-fellow travel.

The case 2d) is similar to 2a), but now γ' travels one step behind γ.

Case 3. *γ and γ' have a common end and both γ and γ' are not geodesic.*

Again, let x and x' be the origins of γ and γ', respectively, so that $|x - x'| = 1$. Again, put $[x, x']$ and z into the apartment Σ. As we know, Σ also contains the parallelograms $\Pi_{xz}, \Pi_{x'z}$, where z is the common end of γ and γ'.



If the initial segments e, e' of γ, γ' are parallel then it is obvious that γ and γ' 1-fellow travel. If e, e' are not parallel, then the angle between e and e' is equal to $\pi/3$. The following lemma shows that this case is impossible.

Lemma. *There are no two edges in the apartment Σ_0 which*

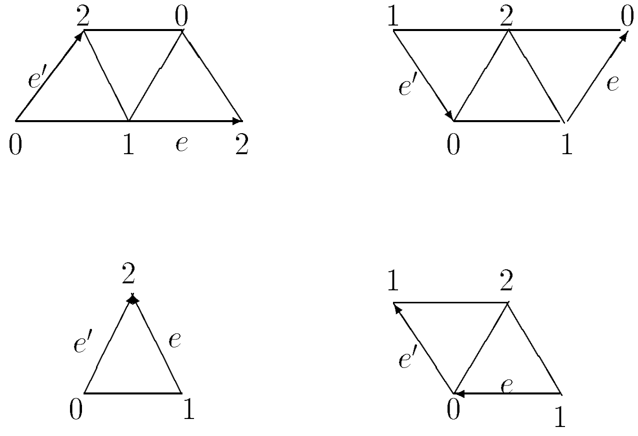
1) are both positive,

2) have an angle of $\pi/3$ between them,

3) have origins, at a distance at most one apart.

Proof. Suppose the contrary. If the edges e, e' begin at the same vertex and have the same type, say 0, then the types of the edges are different and of the form (0,1), (0,2), thus e and e' cannot be both positive.

Now let the origins e and e' lie at distance one apart, so they constitute an edge of type $(0, 1)$, say. Then the following configurations for e and e' are possible:



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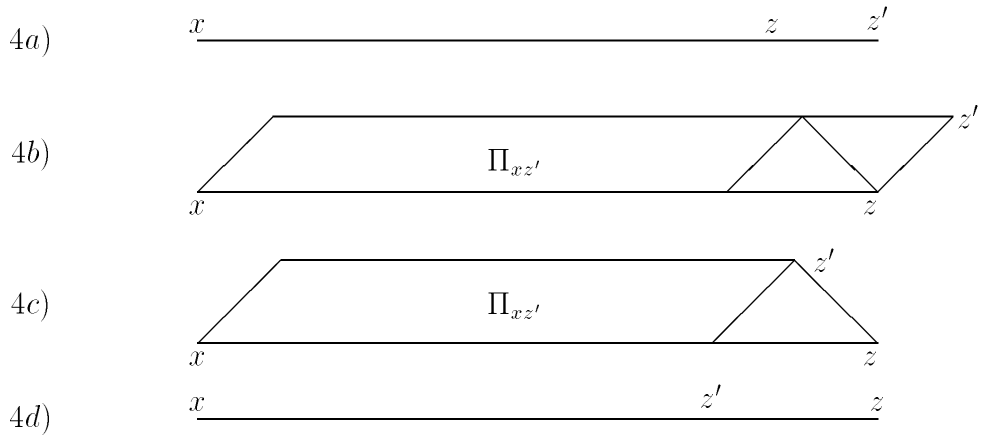
(actually there are more configurations, but they coincide up to isometries of Σ_0 with the pictured ones). In all these cases e and e' are not both positive. This proves the lemma.

Case 4. *γ and γ' have a common origin and γ is geodesic.*

This case is symmetrical to case 2).

Let $\gamma_0 = x, \gamma'_0 = x$ and $\gamma_\infty = z, \gamma'_\infty = z'$ and $|z - z'| = 1$.

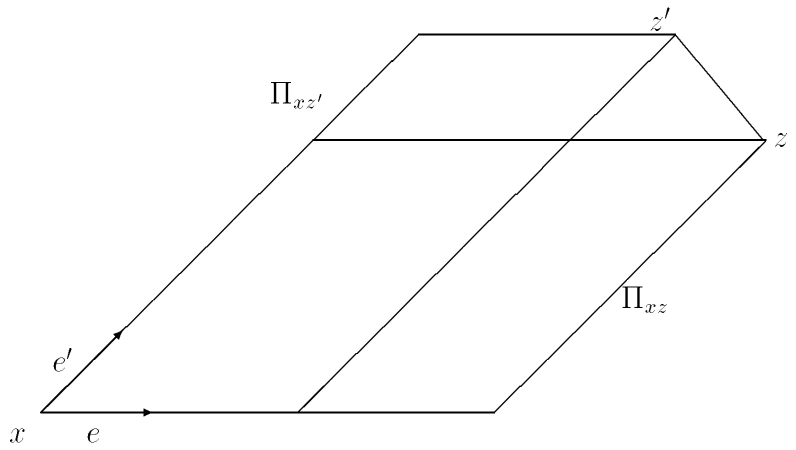
The following configurations are possible in the apartment, containing x and $[z, z']$:



As in Case 2) in all pictured cases γ and γ' 1-travel.

Case 5. *γ and γ' have a common end and both γ and γ' are not geodesic.*

This case is symmetrical to case 3). Let x, z, z' have the same meaning as in the case 4). Put x and $[z, z']$ into apartment Σ. As we know, Σ contains also the parallelograms $\Pi_{xz}, \Pi_{xz'}$



Now let e, e' be the initial segments of γ, γ' respectively, then by definition they are both positive. On the other hand if $e \neq e'$ then the angle between e and e' is equal to $\pi/3$ and consequently e and e' are the faces of a single chamber, so they have opposite signs, a contradiction. The only remaining case is when $e = e'$; then obviously γ and γ' 1-fellow travel.

General case. *Both the origins and the ends of γ and γ' lie at a distance at most one apart.*

Actually only the subcase when $|\gamma_0 - \gamma'_0| = 1$ and $|\gamma_\infty - \gamma'_\infty| = 1$ was not considered. But in this case we can use an "intermediate" path $\delta \in \mathcal{C}$ with origin $\delta(0) = \gamma_0$ and with end $\delta_\infty = \gamma'_\infty$. Then the origins of δ and γ coincide and $|\delta_\infty - \gamma_\infty| = 1$, so $|\delta(t) - \gamma(t)| \leq 1, t \geq 0$, by case 5). The ends of δ and γ also coincide and $|\delta(t) - \gamma'(t)| \leq 1, t \geq 0$, by case 3). It follows from the inequalities above that $|\gamma(t) - \gamma'(t)| \leq 2, t \geq 0$. The theorem is proven.

4 Invariance of the bicombing $\mathcal{C}$ under type-rotating automorphisms

Let φ be a type-rotating automorphism of Δ, that is φ fixes all the types of the vertices or fixes no types at all. Thus φ acts on the additive group $\mathbf{Z/3}$ of types by translation and, correspondingly, it acts on the group $\mathbf{Z/3} \times \mathbf{Z/3}$ of types of directed edges by diagonal translations. In particular φ preserves the sign of every directed edge.

Theorem. *Any type-rotating automorphism φ of Δ leaves the bicombing $\mathcal{C}$ invariant.*

Proof. Since φ is an isometry of Δ it preserves geodesicity and the angles. If $\gamma \in \mathcal{C}$ is geodesic, then $\varphi\gamma$ is geodesic and lies in Δ^1, thus $\varphi\gamma \in \mathcal{C}$, as proven in Theorem 2.3 . Let now $\gamma \in \mathcal{C}$ be not geodesic, then $\gamma = \gamma_+ \cup \gamma_-$ and γ_+ is nontrivial. Consequently $\varphi\gamma$ consists of two geodesics $\varphi\gamma_+, \varphi\gamma_-$ with angle $2\pi/3$ between them.Besides $\varphi\gamma_+$ is positive, since γ_+ is positive. Again $\varphi\gamma_+ \in \mathcal{C}$ in view of Theorem 2.3.

5 Biautomatic structure for groups acting on triangle buildings

Theorem. *If the group G acts simply transitively on the vertices of a locally finite triangle building Δ by type-rotating automorphisms then G admits a biautomatic structure.*

Remarks.1) By definition, a type-rotating automorphism fixes no types or all types. These automorphisms form a subgroup of index ≤ 2 in the full automorphism group. The "type-rotating" condition is actually not very restrictive since, for example the group $\mathbf{PGL_3}(K)$, where K is a locally compact local field with a discrete valuation v, acts by such automorphisms on the corresponding Bruhat-Tits building (more precisely the type of the lattice class corresponding to the lattice with basis v_1, v_2, v_3 is given by $v(\det(v_1, v_2, v_3)) mod 3$ and $g \in \mathbf{PGL_3}(K)$ acts on the group $\mathbf{Z/3}$ of types by translation by $v(\mathrm{det} g) mod 3$).

2) The theorem is proven in [CS] for the case of affine $\tilde{A}_n$-buildings, $n \geq 2$ with the following "thickness condition": every codimension one simplex is a face of at least three chambers. Unfortunately, examples of groups which act simply transitively (and in a type-rotating way) are known only in dimension $n = 2$ [CMSZ,I], $n = 3, 4$ [CMSZ,II] and $n = 5$ [Cartwright,unpublished]. It is proven in [GS1] that if the group G acts freely and cocompactly on a triangle building Δ then G admits a biautomatic structure. It is worth mentioning that the group G in the theorem above possesses a subgroup G_0

of finite index, which acts freely and cocompactly on Δ, so G_0 is biautomatic, but it is not known until now whether biautomacity is inherited by finite extensions or not.

Proof. We will proceed in several steps. Firstly we recall the definition of biautomatic structure, then we identify the Cayley graph of G with Δ, then we associate with the bicombing $\mathcal{C}$ constructed earlier the language of normal forms of elements of G, and finally we apply the theorem about the 2-fellow traveller property for $\mathcal{C}$.

5.1 Biautomatic structure

We adopt the definitions from [NS]. We start with a finitely generated group G and a map from a finite set S into G denoted by $s \mapsto g_s$. Let $\mathcal{S}$ be the free monoid with basis S. It consists of all finite strings $w = s_1 s_2 \dots s_n$ and multiplication is given by concatenation. The map $s \mapsto g_s$ extends to a unique homomorphism π from $\mathcal{S}$ to G. We will assume that this map is onto, that is the set $\{g_s : s \in S\}$ generates G as a semigroup. We will assume that S is equipped with an involution "inversion", denoted by $s \mapsto s^{-1}$, such that $g_{s^{-1}} = g_s^{-1}$ and $s \neq s^{-1}$ for every $s \in S$. Define the *Cayley graph* $\Gamma = \mathcal{C}(\mathcal{G}, \mathcal{S})$ with G as the set of vertices and $G \times S$ as the set of edges. The directed edge (h, s) joins h and hg_s and it is labelled by s. It is convenient to consider (hs, s^{-1}) as the same edge as (h, s) but with the opposite orientation and with label s^{-1}. By making each edge of Γ isometric with the unit interval, we endow Γ with a metric, that is, the distance between two points is defined to be the length of the shortest path connecting them. G acts on γ by left translations, and this action preserves distance. A word $w \in \mathcal{S}$ determines a path in Γ, which we also denote by w, as follows. The path w maps the interval $[0, |w|]$ ($|w|$ is the length of w) into Γ by running at unit speed through the edge path in Γ based at 1 and labelled by w. It is convenient to extend this to a map of $[0, \infty]$ making $w(t)$ stop after $t = |w|$, i.e. by setting $w(t) = w$ for $t \geq |w|$. In the same way we define for every w a path beginning in any $h \in G$.

We call a subset of $\mathcal{S}$ a *language*. A language L is a normal form if $\pi(L) = G$. We will say that a normal form L has the "*fellow traveller property*" if there is a constant k such that for $w, w' \in L$, which begin and end at a distance at most one apart, we have $|w(t) - w'(t)| \leq k$ for all t.

Recall that a *finite state automaton* $\mathcal{M}$ with alphabet A is a finite directed graph on a vertex set S (called the set of *states*) with each edge labelled by an element of A (maybe empty). Moreover, a subset of *start states* $S_0 \subset S$ and a subset of *accepted states* $S_1 \subset S$ are given. By definition, a word w in the alphabet A is in the *language L accepted by* $\mathcal{M}$ iff it defines a path starting from S_0 and ending in an accepted state in this graph. A language is *regular* if it is accepted by some finite state automation.

A *biautomatic structure* for a group G is a regular normal form with the fellow traveller property.

5.2 Simply transitive actions.

Now let G act on Δ as formulated in the theorem. Fix some origin $x_0 \in \Delta^0$ of type 0. Let S be the "sphere of radius 1 in Δ^0 with the center x_0", that is $S = \{x \in \Delta^0; |x - x_0| = 1\}$. For every $x \in S$ let g_x be the (unique) element of G

which takes x_0 to x. Then, by standard arguments we show that the set $\{g_x; x \in S\}$ generates S. Besides this set is closed under inversion. The "type-rotating" condition gives immediately that $g_x^{-1} \neq g_x$ for all $x \in S$ and thus the fixed-point-free involution on S can be defined by $x \mapsto y$ iff $g_x^{-1} = g_y$.

Now one can identify Δ^1 with the Cayley graph $\Gamma = \mathcal{C}(\mathcal{G}, \mathcal{S})$. Indeed one can identify the vertices of Γ with Δ^0 by means of the bijection $g \mapsto gx_0, g \in \Gamma$. In Γ there is a directed edge from g to $gg_s, s \in S$, labelled by s. But to this edge one can associate an edge $[gx_0, gg_sx_0]$ in Δ^1. Indeed,

$$|gx_0 - gg_sx_0| = |x_0 - g_sx_0| = |x_0 - s| = 1.$$

On the other hand every edge $[gx_0, hx_0]$ in Δ^1 is of this form: $1 = |gx_0 - hx_0| = |x_0 - g^{-1}h| = |x_0 - s| = 1 \Longrightarrow g^{-1}h = g_s$ for some $s \in S$.

5.3 Language

Let S be the sphere just defined and let $\mathcal{S}$ be the free monoid with base S. Let $\mathcal{C}$ be the bicombing constructed in 2.1. This bicombing gives rise to a language in $\mathcal{S}$ consisting of the labels of all paths in $\mathcal{C}$ starting at x_0. This language is a normal form since the elements g_s generate the group G.

Lemma. *L is a regular language.*

Proof. We define a finite state automaton $\mathcal{M}$ accepting L as follows. Let $S = S^+ \cup S^-$ be a partition of S into the subsets of vertices of type 1 and 2 respectively. The set of states of $\mathcal{M}$ is S together with an initial state s_0 not in S. The alphabet of $\mathcal{M}$ is S. The set of accepted states of $\mathcal{M}$ is $S \cup s_0$. The edges (directed and labelled) are the following ones:

1) $s_0 \xrightarrow{s} s$ $(s \in S)$

2) $s_1 \xrightarrow{s_2} s_2$, where both s_1 and s_2 lie in S^+ or in S^- and the angle between $[x_0, g_{s_1}x_0]$ and $[g_{s_1}x_0, g_{s_1}g_{s_2}x_0]$ in $g_{s_1}x_0$ is equal to π.

3) $s^+ \xrightarrow{s^-} s^-$, where $s^- \in S^-, s^+ \in S^+$ and the angle between $[x_0, g_{s^+}x_0]$ and $[g_{s^+}x_0, g_{s^+}g_{s^-}x_0$ in $g_{s^+}x_0]$ is equal to $2\pi/3$.

Now travelling through S^+–states we read the words of the form (written as path in Δ):

$x_0, g_{s_1}x_0, g_{s_1}g_{s_2}x_0, \ldots, g_{s_1}g_{s_2}\cdots g_{s_n}x_0,$

where all the edges are positive (here we make use of the condition that Γ acts by type-rotating automorphisms) and any pair of adjacent edges form a geodesic of length 2. But then by 1.4 the path is geodesic, so it belongs to $\mathcal{C}$. Analogously, travelling through S^- – states we read the word which is negative and geodesic, thus belongs to $\mathcal{C}$. Now an edge in $\mathcal{M}$ of type 3) permits us to switch from a positive geodesic to a negative one with the turning angle $2\pi/3$, thus again we obtain a path in $\mathcal{C}$.

The only thing left to prove is the fellow traveller property for L. But with the identification of Γ and Δ^1 the language L is isometric to $\mathcal{C}$ and $\mathcal{C}$ satisfies the 2-fellow traveller property by the theorem of section 3.

References

[AB] Juan M. Alonso and Martin R. Bridson:*Semihyperbolic groups,* Stockholms Universitat, 1992, No 13.

[Bri] Bridson M.:*Geodesics and curvature in metric simplicial complexes,* in " Group theory from a geometrical viewpoint ", (E. Ghys, A. Haefliger, A. Verjovsky, eds.), Proc. ICTP Trieste, World Scientific, 1991.

[Bro] K. S. Brown:*Buildings,* Springer Verlag, 1989.

[CMSZ] D. I. Cartwright, A. M. Mantero, T. Steger and A. Zappa: *Groups acting simply transitively on the vertices of a building of type* $\tilde{A}_2$ *I, II,* Geom.Ded. 47 (1993), 143–166, 167–223.

[C] D. I. Cartwright:*Groups acting simply transitively on the vertices of a building of type* A_n *,* to appear in Proceedings of the 1993 Como conference "Groups of Lie type and their geometries", W. M. Kantor Editor.

[CS] D. I. Cartwright and Michael Shapiro:*Hyperbolic buildings, affine buildings and automatic groups,* Preprint.

[ECHLPT] D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston: *Word processing and group theory,* Jones and Bartlet Publishers, 1992.

[GS] S. M. Gersten and H. B. Short:*Small cancellation theory and automatic groups,II,* Invent.Math., 105(1991), 641-662.

[NS] Walter D.Neumann and Michael Shapiro:*Automatic structures and boundaries for graphs of groups,* Preprint.

Institute of Information Technologies
and Applied Mathematics
Andrianova 28
OMSK, RUSSIA, 644077,