Regularity of invariant measures: the case of non-constant diffusion part

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Abstract

We prove regularity (i.e., smoothness) of measures μ on R^d satisfying the equation $L^*\mu = 0$ where L is an operator of type $Lu = tr(Au'') + B \cdot \nabla u$. Here A is a Lipschitz continuous, uniformly elliptic matrix-valued map and B is merely μ-square integrable. We also treat a class of corresponding infinite dimensional cases where $\mathbb{R}^d$ is replaced by a locally convex topological vector space X. In this cases μ is absolutely continuous w.r.t. a Gaussian measure on X and the square root of the Radon-Nikodym density belongs to the Malliavin test function space $\mathbb{D}^{2,1}$.

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1 Introduction and the main result in finite dimensions

The purpose of this paper is to prove regularity (i.e., smoothness) of measures μ satisfying certain elliptic equations $L^*\mu = 0$ in finite and infinite dimensions where L is a second order operator with non-constant diffusion part and possibly very singular drift. Let us first state our main result in finite dimensions.

Consider the operator

$$\begin{aligned} Lu &= tr(Au'') + (\nabla u, B)\ , \ \ u \in C_b^\infty(\mathbb{R}^d) \\ &= a^{ij}\partial_i\partial_j u + B^i \partial_i u\ . \end{aligned} \tag{1.1}$$

Here we use the summation convention, ∂_i means derivative w.r.t. the i-th coordinate, $(\ ,\ )$ denotes the Euclidean inner product on R^d, $B := (B^i) :\ \mathbb{R}^d \to \mathbb{R}^d$ is Borel-measurable, and $A := (a^{ij}) :\ \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^d)$ is a matrix-valued map which is bounded, uniformly Lipschitz continuous and uniformly strictly elliptic with constant α. Without loss of generality we may assume that $A(x)$ is symmetric for all $x \in \mathbb{R}^d$. Let μ be a probability measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ on $\mathbb{R}^d$. We write

$$L^*\mu = 0 \tag{1.2}$$

if

$$|B| \in L^2(\mu)\ \ (:= L^2(\mathbb{R}^d;\mu))\ . \tag{1.3}$$

and

$$\int Lu\, d\mu = 0 \quad \forall u \in C_b^\infty(\mathbb{R}^d) \tag{1.4}$$

Let dx denote Lebesgue measure on $\mathbb{R}^d$.

Theorem 1.1 *Let μ be a pobability measure on $\mathcal{B}(\mathbb{R}^d)$ such that $|B| \in L^2(\mu)$ and $L^*\mu = 0$. Then*

(i) $\mu = p\,dx$ where $p = \varphi^2$ with $\varphi \in H^{1,2}(\mathbb{R}^d; dx)$ (i.e., the Sobolev space of order 1 in $L^2(\mathrm{R}^d; dx)$). In particular, $p \in L^{d/(d-1)}(\mathbb{R}^d; dx)$.

(ii) The following holds for p, φ :

$$\frac{1}{4}\int \left|\frac{\nabla p}{p}\right|^2 p\,dx = \int |\nabla\varphi|^2 dx \le \frac{1}{4\alpha^2}\int |B + D|^2 d\mu\ , \tag{1.5}$$

where $D := (D^j) := (\partial_i a^{ij})$, $|\ \ |$ denotes Euclidean norm (and the derivatives are meant in the sense of distributions).

(iii) $\frac{\nabla p}{p}$ coincides $\mu - a.e.$ with the orthogonal projection of the vector field $A^{-1}(B - D)$ onto the closure of $\{\nabla u|\ u \in C_0^\infty(\mathbb{R}^d)\}$ in $L^2(\mu, \mathbb{R}^d)$ $(:= L^2(\mathbb{R}^d \to \mathbb{R}^d; \mu))$ equipped with the inner product $(\ ,\ )_2$ defined by

$$(F, G)_2 := \int (AF, G)\, d\mu\ .$$

The proof of Theorem 1.1 will be given in Section 2 below. In Section 3 we state and prove a generalization of this to infinite dimensions, i.e., replace $\mathbb{R}^d$ by a locally convex topological vector space X (cf. Theorem 3.1). More precisely, we assume that there exists a Gaussian Radon measure γ on X with reproducing kernel Hilbert space $H \subset X$ (densely and continuously). Let

$$Lu(x) := \text{trace}_H(A(x) \circ j_H \circ u''(x)) - {}_{X^*}\langle u'(x), x\rangle_X + (j_H(u'(x)), v(x))_H \ , x \in X,$$

where u is a bounded smooth cylinder function, j_H denotes the embedding $X^* \subset H^* \equiv H$, $x \mapsto v(x)$ is a Borel-measurable vector field on X with values in H and $x \mapsto A(x)$ a measurable map from X to the bounded linear operators on H. Under some conditions on A (cf. (A1), A(2) in Section 3) we prove that if μ is a Radon probability measure on $\mathcal{B}(X)$ such that $|v|_H \in L^2(\mu)$, $X^* \subset L^2(\mu)$ and

$$\int Lu \, d\mu = 0 \text{ for all bounded smooth cylinder functions,}$$

then $\mu = \varphi^2 \cdot \gamma$ with φ in the Malliavin test function space of order 1 in $L^2(\gamma)$.

We would like to point out that both Theorem 1.1 and Theorem 3.1 are extensions of results in [BR94b] (cf. Theorems 3.1 resp. 3.5 therein) where the case where A is constant has been treated. In order to keep this paper self-contained we do not refer to techniques or methods developed in [BR94b], but give a complete proof here rather than merely explaining the necessary modifications of the arguments which are quite substantial in the infinite dimensional case.

Finally, we mention that a part of the results in this paper have been announced in [BR94a]. Essentially, all results have been presented in talks at various conferences at Edinburgh, Holzhau, Marseille, Passau, and the International Congress of Mathematicians in Zürich during the spring and summer of 1994.

2 Proof of Theorem 1.1

For the proof of Theorem 1.1 we need the following simple

Lemma 2.1 *Let μ and ν be two probability measures on a measurable commutative group $(G, +)$ and let $f \in L^2(\mu)$. Let g denote the Radon-Nikodym density of $(f\mu) * \nu$ w.r.t. $\mu * \nu$ where*

$$\mu * \nu\,(A) = \int_G \mu(A - x)\,\nu(dx) \ .$$

Then

$$\int_G g^2 \, d(\mu * \nu) \leq \int_G f^2 \, d\mu \ .$$

Proof Let $\Lambda : \ G \times G \to G$ be defined by

$$\Lambda(x, y) = x + y \ .$$

Then, defining $\bar{f} : G \times G \longrightarrow \mathbb{R}$ by $\bar{f}(x, y) = f(x)$, $(x, y) \in G \times G$, we obtain that

$$g \circ \Lambda = E_{\mu\otimes\nu}[\bar{f} \mid \sigma(\Lambda)] ,$$

i.e., the conditional expectation of f w.r.t. the product of μ and ν, given the σ-algebra generated by Λ. Hence Jensen's inequality implies the assertion. $\square$

Proof of Theorem 1.1: Let g be the standard Gaussian density on $\mathbb{R}^d$. Set

$$f_\epsilon(x) := g_\epsilon * \mu(x) = \int g_\epsilon(x-y)\,\mu(dy) ,$$

where $g_\epsilon = \epsilon^{-n} g(x/\epsilon)$. Then each f_ϵ is strictly positive, infinitely differentiable, and tends to zero at infinity together with all its derivatives by Lebesgue's dominated convergence theorem. Furthermore, $\int f_\epsilon\,dx = 1$ for every $\epsilon > 0$ and $f_\epsilon\,dx \underset{\epsilon\to 0}{\longrightarrow} \mu$ vaguely. Assertion (i) is proved by the following claim: there exists a subsequence $(\epsilon_n)_{n\in N}$ with $\epsilon_n \underset{n\to\infty}{\longrightarrow} 0$ and $p \in L^1(\mathrm{R}^d; dx)$ such that $f_{\epsilon_n} \underset{n\to\infty}{\longrightarrow} p$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d; dx)$ and $\varphi := \sqrt{p} \in H^{1,2}(\mathbb{R}^d; dx)$. This claim, however, follows from the estimate

$$\int \left|\nabla\sqrt{f_\epsilon}\right|^2 dx \le const. \tag{2.1}$$

Indeed, if (2.1) holds, then for $\varphi_n := \sqrt{f_{\frac{1}{n}}}$

$$\int \left(\varphi_n^2 + |\nabla\varphi_n|^2\right) dx \le const.$$

Since $H^{1,2}(D; dx) \subset L^2(D; dx)$ compactly for every open ball $D \subset \mathbb{R}^d$, there exists $\varphi \in H^{1,2}(\mathbb{R}^d; dx)$ and a subsequence $(n_k)_{k\in N}$ such that $\varphi_{n_k} \underset{k\to\infty}{\longrightarrow} \varphi$ weakly in $H^{1,2}(\mathbb{R}^d; dx)$ and strongly in $L^2(D; dx)$. Hence take $p := \varphi^2$. So it remains to prove (2.1). For any $u \in C_0^\infty(\mathbb{R}^d)$, integrating by parts and using Fubini's theorem we obtain that

$$\begin{aligned}
& -\int \partial_i u(\partial_j[(a^{ij}\mu) * g_\epsilon] - \partial_i[(B^i\mu) * g_\epsilon])\,dx \\
= & \int \partial_i\partial_j u[(a^{ij}\mu) * g_\epsilon]\,dx + \int \partial_i u[(B^i\mu) * g_\epsilon]\,dx \\
= & \iint \partial_i\partial_j u(x)\, g_\epsilon(x-y)\, a^{ij}(y)\,\mu(dy)\,dx + \iint \partial_i u(x)\, g_\epsilon(x-y)\, B^i(y)\mu(dy)\,dx \\
= & \int a^{ij}\,\partial_i\partial_j(u * g_\epsilon)\,d\mu + \int B^i\,\partial_i(u * g_\epsilon)\,d\mu \\
= & \int L(u * g_\epsilon)\,d\mu = 0
\end{aligned} \tag{2.2}$$

since $u * g_\epsilon \in C_b^\infty(\mathbb{R}^d)$. As is well-known for any $h \in L^1_{\mathrm{loc}}(\mathbb{R}^d; dx)$ with $|\nabla h| \in L^2(\mathbb{R}^d; dx)$ there exist $u_n \in C_0^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$, such that $|\nabla h - \nabla u_n| \underset{n\to\infty}{\longrightarrow} 0$ in $L^2(\mathbb{R}^d; dx)$. Since for $c \in]0,\infty[$, $\log(f_\epsilon + c)$ is bounded and $\nabla\log(f_\epsilon + c) = \nabla f_\epsilon/(f_\epsilon + c) \in L^2(\mathbb{R}^d; dx)$, (2.2) implies that

$$\int \frac{\partial_i f_\epsilon}{f_\epsilon + c}\partial_j[(a^{ij}\mu) * g_\epsilon]\,dx = \int \frac{\partial_i f_\epsilon}{f_\epsilon + c}[(B^i\mu) * g_\epsilon]\,dx . \tag{2.3}$$

(So far we only used that each a^{ij} is bounded. Lipschitz continuity was not required.)

Note that for any bounded function a, which is uniformly Lipschitz continuous with Lipschitz norm λ, we have that for $1 \le j \le d$

$$\partial_j[(a\mu) * g_\epsilon](x) = a(x)\, \partial_j\, f_\epsilon(x) + \int \partial_j g_\epsilon(x-y)[a(y)-a(x)]\, \mu(dy) \tag{2.4}$$

and

$$\begin{aligned}\left|\int \partial_j g_\epsilon(x-y)[a(y)-a(x)]\, \mu(dy)\right| &\le \lambda \int |\partial_j g_\epsilon(x-y)|\, |y-x|\, \mu(dy) \\ &\le \lambda \int \epsilon^{-d} \frac{|x-y|^2}{\epsilon^2}\, g\left(\frac{x-y}{\epsilon}\right)\, \mu(dy) \\ &= \lambda(\mu * q_\epsilon)(x)\end{aligned} \tag{2.5}$$

where $q_\epsilon(x) := \epsilon^{-d} g(x/\epsilon)\, |x/\epsilon|^2$, $x \in \mathbb{R}^d$. Using (2.4) we can rewrite (2.3) as

$$\begin{aligned}&\int \frac{a^{ij}\, \partial_i f_\epsilon\, \partial_j f_\epsilon}{f_\epsilon + c} dx \\ = \;& -\int \frac{\partial_i f_\epsilon}{f_\epsilon + c}(x) \int \partial_j g_\epsilon(x-y)[a^{ij}(y) - a^{ij}(x)]\mu(dy)dx + \int \frac{\partial_i f_\epsilon}{f_\epsilon + c}(x)[(B^i\mu) * g_\epsilon]\, dx\end{aligned} \tag{2.6}$$

whose right hand side by (2.5) and the Cauchy-Schwarz inequality is dominated by

$$d^{3/2}\lambda \left(\int \frac{|\nabla f_\epsilon|^2}{f_\epsilon + c} dx\right)^{1/2} \left(\int \frac{(\mu * q_\epsilon)^2}{f_\epsilon + c} dx\right)^{1/2} + \left(\int \frac{|\nabla f_\epsilon|^2}{f_\epsilon + c} dx\right)^{1/2} \left(\int \frac{\sum_{i=1}^d [(B^i\mu) * g_\epsilon]^2}{f_\epsilon + c} dx\right)^{1/2} . \tag{2.7}$$

By virtue of Lemma 2.1 (or a direct calculation using the Cauchy-Schwarz inequality) we have that

$$\int \frac{[(B^i\mu) * g_\epsilon]^2}{f_\epsilon} dx \le \int (B^i)^2 d\mu \ , \quad 1 \le i \le d \ ,$$

and

$$\int \frac{(\mu * q_\epsilon)^2}{f_\epsilon} dx \le \int |x|^4\, g(x)\, dx \ . \tag{2.8}$$

Using that $a^{ij} x_i x_j \ge \alpha |x|^2$ for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we hence see from (2.7) and (2.8) that

$$\int \frac{|\nabla f_\epsilon|^2}{f_\epsilon + c} dx \le \alpha^{-2} \left[\left(\int |B|^2 dx\right)^{1/2} + d^{3/2}\lambda \left(\int |x|^4 g(x)\, dx\right)^{1/2}\right]^2 \ .$$

Since $|\nabla\sqrt{f_\epsilon}|^2 = \frac{1}{4}|\nabla f_\epsilon|^2\, f_\epsilon^{-1}$, by Fatou's lemma this implies (2.1). The second part of assertion (i) follows by the Gagliardo-Nirenberg embedding theorem (cf. e.g. [S70]).

(ii): We claim that there exist $f_n \in C_0^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$, such that

$$\int \left|\frac{\nabla p}{p} - \nabla f_n\right|^2 d\mu \underset{n\to\infty}{\longrightarrow} 0 \ . \tag{2.9}$$

Indeed, for $n \in \mathbb{N}$ let $g_n(t) = t$, if $t \in [-n, n]$, and $g_n(t) = n \cdot \mathrm{sgn}(t)$, if $|t| > n$. Then each g_n is Lipschitzian on $[0, \infty)$, and hence the functions $\phi_n = g_n(\log(\varphi + \frac{1}{n})^2)$ belong to the domain $D(\mathcal{E}^+_\varphi)$ of the maximal Dirichlet form $\mathcal{E}^+_\varphi(f,f) = \int (\nabla f, \nabla f)\varphi^2\,dx$ (cf. [ARZ93, Sections 2,3]). In addition,

$$\nabla\phi_n = g_n'\left(\log\left(\varphi+\frac{1}{n}\right)^2\right)2\nabla\varphi\Big/\left(\varphi+\frac{1}{n}\right) = I_{\{e^{-n/2}-\frac{1}{n}<\varphi<e^{+n/2}-\frac{1}{n}\}}2\nabla\varphi\Big/\left(\varphi+\frac{1}{n}\right)\ .$$

So, $(\nabla\phi_n)_{n\in\mathbb{N}}$ converges to $2\nabla\varphi/\varphi$ in $L^2(\mu,\mathbb{R}^d)$. It remains to note that by virtue of [RZ94, Proof of Theorem 3.1] one can approximate ϕ_n in the Dirchlet norm by smooth compactly supported functions. This gives L^2-approximations of $2\nabla\varphi/\varphi = \nabla p/p$ by the gradients of such functions.

Let $n \in \mathbb{N}$. Since

$$\int Lf_n\, p\,dx = 0$$

and $p \in H^{1,1}(\mathbb{R}^d; dx)$, we can integrate by parts to obtain that

$$\int a^{ij}\,\partial_i f_n\,\frac{\partial_j p}{p}\,p\,dx = -\int \partial_j a^{ij}\partial_i f_n\,p\,dx + \int B^i\partial_i f_n\,p\,dx\ ,$$

where we used that $\partial_j p = 0$ dx-a.e. on $\{p=0\}$. By (2.9) we can take $n \longrightarrow \infty$ which yields

$$\int a^{ij}\frac{\partial_i p}{p}\frac{\partial_j p}{p}\,p\,dx = -\int \partial_j a^{ij}\frac{\partial_i p}{p}\,p\,dx + \int B^i\frac{\partial_i p}{p}\,p\,dx\ .$$

Using the uniform ellipticity of A and the Cauchy-Schwarz inequality we get

$$\int\left|\frac{\nabla p}{p}\right|^2 d\mu \le \alpha^{-1}\left(\int\left|\frac{\nabla p}{p}\right|^2 d\mu\right)^{1/2}\left(\int|B+D|^2 d\mu\right)^{1/2}$$

which implies assertion (ii).
(iii): By the uniform ellipticity resp. boundness of A and by (2.9) we already know that $\frac{\nabla p}{p}$ is in the closure of $L^2(\mu,\mathbb{R}^d)$ w.r.t. $(\ ,\ )_2^{1/2}$. Hence it suffices to check that for each $u \in C_b^\infty(\mathbb{R}^d)$

$$\int\left(A\nabla u, A^{-1}(B-D)-\frac{\nabla p}{p}\right)d\mu = 0\ .$$

But integrating the left hand side by parts we see that the left hand side is equal to $\int Lu\,d\mu$ which is equal to zero by assumption. □

As an immediate consequence we obtain

Corollary 2.2 *Suppose μ is an invariant measure for the diffusion process $(\xi_t)_{t\ge0}$ satisfying the equation*

$$d\xi_t = A^{1/2}(\xi_t)\,d\omega_t + B(\xi_t)dt\ .$$

Then Theorem 1.1 applies.

Remark 2.3 (i) If one writes L in divergence form, the assertions of Theorem 1.1 can be formulated in a bit simpler and more geometric way. For example there would be no D in (1.5). However, unfortunately this does not simplify the proof. As is well known, in Corollary 2.2 we would then have to consider the corresponding Stratonovich stochastic equation.
(ii) We note that the statement $\varphi \in H^{1,2}(\mathbb{R}^d; dx)$ in Theorem 1.1 (i) is equivalent to : $p \in H^{1,1}(\mathbb{R}^d; dx)$ (i.e., the Sobolev space of order 1 in $L^1(\mathbb{R}^d; dx)$) *and* $|\nabla p|^2 p^{-1} \in L^1(\mathbb{R}^d; dx)$.

3 Infinite dimensional cases

Now we shall discuss analogous questions in an infinite dimensional setting. One of the motivations for this is the study of invariant measures for infinite dimensional diffusion processes. To be more specific let H be a Hilbert space continuously and densely embedded into a locally convex topological vector space X. Let $\mathcal{B}(X)$ denote its Borel σ-algebra and X^* its dual space. We assume that there exists a Radon centred Gaussian measure γ on $\mathcal{B}(X)$ with *reproducing kernel Hilbert space* (=*Cameron-Martin space*) $(H, (\ ,\ )_H)$. Note that by [Ts77] H is automatically separable in this case. One may view this as a generalization of the situation when X is itself a separable Hilbert space and the embedding $H \subset X$ is Hilbert-Schmidt. In this case there exists a nonnegative self-adjoint Hilbert-Schmidt operator T on X such that $H = T(X)$ and $(\cdot,\cdot)_H = (T^{-1}\cdot, T^{-1}\cdot)_X$. Since the restriction of $l \in X^*$ to H is continuous, by Riesz's representation theorem there is a unique element in H, denoted by $j_H(l)$, such that

$$\langle l, h\rangle = (j_H(l), h)_H \quad \text{for all } h \in H,$$

where $\langle\ ,\ \rangle : X^* \times X$ is the natural dualisation of X^* and X. Thus we get the triple

$$j_H(X^*) \subset H \subset X. \tag{3.1}$$

The reader should note that we do not identify H with its dual H^*. Define

$$\mathcal{F}C_b^\infty := \{f(l_1, \dots, l_m)\ |m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), l_1, \dots, l_m \in X^*\}.$$

For $u \in \mathcal{F}C_b^\infty$ let u', u'' denote its first resp. second Frechet derivative. Since H should be considered as the tangent space of X at each $x \in X$, it is natural and very common to set

$$\nabla_H u(x) := j_H(u'(x)),\ x \in X. \tag{3.2}$$

Let $\mathcal{L}(H)$ denote the set of all continuous linear operators on H equipped with the usual operator norm $\|\ \|_{\mathcal{L}(H)}$. Let $A : X \to \mathcal{L}(H)$ be weakly measurable (i.e., $x \mapsto (A(x)h_1, h_2)_H$ is measurable for all $h_1, h_2 \in H$). Let $B : X \to X$ be Borel-measurable. Define the linear operator L with domain $\mathcal{F}C_b^\infty$ by

$$Lu(x) := \operatorname{trace}_H(A(x) \circ j_H \circ u''(x)) + \langle u'(x), B(x)\rangle,\ x \in X, u \in \mathcal{F}C_b^\infty. \tag{3.3}$$

Note that if $u = f(l_1, \dots, l_m), m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m),\ l_1, \dots, l_m \in X^*$, then

$$u'(x) = \sum_{j=1}^m \partial_i f(l_1(x), \dots, l_m(x))\, l_j\ ,\ x \in X,$$

where ∂_j denotes the derivative w.r.t. the j-th coordinate. Hence for $x \in X$

$$\langle u'(x), B(x)\rangle = \sum_{j=1}^{m} \partial_i f(l_1(x), \ldots, l_m(x))\langle l_j, B(x)\rangle,\ x \in X, \tag{3.4}$$

and

$$A(x) \circ j_H \circ u''(x) = \sum_{i,j=1}^{m} \partial_i\partial_j f(l_1(x), \ldots, l_m(x))\ A(x)(j_H(l_j))\, l_i \tag{3.5}$$

and therefore,

$$\mathrm{trace}_H(A(x) \circ j_H \circ u''(x)) = \sum_{i,j=1}^{m} \partial_i\partial_j f(l_1(x), \ldots, l_m(x))\, (A(x)(j_H(l_j)), j_H(l_i))_H. \tag{3.6}$$

¿From (3.6) we see that L is well-defined on $\mathcal{F}C_b^\infty$ (without assuming anything more restrictive on the embedding $H \subset X$) and that as in the finite dimensional case we may assume that $A(x)$ is self-adjoint on H. For our main result (i.e. Theorem 3.1) below we need, however, more specific properties of A and also B. So, from now on we assume that the following conditions hold:

(A1) $A(x)$ is self-adjoint and there exists $\alpha \in]0, \infty[$ such that $\alpha I_H \le A(x) \le \alpha^{-1} I_H$ (in the sense of quadratic forms with I_H the identity operator on H) for all $x \in X$.

A(2) There exists an injective self-adjoint trace class operator $T \in \mathcal{L}(H)$, with $\|T\|_{\mathcal{L}(H)} \le 1$ and a weakly measurable map $S : X \to \mathcal{L}(H)$ with $\sup_{x\in X} \|S(x)\|_{\mathcal{L}(H)} =: q < 1$ and $\sup_{x\in X, h\in H\setminus\{0\}} |h|_H^{-1} \|S(x+h) - S(x)\|_{\mathcal{L}(H)} =: C < \infty$ such that $A(x) = I_H + S(x)T$, $x \in X$.

(B) There exists $v : X \longrightarrow H$ Borel-measurable such that $B = -I_X + v$.

Let μ be a finite positive measure on $\mathcal{B}(X)$. We write $L^*\mu = 0$ if the following three conditions are satisfied:

$$X^* \subset L^2(\mu), \tag{3.7}$$
$$|v|_H \in L^2(\mu), \tag{3.8}$$
$$\int Lu\, d\mu = 0 \quad \text{for all } u \in \mathcal{F}C_b^\infty. \tag{3.9}$$

Here and henceforth $L^p(\mu) := L^p(X \to \mathbb{R}; \mu)$ for $p \in [1, \infty]$. By (3.6), (A1) and (3.7), (3.8) it follows that $Lu \in L^2(\mu)$ for all $u \in \mathcal{F}C_b^\infty$, so (3.9) is meaningful.

Let $H^{1,2}(X; \gamma)$ with norm $\|\ \|_{H^{1,2}(X;\gamma)}$ be the usual Sobolev space over (X, H, γ) of order 1 in $L^2(\gamma)$ (cf. e.g. [RZ92]). We recall that a signed measure ν on $\sigma(X^*)$ ($\subset \mathcal{B}(X)$) of finite total variation is called *differentiable along* $h \in X$ (in the sense of Fomin) and has the (partial) *logarithmic derivative* β_h^ν along h if β_h^ν is a ν-integrable function such that the following integration by parts formula holds

$$\int \frac{\partial u}{\partial h}(x)\ \nu(dx) = -\int u(x)\beta_h^\nu(x)\ \nu(dx) \quad \text{for all } u \in \mathcal{F}C_b^\infty,$$

where $\frac{\partial u}{\partial h} := \frac{d}{ds} u(x+sh)|_{s=0}$. It is known (see [BS90] and also the references therein) that ν is differentiable along $h \in X$ if and only if for all $F \in \mathcal{B}(X)$

$$\lim_{t \to 0} \frac{\mu(F+th) - \mu(F)}{t} \tag{3.10}$$

exists. This limit is the signed measure $\beta_h^\nu \cdot \nu$. Let ν be differentiable along all $h \in j_H(X^*)$ with (partial) logarithmic derivatives $\beta_{j_H(l)}^\nu$, $l \in X^*$. If there exists $\beta_H^\nu : X \longrightarrow X$ ν-measurable such that for all $l \in X^*$

$$\langle l, \beta_H^\nu(x) \rangle = \beta_{j_H(l)}^\nu(x) \quad \text{for } \nu\text{-a.e. } x \in X \tag{3.11}$$

then β_H^ν is called the *vector logarithmic derivative of* ν (*along* H).

It is trivial to verify that if ν is differentiable along $h \in X$ and σ a signed measure on $\sigma(X^*)$ of finite total variation then $\sigma * \nu$ is differentiable along h and

$$\beta_h^{\sigma*\nu} \ \sigma * \nu = \sigma * (\beta_h^\nu \nu). \tag{3.12}$$

Now we can formulate the main result of this section.

Theorem 3.1 *Assume that (A1), (A2) and (B) hold. Let μ be a Radon probability measure on $\mathcal{B}(X)$ such that $L^*\mu = 0$ (i.e., (3.7) – (3.9) hold). Then $\mu = \varphi^2\gamma$ with $\varphi \in H^{1,2}(X;\gamma)$. In particular, μ is differentiable along each $h \in H$ and has a vector logarithmic derivative β_H^μ given by*

$$\beta_H^\mu = -x + 2\varphi^{-1}\nabla_H \varphi.$$

Remark 3.2 In this paper we do not discuss the question whether a solution to $L^*\mu = 0$ exists. We only note that if $\sup_{x\in X} |v(x)|_H < \infty$ then such μ exists. This result is proved in [BRZ95] and generalizes the well-known result of I. Shigekawa in [Sh87] where $A(x) = I_H$ for all $x \in X$ was assumed.

For the proof of Theorem 3.1 we need some preparations.

Lemma 3.3 *Let $\mathcal{F}C^\infty$ be defined as $\mathcal{F}C_b^\infty$ but with $C^\infty(\mathbb{R}^m)$ replacing $C_b^\infty(\mathbb{R}^m), m \in \mathbb{N}$. Define Lu for all $u \in \mathcal{F}C^\infty$ by (3.3). Then for all $l \in X^*$*

$$L(l^2) \in L^1(\mu), \ \int L(l^2) d\mu = 0$$

and

$$\|l\|_{L^2(\mu)} \le \left(\frac{c_2}{2} + \sqrt{c_1 + \frac{c_2^2}{4}} \right) |j_H(l)|_H$$

with $c_1 := \sup_{x\in X} \|A(x)\|_{\mathcal{L}(H)}$, $c_2 := \| |v|_H \|_{L^2(\mu)}$.

Proof. Let $l \in X^*$. For $n \in \mathbb{N}$ let $g_n \in C_b^\infty(\mathbb{R})$ such that $g_n(t) = t$ on $[-n, n]$ and $\sup\{|g_n'(t)| + |g_n''(t)| \mid n \in \mathbb{N}, t \in \mathbb{R}\} < \infty$. Then

$$\begin{aligned} L(g_n(l^2)) &= \left(4g_n''(l^2)l^2 + 2g_n'(l^2)\right)(A(x)j_H(l), j_H(l))_H \\ &\quad -2g_n'(l^2)l^2 + 2g_n'(l^2)\,(j_H(l), v)_H\, l \\ &\underset{n\to\infty}{\longrightarrow} 2(A(x)j_H(l), j_h(l))_H - 2l^2 + 2(j_H(l), v)_H\, l \\ &= L(l^2) \end{aligned}$$

where the convergence takes place in $L^1(\mu)$ by Lebesgue's dominated convergence theorem which applies by (A1), (3.7) and (3.8). Since $g_n(l^2) \in \mathcal{F}C_b^\infty$, the first two assertions follow. To prove the last we note that by the above we know that

$$0 = \frac{1}{2}\int L(l^2)d\mu = \int (A(x)j_H(l), j_H(l))_H d\mu + \int (j_H(l), v)_H\, l\, d\mu - \int l^2 d\mu.$$

Hence the last assertion follows by the Cauchy-Schwarz inequality and an elementary calculation. □

Remark 3.4 (i) An extensive discussion of logarithmic derivatives of measures , including different definitions and existence can be found in [BR94b]. We would only like to mention here that vector logarithmic derivatives are always the drift of a symmetric diffusion on X (see [AR91] and also [BR94b, Theorem 4.1]).
(ii) We emphasize that the following proof of Theorem 3.1 is different from the proof of the corresponding Theorem 3.5 in [BR94b], where $A(x) = I_H$ for all $x \in X$ was assumed. The reason of the necessary modifications is that unlike in [BR94b] (cf. Proposition 3.3 there) we cannot prove that finite dimensional projections of μ are absolutely continuous w.r.t. Lebesgue measure since taking conditional expectations does not preserve the Lipschitz continuity of A and hence we cannot apply the finite dimensional Theorem 1.1 as in [BR94b]. On the other hand the proof below is a direct analogue of the finite dimensional one.
(iii) One additional technical difficulty we have to deal with in the following proof is that T might not have an eigenbasis which belongs to $j_H(X^*)$ (which is in general only contained, but not equal to H).

Proof of Theorem 3.1 Let $\{e_n | n \in \mathbb{N}\}$ be an eigenbasis of T in H. Fix $n \in \mathbb{N}$. By Lemma 3.3 we can find $l_j^{(n)} \in X^*$, $j \in \mathbb{N}$, such that $j_H(l_j^{(n)}) \underset{j\to\infty}{\longrightarrow} e_n$ in H and $(l_j^{(n)})_{j\in\mathbb{N}}$ converges both in $L^2(\mu)$ and $L^2(\gamma)$ as well as both μ-a.e. and γ-a.e. Define

$$X_{e_n}(x) := \liminf_{j\to\infty} l_j^{(n)}(x)\ , \quad x \in X.$$

Note that $X_{e_n}(x) = (e_n, x)_H$ if $x \in H$ and that

$$X_{e_n}(x + y) = (e_n, x)_H + X_{e_n}(y) \quad \text{if } x \in H, y \in X. \tag{3.13}$$

Furthermore, $X_{e_n} \in L^2(\mu)$. For $n \in \mathbb{N}$ let $H_n := \operatorname{span}\{e_1, \cdots, e_n\}$ equipped with the inner product inherited from H and let $P_n : X \longrightarrow H_n$ be defined by

$$P_n := \sum_{j=1}^{n} X_{e_j} e_j \quad n \in \mathbb{N}. \tag{3.14}$$

Note that by definition of $X_{e_j}, j \in \mathbb{N}$, P_n restricted to H is just the orthogonal projection on H with image H_n. Note also that

$$P_n\, P_m = P_n, \text{ if } n \le m, \text{ and } P_n(x+y) = P_n x + P_n y \text{ ,if } x \in H,\, y \in X. \tag{3.15}$$

In the following for simplicity we identify $(H_n, (\ ,\ )_H)$ with $\mathbb{R}^n$ equipped whith the Euclidean inner product $(\ ,\ )$. The corresponding gradient we denote by ∇.

Fix $n \in \mathbb{N}$, $\epsilon \in]0,1]$. Let ν be the centred Gaussian measure on $\mathcal{B}(H)$ with covariance operator T. Set

$$\nu^\epsilon(B) := \nu(B/\epsilon)\ ,\ B \in \mathcal{B}(H),$$

and

$$\mu^\epsilon := \mu * \nu^\epsilon,$$

where we consider ν^ϵ also as a measure on $\mathcal{B}(X)$ extending it by zero. Let g_ϵ denote the Radon-Nikodym derivative of the projection $\nu^\epsilon \circ P_n^{-1}$ of ν^ϵ on $\mathbb{R}^n$ w.r.t. Lebesgue measure dx. It is well-known that ν^ϵ is differentiable along every e_i and that

$$\beta_{e_i}^{\nu^\epsilon} = -\epsilon^{-2} t_i^{-1} X_{e_i}\ ,\ 1 \le i \le n. \tag{3.16}$$

Let $f \in C_b^\infty(\mathbb{R}^n)$ and let $u := f \circ P_n$. By (A1), (B) and (3.4), (3.6) it is straightforward to check that if we define

$$\begin{aligned} Lu(x) \ &:= \ \sum_{i,j=1}^{n} \partial_i \partial_j f(X_{e_1}(x), \cdots, X_{e_n}(x))(A(x)e_j, e_i)_H \\ &- \ \sum_{i=1}^{m} \partial_i f(X_{e_1}(x), \cdots, X_{e_n}(x)) X_{e_i}(x) \\ &+ \ \sum_{i=1}^{m} \partial_i f(X_{e_1}(x), \cdots, X_{e_n}(x))(e_i, v(x))_H\ , \end{aligned}$$

then

$$\int Lu(x)\, \mu(dx) = 0.$$

Now a straightforward calculation and (3.15) imply that

$$\begin{aligned} &- \ \int_X \sum_{i,j=1}^{n} \partial_i f(P_n x)[(a^{ij}\mu) * (\beta_{e_j}^{\nu^\epsilon} \nu^\epsilon)](dx) \\ &+ \ \int_X \sum_{i=1}^{n} \partial_i f(P_n x)[(B^i \mu) * \nu^\epsilon](dx) \\ &= \ \int L((f * g_\epsilon) \circ P_n) d\mu = 0, \end{aligned} \tag{3.17}$$

where $a^{ij}(x) := (A(x)e_i, e_j)_H$, $B^i(x) := -X_{e_i}(x) + (e_i, v(x))_H$, $x \in X$, $1 \le i, j \le n$ (see also the derivation of (2.2)). Let $f_{n,\epsilon}$ denote the Radon-Nikodym derivative of the projection $\mu_n^\epsilon := \mu^\epsilon \circ P_n^{-1}$ of μ^ϵ w.r.t. the standard Gaussian measure γ_n on $\mathbb{R}^n$ (which in turn is the projection of γ on $\mathbb{R}^n$). Clearly, if $g := \frac{d\gamma_n}{dx}$, then

$$f_{n,\epsilon}(x) = g^{-1}(x)\ (g_\epsilon * \mu_n)(x), \quad x \in \mathbb{R}^n, \tag{3.18}$$

where $\mu_n := \mu \circ P_n^{-1}$. Because of the fist part of assumption (A2), $\nu_\epsilon \circ P_n^{-1}$ is a nondegenerate centred Gaussian measure on $\mathbb{R}^n$ (with covariance operator TP_n). Therefore, it is easy to check that because of (3.7)

$$|\nabla \log f_{n,\epsilon}| \in L^2(\mathbb{R}^n; \mu_n^\epsilon) \tag{3.19}$$

Hence by [BR94b, Theorem 2.8]

$$|\nabla f_k - \nabla \log f_{n,\epsilon}| \underset{k\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(\mathbb{R}^n; \mu_n^\epsilon) \text{ for some } f_k \in C_0^\infty(\mathbb{R}^n). \tag{3.20}$$

Our aim is to prove the following

Claim: $\sup_{\epsilon\in]0,1]} \sup_{n\ge N(\epsilon)} \|\, |\nabla \log f_{n,\epsilon}|\, \|_{L^2(\mu_n^\epsilon)} < \infty.$

To this end we first rewrite (3.17). Since $A(x) = I_H + S(x)T$, $x \in X$, and since

$$(\beta_{e_j}^{\nu^\epsilon} \nu^\epsilon) \circ P_n^{-1}(dx) = \partial_j g_\epsilon(x)\, dx, \quad 1 \le j \le n, \tag{3.21}$$

by (3.15), (3.16), and (3.18) equality (3.17) is equivalent to

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\nabla f, \nabla \log f_{n,\epsilon})_{\mathbb{R}^n}\, d\mu_n^\epsilon \\
=\ & \int_X \sum_{i,j=1}^n \partial_i f(P_n x) \left[K^{ij}\left(\mu * (\beta_{e_j}^{\nu^\epsilon} \nu^\epsilon)\right) - (K^{ij}\mu) * (\beta_{e_j}^{\nu^\epsilon} \nu^\epsilon)\right](dx) \\
& - \int_X \sum_{i,j=1}^n \partial_i f(P_n x) K^{ij}(x) \left(\mu * (\beta_{e_j}^{\nu^\epsilon} \nu^\epsilon)\right)(dx) \\
& + \int_X \sum_{i=1}^n \partial_i f(P_n x) \left[X_{e_i}(\mu * \nu^\epsilon) - (X_{e_i}\mu) * \nu^\epsilon\right](dx) \\
& + \int_X \sum_{i=1}^n \partial_i f(P_n x) [(v^i \mu) * \nu^\epsilon](dx),
\end{aligned} \tag{3.22}$$

where $K^{ij} := (S(x)Te_i, e_j)_H$, $v^i(x) = (e_i, v(x))_H$, $x \in X$, $1 \le i, j \le n$. Let us denote the absolute values of the four terms on the right hand side of (3.22) by I, II, III, IV respectively.

Estimate for I*:* By definition of $K^{ij}(x)$, (3.16), and assumption (A2) we have that

$$\begin{aligned}
\mathrm{I} &= \epsilon^{-2} \left| \int_X \int_H (\nabla f(P_n(x+y)), P_n(S(x+y) - S(y))P_n x)\ \nu^\epsilon(dx)\mu(dy) \right| \\
&\le \epsilon^{-2} \left(\int_X \int_H |\nabla f(P_n(x+y))|^2\, \nu^\epsilon(dx)\mu(dy) \right)^{1/2}\ C\ \left(\int_H |x|_H^4 \nu^\epsilon(dx) \right)^{1/2} \\
&\le C\ \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2 \mu_n^\epsilon(dx) \right)^{1/2}\ \left(\int_H |x|_H^4 \nu(dx) \right)^{1/2}.
\end{aligned} \tag{3.23}$$

Estimates for III *and* IV*:* Obviously by (3.15), we have that

$$(\mu * \nu^\epsilon) - (X_{e_i}\mu) * \nu^\epsilon = \mu * (X_{e_i}\nu^\epsilon), \quad 1 \le i \le n.$$

Hence an application of the Cauchy-Schwarz inequality and Lemma 2.1 yields that

$$\mathrm{III} \le \left(\int_{\mathbb{R}^n} |\nabla f(x)|^2 \mu_n^\epsilon(dx)\right)^{1/2} \epsilon \left(\int_H |x|_H^2 \nu(dx)\right)^{1/2}. \tag{3.24}$$

By the same arguments we obtain that

$$\mathrm{IV} \le \left(\int |\nabla f(x)|^2 \mu_n^\epsilon(dx)\right)^{1/2} \left(\int_X |v|_H^2 d\mu\right)^{1/2}. \tag{3.25}$$

Estimate for II*:* Using (3.12), assumption (A2), and the definition of K^{ij} we see that

$$\mathrm{II} = \left|\int \left(S(x+y)^* \nabla f(P_n(x+y)), T\beta_n^{\mu^\epsilon}(x+y)\right)_H \ \nu^\epsilon(dx)\mu(dy)\right|, \tag{3.26}$$

where for $x \in X$

$$\beta_n^{\mu^\epsilon}(x) := \sum_{i=1}^n \beta_{e_i}^{\mu^\epsilon}(x)\ e_i \quad (\in H). \tag{3.27}$$

Hence, since $\|S(x+y)^*\|_{\mathcal{L}(H)} = \|S(x+y)\|_{\mathcal{L}(H)} \le q$, it follows by the Cauchy-Schwarz inequality that

$$\mathrm{II} \le q \left(\int |\nabla f(x)|^2 \ \mu_n^\epsilon(dx)\right)^{1/2} \left(\int |T\beta_n^{\mu^\epsilon}(x)|_H^2 \ \mu^\epsilon(dx)\right)^{1/2}. \tag{3.28}$$

Now let us estimate the second factor on the right hand side of (3.28). First we note that using (3.12) and (3.16) we get by the proof of Lemma 2.1 that

$$\beta_{e_i}^{\mu^\epsilon} \circ \Lambda = E_{\nu^\epsilon \otimes \mu}\left[(-\epsilon^{-2} t_i^{-1} \bar{X}_{e_i}) | \sigma(\Lambda)\right] \quad \nu^\epsilon \otimes \mu\text{-a.e.}, 1 \le i \le n, \tag{3.29}$$

where $\bar{X}_{e_i}(x,y) = X_{e_i}(x)$, and $\Lambda : X \times X \to X$ is defined by $\Lambda(x,y) = x+y$, $(x,y) \in X \times X$. In particular,

$$\sum_{i=1}^\infty \int \left(t_i \beta_{e_i}^{\mu^\epsilon}(x)\right)^2 \ \mu^\epsilon(dx) = \epsilon^{-4} \int |x|_H^2 \nu^\epsilon(dx) < \infty.$$

Hence $(T\beta_n^{\mu^\epsilon})_{n\in\mathbb{N}}$ and therefore also $(E_{\mu^\epsilon}[T\beta_n^{\mu^\epsilon}|\sigma(P_n)])_{n\in\mathbb{N}}$ converges in $L^2(X \to H; \mu^\epsilon)$ and both have the same limit. Since

$$\begin{aligned} E_{\mu^\epsilon}[\beta_{e_i}^{\mu^\epsilon}|\sigma(P_n)] &= \partial_i(\log(g_\epsilon * \mu_n)) \circ P_n \\ &= \partial_i(\log f_{n,\epsilon}) \circ P_n - X_{e_i}, \quad \mu^\epsilon\text{-a.e.}, 1 \le i \le n, \end{aligned} \tag{3.30}$$

(as is easy to check), for $c \in]0,1[$ such that $q_1 := (1+c)q < 1$ and sufficiently big $n \ge N(\epsilon)$ we therefore obtain that

$$\|\,|T\beta_n^{\mu^\epsilon}|_H\,\|_{L^2(\mu^\epsilon)} \le (1+c)\left(\|\,|\nabla \log f_{n,\epsilon}|\,\|_{L^2(\mu_n^\epsilon)} + \|\,|TP_n|_H\,\|_{L^2(\mu^\epsilon)}\right) \tag{3.31}$$

because $\|T\|_{\mathcal{L}(H)} \leq 1$. We claim that

$$\sup_{\substack{\epsilon\in]0,1] \\ n\in\mathbb{N}}} \| \, |TP_n|_H \, \|_{L^2(\mu^\epsilon)} < \infty. \tag{3.32}$$

Indeed,

$$\begin{aligned} \int |TP_n(x)|^2 \mu^\epsilon(dx) &\leq 2\int |TP_n(x)|_H^2 \mu(dx) + 2\int |Tx|_H^2 \nu^\epsilon(dx) \\ &\leq 2\sum_{i=1}^\infty t_i^2 \int l_i^2 d\mu + 2\epsilon^2 \int |Tx|_H^2 \nu(dx) \end{aligned}$$

which is finite by Lemma 3.3 since T is trace class. Hence by (3.28) and (3.31)we can find $c_3 \in]0,\infty[$ independent of f, n, ϵ such that

$$\mathrm{II} \leq \| \, |\nabla f| \, \|_{L^2(\mu_n^\epsilon)}^{1/2} \left(q_1 \| \, |\nabla \log f_{n,\epsilon}| \, \|_{L^2(\mu_n^\epsilon)}^{1/2} + c_3 \right). \tag{3.33}$$

Combining all four estimates (3.23)–(3.25), (3.33) with (3.19), (3.20), (3.22) and taking into account that $q_1 < 1$ we deduce our claim, i.e., that

$$K_1 := \sup_{\epsilon\in]0,1]} \sup_{n\geq N(\epsilon)} \| \, |\nabla \log f_{n,\epsilon}| \, \|_{L^2(\mu_n^\epsilon)} < \infty. \tag{3.34}$$

We shall now use L. Gross' well-known logarithmic Sobolev inequality (see [G76]), i.e.,

$$\int F^2(x) \log F(x) \gamma(dx) \leq \int \|\nabla_H F(x)\|_H^2 \gamma(dx) + \|F\|_{L^2(\gamma)}^2 \ln \|F\|_{L^2(\gamma)}^2.$$

Fix $\epsilon \in]0,1]$ and let $n \in \mathbb{N}$. (3.19) implies that $\sqrt{f_{n,\epsilon}} \in H^{1,2}(\mathbb{R}^n;\gamma_n)$ (see (3.35) below), hence

$$\varphi_{n,\epsilon} := \sqrt{f_{n,\epsilon}} \circ P_n \in H^{1,2}(X;\gamma)$$

and we can apply the logarithmic Sobolev inequality to $\varphi_{n,\epsilon}$. Moreover,

$$\begin{aligned} \|\varphi_{n,\epsilon}\|_{L^2(\gamma)}^2 &= \int_X f_{n,\epsilon}(P_n x)\gamma(dx) = \int_{\mathbb{R}^n} f_{n,\epsilon}(x)\gamma_n(dx) \\ &= \mu_n^\epsilon(\mathbb{R}^n) = 1, \end{aligned}$$

and

$$\begin{aligned} 4\int_X |\nabla_H \varphi_{n,\epsilon}(x)|_H^2 \gamma(dx) &= \int_{\mathbb{R}^n} \left| \frac{\nabla f_{n,\epsilon}(x)}{\sqrt{f_{n,\epsilon}(x)}} \right|^2 \gamma_n(dx) \\ &= \int_{\mathbb{R}^n} |\nabla \log f_{n,\epsilon}|^2 \mu_n^\epsilon(dx). \end{aligned} \tag{3.35}$$

By virtue of (3.34) now we conclude that

$$\sup_{\epsilon\in]0,1]} \sup_{n\geq N(\epsilon)} \int_X \varphi_{n,\epsilon}^2(x) \log \varphi_{n,\epsilon}(x) \; \gamma(dx) < \infty.$$

Hence $(\varphi^2_{n,\epsilon})_{n,\epsilon}$ is uniformly integrable on $(X, \mathcal{B}(X), \gamma)$. Furthermore, $(\varphi^2_{n,\epsilon})_{n\in\mathbb{N}}$ is a $(\sigma(P_n))_{n\in\mathbb{N}}$-martingale under γ (since the projection of μ^ϵ_{n+1} onto $\mathbb{R}^n$ coincides with μ^ϵ_n and because of (3.15)), thus it converges to some function $p_\epsilon \in L^1(\gamma)$. Let $\varphi_\epsilon := \sqrt{p_\epsilon}$. Therefore, if $Y := c_1 X_{e_1} + \cdots + c_m X_{e_m}$, one has that

$$\int_X \exp(iY(x))\varphi^2_{n,\epsilon}(x)\gamma(dx) \underset{n\to\infty}{\longrightarrow} \int_X \exp(iY(x))\varphi^2_\epsilon(x)\gamma(dx).$$

On the other hand for $n \geq m$,

$$Y \circ P_n = Y,$$

hence

$$\begin{aligned} &\int_X \exp\left(iY(x)\right)\varphi^2_{n,\epsilon}(x)\gamma(dx) = \int_X \exp\left(iY(P_n x)\right) f_{n,\epsilon}(P_n x)\gamma(dx) \\ = &\int_{\mathbb{R}^n} \exp\left(iY(x)\right)\mu^\epsilon_n(dx) = \int_X \exp\left(iY(x)\right)\mu^\epsilon(dx). \end{aligned}$$

Consequently, $\int_X \exp\left(iY(x)\right)\varphi^2_\epsilon(x)\gamma(dx) = \int_X \exp\left(iY(x)\right)\mu^\epsilon(dx)$. Since for each $l \in X^*$, $j_H(l)$ can be approximated by finite linear combinations of e_n, $n \in \mathbb{N}$, it follows immediately by Lemma 3.3 that for each $l \in X^*$ there exists Y_n, $n \in \mathbb{N}$, where each Y_n is a finite linear combination of X_{e_i}, $i \in \mathbb{N}$, such that $Y_n \underset{n\to\infty}{\longrightarrow} l$ both in $L^2(\mu^\epsilon)$ and $L^2(\gamma)$ as well as both μ^ϵ-a.e. and γ-a.e. Hence the last equality implies that

$$\int_X \exp\left(i\, l(x)\right)\ \varphi^2_\epsilon(x)\gamma(dx) = \int_X \exp\left(i\, l(x)\right)\mu^\epsilon(dx).$$

Since both $\varphi^2_\epsilon\gamma$ and μ^ϵ are Radon measures, it follows that

$$\mu^\epsilon = \varphi^2_\epsilon\ \gamma \quad \text{on } \mathcal{B}(X). \tag{3.36}$$

Since $\varphi_{n,\epsilon} \underset{n\to\infty}{\longrightarrow} \varphi_\epsilon$ in γ-measure, by a standard argument (3.35) and (3.34) imply that

$$\varphi_\epsilon \in H^{1,2}(X;\gamma) \quad \text{and} \sup_{\epsilon\in]0,1]} \|\varphi_\epsilon\|^2_{H^{1,2}(X;\gamma)} \leq 1 + K^2. \tag{3.37}$$

It now follows (cf. the argument in [BR94b] which led to (3.17)) that μ^ϵ is differentiable along each $h \in H$ and that

$$\beta^{\mu^\epsilon}_h = \beta^\gamma_h + (h, 2\varphi^{-1}_\epsilon \nabla_H \varphi_\epsilon)_H \tag{3.38}$$

For $h = \sum_{i=1}^m c_i e_i$ one has that

$$\beta^\gamma_h = -\sum_{i=1}^m c_i X_{e_i}$$

(cf. (3.16)). Hence by (3.15)

$$\begin{aligned} \int_X \beta^\gamma_h(x)^2\,\mu^\epsilon(dx) &\leq 2\int_X \beta^\gamma_h(x)^2\,\mu(dx) + 2\int \beta^\gamma_h(x)^2 \nu^\epsilon(dx) \\ &\leq 2\int_X \left(\sum_{i=1}^m c_i X_{e_i}(x)\right)^2 \mu(dx) + 2\int \left(\sum_{i=1}^m c_i X_{e_i}(x)\right)^2 \nu(dx) \end{aligned}$$

which by Lemma 3.3 is uniformly bounded in $\epsilon \in]0,1]$ and $h \in \{h \in H \mid |h|_H \le 1\}$. Combining this with (3.38) and (3.34) we conclude that

$$\sup_{\epsilon\in]0,1]} \sup_{|h|_H\le 1} \int \beta_h^{\mu^\epsilon}(x)^2 \mu^\epsilon(dx) < \infty.$$

Since the Fourier transforms of μ^ϵ converge pointwise to the Fourier transforms of μ, [B93, Proposition 3.6] implies that μ is differentiable along each $h \in H$ and

$$K_2 := \sup_{|h|_H\le 1} \int \beta_h^{\mu}(x)^2 \mu(dx) \; < \infty. \tag{3.39}$$

By (3.10) for $h \in H$ and $F \in \mathcal{B}(X)$

$$|\mu(F+h) - \mu(F)| \le \|\beta_h^\mu\|_{L^1(\mu)},$$

hence for all $\epsilon \in]0,1]$

$$\begin{aligned} |\mu^\epsilon(F) - \mu(F)| &\le \int_H |\mu(F+\epsilon h) - \mu(F)| \nu(dh) \\ &\le \epsilon \int \|\beta_h^\mu(x)\|_{L^1(\mu)} \nu(dh) \\ &\le \epsilon\, K_2^{1/2} \int |h|_H \nu(dh) \; < \infty, \end{aligned}$$

where we used (3.39) in the last step. Because of (3.36) this implies that $\mu = p\,\gamma$ and that $\varphi_\epsilon^2 \underset{\epsilon\to 0}{\longrightarrow} p$ in $L^1(\gamma)$. Hence $\varphi_\epsilon \underset{\epsilon\to 0}{\longrightarrow} \varphi := \sqrt{p}$ in $L^2(\gamma)$ (since φ_ϵ^2, p are probability densities) and thus by (3.37) $\varphi \in H^{1,2}(X;\gamma)$ (cf. above). The rest of the assertion of Theorem 3.1 follows as in the derivation of (3.38). □

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References

[AR91] S. Albeverio, M. Röckner: Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms. Probab. Theor. Rel. Fields 89, 347-386 (1991).

[ARZ93] S. Albeverio, M. Röckner, T.S. Zhang: Markov uniqueness for a class of infinite dimensional Dirichlet operators. In: Stochastic Processes and Optimal Control. Stochastic Monographs Vol. 7, 1-26, Editors: H.J. Engelbert et al. Gordon & Breach 1993.

[B93] V.I. Bogachev: Remarks on integration by parts in infinite dimensions. Acta Univ. Carolina 34, 11-29 (1993).

[BR94a] V.I. Bogachev, M. Röckner: Hypoellipticity and invariant measures of infinite-dimensional diffusions. C.R. Acad. Sci. Paris, Ser. I, 318, 553-558 (1994).

[BR94b] V.I. Bogachev, M. Röckner: Regularity of invariant measures on finite and infinite dimensional spaces and applications. SFB-256-Bonn Preprint, June (1994).

[BRZ95] V.I. Bogachev, M. Röckner, T.S. Zhang: in preparation.

[BS90] V.I. Bogachev, O.G. Smolyanov: Analytic properties of infinite dimensional distributions. Russian Math. Surveys 45, n 3, 1-104 (1990).

[G76] L. Gross: Logarithmic Sobolev inequalities. Amer.J.Math. 97, 1061-1083 (1976).

[RZ92] M. Röckner, T.S. Zhang: Uniqueness of generalized Schrödinger operators and applications. J.Funct.Anal. 105, n 1, 187-231 (1992).

[RZ94] M. Röckner, T.S Zhang: Uniqueness of generalized Schrödinger operators – Part II. J.Funct.Anal. 119, 455-467 (1994).

[S70] E. Stein: Singular integrals and differentiability properties of functions. Princeton University Press; Princeton, NJ, 1970.

[Sh87] I. Shigekawa: Existence of invariant measures of diffusion on an abstract Wiener space. Osaka J.Math. 24, n 1, 37-59 (1987).

[Ts77] B.S. Tsirelson: Supplement to: A natural modification of a random process and its application to series of random functions and to Gaussian measures. Zap.Nauchn.Sem. Leningrad. Otdel.Mat.Inst.Steklov (LOMI), 72, 202-211 (1977).