# FIRST ORDER PERTURBATIONS OF DIRICHLET OPERATORS: EXISTENCE AND UNIQUENESS

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Abstract. We study perturbations of type $B \cdot \nabla$ of Dirichlet operators $(L^0, D(L^0))$ associated with Dirichlet forms of type $\mathcal{E}^0(u,v) = 1/2 \int \langle \nabla u, \nabla v\rangle_H \, d\mu$ on $L^2(E,\mu)$ where $E$ is a finite or infinite dimensional Banach space E. Here $H$ denotes a Hilbert space densely and continuously embedded in $E$. Assuming quasi–regularity of $(\mathcal{E}^0, D(\mathcal{E}^0))$ we show that there always exists a closed extension of $Lu := L^0 u + \langle B, \nabla u\rangle_H$ that generates a sub–Markovian $C_0$–semigroup of contractions on $L^2(E,\mu)$ (resp. $L^1(E,\mu)$), if $B \in L^2(E; H, \mu)$ and $\int \langle B, \nabla u\rangle_H \, d\mu \le 0, \ u \ge 0$. If $D$ is an appropriate core for $(L^0, D(L^0))$ we show that there is only one closed extension of $(L, D)$ in $L^1(E,\mu)$ generating a strongly continuous semigroup. In particular we apply our results to operators of type $\Delta_H + B \cdot \nabla$, where $\Delta_H$ denotes the Gross–Laplacian on an abstract Wiener space $(E, H, \gamma)$ and $B = -\mathrm{id}_E + v$, where $v$ takes values in the Cameron–Martin space $H$.

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# 0. Introduction

In this paper we are concerned with the problem whether certain (non–symmetric) first order perturbations of Dirichlet operators on finite or in particular infinite dimensional state spaces have extensions which generate sub–Markovian $C_0$–semigroups and, if so, whether there is only one such extension.

In order to explain our results more precisely we first consider the following special finite dimensional example: Denote the closure of

$$1/2\int\langle\nabla u,\nabla v\rangle\varphi^2dx;\ u,v\in C_0^\infty(\mathbb{R}^d)$$

in $L^2(\mathbb{R}^d,\varphi^2dx)$ by $(\mathcal{E}^0,D(\mathcal{E}^0))$, where $\varphi\in L^2(\mathbb{R}^d,dx)$ with $|\nabla\varphi|\in L^2_{\text{loc}}(\mathbb{R}^d,dx)$. Denote the associated generator by $(L^0,D(L^0))$ and let $T_t^0=e^{tL^0}$, $t\geq 0$, be the corresponding semigroup. Let $B$ be a measurable vector field on $\mathbb{R}^d$ and assume that $|B|\in L^2(\mathbb{R}^d,\varphi^2dx)$. We can then define the linear operator

$$Lu:=L^0u+\langle B,\nabla u\rangle;\ u,v\in C_0^\infty(\mathbb{R}^d).$$

Question 1: Is there a linear operator $(L,D(L))$ on $L^2(\mathbb{R}^d,\varphi^2dx)$ (resp. $L^1(\mathbb{R}^d,\varphi^2dx)$) extending $(L,C_0^\infty(\mathbb{R}^d))$ such that $(L,D(L))$ generates a $C_0$–semigroup $(T_t)_{t\geq 0}$ which is sub–Markovian (i.e. $0\leq u\leq 1$ implies $0\leq T_tu\leq 1$ for all $t\geq 0$) ?
Question 2: If the answer to Question 1 is yes, is $(L,D(L))$ having these properties unique ?

Assuming $\int|x|^2\varphi^2(x)\,dx<\infty$ the answer to Question 1 is positive under the following condition

$$\int\langle B,\nabla u\rangle\,\varphi^2dx\leq 0\quad\forall u\in C_0^\infty(\mathbb{R}^d),\ u\geq 0\,. \tag{0.1}$$

The proof relies on the application of a method due to E. A. Carlen (cf. [Ca]) which he used in the construction of conservative diffusions and which is partly generalized to our situation. We approximate $B$ by bounded vector fields $B^n$, $n\geq 1$, such that $|B-B^n|\to 0$ in $L^2(\mathbb{R}^d,\varphi^2dx)$. For every $n$ we can define the operator $L^nu:=L^0u+\langle B^n,\nabla u\rangle$, $u\in D(L^0)$, that obviously generates a strongly continuous sub–Markovian semigroup $(T_t^n)_{n\geq 1}$ on $L^2(\mathbb{R}^d,\varphi^2dx)$. By (0.1) it is then easy to see that $(T_t^nu)_{t\geq 0}$ is an $L^2(\mathbb{R}^d,\varphi^2dx)$–Cauchy sequence for bounded $u$ and that the limit $T_tu$ can be extended to a sub–Markovian contraction operator on $L^2(\mathbb{R}^d,\varphi^2dx)$ as well as on $L^1(\mathbb{R}^d,\varphi^2dx)$. However, in order to prove that the family $(T_t)_{t\geq 0}$ is a strongly continuous semigroup whose generator indeed extends $(L,C_0^\infty(\mathbb{R}^d))$ we have to use probabilistic tools, i.e. diffusions $\mathbf{M}^0=(\Omega,\mathcal{M}^0,(X_t)_{t\geq 0},(P_x^0)_{x\in\mathbb{R}^d})$ and $\mathbf{M}^n=(\Omega,\mathcal{M}^n,(X_t)_{t\geq 0},(P_x^n)_{x\in\mathbb{R}^d})$ on $\mathbb{R}^d$ with transition probabilities given by $(T_t^0)_{t\geq 0}$ resp. $(T_t^n)_{t\geq 0}$. The family $(T_t)_{t\geq 0}$ can then be identified as the transition semigroup of some diffusion process $\mathbf{M}^B=(\Omega,\mathcal{M}^B,(X_t)_{t\geq 0},(Q_x)_{x\in\mathbb{R}^d})$ obtained as the Girsanov–transform of $\mathbf{M}^0$.

We would like to emphasize that the role of the Laplacian in the above discussion can be replaced by a more general second order strongly elliptic operator.

Assuming the answer to Question 1 is positive, applying a method we learnt from a paper by V. A. Liskevich and Yu. A. Semenov (cf. [LiS]), we show that $(L, D(L))$ is the unique operator on $L^1(\mathbb{R}^d, \varphi^2 dx)$ that extends $(L, C_0^\infty(\mathbb{R}^d))$ and generates a $C_0$–semigroup, provided $C_0^\infty(\mathbb{R}^d)$ is a domain of essential self–adjointness for the unperturbed operator $(L^0, D(L^0))$. This already implies that $C_0^\infty(\mathbb{R}^d)$ is an operator core for $(L, D(L))$. We want to thank Andreas Eberle for pointing out this connection to us.

The above programme is carried out (both for Question 1 and Question 2) in the general framework of gradient Dirichlet forms on (finite and infinite dimensional) Banach spaces (cf. [AR], [MR]). The existence result is formulated in Theorem 1.1 and the uniqueness result in Theorem 1.2.

As an application we discuss in particular the case of an abstract Wiener space. More precisely we apply our results to operators of type $L = 1/2(\Delta_H + B \cdot \nabla)$, where $\Delta_H$ is the Gross–Laplacian on some abstract Wiener space $(E, H, \gamma)$ and $B = -\mathrm{id}_E + v$, where $v \in L^2(E; H, \mu)$ (i.e., $L^0$ above is the generalized Schrödinger operator $1/2(\Delta_H - \mathrm{id}_E) + \nabla\varphi/\varphi)$ (cf. [RZ]). Assuming that $\mu$ is a finite positive measure such that

$$\int Lu\, d\mu = 0 \quad \forall u \in \mathcal{F}C_b^\infty ,$$

we can show, by using a result of Bogachev and Röckner (cf. [BR]), that there always exists a closed extension of $L$ that generates a strongly continuous contraction semigroup that is associated with some $\mu$–tight standard process on $E$.

Finally, we would like to mention that in the special finite dimensional case illustrated above under the stronger assumptions that $\varphi, |\nabla\varphi| \in L^2(\mathbb{R}^d, dx)$ and $\int \langle B, \nabla u\rangle \varphi^2 dx = 0 \, \forall u \in C_0^\infty(\mathbb{R}^d)$ it follows from a general result obtained by P. Cattiaux and C. Leonard with different techniques (cf. [CL]) that there exists a strong Markov probability measure $\overline{Q}$ on $C([0, T], \mathbb{R}^d)$ (= the space of all continuous functions $\omega : [0, T] \to \mathbb{R}^d$) that solves the martingale problem corresponding to $(L, C_0^\infty(\mathbb{R}^d))$, which should also imply the existence of the generator $(L, D(L))$ as above. However, it seems that their method does not imply the existence of the "full" diffusion process $\mathbf{M}^B = (\Omega, \mathcal{M}^B, (X_t)_{t\geq 0}, (Q_x)_{x\in\mathbb{R}^d})$ rather than only the probability measure $\overline{Q}$ mentioned above which coincides with $Q_{\varphi^2 dx} = \int Q_x \varphi^2(x) dx$.

## 1. Main results

### (a) Preliminaries.

Let $E$ be a separable real Banach space, $H$ a separable real Hilbert space such that $H \subset E$ densely and continuously. Identifying $H$ with its dual, we obtain $E' \subset H \subset E$ densely and continuously. Let

$$\mathcal{F}C_b^\infty := \{ f(\ell_1, \ldots, \ell_m) \mid m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), \ell_1, \ldots, \ell_m \in E' \} ,$$

where $C_b^\infty(\mathbb{R}^m)$ denotes the space of all smooth functions $f$ on $\mathbb{R}^m$ with $f$ and all partial derivatives bounded.

For $u \in \mathcal{F}C_b^\infty$ and $k \in E$ define $\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z+sk)|_{s=0}$, $z \in E$. Clearly, for $u = f(\ell_1, \dots, \ell_m)$ and $k \in H$ we have that

$$\begin{aligned}\frac{\partial u}{\partial k}(z) &= \sum_{i=1}^m \frac{\partial f}{\partial x_i}(\ell_1(z), \dots, \ell_m(z))\ell_i(k) \\ &= \sum_{i=1}^m \frac{\partial f}{\partial x_i}(\ell_1(z), \dots, \ell_m(z))\langle \ell_i, k\rangle_H \ .\end{aligned}$$

Consequently $k \mapsto \frac{\partial u}{\partial k}(z)$ is continuous on $H$ and we can define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k\rangle_H = \frac{\partial u}{\partial k}(z) \ .$$

Let $\mu$ be a finite positive measure on the Borel $\sigma$–algebra $\mathcal{B}(E)$ of $E$ with supp $\mu = E$. Define the bilinear form $(\mathcal{E}^0, \mathcal{F}C_b^\infty)$ by

$$\mathcal{E}^0(u,v) = 1/2 \int \langle \nabla u, \nabla v\rangle_H \, d\mu \ ; \ u, v \in \mathcal{F}C_b^\infty \ .$$

The existence of the integral is well–known (cf. [MR, p. 57, Remark II.3.7]). Suppose that $(\mathcal{E}^0, \mathcal{F}C_b^\infty)$ is closable in $L^2(E,\mu)$ (cf. [MR, Section I.3]). It is easy to see that the closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a (symmetric) Dirichlet form (cf. [MR, Section I.4]). Denote by $(L^0, D(L^0))$ the corresponding generator and by $(T_t^0)_{t\geq 0}$ the corresponding semigroup. We assume that $\mathcal{F}C_b^\infty \cap D(L^0) \subset L^2(E,\mu)$ dense. Closability of $(\mathcal{E}^0, \mathcal{F}C_b^\infty)$ has been investigated by Albeverio and Röckner in [AR]. For a sufficient condition we refer to [MR, p. 57, Proposition II.3.8]. We define $\mathcal{E}^0$–nests, $\mathcal{E}^0$–exceptional sets as well as the notions $\mathcal{E}^0$–quasi everywhere ($\mathcal{E}^0$–q.e.) and $\mathcal{E}^0$–quasi continuous ($\mathcal{E}^0$–q.c.) as in [MR, Chapter III]. These notions coincide with the corresponding ones in [FOT, Chapter 2] since $\mathcal{E}^0$ is symmetric and $\mathbb{1} \in D(\mathcal{E}^0)$.

Suppose that $B : E \to H$ is a measurable vector field in $L^2(E; H, \mu)$, i.e.

$$\int_E \|B(z)\|_H^2 \mu(dz) < +\infty \ .$$

Define the linear operator $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ on $L^2(E,\mu)$ by

$$Lu := L^0 u + \langle B, \nabla u\rangle_H \ .$$

This paper now deals with questions about existence and uniqueness of (closed) extensions of $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ that generate (sub–Markovian) strongly continuous contraction semigroups on $L^2(E,\mu)$ (resp. $L^1(E,\mu)$).

**(b) An existence result.**

In this subsection we assume that

(E.1) $$\int \ell^2(z)\mu(dz) < +\infty \quad \forall \ell \in K \, ,$$

where $K$ is a dense subspace of $E'$.

(E.2) $(\mathcal{E}^0, D(\mathcal{E}^0))$ is *quasi–regular* (cf. [MR, Section IV.3]).

(E.3) $$\int_E \langle B(z), \nabla u(z)\rangle_H \mu(dz) \le 0 \quad \forall u \in \mathcal{F}C_b^\infty, u \ge 0 \ .$$

By the general theory of Dirichlet forms there exists a $\mu$–tight special standard process $\mathbf{M}^0 = (\Omega^0, \mathcal{M}^0, (X_t^0)_{t\ge 0}, (P_x^0)_{x\in E_\Delta})$ properly associated with $(\mathcal{E}^0, D(\mathcal{E}^0))$, i.e. $E.^0[f(X_t^0)]$ is an $\mathcal{E}^0$–q.c. $\mu$–version of $T_t^0 f$ for any $f \in \mathcal{B}_b(E)$, $t > 0$, where $\mathcal{B}_b(E)$ denotes the set of all bounded $\mathcal{B}(E)$–measurable functions on $E$ (cf. [MR, Chapter IV] for the notion of a $\mu$–tight special standard process).

Since $\nabla u^2 = 2u \cdot \nabla u$, $u \in \mathcal{F}C_b^\infty$, we have that

(1.1) $$\int_E \langle B(z), \nabla u(z)\rangle_H u(z)\mu(dz) \le 0 \quad \forall u \in \mathcal{F}C_b^\infty \ .$$

Note that the operator $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ is dissipative, i.e.

$$(Lu, u)_{L^2(E,\mu)} = (L^0 u, u) + \int_E \langle B, \nabla u\rangle_H u \, d\mu \le 0 \ ; \ u \in \mathcal{F}C_b^\infty \cap D(L^0) \ ,$$

due to the fact that $(L^0, D(L^0))$ is dissipative and (1.1).

We know by a Theorem of Phillips (cf. [P] or [D, Section 6.1]) that there is some maximal dissipative extension of $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ in $L^2(E,\mu)$ that generates a strongly continuous contraction semigroup. Three questions now arise. The first question is, whether there is some maximal extension that generates a *sub–Markovian* strongly continuous contraction semigroup $(T_t)_{t\ge 0}$ on $L^2(E,\mu)$, the second one, whether there is a Markov process $\mathbf{M}^B = (\Omega, \mathcal{M}^B, (X_t)_{t\ge 0}, (Q_x)_{x\in E_\Delta})$ with nice sample path properties that is associated with $(T_t)_{t\ge 0}$ (i.e. $E.^Q[f(X_t)]$ is a $\mu$–version of $T_t f$ for all $f \in \mathcal{B}_b(E)$, $t \ge 0$, and the third one is about connections between $\mathbf{M}^B$ and Markov processes $\mathbf{M}^0$ that are properly associated with $(\mathcal{E}^0, D(\mathcal{E}^0))$.

Under the assumptions listed above we show the following

**Theorem 1.1.** *There exists a $\mu$–tight standard process $\mathbf{M}^B = (\Omega, \mathcal{M}^B, (X_t)_{t\ge 0}, (Q_x)_{x\in E_\Delta})$ having $\mu$ as excessive measure (i.e. $\int E_x^Q[f(X_t)]\,\mu(dx) \le \int f\,d\mu \,\forall f \in \mathcal{B}^+(E)$) that is a conservative diffusion in the sense that $Q_x[\zeta = +\infty] = 1$ $\mu$–a.e. and $Q_x[t \mapsto X_t$ is continuous on $[0,\zeta)] = 1$ for any $x \in E$, whose transition semigroup induces a strongly continuous sub–Markovian contraction semigroup on $L^2(E,\mu)$. The corresponding generator $(L, D(L))$ is an extension of $(L, \mathcal{F}C_b^\infty \cap D(L^0))$.*

*Moreover, there exists a $\mu$–tight standard process $\mathbf{M} = (\Omega, \mathcal{M}, (X_t)_{t\ge 0}, (P_x)_{x\in E_\Delta})$ that is properly associated with $(\mathcal{E}^0, D(\mathcal{E}^0))$ such that*

$$Q_x \ll P_x \text{ on } \sigma(X_s \mid s \le t) \cap \{\zeta > t\} \ \forall t \ge 0,\ x \in E.$$

*Especially* $$Q_\mu \ll P_\mu \text{ on } \sigma(X_s \mid s \le t) \ \forall t \ge 0.$$

**Proof.** See Section 2 below.

**Remark.** If $\mathcal{F}C_b^\infty \cap D(L^0)$ is a domain of essential self–adjointness for the unperturbed operator $(L^0, D(L^0))$ (i.e. $\mathcal{F}C_b^\infty \cap D(L^0) \subset D(L^0)$ is dense w.r.t. the graph norm), the process $\mathbf{M}^B$ is unique in the following sense: Let $\tilde{\mathbf{M}}^B$ with transition semigroup $(\tilde{q}_t)_{t\geq 0}$ be another right process having $\mu$ as excessive measure. If the corresponding $L^2$–generator is an extension of $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ we obtain by the uniqueness result in subsection (c) below that $\tilde{q}_t f$ is a $\mu$–version of $T_t f$ too for all $f \in \mathcal{B}_b(E)$.

**(c) A uniqueness result.**

In this subsection we assume that

U.1 There is a (closed) extension $(L', D(L'))$ of $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ in $L^1(E,\mu)$ that generates a strongly continuous semigroup $(T_t)_{t\geq 0}$.

We are able to show the following:

**Theorem 1.2.** *Let $D \subset \mathcal{F}C_b^\infty \cap D(L^0)$ be a domain of essential self–adjointness for $(L^0, D(L^0))$. Then:*

*(i) If $(B^n)_{n\geq 1}$ is a sequence of bounded measurable vector fields in $L^2(E;H,\mu)$ such that $|B^n - B|_H \to 0$ in $L^2(E,\mu)$ and if $(T_t^n)_{t\geq 0}$ denotes the strongly continuous semigroup corresponding to the generator $(L^n, D(L^0))$ (where $L^n u := L^0 u + \langle B^n, \nabla u\rangle_H$) then*

$$\lim_{n\to\infty} T_t^n u = T_t u \;\forall u \in D(L^0) \cap L^\infty(E,\mu) \text{ in } L^1(E,\mu) \;.$$

*(ii) $(T_t)_{t\geq 0}$ is sub–Markovian.*

*(iii) $(L', D(L'))$ is the only (closed) extension of $(L, D)$ in $L^1(E,\mu)$ which generates a strongly continuous semigroup. Moreover, $D \subset D(L')$ dense w.r.t. the graph norm, i.e. $(L', D(L'))$ is the closure of $(L, D)$ in $L^1(E,\mu)$.*

*(iv) $(T_t)_{t\geq 0}$ induces a strongly continuous semigroup on $L^2(E,\mu)$, again denoted with $(T_t)_{t\geq 0}$, whose generator $(L, D(L))$ is just the part of $(L', D(L'))$ on $L^2(E,\mu)$, i.e. $Lu = L'u$ on $D(L)$ and*

$$D(L) = \{u \in D(L') \cap L^2(E,\mu) \mid L'u \in L^2(E,\mu)\} \;,$$

*and therefore an extension of $(L, D)$.*

**Proof.** See Section 3 below.

**Remark.** Note that U.1 is fulfilled in the situation of Subsection (b). Due to Theorem 1.1 there is an extension $(L, D(L))$ of $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ that generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^2(E,\mu)$ that induces a strongly continuous contraction semigroup on $L^1(E,\mu)$ as well. Hence, if $D \subset \mathcal{F}C_b^\infty \cap D(L^0)$ is a domain of essential self–adjointness for the "unperturbed" operator $(L^0, D(L^0))$

we obtain that $(L, D(L))$ is the only closed extension of $(L, D)$ that generates a strongly continuous semigroup $(T_t)_{t\geq 0}$ on $L^2(E,\mu)$ such that

$$\int |T_t u|\, d\mu \leq \int |u|\, d\mu \ , \ \forall u \in L^2(E,\mu)\, .$$

**Remark.** Sufficient conditions for a subset $D \subset D(L^0)$ to be a domain of essential self–adjointness for $(L^0, D(L^0))$ have been given in many situations by several authors; for example S. Albeverio, Yu. G. Kondratiev, M. Röckner, I. Shigekawa and N. Wielens (cf. [AKR 1] and [AKR 2] for references and corresponding results).

**(d) Application to invariant measures.**

In this subsection we give some connections of our result to the work of Bogachev and Röckner (cf. [BR]) from which our work started (cf. [BR, Remark 3.0 and Remark 3.2]). Suppose we are given an abstract Wiener space $(E, H, \gamma)$, i.e. $E$ is a separable real Banach space, $H$ a separable real Hilbert space continuously and densely embedded into $E$ and $\gamma$ a Gaussian measure on $\mathcal{B}(E)$ with covariance $\langle \cdot, \cdot \rangle_H$. Suppose also that we are given another probability measure $\mu$ on $\mathcal{B}(E)$ with supp $\mu = E$ and $\int \ell^2(x)\mu(dx) < +\infty$ for any $\ell \in E'$.
Assume that $B^* : E \to E$ is a measurable vector field such that $B^*(x) = -x + v(x)$ and $v \in L^2(E; H, \mu)$.
Then the operator

$$Lu := 1/2\Delta_H u + 1/2\, {}_{E'}\langle \nabla u, B^* \rangle_E ;\, u \in \mathcal{F}C_b^\infty$$

is well–defined on $L^2(E,\mu)$. Here ${}_{E'}\langle \cdot, \cdot \rangle_E$ denotes the dualization between $E'$ and $E$ and $\Delta_H$ the Gross–Laplacian, i.e. $\Delta_H u(x) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\ell_1, \ldots, \ell_n)(x)\langle \ell_i, \ell_j \rangle_H$ for $u = f(\ell_1, \ldots, \ell_n) \in \mathcal{F}C_b^\infty$.
Assume furthermore, that $L$ is $\mu$–harmonic, i.e.

$$\int Lu\, d\mu = 0\ \forall u \in \mathcal{F}C_b^\infty\ . \tag{1.2}$$

Then we can show the following

**Proposition 1.3.** *There exists a closed extension $(L, D(L))$ of $(L, \mathcal{F}C_b^\infty)$ that generates a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ on $L^2(E,\mu)$. $\mu$ is invariant for $(T_t)_{t\geq 0}$. Moreover, $(T_t)_{t\geq 0}$ is associated with a $\mu$–tight standard process $\mathbf{M}^{B^*} = (\Omega, \mathcal{M}^{B^*}, (X_t)_{t\geq 0}, (Q_x)_{x\in E_\Delta})$ with life time $\zeta$ that is a conservative diffusion in the sense that $Q_x[\zeta = +\infty] = 1$ $\mu$–a.e. and $Q_x[t \mapsto X_t$ is continuous on $[0, \zeta)] = 1$ for any $x \in E$.*

**Proof.** By [BR, Theorem 3.5] $\mu$ is absolutely continuous w.r.t. $\gamma$ and its density $\rho$ admits a representation $\rho = \varphi^2$ with $\varphi \in H^{1,2}(\gamma)$ (where $H^{1,2}(\gamma)$ is just the domain of the Dirichlet form $\mathbb{D}$ which is given by the closure of $\int \langle \nabla u, \nabla v \rangle_H\, d\gamma$ on $\mathcal{F}C_b^\infty$

in $L^2(E,\gamma)$). Furthermore, the logarithmic derivative $\beta_H^\mu$ of $\mu$ associated with $H$ exists and admits a representation $\beta_H^\mu(x) = -x + 2\frac{\nabla\varphi}{\varphi}(x)$ (for the terminology we refer to [BR]). Hence we can define the form $(\mathcal{E}^0, \mathcal{F}C_b^\infty)$ on $L^2(E,\mu)$ by

$$\mathcal{E}^0(u,v) = 1/2 \int \langle \nabla u, \nabla v\rangle_H \, d\mu \; .$$

Since every $h \in E'$ is well–$\mu$–admissible (cf. [MR, Section II.3]) the form $(\mathcal{E}^0, \mathcal{F}C_b^\infty)$ is closable on $L^2(E,\mu)$ and its closure $(\mathcal{E}^0, D(\mathcal{E}^0))$ a symmetric Dirichlet form on $L^2(E,\mu)$. Denote by $(L^0, D(L^0))$ the associated generator. It is easy to see that $\mathcal{F}C_b^\infty \subset D(L^0)$ and

$$L^0 u = 1/2\Delta_H u + 1/2 {}_{E'}\langle \nabla u, \beta_H^\mu\rangle_E ; u \in \mathcal{F}C_b^\infty \; .$$

Set $B := 1/2(B^* - \beta_H^\mu) = 1/2v - \frac{\nabla\varphi}{\varphi}$. Clearly $B \in L^2(E;H,\mu)$ and

$$\int \langle B, \nabla u\rangle_H \, d\mu = \int Lu \, d\mu - \int L^0 u \, d\mu = 0; \; u \in \mathcal{F}C_b^\infty \tag{1.3}$$

by (1.2) and the fact that $\int L^0 u \, d\mu = -\mathcal{E}^0(u,1) = 0$. Note that $L^0 u + \langle B, \nabla u\rangle_H = Lu, u \in \mathcal{F}C_b^\infty$.

In order to apply Theorem 1.1 we therefore only have to check that $(\mathcal{E}^0, D(\mathcal{E}^0))$ is quasi–regular. This follows immediately from [MR, Section IV.4] because

$$\mathcal{E}^0(u,v) = \sum_{k \in K_0} \int \frac{\partial u}{\partial k}\frac{\partial v}{\partial k} \, d\mu; \; u,v \in \mathcal{F}C_b^\infty \tag{1.4}$$

where $K_0 \subset E'$ is such that $K_0$ is an orthonormal basis of $H$ and

$$\sum_{k \in K_0} {}_{E'}\langle \ell, k\rangle_E^2 = \sum_{k \in K_0} \langle \ell, k\rangle_H^2 = \|\ell\|_H^2 \le C\|\ell\|_{E'}^2 \tag{1.5}$$

for some constant $C > 0$, since the embedding $E' \subset H$ is continuous. The $\mu$–invariance of $(T_t)_{t\ge 0}$ follows from Lemma 2.12 (iii). □

**Remark.** Note that also $\int \langle -B, \nabla u\rangle_H \, d\mu = 0, \; \forall u \in \mathcal{F}C_b^\infty$. Due to Theorem 1.1 again there exists a $\mu$–tight standard process $\mathbf{M}^{-B^*}$ whose transition semigroup induces a sub–Markovian $C_0$–semigroup of contractions $(\widehat{T}_t)_{t\ge 0}$ on $L^2(E,\mu)$ whose generator $(\widehat{L}, D(\widehat{L}))$ is a closed extension of $(\widehat{L}, \mathcal{F}C_b^\infty)$ where

$$\widehat{L}u := L^0 u - \langle B, \nabla u\rangle_H = 1/2\Delta_H u + {}_{E'}\langle \nabla u, \beta_H^\mu - 1/2B^*\rangle_E ; \; u \in \mathcal{F}C_b^\infty \; .$$

Note that for $u, v \in \mathcal{F}C_b^\infty$

$$\begin{aligned}
(Lu, v)_{L^2(E,\mu)} &= (L^0 u, v)_{L^2(E,\mu)} + \int \langle B, \nabla u\rangle_H v \, d\mu \\
&= (u, L^0 v)_{L^2(E,\mu)} + \int \langle B, \nabla(uv)\rangle_H \, d\mu - \int \langle B, \nabla v\rangle_H u \, d\mu \\
&= (u, \widehat{L}v)_{L^2(E,\mu)} \; .
\end{aligned}$$

This does not imply that $(\widehat{L}, D(\widehat{L}))$ is the adjoint operator of $(L, D(L))$. Unfortunately we cannot show this, since we do not know, that $\mathcal{F}C_b^\infty$ is dense in $D(L)$ w.r.t. the operator norm.

## 2. Proof of Theorem 1.1

**(a) Approximation of B.**

Let $(e_n)_{n\geq 1}$ be a complete orthonormal system in $H$ such that $e_n \in K$ $\forall n$. If $\dim H < \infty$ let $e_n = 0$ for $n > \dim H$. Define $\phi_n : E \to \mathbb{R}$ by $\phi_n(z) = {}_{E'}\langle e_n, z\rangle_E$. Clearly $\phi_n \in L^2(E,\mu)$. It is well–known that $\phi_n \in D(\mathcal{E}^0)$ and $\nabla\phi_n = e_n$.

For a measurable vector field $W : E \to H$ denote by $W_n$ the measurable function $W_n(z) = \langle W(z), e_n\rangle_H$. Clearly $W_n \in L^2(E,\mu)$ for $|W|_H \in L^2(E,\mu)$ and $\|W_n\|_{L^2(E,\mu)} \leq \||W|_H\|_{L^2(E,\mu)}$. We say that $W$ is bounded if $|W|_H \in L^\infty(E,\mu)$.

For $n$ take bounded measurable functions $b_{n,i} : E \to \mathbb{R}$, $1 \leq i \leq n$, such that

$$\|b_{n,i} - B_i\|_{L^2(E,\mu)} \leq \frac{1}{n} \ .$$

It follows that

$$B^n(z) := \sum_{i=1}^n b_{n,i}(z)e_i$$

is a bounded measurable vector field such that

$$\begin{aligned}\int_E |B(z) - B^n(z)|_H^2\mu(dz) &= \sum_{i=1}^n \int_E |B_i(z) - b_{n,i}(z)|^2\mu(dz) \\ &\quad + \sum_{i=n+1}^\infty \int_E B_i(z)^2\mu(dz) \\ &\leq \frac{1}{n} + \sum_{i=n+1}^\infty \int_E B_i(z)^2\mu(dz) \to 0 \text{ for } n\to\infty \ .\end{aligned}$$

Since $|B^n(z)|_H$ is bounded and $D(L^0) \subset D(\mathcal{E}^0)$ we can define the operator $(L^n, D(L^0))$ by $L^n u = L^0 u + \langle B^n, \nabla u\rangle_H$.

**(b) Existence of a strongly continuous semigroup $(T_t^n)_{t\geq 0}$ with corresponding generator $(L^n, D(L^0))$.**

**Proposition 2.1.** *$(L^n, D(L^0))$ generates a strongly continuous semigroup $(T_t^n)_{t\geq 0}$ on $L^2(E,\mu)$ such that*

$$T_t^n(L^2(E,\mu)) \subset D(L^0) \ \forall t > 0, n \geq 1 \ .$$

For the proof we need the following

**Lemma 2.2.** *Suppose $W : E \to H$ is a bounded vector field. Then for arbitrary $\varepsilon > 0$ there exists a constant $C_\varepsilon$ such that*

$$\left|\int \langle W, \nabla u\rangle_H u \, d\mu\right| \leq \varepsilon \ \mathcal{E}^0(u,u) + C_\varepsilon \|u\|^2_{L^2(E,\mu)}, \ \forall u \in D(\mathcal{E}^0) \ .$$

**Proof.** Suppose there is some $\epsilon > 0$ such that for suitable $(\tilde{u}_n)_{n\geq 1} \subset D(\mathcal{E}^0)$ we have that

$$\left|\int \langle W, \nabla \tilde{u}_n\rangle_H \tilde{u}_n \, d\mu\right| > \varepsilon\ \mathcal{E}^0(\tilde{u}_n, \tilde{u}_n) + n\|\tilde{u}_n\|^2_{L^2(E,\mu)}\ \forall n\ .$$

Define $u_n := \frac{\tilde{u}_n}{\|\tilde{u}_n\|_{L^2(E,\mu)}}$. It follows

$$\varepsilon\ \mathcal{E}^0(u_n, u_n) < 2^{1/2}\||W|_H\|_\infty\ \mathcal{E}^0(u_n, u_n)^{1/2}\ .$$

Consequently $\sup\limits_{n\geq 1} \mathcal{E}^0(u_n, u_n) < +\infty$. But on the other hand for all $m$

$$m < 2^{1/2}\||W|_H\|_\infty\ \mathcal{E}^0(u_m, u_m)^{1/2}$$

which is a contradiction. $\square$

**Proof of Proposition 2.1.** Fix $n \in \mathbb{N}$. Denote by $(\mathcal{E}^n, D(\mathcal{E}^0))$ the form

$$\mathcal{E}^n(u, v) := \mathcal{E}^0(u, v) - \int \langle B^n, \nabla u\rangle_H v \, d\mu\ ;\ u, v \in D(\mathcal{E}^0)\ .$$

Due to Lemma 2.2 there is a constant $\alpha$ such that

$$\left|\int \langle B^n, \nabla u\rangle_H u \, d\mu\right| \leq 1/2\ \mathcal{E}^0(u, u) + \alpha\|u\|^2_{L^2(E,\mu)},\ \ \forall u \in D(\mathcal{E}^0)\ .$$

Consequently

$$\begin{aligned}\mathcal{E}^n(u, u) &\geq \mathcal{E}^0(u, u) - \left|\int \langle B^n, \nabla u\rangle u \, d\mu\right| \\ &\geq 1/2\ \mathcal{E}^0(u, u) - \alpha\|u\|^2_{L^2(E,\mu)}\,.\end{aligned}$$

Thus

$$\begin{aligned}\left|\mathcal{E}^n_{\alpha+1}(u, v)\right| &\leq \left|\mathcal{E}^0_{\alpha+1}(u, v)\right| + \left|\int \langle B^n, \nabla u\rangle v \, d\mu\right| \\ &\leq \mathcal{E}^0_{\alpha+1}(u, u)^{1/2}\ \mathcal{E}^0_{\alpha+1}(v, v)^{1/2} + 2^{1/2}\||B^n|_H\|_\infty\ \mathcal{E}^0(u, u)^{1/2}\|v\|_{L^2(E,\mu)} \\ &\leq \left(1 + 2^{1/2}\||B^n|_H\|_\infty\right)\ \mathcal{E}^0_{\alpha+1}(u, u)^{1/2}\ \mathcal{E}^0_{\alpha+1}(v, v)^{1/2} \\ &\leq 2\left(1 + 2^{1/2}\||B^n|_H\|_\infty\right)\ \mathcal{E}^n_{2\alpha+1}(u, u)^{1/2}\ \mathcal{E}^n_{2\alpha+1}(v, v)^{1/2} \\ &\leq 2\left(1 + 2^{1/2}\||B^n|_H\|_\infty\right)(1 + \alpha)\ \mathcal{E}^n_{\alpha+1}(u, u)^{1/2}\ \mathcal{E}^n_{\alpha+1}(v, v)^{1/2}\ .\end{aligned}$$

It follows that $(\mathcal{E}^n_\alpha, D(\mathcal{E}^0))$ is a coercive closed form on $L^2(E, \mu)$. Denote the corresponding generator by $(L^n_\alpha, D(L^n_\alpha))$. Note that $v \mapsto \int \langle B^n, \nabla u\rangle_H v \, d\mu$ is continuous on $L^2(E, \mu)$ for all $u \in D(\mathcal{E}^0)$. Consequently $D(L^n_\alpha) = D(L^0)$ and $L^n_\alpha u = L^n u - \alpha \cdot u$ (cf. [MR, p. 23, Proposition 2.16]). By [MR, p. 25, Theorem I.2.20 and Corollary I.2.21] $(L^n_\alpha, D(L^0))$ is the generator of a strongly continuous contraction semigroup $(T^{n,\alpha}_t)_{t\geq 0}$ on $L^2(E, \mu)$ such that $T^{n,\alpha}_t(L^2(E, \mu)) \subset D(L^0)$ for all $t > 0$. Define $T^n_t u := e^{\alpha t}T^{n,\alpha}_t u$. Then $(T^n_t)_{t\geq 0}$ is a strongly continuous semigroup with generator $(L^n, D(L^0))$ and $T^n_t(L^2(E, \mu)) \subset D(L^0)$ $\forall t > 0$, which proves the assertion. $\square$

**(c) Identification of $(T_t^n)_{t\geq 0}$ as Girsanov–Transform of $(T_t^0)_{t\geq 0}$.**
For the terminology in this section we refer to [MR, Chapter VI]. Since $(\mathcal{E}^0, D(\mathcal{E}^0))$ is quasi–regular there exists an $\mathcal{E}^0$–nest $(E_k)_{k\geq 1}$ of compact metrizable sets in $E$ and a locally compact separable metric space $\overline{E}$ such that $Y := \bigcup_{k\geq 1} E_k$ is a dense subset of $\overline{E}$ and the image $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ of $(\mathcal{E}^0, D(\mathcal{E}^0))$ under the inclusion map $i : Y \subset \overline{E}$ is a regular Dirichlet form on $L^2(\overline{E}, \overline{\mu})$ with associated generator $(\overline{L}, D(\overline{L}))$, where $\overline{\mu} := \mu \circ i^{-1}$ is a positive Radon measure on $\overline{E}$ (cf. [MR, p. 174, Theorem VI.1.2]).

**Notation.** For a $\mathcal{B}(E)$–measurable function $f : E \to \mathbb{R}$ define $\overline{f} : \overline{E} \to \mathbb{R}$ by $\overline{f}(z) = f(z)$ for $z \in Y$ and $\overline{f}(z) = 0$ otherwise. For a $\mathcal{B}(E)$–measurable vector field $W : E \to H$ define $\overline{W} : \overline{E} \to H$ by $\overline{W}(z) = W(z)$ for $z \in Y$ and $\overline{W}(z) = 0$ otherwise.

**Remark 2.3.** (i) Since $Y \in \mathcal{B}(\overline{E})$ (cf. [MR, p. 174, Theorem VI. 1.2]) $\overline{f}$ and $\overline{W}$ are measurable

(ii) $\overline{|W|_H} = |\overline{W}|_H$.

(iii) $\int \overline{f}\, d\overline{\mu} = \int f\, d\mu$.

(iv) $f$ is $\mathcal{E}^0$–quasi continuous if and only if $\overline{f}$ is $\overline{\mathcal{E}}$–quasi continuous.

(v) $f \in \mathcal{B}(E) \cap D(L^0)$ implies $\overline{f} \in \mathcal{B}(\overline{E}) \cap D(\overline{L})$ and $\overline{L^0 f}$ is a $\overline{\mu}$–version of $\overline{L}\,\overline{f}$.

(vi) $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ has the local property.

By the general theory of Dirichlet forms there exists a Hunt process $\overline{\mathbf{M}} = (\overline{\Omega}, \overline{\mathcal{F}}, (\overline{X}_t)_{t\geq 0}, (\overline{P}_x)_{x\in \overline{E}_\Delta})$ properly associated with $(\overline{\mathcal{E}}, D(\overline{\mathcal{E}}))$ (cf. [FOT] or [MR]) which is a conservative diffusion (i.e. $\overline{P}_x[\zeta = \infty] = 1$ and $\overline{P}_x[t \mapsto \overline{X}_t$ is continuous on $[0,\zeta)] = 1$ for all $x \in \overline{E}$). Here $\overline{E}_\Delta$ is to be taken as the one–point compactification of $\overline{E}$ if $\overline{E}$ is non–compact. If $\overline{E}$ is already compact, $\Delta$ is adjoined as an isolated point. As usual we set $f(\Delta) = 0$ for any $f \in \mathcal{B}(\overline{E})$. For technical reasons we choose

$$\overline{\Omega} = \big\{\omega \mid \omega : [0,\infty) \to \overline{E}_\Delta \text{ is right continuous and has left–limits in } \overline{E} \\ \text{on } (0,\zeta(\omega)), \omega(t) = \Delta \text{ for } t \geq \zeta(\omega)\big\}\ ,$$

where $\zeta(\omega) = \inf\{t \geq 0 \mid \omega(t) = \Delta\}$ denotes the lifetime of $\omega$, and $\overline{X}_t(\omega) = \omega(t)$. Let $\overline{\mathcal{F}}_t^0 = \sigma(\overline{X}_s \mid s \leq t)$ and denote by $(\overline{\mathcal{F}}_t)_{t\geq 0}$ the natural filtration corresponding to $\overline{\mathbf{M}}$. Let $(\overline{p}_t)_{t\geq 0}$ be the transition semigroup and $(\overline{R}_\alpha)_{\alpha>0}$ the resolvent corresponding to $\overline{\mathbf{M}}$.

For the terminology in the next paragraphs we refer to [FOT, Chapter 5]. Recall that an additive functional (= AF) $(A_t)_{t\geq 0}$ of $\overline{\mathbf{M}}$ is called a *continuous additive functional* (= CAF) if $t \mapsto A_t(\omega)$ is continuous on $[0,\infty)$ and a *positive continuous additive functional* (= PCAF) if in addition $A_t(\omega) \geq 0$ for all $t \geq 0$ and $\omega$ in a defining set. Two additive functionals $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are called *equivalent* if there exists a common defining set $\Lambda$ and a common exceptional set $N$ with $A_t(\omega) = B_t(\omega)\ \forall t \geq 0, \omega \in \Lambda$.

As usual we define the *energy* $e(A)$ of an $AF$ $A = (A_t)_{t\geq 0}$ by

$$e(A) = \lim_{t\downarrow 0}(1/2t)\overline{E}_{\overline{\mu}}[A_t^2]\,,$$

whenever this limit exists. Note that $e(A) = \sup_{t>0}(1/2t)\overline{E}_{\overline{\mu}}[A_t^2]$ if $(A_t)_{t\geq 0}$ is a *martingale additive functional* ($= MAF$) (i.e. an $AF$ such that $\overline{E}_x[A_t^2] < \infty, \overline{E}_x[A_t] = 0\ \forall t \geq 0\ \overline{\mathcal{E}}$–q.e.), since $t \mapsto \overline{E}_{\overline{\mu}}[A_t^2]$ is subadditive. Denote with $\overset{\circ}{\mathcal{M}}$ the set of all $MAF$ of finite energy.

For an $\mathcal{E}^0$–quasi continuous function $u$ in $D(\mathcal{E}^0)$ denote by $(A_t^{[\overline{u}]})_{t\geq 0}$ the $CAF$ $(\overline{u}(\overline{X}_t) - \overline{u}(\overline{X}_0))_{t\geq 0}$. Note that $\overline{u}$ is $\overline{\mathcal{E}}$–quasi continuous due to Remark 2.3 (iv). By [FOT, p. 203, Theorem 5.2.2] $(A_t^{[\overline{u}]})_{t\geq 0}$ can be decomposed uniquely into the sum of a $MAF$ $(M_t^{[\overline{u}]})_{t\geq 0}$ of finite energy and a $CAF$ $(N_t^{[\overline{u}]})_{t\geq 0}$ of zero energy. Since $(A_t^{[\overline{u}]})_{t\geq 0}$ and $(N_t^{[\overline{u}]})_{t\geq 0}$ are continuous, so is $(M_t^{[\overline{u}]})_{t\geq 0}$.

For $M \in \overset{\circ}{\mathcal{M}}$ there exists (up to equivalence) a unique $PCAF\langle M\rangle = (\langle M\rangle_t)_{t\geq 0}$ such that $\overline{E}_x[M_t^2] = \overline{E}_x[\langle M\rangle_t]\ \forall t \geq 0,\ \overline{\mathcal{E}}$–q.e. Denote with $\mu_{\langle M\rangle}$ the uniquely determined smooth measure corresponding to $\langle M\rangle$ via the Revuz correspondence (cf. [FOT, p. 188, Theorem 5.1.4]). Let $\mu_{\langle\overline{u}\rangle} := \mu_{\langle M^{[\overline{u}]}\rangle}$. Note that every finite positive measure $\nu$ which is absolutely continuous w.r.t. $\overline{\mu}$ is *smooth*.

Denote by $S_0$ the set of all positive Radon measures of finite energy integral, i.e. for any $\nu \in S_0$ there is a constant $C > 0$ such that

$$\int |v|\, d\nu \leq C \cdot\ \overline{\mathcal{E}}_1(v,v)^{1/2};\ \forall v \in D(\overline{\mathcal{E}}) \cap C_0(\overline{E})\ .$$

Denote by $U_1\nu$ the unique function in $D(\overline{\mathcal{E}})$ with

$$\overline{\mathcal{E}}_1(U_1\nu, v) = \int v\, d\nu;\ \forall v \in D(\overline{\mathcal{E}}) \cap C_0(\overline{E})\ .$$

(cf. [FOT, Chapter 2]). Let $S_{00}$ be the subset of all finite measures $\nu \in S_0$ such that $\|U_1\nu\|_\infty < \infty$.

**Remark 2.4.** For any $f \in \mathcal{B}(\overline{E}) \cap L^1(\overline{E}, \overline{\mu})$ we have that

$$\overline{P}_x[\int_0^t |f|(\overline{X}_s)\, ds < +\infty,\, t \geq 0] = 1 \quad \overline{\mathcal{E}}\text{–q.e.}$$

**Proof.** Note that for $\nu \in S_{00}$

$$\begin{aligned}(2.1)\qquad \int \overline{R}_1|f|\, d\nu &= \sup_{n\geq 1}\int \overline{R}_1(|f| \wedge n)\, d\nu = \sup_{n\geq 1}\overline{\mathcal{E}}_1(U_1\nu, \overline{R}_1(|f| \wedge n)) \\ &= \sup_{n\geq 1}\int U_1\nu \cdot |f| \wedge n\, d\overline{\mu} \leq \|U_1\nu\|_\infty \|f\|_{L^1(\overline{E},\overline{\mu})}\ .\end{aligned}$$

Hence $\overline{R}_1|f| < +\infty$ $\overline{\mathcal{E}}$–q.e. by [FOT, p. 79, Theorem 2.2.3] and therefore

$$\overline{E}_x[\int_0^t |f|(\overline{X}_s)\,ds] \le e^t \overline{R}_1|f|(x) < \infty \ \overline{\mathcal{E}}\text{–q.e.} \tag{2.2}$$

which implies the assertion. □

Due to the last remark, the process $(N_t^f)_{t\ge 0}$ given by $N_t^f := \int_0^t f(\overline{X}_s)\,ds$, is a CAF for any $f \in \mathcal{B}(\overline{E}) \cap L^1(\overline{E},\overline{\mu})$.

**Lemma 2.5.** *Let $f_n, f \in \mathcal{B}(\overline{E}) \cap L^1(\overline{E},\overline{\mu})$ such that $f_n \to f$ in $L^1(\overline{E},\overline{\mu})$. Then $N_t^{f_{n_k}} \to N_t^f$ in $L^1(\overline{P}_x)$, $t \ge 0$, $\overline{\mathcal{E}}$–q.e. along some subsequence $(n_k)_{k\ge 1}$.*

**Proof.** Take a subsequence $(n_k)_{k\ge 1}$ with $\|f - f_{n_k}\|_{L^1(\overline{E},\overline{\mu})} \le 1/2^k$. By (2.2)

$$\overline{E}_x\left[ |\int_0^t f(\overline{X}_s) - f_{n_k}(\overline{X}_s)\,ds| \right] \le e^t \overline{R}_1|f - f_{n_k}|(x)\,.$$

Since by (2.1)

$$\int \overline{R}_1|f - f_{n_k}|\,d\nu \le \|U_1\nu\|_\infty/2^k$$

for any $\nu \in S_{00}$ it follows from [FOT, p. 79, Theorem 2.2.3] that the set $\{x \mid \lim_{k\to\infty} \overline{R}_1|f - f_{n_k}|(x) > 0\}$ is $\overline{\mathcal{E}}$–exceptional which implies the assertion. □

The next proposition can be proved as Proposition 4.5 in [AR 2]. We include the proof for the readers convenience.

**Proposition 2.6.** *Let $u \in D(\mathcal{E}^0)$, then $\langle M^{[\overline{u}]}\rangle_t = \int_0^t \overline{|\nabla u|}_H^2(\overline{X}_s)\,ds, \quad t \ge 0.$*

**Proof.** Let $u \in \mathcal{F}C_b^\infty$. By [FOT, p. 206, Theorem 5.2.3]

$$\begin{aligned}
\int \overline{f}\,d\mu_{\langle\overline{u}\rangle} &= 2\overline{\mathcal{E}}(\overline{u}\overline{f},\overline{u}) - \overline{\mathcal{E}}(\overline{u}^2,\overline{f}) \\
&= 2\mathcal{E}^0(uf,u) - \mathcal{E}^0(u^2,f) \\
&= \int f|\nabla u|_H^2\,d\mu = \int \overline{f}\,\overline{|\nabla u|}_H^2\,d\overline{\mu}
\end{aligned}$$

for $f \in D(\mathcal{E}^0)$ bounded. Hence $\mu_{\langle\overline{u}\rangle} = \overline{|\nabla u|}_H^2\,d\overline{\mu}$. On the other hand it is easy to see that $(N_t^{\overline{|\nabla u|}_H^2})_{t\ge 0}$ is a PCAF with associated smooth measure $\overline{|\nabla u|}_H^2\,d\overline{\mu}$. Thus $N^{\overline{|\nabla u|}_H^2}$ and $\langle M^{[\overline{u}]}\rangle$ must be equivalent.

For general $u \in D(\mathcal{E}^0)$ take a sequence $(u_n)_{n\ge 1} \subset \mathcal{F}C_b^\infty$ with $\mathcal{E}^0(u-u_n,u-u_n) \le 1/2^k$. Since for $\nu \in S_{00}$

$$\begin{aligned}
\overline{E}_\nu[|\langle M^{[\overline{u}]}\rangle_t - \langle M^{[\overline{u_n}]}\rangle_t|] &\le (\overline{E}_\nu[\langle M^{[\overline{u}]}\rangle_t]^{1/2} + \overline{E}_\nu[\langle M^{[\overline{u_n}]}\rangle_t]^{1/2}) \\
&\qquad \overline{E}_\nu[\langle M^{[\overline{u-u_n}]}\rangle_t]^{1/2} \\
&\le 2(1+t)\|U_1\nu\|_\infty(\mathcal{E}^0(u,u)^{1/2} + \mathcal{E}^0(u_n,u_n)^{1/2}) \\
&\qquad \mathcal{E}^0(u-u_n,u-u_n)^{1/2}
\end{aligned}$$

converges to zero as $n \to \infty$ (cf. [FOT, p. 190, Lemma 5.1.9]), it follows as in the proof of Lemma 2.5 that $\langle M^{[\overline{u_n}]}\rangle_t \to \langle M^{[\overline{u}]}\rangle_t$, $t \geq 0$, in $L^1(\overline{P}_x)\,\overline{\mathcal{E}}$–q.e. Since also $N_t^{\overline{|\nabla u_{n_k}|_H^2}} \to N_t^{\overline{|\nabla u|_H^2}}$, $t \geq 0$, in $L^1(\overline{P}_x)\,\overline{\mathcal{E}}$–q.e. along some subsequence $(n_k)_{k\geq 1}$, due to Lemma 2.5, this implies the assertion. $\square$

For $u, v \in D(\mathcal{E}^0)$ it follows that

$$\langle M^{[\overline{u}]}, M^{[\overline{v}]}\rangle_t = \int_0^t \overline{\langle \nabla u, \nabla v\rangle_H}(\overline{X}_s)\,ds$$

and consequently $\langle M^i, M^j\rangle_t = \delta_{ij}\cdot t$, where $M^i := M^{[\overline{\phi}_i]}$.

By [FOT, p. 241, Theorem 5.6.1] and the last proposition there exists the stochastic integral $\overline{b}_{n,i}\cdot M^i \in \overset{\circ}{\mathcal{M}}$ such that

$$\langle \overline{b}_{n,i}\cdot M^i, L\rangle_t = \int_0^t \overline{b}_{n,i}(\overline{X}_s)d\langle M^i, L\rangle_s \ ; \ L \in \overset{\circ}{\mathcal{M}} \ . \tag{2.3}$$

Define $Y^n := \sum_{i=1}^n \overline{b}_{n,i}\cdot M^i \in \overset{\circ}{\mathcal{M}}$. Since $M^i$ is continuous, $Y^n$ is continuous again.

Note that for $u \in D(\mathcal{E}^0)$ we obtain from (2.3)

$$\begin{aligned}\langle Y^n, M^{[\overline{u}]}\rangle_t &= \sum_{i=1}^n \int_0^t \overline{b}_{n,i}(\overline{X}_s)\overline{\langle e_i, \nabla u\rangle}_H(\overline{X}_s)\,ds \\ &= \int_0^t \overline{\langle B^n, \nabla u\rangle}_H(\overline{X}_s)\,ds\,.\end{aligned} \tag{2.4}$$

Especially by (2.4)

$$\langle Y^n\rangle_t = \sum_{i=1}^n \int_0^t \overline{b}_{n,i}(\overline{X}_s)\overline{\langle B^n, e_i\rangle}_H(\overline{X}_s)\,ds = \int_0^t \overline{|B^n|_H^2}(\overline{X}_s)\,ds\,. \tag{2.5}$$

Define $Z_t^n := \exp(Y_t^n - (1/2)\langle Y^n\rangle_t)$. It is well–known that $(Z_t)_{t\geq 0}$ is a continuous non–negative local martingale, hence a supermartingale $\overline{\mathcal{E}}$–q.e. Since $\langle Y^n\rangle_t \leq t\|\,|B^n|_H^2\|_\infty$ it is in fact a square–integrable martingale $\overline{\mathcal{E}}$–q.e. with

$$\overline{E}_x[(Z_t^n)^2] \leq \exp(t\|\,|B^n|_H^2\|_\infty). \tag{2.6}$$

For $f \in \mathcal{B}_b(E)$ define

$$P_t^n f(x) := \overline{E}_x[\overline{f}(\overline{X}_t)Z_t^n];\ x \in Y\,.$$

**Lemma 2.7.** *The family of operators $P_t^n : f \mapsto P_t^n f$ induces a strongly continuous semigroup on $L^2(E,\mu)$ - again denoted with $(P_t^n)_{t\geq 0}$.*

**Proof.** By (2.6) and the $\overline{\mu}$–symmetry of $\overline{\mathbf{M}}$ we have for $f \in \mathcal{B}_b(\overline{E})$

$$\begin{aligned}
\int \overline{E}_x[f(\overline{X}_t)Z_t^n]^2\,\overline{\mu}(dx) &\leq \int \overline{E}_x[f^2(\overline{X}_t)]\overline{E}_x[(Z_t^n)^2]\,\mu(dx)\\
&\leq \exp(t\|\,|B^n|_H^2\|_\infty)\int \overline{p}_t(f^2)\,d\overline{\mu}\\
&= \exp(t\|\,|B^n|_H^2\|_\infty)\|f\|^2_{L^2(\overline{E},\overline{\mu})}\,.
\end{aligned}\tag{2.7}$$

(2.7) implies $\overline{E}_x[f(\overline{X}_t)Z_t^n] = \overline{E}_x[f'(\overline{X}_t)Z_t^n]$ $\mu$-a.e. for any two functions $f, f'$ with $f_{|Y} = f'_{|Y}$ $\mu$-a.e. and also

$$\int (P_t^n f)^2\,d\mu \leq \exp(t\|\,|B^n|_H^2\|_\infty)\|f\|^2_{L^2(E,\mu)};\ f\in\mathcal{B}_b(E)\,.$$

Thus $f \mapsto P_t^n f$ induces a bounded linear operator on $L^2(E,\mu)$ again denoted with $P_t^n$.

Since $(Z_t^n)_{t\geq 0}$ is a multiplicative functional and $\overline{P_s^n f}(x) = \overline{E}_x[\overline{f}(\overline{X}_s)Z_s^n]$ for $x \in Y$ it follows for bounded $f$ and $\mu$-a.e. $x \in Y$

$$\begin{aligned}
P_{t+s}^n f(x) &= \overline{E}_x[\overline{f}(\overline{X}_{t+s})Z_{t+s}^n] = \overline{E}_x[\overline{E}_{\overline{X}_t}[\overline{f}(\overline{X}_s)Z_s^n]Z_t^n]\\
&= \overline{E}_x[\overline{P_s f}(\overline{X}_t)Z_t^n] = P_t^n(P_s^n f)(x)\,.
\end{aligned}$$

Hence the semigroup property holds for $(P_t^n)_{t\geq 0}$.

Since by the $\overline{\mu}$–symmetry of $\overline{\mathbf{M}}$

$$\begin{aligned}
\int (P_t^n f - f)^2\,d\mu &= \int \overline{E}_x[(\overline{f}(\overline{X}_t) - \overline{f}(\overline{X}_0))Z_t^n]^2\,\overline{\mu}(dx)\\
&\leq \exp(t\|\,|B^n|_H^2\|_\infty)\int \overline{E}_x[(\overline{f}(\overline{X}_t) - \overline{f}(\overline{X}_0))^2]\,\overline{\mu}(dx)\\
&= \exp(t\|\,|B^n|_H^2\|_\infty)\int (\overline{p}_t\overline{f}^2 - 2\overline{f}\,\overline{p}_t\overline{f} + \overline{f}^2)\,d\overline{\mu}\\
&\leq 2\exp(t\|\,|B^n|_H^2\|_\infty)\|f\|_{L^2(E,\mu)}\|f - T_t^0 f\|_{L^2(E,\mu)}
\end{aligned}$$

for $f \in \mathcal{B}_b(E)$, the strong continuity of $(P_t^n)_{t\geq 0}$ is now an easy consequence. □

Denote the infinitesimal generator corresponding to $(P_t^n)_{t\geq 0}$ with $(A^n, D(A^n))$.

**Proposition 2.8.** $(A^n, D(A^n)) = (L^n, D(L^0))$.

**Proof.** Let $u$ be an $\mathcal{E}^0$–q.c. $\mu$–version of some element in $D(L^0)$. Fix $t > 0$ and $x \in Y$ such that $Z_s^n$, $0 \leq s \leq t$, is an $L^2(\overline{P}_x)$–integrable continuous martingale.

Denote by $\overline{Q}_x$ the measure $Z_t^n \overline{P}_x$ on $\overline{\mathcal{F}}_t$. Note that any $\overline{\mathcal{F}}_t$–measurable, $L^2(\overline{P}_x)$–integrable random variable $F$ is $\overline{Q}_x$–integrable, since

$$E_x^{\overline{Q}}[|F|] = \overline{E}_x[|F| Z_t^n] \leq \overline{E}_x[F^2]^{1/2} \overline{E}_x[(Z_t^n)^2]^{1/2}.$$

From the general Girsanov Theorem (cf. [IW, p. 177, Theorem 4.1]) we obtain a sequence of stopping times $(T_n)_{n\geq 1}$ with $T_n \uparrow t$ such that $M_{T_n\wedge s}^{[\overline{u}]} - \langle M^{[\overline{u}]}, Y^n \rangle_{T_n \wedge s}$, $0 \leq s \leq t$, is an $L^2(\overline{Q}_x)$–integrable martingale. Consequently by (2.4) and Remark 2.3 (v)

$$\begin{aligned}
P_t^n u(x) - u(x) &= \overline{E}_x[(\overline{u}(\overline{X}_t) - \overline{u}(\overline{X}_0)) Z_t^n] \\
&= \overline{E}_x^{\overline{Q}}[M_t^{[\overline{u}]} + \int_0^t \overline{L^0 u}(\overline{X}_s)\, ds] \\
&= \lim_{n\to\infty} \overline{E}_x^{\overline{Q}}[M_{T_n\wedge t}^{[\overline{u}]} + \int_0^{T_n\wedge t} \overline{L^0 u}(\overline{X}_s)\, ds] \\
&= \lim_{n\to\infty} \overline{E}_x^{\overline{Q}}[\int_0^{T_n\wedge t} \overline{\langle B^n, \nabla u\rangle_H}(\overline{X}_s) + \overline{L^0 u}(\overline{X}_s)\, ds\,] \\
&= \int_0^t P_s^n(\langle B^n, \nabla u\rangle_H + L^0 u)(x)\, ds\,.
\end{aligned}$$

Since this reasoning holds for $\overline{\mathcal{E}}$–q.e. $x \in Y$ we conclude that $P_t^n u - u = \int_0^t P_s^n(\langle B^n, \nabla u\rangle_H + L^0 u)\, ds$. Therefore $u \in D(A^n)$ and $A^n u = \langle B^n, \nabla u\rangle_H + L^0 u = L^n u$. Thus $(A^n, D(A^n))$ is an extension of $(L^n, D(L^0))$. Since $(L^n, D(L^0))$ is maximal (i.e. there is no real extension of $(L^n, D(L^0))$ that generates a strongly continuous semigroup) the assertion follows. $\square$

**Remark 2.9.** Note that as a consequence of the last proposition we obtain that the semigroups $(P_t^n)_{t\geq 0}$ and $(T_t^n)_{t\geq 0}$ coincide. Especially $(T_t^n)_{t\geq 0}$ is sub–Markovian and $T_t \mathbb{1} = \mathbb{1}$. But there is also a pure analytic proof for these two properties of $(T_t^n)_{t\geq 0}$:

**1. Step:** $(\mathcal{E}_\alpha^n, D(\mathcal{E}^0))$ is positivity preserving. To this end it suffices to prove that $\mathcal{E}_\alpha^n(u^+, u^-) \leq 0$ for all $u \in D(\mathcal{E}^0)$ (cf. [MR 2, p. 6, Proposition 1.3]). But this follows from the fact that $\int \langle B^n, \nabla u^+\rangle_H u^-\, d\mu = 0$, since $\nabla u^+ = \mathbb{1}_{\{u\geq 0\}} \nabla u$, and from the fact that $\mathcal{E}_\alpha^0(u^+, u^-) \leq 0$, since $(\mathcal{E}^0, D(\mathcal{E}^0))$ is a Dirichlet form.
**2. Step:** Due to [MR 2, p. 7, Theorem 1.5] $(T_t^{n,\alpha})_{t\geq 0}$ is positivity preserving and so is $(T_t^n)_{t\geq 0}$, because $T_t^n = e^{\alpha t} T_t^{n,\alpha}$. Since $\mathbb{1} \in D(L^0)$ and $\nabla \mathbb{1} = 0$ we have that $L^n \mathbb{1} = 0$ and therefore $T_t^n \mathbb{1} = \mathbb{1}$; $\forall t \geq 0$. This implies that $(T_t^n)_{t\geq 0}$ is sub–Markovian.

**(d) The limit $T_t^n u$ as $n \to \infty$ for bounded $u$.**

**Lemma 2.10.**

*(i)* $\int \langle B, \nabla u\rangle_H u\, d\mu \leq 0\, \forall u \in D(\mathcal{E}^0) \cap L^\infty(E, \mu)$.

*(ii)* $\int\langle B,\nabla u\rangle_H u\,d\mu=0\,\forall u\in D(\mathcal{E}^0)\cap L^\infty(E,\mu)$ *if* $\int\langle B,\nabla u\rangle_H\,d\mu=0\,\forall u\in\mathcal{F}C_b^\infty$.

**Proof.** First note that E.3 implies $\int\langle B,\nabla u\rangle_H\,d\mu\le 0$ (resp. $=0$ in (ii)) $\forall u\in D(\mathcal{E}^0)$, $u\ge 0$. Since $\nabla(u^2)=2u\nabla u$ for $u\in D(\mathcal{E}^0)\cap L^\infty(E,\mu)$ the assertion follows. $\square$

**Lemma 2.11.** *Let* $u\in L^\infty(E,\mu)$, *then* $(T_t^n u)_{n\ge 1}$ *is an* $L^2(E,\mu)$*–Cauchy sequence.*

**Proof. 1. Step:** Since $(T_t^n)_{t\ge0}$ is sub–Markovian we have that

$$\begin{aligned}\mathcal{E}^0(T_s^n u,T_s^n u)&=(-L^nT_s^nu,T_s^nu)_{L^2(E,\mu)}+(\langle B^n,\nabla T_s^nu\rangle_H,T_s^nu)_{L^2(E,\mu)}\\&\le(-L^nT_s^nu,T_s^nu)_{L^2(E,\mu)}+\|u\|_\infty\|\,|B^n|_H\|_{L^2(E,\mu)}(2\mathcal{E}^0(T_s^nu,T_s^nu))^{1/2}\,.\end{aligned}$$

Note that $s\mapsto T_s^nu$ is strongly measurable on $(0,\infty)$ w.r.t. the $\mathcal{E}^0$–norm, since $s\mapsto\mathcal{E}^0(v,T_s^nu)$ is measurable for all $v\in D(L^0)$, hence all $v\in D(\mathcal{E}^0)$, and $D(\mathcal{E}^0)$ is a separable Hilbert space. Thus for $t>\epsilon>0$

$$\begin{aligned}&\int_\epsilon^t\mathcal{E}^0(T_s^nu,T_s^nu)\,ds\\&\le\int_\epsilon^t(-L^nT_s^nu,T_s^nu)_{L^2(E,\mu)}\,ds+\|u\|_\infty\|\,|B^n|_H\|_{L^2(E,\mu)}\int_\epsilon^t(2\mathcal{E}^0(T_s^nu,T_s^nu))^{1/2}\,ds\\&\le(1/2)\|T_\epsilon^nu\|^2_{L^2(E,\mu)}+(2t)^{1/2}\|u\|_\infty\|\,|B^n|_H\|_{L^2(E,\mu)}(\int_\epsilon^t\mathcal{E}^0(T_s^nu,T_s^nu)\,ds)^{1/2}\end{aligned}$$

which implies

$$(\int_\epsilon^t\mathcal{E}^0(T_s^nu,T_s^nu)\,ds)^{1/2}\le2^{-1/2}\|T_\epsilon^nu\|_{L^2(E,\mu)}+(2t)^{1/2}\|u\|_\infty\|\,|B^n|_H\|_{L^2(E,\mu)}$$

and for $\epsilon\to0$

$$(\int_0^t\mathcal{E}^0(T_s^nu,T_s^nu)\,ds)^{1/2}\le2^{-1/2}\|u\|_{L^2(E,\mu)}+(2t)^{1/2}\|u\|_\infty\|\,|B^n|_H\|_{L^2(E,\mu)}\,.$$

Hence $M:=\sup_{n\ge1}(\int_0^t\mathcal{E}^0(T_s^nu,T_s^nu)\,ds)^{1/2}<\infty$.

**2.Step:** Note that

$$\begin{aligned}(L^nT_s^nu-L^mT_s^mu,T_s^nu-T_s^mu)_{L^2(E,\mu)}&=(L^0(T_s^nu-T_s^mu),T_s^nu-T_s^mu)_{L^2(E,\mu)}\\&\quad+(\langle B^n,\nabla T_s^nu\rangle_H-\langle B^m,\nabla T_s^mu\rangle_H,T_s^nu-T_s^mu)_{L^2(E,\mu)}\\&\le(\langle B^n-B,\nabla T_s^nu\rangle_H+\langle B-B^m,\nabla T_s^mu\rangle_H,T_s^nu-T_s^mu)_{L^2(E,\mu)}\\&\le2\|u\|_\infty(\|\,|B^n-B|_H\|_{L^2(E,\mu)}(2\mathcal{E}^0(T_s^nu,T_s^nu))^{1/2}\\&\quad+\|\,|B-B^m|_H\|_{L^2(E,\mu)}(2\mathcal{E}^0(T_s^mu,T_s^mu))^{1/2})\,.\end{aligned}$$

Thus for $t > \epsilon > 0$

$$
\begin{aligned}
&\|T_t^n u - T_t^m u\|^2_{L^2(E,\mu)} - \|T_\epsilon^n u - T_\epsilon^m u\|^2_{L^2(E,\mu)} \\
&= 2\int_\epsilon^t (L^n T_s^n u - L^m T_s^m u, T_s^n u - T_s^m u)_{L^2(E,\mu)}\, ds \\
&\le 4\|u\|_\infty (\|\,|B^n - B|_H\|_{L^2(E,\mu)} \int_\epsilon^t (2\mathcal{E}^0(T_s^n u, T_s^n u))^{1/2}\, ds \\
&\qquad + \|\,|B - B^m|_H\|_{L^2(E,\mu)} \int_\epsilon^t (2\mathcal{E}^0(T_s^m u, T_s^m u))^{1/2}\, ds)
\end{aligned}
$$

which implies for $\epsilon \to 0$

$$
\begin{aligned}
\|T_t^n u - T_t^m u\|^2_{L^2(E,\mu)} \le 4\|u\|_\infty (&\|\,|B^n - B|_H\|_{L^2(E,\mu)} \\
&+ \|\,|B - B^m|_H\|_{L^2(E,\mu)})(2t)^{1/2} M\,.
\end{aligned}
$$

Clearly the right hand side of the last inequality converges to zero as $n, m \to \infty$. $\square$

For $u \in L^\infty(E,\mu)$ let $T_t u := \lim_{n\to\infty} T_t^n u$.

**Lemma 2.12.** *Let $u \in L^\infty(E,\mu)$. Then*

*(i)* $\|T_t u\|_{L^2(E,\mu)} \le \|u\|_{L^2(E,\mu)}$.

*(ii)* $\int T_t u\, d\mu \le \int u\, d\mu$ *if* $u \ge 0$.

*(iii)* $\int T_t u\, d\mu = \int u\, d\mu$ *if* $\int \langle B, \nabla u\rangle\, d\mu = 0\, \forall u \in \mathcal{F}C_b^\infty$.

**Proof.** Let $M := \sup_{n\ge 1} (\int_0^t \mathcal{E}^0(T_s^n u, T_s^n u)\, ds)^{1/2}$. Clearly

$$
\begin{aligned}
(L^n T_s^n u, T_s^n u)_{L^2(E,\mu)} &= (L^0 T_s^n u, T_s^n u)_{L^2(E,\mu)} + (\langle B^n, \nabla T_s^n u\rangle_H, T_s^n u)_{L^2(E,\mu)} \\
&\le (\langle B^n - B, \nabla T_s^n u\rangle_H, T_s^n u)_{L^2(E,\mu)} \\
&\le \|u\|_\infty \|\,|B^n - B|_H\|_{L^2(E,\mu)} (2\mathcal{E}^0(T_s^n u, T_s^n u))^{1/2}\,.
\end{aligned}
$$

Thus for $t > \epsilon > 0$

$$
\begin{aligned}
\|T_t^n u\|^2_{L^2(E,\mu)} - \|T_\epsilon^n u\|^2_{L^2(E,\mu)} &= 2\int_\epsilon^t (L^n T_s^n u, T_s^n u)_{L^2(E,\mu)}\, ds \\
&\le 2\|u\|_\infty \|\,|B^n - B|_H\|_{L^2(E,\mu)} (2t)^{1/2} M
\end{aligned}
$$

hence for $\epsilon \to 0$

$$
\|T_t^n u\|^2_{L^2(E,\mu)} - \|u\|^2_{L^2(E,\mu)} \le 2\|u\|_\infty \|\,|B^n - B|_H\|_{L^2(E,\mu)} (2t)^{1/2} M
$$

which implies

$$
\|T_t u\|^2_{L^2(E,\mu)} - \|u\|^2_{L^2(E,\mu)} = \lim_{n\to\infty} \|T_t^n u\|^2_{L^2(E,\mu)} - \|u\|^2_{L^2(E,\mu)} \le 0\,.
$$

For the proof of (ii) and (iii) note that for $u \geq 0$

$$
\begin{aligned}
\int L^n T_s^n u \, d\mu &= \int L^0 T_s^n u \, d\mu + \int \langle B^n, \nabla T_s^n u \rangle_H \, d\mu \\
&\leq \int \langle B^n - B, \nabla T_s^n u \rangle_H \, d\mu \\
&\leq \| \, |B^n - B|_H \|_{L^2(E,\mu)} (2\mathcal{E}^0(T_s^n u, T_s^n u))^{1/2}
\end{aligned}
$$

since $L^0 \mathbb{1} = 0$, hence $\int L^0 u \, d\mu = 0 \, \forall u \in D(L^0)$. Consequently for $t > \epsilon > 0$

$$
\begin{aligned}
\int T_t^n u - T_\epsilon^n u \, d\mu &= \int_\epsilon^t \int L^n T_s^n u \, d\mu \, ds \\
&\leq \| \, |B^n - B|_H \|_{L^2(E,\mu)} \int_\epsilon^t (2\mathcal{E}^0(T_s^n u, T_s^n u))^{1/2} \, ds
\end{aligned}
$$

which implies

$$
\begin{aligned}
\int T_t^n u - u \, d\mu &= \lim_{\epsilon \downarrow 0} \int T_t^n u - T_\epsilon^n u \, d\mu \\
&\leq \| \, |B^n - B|_H \|_{L^2(E,\mu)} (2t)^{1/2} M
\end{aligned}
$$

and therefore

$$
\int T_t u - u \, d\mu = \lim_{n \to \infty} \int T_t^n u - u \, d\mu \leq 0 \, .
$$

In the case $\int \langle B, \nabla u \rangle_H \, d\mu = 0 \, \forall u \in \mathcal{F}C_b^\infty$ clearly $\int T_t u - u \, d\mu = 0$. $\square$

Due to Lemma 2.12 every operator $T_t$ can be extended to a contraction on $L^2(E, \mu)$ again denoted with $T_t$.

**(e) Identification of $(T_t)_{t\geq 0}$ as Girsanov–transform of $(T_t^0)_{t\geq 0}$.**

**Lemma 2.13.** *The sequence $(Y^n)_{n\geq 1}$ of continuous MAF is an e–Cauchy sequence.*

**Proof.** From (2.5) it follows that

$$
\overline{E}_{\overline{\mu}} \left[ (Y_t^n)^2 \right] = \overline{E}_{\overline{\mu}} \left[ \langle Y^n \rangle_t \right] = \overline{E}_{\overline{\mu}} \left[ \int_0^t \overline{|B^n|_H^2} \left( \overline{X}_s \right) \, ds \right] \quad \forall n.
$$

Moreover by (2.4)

$$
\begin{aligned}
\overline{E}_{\overline{\mu}} \left[ Y_t^n Y_t^m \right] &= \overline{E}_{\overline{\mu}} \left[ \langle Y^n, Y^m \rangle_t \right] = \overline{E}_{\overline{\mu}} \left[ \sum_{i=1}^n \langle \overline{b}_{n,i} \cdot M^i, Y^m \rangle_t \right] \\
&= \overline{E}_{\overline{\mu}} \left[ \sum_{i=1}^n \int_0^t \overline{b}_{n,i}(\overline{X}_s) \langle \overline{e_i, B^m} \rangle_H (\overline{X}_s) \, ds \right] \\
&= \overline{E}_{\overline{\mu}} \left[ \int_0^t \langle \overline{B^n, B^m} \rangle_H \left( \overline{X}_s \right) \, ds \right] .
\end{aligned}
$$

Consequently

$$\begin{aligned}\overline{E}_{\overline{\mu}}[(Y_t^n - Y_t^m)^2] &= \overline{E}_{\overline{\mu}}[\int_0^t \overline{|B^n - B^m|}_H^2(\overline{X}_s)\,ds] \\ &= \int_0^t \int \overline{p}_s \overline{|B^n - B^m|}_H^2 \,d\overline{\mu}\,ds \\ &= t\,\| |B^n - B^m|_H \|_{L^2(E,\mu)}^2 \ ,\end{aligned}$$

and thus $e(Y^n - Y^m) = (1/2)\,\| |B^n - B^m|_H \|_{L^2(E,\mu)}^2$. Since $|B^n - B^m|_H$ is an $L^2(E,\mu)$–Cauchy sequence, the assertion follows. $\square$

By [FOT, p.202, Theorem 5.2.1] there is some MAF $Y$ of finite energy such that $\lim_{n\to\infty} e(Y^n - Y) = 0$. Moreover, $Y$ is continuous since there is some subsequence $(n_k)_{k\geq 1}$ with

$$\overline{P}_x[\lim_{k\to\infty} Y_t^{n_k} = Y_t \text{ uniformly on } [0,T]\ \forall T\geq 0] = 1 \quad \overline{\mathcal{E}}\text{–q.e.}$$

**Lemma 2.14.**

*(i)* $\langle Y\rangle_t = \int_0^t \overline{|B|}_H^2(\overline{X}_s)\,ds, \quad t\geq 0\,.$

*(ii) Let* $u\in D(\mathcal{E}^0)$, *then* $\langle Y, M^{[\overline{u}]}\rangle_t = \int_0^t \langle \overline{B,\nabla u}\rangle_H(\overline{X}_s)\,ds\,, \quad t\geq 0,\ \overline{P}_{\overline{\mu}}$*–a.s.*

**Proof.** (i) Since $Y_t^n \to Y_t$ in $L^2(\overline{P}_{\overline{\mu}})$ by Lemma 2.13 we obtain that $\langle Y^n\rangle_t \to \langle Y\rangle_t$ in $L^1(\overline{P}_{\overline{\mu}})$. But $\langle Y^n\rangle_t = \int_0^t \overline{|B^n|}_H^2(\overline{X}_s)\,ds$ by (2.5) and $\overline{E}_{\overline{\mu}}[\int_0^t \overline{|B^n - B|}_H^2(\overline{X}_s)\,ds]$ converges to zero for $n\to\infty$. Hence $\langle Y\rangle_t = \int_0^t \overline{|B|}_H^2(\overline{X}_s)\,ds\,,\ t\geq 0\ \overline{P}_{\overline{\mu}}$–a.s. Thus the corresponding Revuz measures coincide which implies that $\langle Y\rangle$ and $\int_0^{\cdot} \overline{|B|}_H^2(\overline{X}_s)\,ds$ must be equivalent.
(ii) Since $\langle L, M^{[\overline{u}]}\rangle_t \leq \langle L\rangle_t^{1/2}\langle M^{[\overline{u}]}\rangle_t^{1/2}$ by the inequality of Cauchy–Schwarz, for any $L\in \overset{\circ}{\mathcal{M}}$, and $\langle Y^n\rangle_t \to \langle Y\rangle_t$ in $L^1(\overline{P}_{\overline{\mu}})$ it follows from (2.4)

$$\langle Y, M^{[\overline{u}]}\rangle_t = \lim_{n\to\infty}\langle Y^n, M^{[\overline{u}]}\rangle_t = \int_0^t \langle\overline{B,\nabla u}\rangle_H(\overline{X}_s)\,ds$$

in $L^1(\overline{P}_{\overline{\mu}})$. $\square$

It is easy to construct an $(\overline{\mathcal{F}}_t^0)$–adapted version $\overline{Y}$ of $Y$ which is indistinguishable from $Y$ in the sense that $\overline{P}_x[\overline{Y}_t = Y_t\ \forall t\geq 0] = 1 \quad \overline{\mathcal{E}}$–q.e. Hence there exists some $\overline{\mathcal{E}}$–exceptional set $N\in\mathcal{B}(\overline{E})$ such that

(2.9) $(\overline{Y}_t)_{t\geq 0}$ is an $L^2(\overline{P}_x)$ –integrable continuous martingale with $\overline{Y}_0 = 0$, and $\langle\overline{Y}\rangle_t = \int_0^t \overline{|B|}_H^2(\overline{X}_s)\,ds, \quad t\geq 0\,,\forall x\in\overline{E}\setminus N$.

(2.10) $\overline{Y}_{t+s} = \overline{Y}_t\circ\theta_s + \overline{Y}_s \quad \overline{P}_x$-a.s. for all $s,t\geq 0,\ x\in\overline{E}\setminus N$.

Define the process $Z_t := \exp((\overline{Y}_t - 1/2\langle\overline{Y}\rangle_t)\mathbb{1}_{\{\overline{X}_0\in\overline{E}\setminus N\}})$. Clearly $(Z_t)_{t\geq 0}$ is $(\overline{\mathcal{F}}^0_t)_{t\geq 0}$–adapted, $\overline{P}_x[Z_t \equiv 1\,\forall t\geq 0] = 1$ for $x\in N$ and $(Z_t)_{t\geq 0}$ is a continuous non-negative local martingale. Thus the following clearly holds for every $x\in\overline{E}$ due to (2.9) and (2.10):

(2.11) $Z_{t+s} = Z_t\circ\theta_s\cdot Z_s$ $\overline{P}_x$-a.s. $\forall s,t\geq 0$

(2.12) $Z_t$ is continuous $\overline{P}_x$-a.s.

(2.13) $\overline{E}_x[Z_t]\leq 1 \quad \forall t\geq 0$ and $\overline{E}_x[Z_0] = 1$.

Due to (2.11) - (2.13) we can apply [KW, p.189, Theorem] to obtain a standard process $\overline{\mathbf{M}}^B = (\overline{\Omega}, \bigvee_{t\geq 0}\overline{\mathcal{F}}^B_t, (\overline{X}_t)_{t\geq 0}, (\overline{Q}_x)_{x\in\overline{E}_\Delta})$ w.r.t. the filtration $(\overline{\mathcal{F}}^B_t)_{t\geq 0}$, where $\overline{\mathcal{F}}^B_t$ denotes the universal completion of $\overline{\mathcal{F}}^0_t$ (in $\overline{\mathcal{F}}^0_t$) w.r.t. all measures $\int\overline{Q}_x\nu(dx)$, where $\nu$ is a finite measure on $(\overline{E},\mathcal{B}(\overline{E}))$, such that $\overline{Q}_x[A\cap\{\tau<\zeta\}] = \overline{E}_x[\mathbb{1}_{A\cap\{\tau<\zeta\}}Z_\tau]$ for any $(\overline{\mathcal{F}}^0_t)_{t\geq 0}$–stopping time $\tau$ with $\tau\leq\zeta$ and $A\in\overline{\mathcal{F}}^0_\tau$. This implies

$$\overline{Q}_x[A\cap\{\tau<\zeta\}] = \overline{E}_x[\mathbb{1}_{A\cap\{\tau<\zeta\}}Z_\tau] \tag{2.14}$$

for an arbitrary $(\overline{\mathcal{F}}^0_t)_{t\geq 0}$–stopping time $\tau$ and $A\in\overline{\mathcal{F}}^0_\tau$, since

$$\begin{aligned}\overline{Q}_x[A\cap\{\tau<\zeta\}] &= \overline{Q}_x[A\cap\{\tau<\zeta\}\cap\{\tau\wedge\zeta<\zeta\}] = \overline{E}_x[\mathbb{1}_{A\cap\{\tau<\zeta\}}Z_{\tau\wedge\zeta}]\\ &= \overline{E}_x[\mathbb{1}_{A\cap\{\tau<\zeta\}}Z_\tau].\end{aligned}$$

Since $(E_k)_{k\geq 1}$ is also an $\overline{\mathcal{E}}$–nest, there exists some $\overline{\mathcal{E}}$–exceptional set $N_1\supset N$ such that

$$\overline{P}_x[\lim_{k\to\infty}\sigma_{\overline{E}\setminus E_k}<\zeta] = 0 \quad \forall x\in\overline{E}\setminus N_1 \tag{2.15}$$

where $\sigma_A = \inf\{t>0 \mid \overline{X}_t\in A\}$ for Borel sets $A$ (cf. [MR, p. 139, Proposition IV.5.30]).

Since the trace topologies on $E_k$ induced by $\overline{E}$ and $E$ coincide (cf. [MR, p. 174, Theorem VI.1.2]) we have that $t\to\overline{X}_t$ is continuous on $[0,\zeta)$ w.r.t. the original topology if and only if it is continuous on $[0,\zeta)$ w.r.t. the topology induced by $\overline{E}$ $\overline{P}_x$–a.s., $x\in\overline{E}\setminus N_1$. By [MR, p. 141, Corollary IV.6.5] we can find some $\overline{\mathcal{E}}$–exceptional set $N_0\in\mathcal{B}(\overline{E})$ with $N_0\supset N_1$ such that $\overline{E}\setminus N_0$ is $\overline{\mathbf{M}}$–invariant, i.e.

$$\overline{\Omega}_{\overline{E}\setminus N_0} := \{\omega \mid \omega(t)\in\overline{E}\setminus N_0, \omega(t-)\in\overline{E}\setminus N_0\,\forall t\in[0,\zeta(\omega))\}$$

satisfies $\overline{P}_x[\overline{\Omega}_{\overline{E}\setminus N_0}] = 1 \quad \forall x\in\overline{E}\setminus N_0$.

**Lemma 2.15.**

*(i)* $\overline{Q}_x[t\to\overline{X}_t$ *is continuous on* $[0,\zeta)] = 1 \quad \forall x\in\overline{E}$.

*(ii)* $\overline{Q}_x[\Omega_{\overline{E}\setminus N_0}] = 1 \quad \forall x \in \overline{E} \setminus N_0$, *i.e.* $\overline{E} \setminus N_0$ *is also* $\overline{\mathbf{M}}^B$*–invariant.*

*(iii)* $\overline{Q}_x[\sup_{k\geq 1} \sigma_{\overline{E}\setminus E_k} < \zeta] = 0 \quad \forall x \in \overline{E} \setminus N_1$ .

**Proof.** (i) Obvious, since $\overline{Q}_x \ll \overline{P}_x$ on $\overline{\mathcal{F}}^0_t \cap \{t < \zeta\}$.

(ii) Fix $x \in \overline{E} \setminus N_0$. Since $\overline{\mathbf{M}}$ is a Hunt process there exists a decreasing sequence of open sets $(G_n)_{n\geq 1}$ containing $N_0$ such that $\sigma_{G_n} \to \sigma_{N_0}$ $\overline{P}_x$–a.s. (cf. [FOT, p. 318, Theorem A.2.4]). Note that $\{\sigma_A < t\} \in \overline{\mathcal{F}}^0_t$ for open sets $A$. Hence for $t \geq 0$

$$\begin{aligned}\overline{Q}_x[\sigma_{N_0} < t, t < \zeta] &\leq \lim_{n\to\infty} \overline{Q}_x[\sigma_{G_n} < t, t < \zeta] \\ &= \lim_{n\to\infty} \overline{E}_x[\mathbb{1}_{\{\sigma_{G_n} < t,\, t<\zeta\}} Z_t] \\ &= \overline{E}_x[\mathbb{1}_{\{\sigma_{N_0} < t,\, t<\zeta\}} Z_t] = 0.\end{aligned}$$

Since $\bigcup_{t\in\mathbb{Q}^+\cap[0,\infty)}\{\sigma_{N_0} < t,\, t < \zeta\} = \{\sigma_{N_0} < \zeta\}$ it follows $\overline{Q}_x[\sigma_{N_0} < \zeta] = 0$. Since $\overline{\Omega}_{\overline{E}\setminus N_0} = \{\omega \mid \omega(t) \in \overline{E} \setminus N_0 \,\forall t \in [0, \zeta(\omega))\}$ $\overline{Q}_x$–a.s. by (i) and therefore $\overline{\Omega} \setminus \overline{\Omega}_{\overline{E}\setminus N_0} \subset \{\sigma_{N_0} < \zeta\}$ $\overline{Q}_x$–a.s. the assertion follows.

(iii) Fix $x \in \overline{E} \setminus N_1$. Then for $t \geq 0$

$$\begin{aligned}\overline{Q}_x[\sup_{k\geq 1} \sigma_{\overline{E}\setminus E_k} < t, t < \zeta] &\leq \lim_{k\to\infty} \overline{Q}_x[\sigma_{\overline{E}\setminus E_k} < t, t < \zeta] \\ &= \lim_{k\to\infty} \overline{E}_x[\mathbb{1}_{\{\sigma_{\overline{E}\setminus E_k} < t,\, t<\zeta\}} Z_t] \\ &= \overline{E}_x[\mathbb{1}_{\{\sup_{k\geq 1}\sigma_{\overline{E}\setminus E_k} < t,\, t<\zeta\}} Z_t] = 0.\end{aligned}$$

It follows as in the proof of (ii) that $\overline{Q}_x[\sup_{k\geq 1} \sigma_{\overline{E}\setminus E_k} < \zeta] = 0$. $\square$

Denote by $\mathbf{M}_{\overline{E}\setminus N_0}$ the restriction of $\overline{\mathbf{M}}$ to $\overline{\Omega}_{\overline{E}\setminus N_0}$ and by $\mathbf{M}^B{}_{\overline{E}\setminus N_0}$ the restriction of $\overline{\mathbf{M}}^B$ to $\overline{\Omega}_{\overline{E}\setminus N_0}$. Clearly, $\mathbf{M}_{\overline{E}\setminus N_0}$ and $\mathbf{M}^B{}_{\overline{E}\setminus N_0}$ are standard processes on $\overline{E} \setminus N_0$ also w.r.t. the original topology on $\overline{E} \setminus N_0$.

Let $\mathbf{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be the trivial extension of $\mathbf{M}_{\overline{E}\setminus N_0}$ to $E$ and $\mathbf{M}^B = (\Omega, \mathcal{M}^B, (X_t)_{t\geq 0}, (Q_x)_{x\in E_\Delta})$ the trivial extension of $\mathbf{M}^B{}_{\overline{E}\setminus N_0}$ to $E$ (cf. [MR, p.117/118]).

(2.16) $\mathbf{M}$ and $\mathbf{M}^B$ are $\mu$–tight standard processes again and $\mathbf{M}$ is properly associated with $(\mathcal{E}^0, D(\mathcal{E}^0))$. The $\mu$–tightness is implied by (2.15) and Lemma 2.15 (iii).

**Lemma 2.16.** *Let* $f \in \mathcal{B}^+(E)$*, then* $E^Q_\mu[f(X_t)] \leq \int f\, d\mu$*.*

**Proof.** First take $f \in \mathcal{B}^+_b(E)$. Since $Y^n_t \to \overline{Y}_t$ in $L^2(\overline{P}_{\overline{\mu}})$ and $\langle Y^n_t\rangle_t \to \langle\overline{Y}\rangle_t$ in $L^1(\overline{P}_{\overline{\mu}})$ by Lemma 2.13 there is a subsequence $(n_k)_{k\geq 1}$ with $Z^{n_k}_t \to Z_t$ $\overline{P}_{\overline{\mu}}$–a.s. Note that for $x \in E \setminus N_0$

$$\begin{aligned}E^Q_x[f(X_t)] &= \overline{E}^Q_x[\overline{f}(\overline{X}_t)] = \overline{E}^Q_x[\overline{f}(\overline{X}_t)\mathbb{1}_{\{t<\zeta\}}] \\ &= \overline{E}_x[\overline{f}(\overline{X}_t)\mathbb{1}_{\{t<\zeta\}} Z_t] = \overline{E}_x[\overline{f}(\overline{X}_t) Z_t]\,.\end{aligned}$$

Thus by the Lemma of Fatou, Remark 2.9 and Lemma 2.12

$$\begin{aligned} E_\mu^Q[f(X_t)] &= \overline{E}_{\overline{\mu}}[\overline{f}(\overline{X}_t)Z_t] \\ &\le \liminf_{k\to\infty} \overline{E}_{\overline{\mu}}[\overline{f}(\overline{X}_t)Z_t^{n_k}] \\ &= \liminf_{k\to\infty} \int T_t^{n_k} f \, d\mu \\ &= \int T_t f \, d\mu \le \int f \, d\mu \,. \end{aligned}$$

For arbitrary $f \in \mathcal{B}^+(E)$ the assertion now follows by the monotone convergence theorem. $\square$

Let $\tau_n := \inf\{t \ge 0 \mid \int_0^t \overline{|B|}_H^2(\overline{X}_s)\, ds \ge n\}$ and $\tau := \sup_{n\ge 1} \tau_n$. Note that $\tau_n$ and $\tau$ are $(\overline{\mathcal{F}}_t^0)_{t\ge 0}$–stopping times. Since $(Z_{\tau_n \wedge t})_{t\ge 0}$ is a square–integrable martingale for $x \in E \setminus N_0$ it follows from (2.14)

$$\begin{aligned} Q_x[\tau_n \wedge t < \zeta] &= \overline{Q}_x[\tau_n \wedge t < \zeta] = \overline{E}_x[\mathbb{1}_{\{\tau_n\wedge t<\zeta\}} Z_{\tau_n\wedge t}] \\ &= \overline{E}_x[Z_{\tau_n\wedge t}] = 1 \quad \text{for } x \in E \setminus N_0 \,. \end{aligned}$$

Thus $Q_x[\tau \wedge t \le \zeta] = 1 \, \forall t \ge 0$ and therefore

$$Q_x[\tau \le \zeta] = 1 \, \forall x \in E \setminus N_0 \,. \tag{2.17}$$

On the other hand Lemma 2.16 implies that

$$E_\mu^Q[\int_0^t |B|_H^2(X_s)\, ds] = \int_0^t E_\mu^Q[|B|_H^2(X_s)]\, ds \le t \int |B|_H^2 \, d\mu$$

and thus $Q_\mu[\tau = \infty] = 1$. Hence $Q_\mu[\zeta = \infty] = 1$ by (2.17) and

$$E_\mu[Z_t] = Q_\mu[t < \zeta] = 1 \, \forall t \ge 0. \tag{2.18}$$

**Lemma 2.17.** *$Q_x[f(X_t)]$ is a $\mu$–version of $T_t f$ for every $f \in \mathcal{B}_b(E)$.*

**Proof.** Since $Z_t^{n_k} \to Z_t$ $P_\mu$-a.s. for some subsequence $(n_k)_{k\ge 1}$ (cf. the proof of Lemma 2.16) and $E_\mu[Z_t^n] = 1 = E_\mu[Z_t]$ it follows $Z_t^{n_k} \to Z_t$ in $L^1(P_\mu)$. Thus for $f \in \mathcal{B}_b(E)$

$$\begin{aligned} \int |T_t f - E_\cdot[f(X_t)Z_t]| \, d\mu &\le \liminf_{k\to\infty} \int |T_t^{n_k} f - E_\cdot[f(X_t)Z_t]| \, d\mu \\ &\le \liminf_{k\to\infty} \int E_x[|f(X_t)||Z_t^{n_k} - Z_t|] \, d\mu = 0 \,, \end{aligned}$$

which proves the assertion. $\square$

**Lemma 2.18.** *$(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(E,\mu)$.*

**Proof.** Clearly $T_tT_sf = T_{t+s}f\,\forall f \in \mathcal{B}_b(E)$ by the Markov–property of $\mathbf{M}^B$. Since $f(X_t) \to f(X_0)$ for $f \in C_b(E)$ it follows from Lebesgues theorem that

$$\begin{aligned}\lim_{t\downarrow 0}\int |T_tf - f|^2\,d\mu &= \lim_{t\downarrow 0}\int |Q_x[f(X_t) - f(X_0)]|^2\,\mu(dx)\\ &\leq \limsup_{t\downarrow 0}\int Q_x[(f(X_t)-f(X_0))^2]\,\mu(dx)\\ &= \limsup_{t\downarrow 0} Q_\mu[(f(X_t)-f(X_0))^2] = 0\,.\end{aligned}$$

Since each $T_t$ is a contraction, the semigroup property and the strong continuity extends to every $f \in L^2(E,\mu)$. $\square$

Denote the generator corresponding to $(T_t)_{t\geq 0}$ with $(L, D(L))$.

**Remark 2.19.** Due to Lemma 2.12 and the fact that $(T_t)_{t\geq 0}$ is sub–Markovian the semigroup extends to a strongly continuous contraction semigroup on $L^1(E,\mu)$, since

$$\begin{aligned}\int |T_tf|\,d\mu = \int |T_tf^+ - T_tf^-|\,d\mu &\leq \int |T_tf^+| + |T_tf^-|\,d\mu\\ &\leq \int f^+ + f^-\,d\mu = \int |f|\,d\mu\,,\end{aligned}$$

for all $f \in L^\infty(E,\mu)$.

**Proposition 2.20.** *$(L, D(L))$ is an extension of $(L, \mathcal{F}C_b^\infty \cap D(L^0))$.*

**Proof.** Let $u$ be an $\mathcal{E}^0$–q.c. $\mu$–version of some element in $(L, \mathcal{F}C_b^\infty \cap D(L^0))$. Take a function $g \in \mathcal{B}_b^+(E)$ with $\int g\,d\mu = 1$. By the Girsanov Theorem there exists a sequence of stopping times $(T_n)_{n\geq 1}$ such that $T_n \uparrow \infty$ and $M^{[\overline{u}]}_{T_n\wedge t} - \langle M^{[\overline{u}]}, \overline{Y}\rangle_{T_n\wedge t}$, $t \geq 0$, is an $L^2(Q_{g\mu})$–integrable martingale. Due to Lemma 2.16 we have that $E^Q_{g\mu}[(\int_0^t L^0u(X_s)\,ds)^2] \leq E^Q_{g\mu}[t\int_0^t (L^0u)^2(X_s)\,ds] \leq t^2\|g\|_\infty \|L^0u\|^2_{L^2(E,\mu)}$. Thus $(M_t^{[\overline{u}]})_{t\geq 0}$ is $L^2(Q_{g\mu})$–integrable. Note that $M_t^{[\overline{u}]} = u(X_t) - u(X_0)$ $- \int_0^t L^0u(X_s)\,ds$ $P_{g\mu}$-a.s. and thus $Q_{g\mu}$-a.s. Hence by Lemma 2.14 (ii)

$$\begin{aligned}\int (T_tu - u)g\,d\mu &= \int E_x^Q[u(X_t) - u(X_0)]g(x)\,\mu(dx)\\ &= E^Q_{g\mu}[u(X_t) - u(X_0)]\\ &= E^Q_{g\mu}[M_t^{[\overline{u}]} + \int_0^t L^0u(X_s)\,ds]\\ &= \lim_{n\to\infty} E^Q_{g\mu}[M^{[\overline{u}]}_{T_n\wedge t} + \int_0^{T_n\wedge t} L^0u(X_s)\,ds]\\ &= \lim_{n\to\infty} E^Q_{g\mu}[\int_0^{T_n\wedge t} (\langle B, \nabla u\rangle_H(X_s) + L^0u(X_s))\,ds]\,.\end{aligned}$$

Due to Lemma 2.16 again we have that

$$E^Q_{g\mu}[\int_0^t |\langle B, \nabla u\rangle_H|(X_s)\,ds] \le t\|g\|_\infty \|\langle B, \nabla u\rangle_H\|_{L^1(E,\mu)}$$

and thus by Lemma 2.17

$$\begin{aligned}\lim_{n\to\infty} E^Q_{g\mu}[&\int_0^{T_n\wedge t} (\langle B, \nabla u\rangle_H(X_s) + L^0 u(X_s))\,ds]\\ &= E^Q_{g\mu}[\int_0^t (\langle B, \nabla u\rangle_H(X_s) + L^0 u(X_s))\,ds]\\ &= \int\int_0^t T_s(\langle B, \nabla u\rangle_H + L^0 u)\,ds\, g\,d\mu\,.\end{aligned}$$

Thus $T_t u - u = \int_0^t T_s(\langle B, \nabla u\rangle_H + L^0 u)\,ds$ and consequently $u \in D(L)$ and $Lu = L^0 u + \langle B, \nabla u\rangle_H$ which implies the assertion. $\square$

Theorem 1.1 is now an easy consequence of (2.14), (2.16), Lemma 2.15, Lemma 2.16, Lemma 2.17, Lemma 2.18, and Proposition 2.20.

**Remark 2.21.** It is also possible to show that $(T_t)_{t\ge 0}$ is a strongly continuous contraction semigroup and that the corresponding generator $(L, D(L))$ is an extension of $(L, \mathcal{F}C_b^\infty \cap D(L^0))$ without using the transformation of a Markov process by means of a multiplicative functional. It can be shown directly that $\overline{E}_x[Z_t] = 1\,\overline{\mu}$–a.e. and hence $(Z_t)_{t\ge 0}$ is an $(\overline{\mathcal{F}}_t)_{t\ge 0}$–martingale w.r.t. $\overline{P}_{g\overline{\mu}}$, where $g \in \mathcal{B}^+(\overline{E})$ with $\int g\,d\mu = 1$. Note that we have not shown the existence of an associated standard process in this case.

**Proof. 1. Step:** $\overline{E}_{\overline{\mu}}[\overline{f}(\overline{X}_t)Z_t] \le \int f\,d\mu \quad \forall f \in \mathcal{B}^+(E)$.

Since $Z_t^{n_k} \to Z_t$ $\overline{P}_{\overline{\mu}}$-a.s. along some subsequence $(n_k)_{k\ge 1}$ (cf. the proof of Lemma 2.16) we have from Fatous Lemma, Remark 2.9 and Lemma 2.12

$$\begin{aligned}\overline{E}_{\overline{\mu}}[\overline{f}(\overline{X}_t)Z_t] &\le \liminf_{k\to\infty} \overline{E}_\mu[\overline{f}(\overline{X}_t)Z_t^{n_k}]\\ &= \liminf_{k\to\infty} \int \overline{E}_x[\overline{f}(\overline{X}_t)Z_t^{n_k}]\,\mu(dx)\\ &= \liminf_{k\to\infty} \int T_t^{n_k} f\,d\mu = \int T_t f\,d\mu \le \int f\,d\mu\end{aligned} \tag{2.19}$$

for all $f \in \mathcal{B}_b^+(E)$ and subsequently all $f \in \mathcal{B}^+(E)$.

**2. Step:** Let $T > 0$ and $\tau$ be an arbitrary $(\overline{\mathcal{F}}_t)$–stopping time with $\tau \le T$. Then $\overline{E}_{\overline{\mu}}[\langle \overline{Y}\rangle_\tau Z_\tau] \le T\|\,|B|_H\|^2_{L^2(E,\mu)}$.

**Proof.**

(i) Clearly

$$\begin{aligned}\overline{E}_{\overline{\mu}}[\langle \overline{Y}\rangle_\tau Z_\tau] &= \overline{E}_{\overline{\mu}}[\int_0^\tau \overline{|B|}_H^2(\overline{X}_s)\,ds Z_\tau] \\ &= \overline{E}_{\overline{\mu}}[\int_0^T \overline{|B|}_H^2(\overline{X}_s) Z_\tau \mathbb{1}_{\{s\le\tau\}}\,ds] \\ &= \int_0^T \overline{E}_{\overline{\mu}}[\overline{|B|}_H^2(\overline{X}_s) Z_\tau \mathbb{1}_{\{s\le\tau\}}]\,ds\,.\end{aligned}$$

(ii) By the optional sampling theorem and (2.19)

$$\begin{aligned}\overline{E}_{\overline{\mu}}[\overline{|B|}_H^2(\overline{X}_s) Z_\tau \mathbb{1}_{\{s\le\tau\}}] &\le \overline{E}_{\overline{\mu}}[\overline{|B|}_H^2(\overline{X}_s) Z_{\tau\wedge s} \mathbb{1}_{\{s\le\tau\}}] \\ &= \overline{E}_{\overline{\mu}}[\overline{|B|}_H^2(\overline{X}_s) Z_s \mathbb{1}_{\{s\le\tau\}}] \\ &\le \overline{E}_{\overline{\mu}}[\overline{|B|}_H^2(\overline{X}_s) Z_s] \\ &\le \|\,|B|_H\|_{L^2(E,\mu)}^2\,.\end{aligned} \tag{2.20}$$

Consequently $\overline{E}_{\overline{\mu}}[\langle \overline{Y}\rangle_\tau Z_\tau] \le \int_0^T \|\,|B|_H\|_{L^2(E,\mu)}^2\,ds$.

**3. Step:** Let $\tau_n := \inf\{t\ge 0 \mid \langle\overline{Y}\rangle_t \vee |\overline{Y}_t| \ge n\}$. Then $\tau_n$ is an $(\overline{\mathcal{F}}_t)$–stopping time and $(Z_{\tau_n\wedge t})_{t\ge 0}$ a continuous square–integrable martingale w.r.t. $\overline{P}_{\overline{\mu}}$. By Girsanovs Theorem the process $\overline{Y}_{\tau_n\wedge t} - \langle\overline{Y}\rangle_{\tau_n\wedge t}$, $t\ge 0$, is a continuous local martingale w.r.t. the measure $\overline{Q}_{\overline{\mu}}^n = Z_{\tau_n}\overline{P}_\mu$ on $(\overline{\Omega}, \bigvee_{t\ge 0}\overline{\mathcal{F}}_t)$.

Since $\overline{Y}_{\tau_n\wedge t}$ and $\langle\overline{Y}\rangle_{\tau_n\wedge t}$ are uniformly bounded, $\overline{Y}_{\tau_n\wedge t} - \langle\overline{Y}\rangle_{\tau_n\wedge t}$ is in fact a martingale, and therefore

$$\overline{E}_{\overline{\mu}}[\overline{Y}_{\tau_n\wedge t} Z_{\tau_n\wedge t}] = \overline{E}_{\overline{\mu}}[\langle\overline{Y}\rangle_{\tau_n\wedge t} Z_{\tau_n\wedge t}] \tag{2.21}$$

and thus by the 2. Step

$$\begin{aligned}\overline{E}_{\overline{\mu}}[Z_{\tau_n\wedge t}\log Z_{\tau_n\wedge t}] &= \overline{E}_{\overline{\mu}}[Z_{\tau_n\wedge t}(\overline{Y}_{\tau_n\wedge t} - (1/2)\langle\overline{Y}\rangle_{\tau_n\wedge t})] \\ &= (1/2)\overline{E}_{\overline{\mu}}[\langle\overline{Y}\rangle_{\tau_n\wedge t} Z_{\tau_n\wedge t}] \\ &\le (t/2)\|\,|B|_H\|_{L^2(E,\mu)}^2\,,\end{aligned} \tag{2.22}$$

which implies the uniform integrability of $(Z_{\tau_n\wedge t})_{n\ge 1}$. On the other hand note that $\tau_n \uparrow \infty$ $\overline{P}_{\overline{\mu}}$–a.s. since $\langle\overline{Y}\rangle_t < \infty$ $\overline{P}_{\overline{\mu}}$–a.s. and $\overline{P}_{\overline{\mu}}[\sup_{0\le s\le t}|\overline{Y}_s| > \lambda] \le (1/\lambda^2)\overline{E}_{\overline{\mu}}[\overline{Y}_t^2]$ by Doobs maximal inequality. Therefore $Z_{\tau_n\wedge t} \to Z_t$ in $L^1(\overline{P}_{\overline{\mu}})$ which implies $\overline{E}_{\overline{\mu}}[Z_t] = \lim_{n\to\infty}\overline{E}_{\overline{\mu}}[Z_{\tau_n\wedge t}] = 1$. Since $\overline{E}_x[Z_t] \le 1$ $\mu$-a.e. it follows that $\overline{E}_x[Z_t] = 1$ $\mu$-a.e.

**4. Step:** We can show exactly in the same way as in the proof of Lemma 2.17 that $\overline{E}_x[\overline{f}(\overline{X}_t)Z_t]$ is a $\mu$–version of $T_t f$ for every $f \in \mathcal{B}_b(E)$ and in a similar way as in the proof of Lemma 2.18 that $(T_t)_{t\ge 0}$ is a strongly continuous contraction semigroup on $L^2(E,\mu)$. Since we can also define the measure $Q_{g\mu}$ on $\overline{\mathcal{F}}_t$ via $Q_{g\mu}[A] := \int \overline{E}_x[\mathbb{1}_A Z_t] g(x)\,\mu(dx)$; $A \in \overline{\mathcal{F}}_t$, the proof of Proposition 2.20 can be done exactly in the same way as above. $\square$

**Remark 3.1.** First note that $D(L^0) \subset D(L')$ and $L'u = L^0u + \langle B, \nabla u\rangle_H; u \in D(L^0)$, since for $u \in D(L^0)$ there exists a sequence $(u_n)_{n\geq 1} \subset D$ such that $u_n \to u$ and $L^0u_n \to L^0u$ in $L^2(E,\mu)$ hence in $L^1(E,\mu)$. Clearly $\mathcal{E}^0(u_n-u, u_n-u) = (-L^0(u_n-u),(u_n-u))_{L^2(E,\mu)} \leq \|L^0(u_n-u)\|_{L^2(E,\mu)}\|u_n-u\|_{L^2(E,\mu)}$. Thus $\nabla u_n \to \nabla u$ in $L^2(E;H,\mu)$ which implies $\langle B, \nabla u_n\rangle_H \to \langle B,\nabla u\rangle_H$ in $L^1(E,\mu)$. Therefore $u$ lies in the minimal closed extension of $(L,D)$ that is obviously contained in the closed extension $(L', D(L'))$.

Now let $(B^n)_{n\geq 1}$ be a sequence of bounded vector fields as in Theorem 1.2 (i). Denote by $(T_t^n)_{t\geq 0}$ the strongly continuous semigroup corresponding to the generator $(L^n, D(L^0))$.

**Lemma 3.2.** $\lim\limits_{n\to\infty} T_t^n u = T_t u \ \forall u \in D(L^0)\cap L^\infty(E,\mu)$ *in* $L^1(E,\mu)$.

**Proof.** By the strong continuity of $(T_t)_{t\geq 0}$ there exist constants $C, M \geq 0$ such that $\|T_tu\|_{L^1(E,\mu)} \leq Ce^{Mt}\|u\|_{L^1(E,\mu)}$, $t\geq 0, u \in L^1(E,\mu)$. By Lemma 2.11 (1. Step) and Remark 2.9 it follows that $\sup\limits_{n\geq 1}\left(\int_0^t \mathcal{E}^0(T_s^nu, T_s^nu)\,ds\right)^{1/2} < +\infty$ (Note that we do not use E.3 in the 1. Step of the proof of Lemma 2.11). Since $T_t^n(L^\infty(E,\mu)) \subset D(L^0)$ (cf. Proposition 2.1) and $T_tu - T_t^nu = \int_0^t T_{t-s}(L^n - L')T_s^nu\,ds$ we obtain that

$$
\begin{aligned}
\|T_tu - T_t^nu\|_{L^1(E,\mu)} &\leq \int_0^t Ce^{M(t-s)}\|\langle B^n - B, \nabla T_s^nu\rangle_H\|_{L^1(E,\mu)}\,ds \\
&\leq Ce^{Mt}\int_0^t \||B^n-B|_H\|_{L^2(E,\mu)}\||\nabla T_s^nu|_H\|_{L^2(E,\mu)}\,ds \\
&\leq Ce^{Mt}\||B^n-B|_H\|_{L^2(E,\mu)}\sqrt{t}\left(2\int_0^t \mathcal{E}^0(T_s^nu,T_s^nu)\,ds\right)^{1/2} \\
&\leq Ce^{Mt}\sqrt{t}\sup_{\ell\geq 1}\left(2\int_0^t \mathcal{E}^0(T_s^\ell u, T_s^\ell u)\,ds\right)^{1/2}\||B^n-B|_H\|_{L^2(E,\mu)}\,.
\end{aligned}
$$

Clearly, the right hand side converges to zero for $n\to\infty$. $\square$

Thus Theorem 1.2 (i) is proved. (ii) is an easy consequence, since each $(T_t^n)_{t\geq 0}$ is sub–Markovian (cf. Remark 2.9) and $L^1$–convergence implies $\mu$–a.s. convergence along some subsequence. For the proof of (iii) suppose that $(L'', D(L''))$ is another extension of $(L,D)$ that generates a strongly continuous semigroup $(T_t'')_{t\geq 0}$ on $L^1(E,\mu)$. Since $D(L^0)\subset D(L'')$ and $L''u = L^0u + \langle B,\nabla u\rangle_H$ on $D(L^0)$ again, by the proof of Lemma 3.2 $T_t^nu \to T_t''u$ in $L^1(E,\mu)$ for all $u\in D(L^0)\cap L^\infty(E,\mu)$ also, hence $T_t''u = T_tu$, which implies $T_t'' = T_t$.

Thus $(L', D(L'))$ is the only (closed) extension of $(L,D)$ that generates a strongly continuous semigroup in $L^1(E,\mu)$. By [Na, Theorem A-II, 1.33, p. 46] $D$ is a core for $(L', D(L'))$ which proves Theorem 1.2 (iii).

For the proof of Theorem 1.2 (iv) we need the following

**Lemma 3.3.** *Suppose that $(U'_t)_{t\geq 0}$ is a sub–Markovian strongly continuous semigroup on $L^1(E,\mu)$ with corresponding generator $(A', D(A'))$. Then $(U'_t)_{t\geq 0}$ can be restricted to a strongly continuous semigroup $(U_t)_{t\geq 0}$ on $L^2(E,\mu)$. The corresponding generator $(A, D(A))$ is just the part of $(A', D(A'))$ on $L^2(E,\mu)$.*

**Proof.** By the strong continuity there exist constants $C, M \geq 0$ with $\|U'_t f\|_{L^1(E,\mu)} \leq Ce^{Mt}\|f\|_{L^1(E,\mu)}$; $\forall f \in L^1(E,\mu), t \geq 0$. Since $(U'_t)_{t\geq 0}$ is sub–Markovian $\|U'_t f\|_{L^\infty(E,\mu)} \leq \|f\|_{L^\infty(E,\mu)}$; $\forall f \in L^\infty(E,\mu), t \geq 0$. Hence $U'_t f \in L^2(E,\mu)$; $\forall f \in L^\infty(E,\mu), t \geq 0$, and

$$\|U'_t f\|_{L^2(E,\mu)} \leq C^{1/2}e^{(M/2)t}\|f\|_{L^2(E,\mu)} \ , \ \forall f \in L^\infty(E,\mu),\ t \geq 0$$

by the Riesz–Thorin interpolation Theorem (cf. [ReSi, Theorem IX.17, p. 27]), and $(U'_t)_{t\geq 0}$ can be restricted to a semigroup of bounded linear operators $(U_t)_{t\geq 0}$ on $L^2(E,\mu)$.

We want to show that $(U_t)_{t\geq 0}$ is weakly continuous on $L^2(E,\mu)$. To this end choose $f \in L^2(E,\mu)$ and an arbitrary sequence $(t_n)_{n\geq 1} \subset [0,\infty)$ such that $t_n \to 0$. Clearly $\sup\limits_{n\geq 1}\|U_{t_n} f\|_{L^2(E,\mu)} < +\infty$ and thus by the Banach–Alaoglu Theorem (cf. [MR, Theorem A.2.1, p. 184]) there exists some $v \in L^2(E,\mu)$ such that $U_{t_{n_k}} f \to v$ weakly in $L^2(E,\mu)$ along some subsequence $(t_{n_k})_{k\geq 1}$. Since $U_{t_{n_k}} f \to f$ in $L^1(E,\mu)$ we have that $f = v$. Hence $U_{t_n} f \to f$ weakly in $L^2(E,\mu)$ since this reasoning holds for every subsequence. Due to [Y, p. 233, Theorem] $(U_t)_{t\geq 0}$ is strongly continuous. Denote the corresponding generator by $(A, D(A))$.

Clearly $D(A) \subset \{u \in L^2(E,\mu) \cap D(A') \mid A'u \in L^2(E,\mu)\}$ and $Au = A'u$ since $L^2$–convergence implies $L^1$–convergence. On the other hand, if $u \in L^2(E,\mu) \cap D(A')$ such that $A'u \in L^2(E,\mu)$, clearly

$$\frac{1}{t}(U_t u - u) = \frac{1}{t}(U'_t u - u) = \frac{1}{t}\int_0^t U'_s A'u\, ds = \frac{1}{t}\int_0^t U_s A'u\, ds \ . \tag{3.1}$$

The right hand side of (3.1) converges to $A'u$ in $L^2(E,\mu)$ as $t \to \infty$ which proves $u \in D(A)$ and $Au = A'u$. $\square$

Applying the last lemma to the semigroup $(T_t)_{t\geq 0}$ gives Theorem 1.2 (iv).

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