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ERGODICITY OF L^2–SEMIGROUPS AND EXTREMALITY OF GIBBS STATES

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Abstract. We extend classical results of Holley–Stroock on the characterization of extreme Gibbs states for the Ising model in terms of the irreducibility (resp. ergodicity) of the corresponding Glauber dynamics to the case of lattice systems with unbounded (linear) spin spaces. We first develop a general framework to discuss questions of this type using classical Dirichlet forms on infinite dimensional state spaces and their associated diffusions. We then describe concrete applications to lattice models with polynomial interactions (i.e., the discrete $P(\varphi)_d$–models of Euclidean quantum field theory). In addition, we prove the equivalence of extremality and shift–ergodicity for tempered Gibbs states of these models and also discuss this question in the general framework.

1. Introduction and preliminaries

Since the classical work of Holley and Stroock (cf. [HSt 76]) it is well–known that in the Ising model a Gibbs state μ is *extremal* if and only if the semigroup of operators given by the corresponding Glauber dynamics considered as an $L^2(\mu)$–semigroup is *irreducible* (or equivalently *ergodic*, see Section 2 below). The purpose of this paper is to prove this equivalence for classes of lattice models with non–compact, but linear spin space. The irreducibility of a symmetric $L^2(\mu)$–semigroup is equivalent to the irreducibility of the corresponding Dirichlet form, which in our cases are classical Dirichlet forms in the sense of [AR 90 a]. We thus also obtain a wide variety of examples of irreducible classical Dirichlet forms on infinite dimensional state space of which so far only a few (mainly Gaussian type) examples have been known.

To explain our main results more explicitly we need some preparations. We start with introducing the notion of Gibbs states associated with a "vector field" $b = (b_k)$.

Let E be a locally convex topological vector space (over the real numbers $\mathbb{R}$) which for simplicity is assumed to be Souslinean (i.e., the continuous image of a complete

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separable metric space). Let $\mathcal{B}(E)$ denote its Borel σ–field and E' its topological dual. Define for $K_0 \subset E'$

$$\mathcal{F}C_b^\infty(K_0) := \{f(\ell_1, \dots, \ell_m) \mid m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), \ell_1, \dots, \ell_m \in K_0\}$$

and set

$$\mathcal{F}C_b^\infty := \mathcal{F}C_b^\infty(E') \ .$$

Elements in $\mathcal{F}C_b^\infty$ are called *finitely based smooth bounded functions* on E. For $K \subset E$, $\mathcal{B}(E)$–measurable real–valued functions $b_k, k \in K$, and $b := (b_k)_{k \in K}$ we define $\mathcal{G}^b$ to be the set of all probability measures μ on $\mathcal{B}(E)$ such that for all $k \in K$, $b_k \in L^2(E;\mu)$ and the following integration by parts formula holds:

$$\int \frac{\partial u}{\partial k}\, d\mu = -\int u\, b_k\, d\mu \text{ for all } u \in \mathcal{F}C_b^\infty \ , \tag{1.1}$$

where $\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk)|_{s=0}, z \in E$. Elements in $\mathcal{G}^b$ are called *Gibbs states associated with* b. $\mathcal{G}^b$ is obviously a convex set (which, however, might be empty). For sufficient conditions on b so that $\mathcal{G}^b \neq \emptyset$ we refer to [BoR 94, Sect. 5, in particular Remark 5.5 (ii)]. Let $\mathcal{G}^b_{ext}$ denote the set of all extreme points of $\mathcal{G}^b$. Fix K, b, and $\mu \in \mathcal{G}^b$. For any $k \in K$ we can consider its associated Dirichlet form $(\mathcal{E}_{\mu,k}, D(\mathcal{E}_{\mu,k}))$ defined as the closure on $L^2(E;\mu)$ of

$$\mathcal{E}_{\mu,k}(u,v) = \int \frac{\partial u}{\partial k} \frac{\partial v}{\partial k}\, d\mu \ ; \ u, v \in \mathcal{F}C_b^\infty$$

(cf. [AR 90 a, Sect. 3], and for closability [AKR 90, Proposition 2.2]). For the general definition of a Dirichlet form (not needed here) we refer to [F 80], [MR 92] or [FOT 94].

Remark 1.1. (i) Note that since (1.1) holds, it is clear that $u = v$ μ–a.e. implies $\frac{\partial u}{\partial k} = \frac{\partial v}{\partial k}$ μ–a.e. for all $u, v \in \mathcal{F}C_b^\infty$. Hence $\mathcal{E}_{\mu,k}$ is well–defined on the μ–classes $\widetilde{\mathcal{F}C}_b^\infty$ induced by $\mathcal{F}C_b^\infty$.

(ii) $(\frac{\partial}{\partial k}, \widetilde{\mathcal{F}C}_b^\infty)$ is a closable operator on $L^2(E;\mu)$. We denote its closure by $\frac{\partial^\mu}{\partial k}$. In particular, for all $u, v \in D(\mathcal{E}_{\mu,k})$

$$\mathcal{E}_{\mu,k}(u,v) = \int \frac{\partial^\mu u}{\partial k} \frac{\partial^\mu v}{\partial k}\, d\mu \ .$$

(iii) It is easy to see that (1.1) extends to all $u \in D(\mathcal{E}_{\mu,k})$ with $\frac{\partial^\mu}{\partial k}$ replacing $\frac{\partial}{\partial k}$ by approximation (see also [AKR 90, Theorem 2.5]).

As in [AR 90 a, Sect. 3] we now define

$$D(\mathcal{E}_\mu) := \left\{ u \in \bigcap_{k \in K} D(\mathcal{E}_{\mu,k}) \mid \sum_{k \in K} \mathcal{E}_{\mu,k}(u,u) < \infty \right\} \tag{1.2}$$

and

$$\mathcal{E}_\mu(u,v) := \frac{1}{2} \sum_{k \in K} \mathcal{E}_{\mu,k}(u,v) \ ; \ u, v \in D(\mathcal{E}_\mu) \ . \tag{1.3}$$

From now on we assume that K is countable and such that

$$\sum_{k \in K} |\ell(k)|^2 < \infty \text{ for all } \ell \in E' \ . \tag{1.4}$$

(1.4) clearly implies that $\widetilde{\mathcal{F}C}_b^\infty \subset D(\mathcal{E}_\mu)$ and, since $\widetilde{\mathcal{F}C}_b^\infty$ is dense in $L^2(E;\mu)$, it then follows that $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is densely defined and by [AR 90 a, Theorem 3.8] a Dirichlet form on $L^2(E;\mu)$. We recall that $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is called *irreducible* if for any $u \in D(\mathcal{E}_\mu)$ with $\mathcal{E}_\mu(u,u) = 0$ it follows that u is constant μ–a.e.. Let $(L_\mu, D(L_\mu))$ be the generator corresponding to $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$, i.e., the unique negative definite self–adjoint operator such that

$$\mathcal{E}_\mu(u,v) = \left(\sqrt{-L_\mu}u, \sqrt{-L_\mu}v\right)_{L^2(E;\mu)} \quad \text{for all } u,v \in D(\mathcal{E}_\mu) = D\left(\sqrt{-L_\mu}\right) \ . \tag{1.5}$$

Let $T_t^\mu := e^{tL_\mu}, t \geq 0$, denote the corresponding strongly continuous semigroup on $L^2(E;\mu)$ (cf. e.g. [MR 92, Chap. I, Sections 1,2]) for details. The irreducibility of $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is equivalent to the *irreducibility* of $(T_t^\mu)_{t\geq 0}$ (i.e., if $u \in L^2(E;\mu)$ such that $T_t^\mu(uf) = uT_t^\mu f$ for all $f \in L^\infty(E;\mu), t > 0$, then u is a constant). For this and other elementary characterizations of irreducibility (e.g. in terms of $(L_\mu, D(L_\mu))$) we refer to Proposition 2.3 below resp. the Appendix. Our first main result (which is in particular a consequence of Theorem 6.15 (i) in [BoR 94]) is the following general structure result:

Theorem 1.2. *Let K, b be as above such that (1.4) holds and let $\mu \in \mathcal{G}^b$. Then $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is irreducible if and only if $\mu \in \mathcal{G}^b_{ext}$.*

The proof of Theorem 1.2 will be given among other things in the next section.

In Section 3 we discuss the relation of the irreducibility of $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ with K–*ergodicity* (i.e., ergodicity w.r.t. shifts in K) of μ. We show that irreducibility always implies K–ergodicity (cf. Theorem 3.4) while the converse holds only under the additional assumption (3.2) (cf. Theorem 3.7 and Remark 3.9 (ii)) which is, however, fulfilled in many cases (cf. Remark 3.9 (iii) and Section 5). Under assumption (3.2) one can also prove that $\mathcal{G}^b_{ext} \neq \emptyset$ provided $\mathcal{G}^b \neq \emptyset$ and that one has an integral representation of an arbitrary element in $\mathcal{G}^b$ in terms of extreme ones (cf. Remark 3.9 (iv) below). In particular, one obtains the above mentioned irreducible classical Dirichlet forms.

In Section 4 the connection of the irreducibility of $(\mathcal{E}^\mu, D(\mathcal{E}^\mu))$ with the corresponding diffusion process $\mathbb{M}$ is studied provided it exists. We particularly point out in detail the essential role which is played by the so–called *Markov uniqueness problem*. The main results are summarized by Theorem 4.9. We would also like to draw attention to Remark 4.10 about S. Mück's result in [Mü 92], [Mü 94] on large deviations from ergodic behaviour which by Theorem 4.9 applies to $\mathbb{M}$ provided $\mu \in \mathcal{G}^b_{ext}$.

In Section 5 we discuss applications. The main technique is simply to prove that the (tempered) Gibbs states $\mathcal{G}^{\mathcal{V}}$ for some local specification $\mathcal{V}$ of some lattice model in statistical mechanics coincide with $\mathcal{G}^b$. In this paper for simplicity and to illustrate the basic ideas we restrict ourselves to lattice models and interactions of polynomial type which are approximations for the so–called $P(\varphi)_d$*–models* in Euclidean quantum field theory. Applications to more general models will be discussed in the forthcoming paper [AKoR 95]. For the $P(\varphi)_d$–lattice models all result in Sections 1 – 4 apply and we obtain Theorem 5.15 characterizing tempered extreme Gibbs states in four different ways (giving an extended analogue of [HSt 76, Theorem (1.12)]).

In the last subsection of Section 5 we discuss an alternative construction of the diffusion process $\mathbb{M}$ of Section 4 which is (unlike before) independent of the a priori given Gibbs measure μ and is just universal w.r.t. all Gibbs measures. Such a construction is possible due to the regularity of the interaction potentials for the $P(\Phi)_d$–lattice models. As a consequence, we finally prove that two distinct tempered extreme Gibbs measures are not only (as is well–known) singular in the sense of measures, but even in the sense of the corresponding exceptional sets (resp. nests; cf. Theorem 5.16).

The results of this paper have been announced at various conferences in Bielefeld, Marseille and Warwick during the spring and summer of 1994 as well as in several invited talks e.g. at the University of California, San Diego, October 1994, and the Mittag–Leffler–Institute, Stockholm, January 1995.

2. Proof of Theorem 1

Let E, K, b be as in Section 1 such that (1.4) holds. The proof of Theorem 1.1 is a consequence of the following result.

Theorem 2.1. *Let $\mu \in \mathcal{G}^b$ and let $\mathcal{G}^b_{\mu,ac}$ denote all $\nu \in \mathcal{G}^b$ which are absolutely continuous w.r.t. μ. Then $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is irreducible if and only if $\mathcal{G}^b_{\mu,ac} = \{\mu\}$.*

Proof. This is a straightforward adaptation of the proof of [BoR 94, Theorem 6.15 (i)], to the present framework. $\square$

As a consequence we obtain the

Proof of Theorem 1.1. Suppose that $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is irreducible and let $\mu_1, \mu_2 \in \mathcal{G}^b$, $t \in]0,1[$ such that $\mu = t\mu_1 + (1-t)\mu_2$. Then $\mu_1, \mu_2 \in \mathcal{G}^b_{\mu,ac}$, hence $\mu_1 = \mu_2 = \mu$ by Theorem 2.1. Consequently, $\mu \in \mathcal{G}^b_{ex}$.
Suppose that $\mu \in \mathcal{G}^b_{ex}$. Let $\nu \in \mathcal{G}^b_{\mu,ac}$, i.e., $\nu = \rho \cdot \mu$ with $\int \rho \, d\mu = 1$. By [BoR 94, Lemma 6.14] it follows that $\sqrt{\rho} \in D(\mathcal{E}_\mu)$ and $\mathcal{E}_\mu(\sqrt{\rho}, \sqrt{\rho}) = 0$. Define

$$\mu_1 := \frac{\alpha^{-1}}{1+\sqrt{\rho}}\,\mu \qquad , \; \mu_2 := \frac{\beta^{-1}\sqrt{\rho}}{1+\sqrt{\rho}}\,\mu$$
$$\text{where } \alpha := \int \frac{1}{1+\sqrt{\rho}}\,d\mu \quad \text{ and } \; \beta := \int \frac{\sqrt{\rho}}{1+\sqrt{\rho}}\,d\mu\ .$$

By the chain rule or [MR 92, Exercise 4.16] we have that $(1+\sqrt{\rho})^{-1}, \sqrt{\rho}/(1+\sqrt{\rho}) \in D(\mathcal{E}_\mu)$ and, since $\mathcal{E}_\mu(\sqrt{\rho}, \sqrt{\rho}) = 0$,

$$\mathcal{E}_\mu\left(\frac{1}{1+\sqrt{\rho}}, \frac{1}{1+\sqrt{\rho}}\right) = 0 = \mathcal{E}_\mu\left(\frac{\sqrt{\rho}}{1+\sqrt{\rho}}, \frac{\sqrt{\rho}}{1+\sqrt{\rho}}\right)\ .$$

By Remark 1.1 (iii) and the product rule (cf. [AKR 90, Theorem 2.5]) it follows that $\mu_1, \mu_2 \in \mathcal{G}^b$. Since $\alpha, \beta \in]0,1[$, $\alpha + \beta = 1$, and $\mu = \alpha\mu_1 + \beta\mu_2$, this and the assumption imply that $\mu = \mu_1 = \mu_2$, hence $\sqrt{\rho}$ must be constant μ–a.e., therefore $\rho \equiv 1$. Consequently, $\mathcal{G}^b_{\mu,ac} = \{\mu\}$ and thus Theorem 2.1. implies that $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is irreducible. $\square$

Example 2.2. Except for the simple case where K is finite, condition (1.4) seems difficult to check. But a large class of examples, for which (1.4) holds, are given as follows: Let H be a separable real Hilbert space such that $H \subset E$ continuously and densely. Hence $E' \subset H'$ continuously (by restriction) and we can define a continuous one–two–one map $j_H : E' \to H$ with dense image as follows. For any $\ell \in E'$ by the Riesz representation theorem there exists a unique $j_H(\ell) \in H$ such that

$$\ell(h) = \langle j_H(\ell), h\rangle_H \text{ for all } h \in H \ .$$

Now suppose that $K_0 \subset E'$ is such that $j_H(K_0)$ is an orthonormal base in H (cf. [AR 91, Lemma 5.6] for existence). Then (1.4) is trivially satisfied if $K = j_H(K_0)$. In this case, if $\mu \in \mathcal{G}^b$, then

$$\mathcal{E}_\mu(u,v) = \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_H \, d\mu \text{ for all } u, v \in \mathcal{F}C_b^\infty \ ,$$

where for $z \in E$, $\nabla u(z) := j_H(\ell \mapsto \frac{\partial u}{\partial j_H(\ell)}(z), \ell \in E')$, i.e., $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is a *gradient Dirichlet form* (cf. [RZ 92, p. 197]). In this case for $u \in \mathcal{F}C_b^\infty(K_0)$, $u = f(\ell_1, \dots, \ell_m)$, we have that $u \in D(L_\mu)$ and

$$L_\mu u = \frac{1}{2}\sum_{i=1}^m \partial_i^2 f(\ell_1,\dots,\ell_m) + \frac{1}{2}\sum_{i=1}^m b_{j_H(\ell_i)}\partial_i f(\ell_1,\dots,\ell_m) \tag{2.1}$$

(cf. e.g. [RZ 92, p. 198]) where ∂_i denotes the derivative of f w.r.t. its i–th variable. In particular, for such functions u, L_μ is independent of μ. However, (2.1) does not determine $(L_\mu, D(L_\mu))$ as an operator on $L^2(E;\mu)$ since functions u of the above kind (or $\widetilde{\mathcal{F}C}_b^\infty$) might not be dense in $D(L_\mu)$ w.r.t. the graph norm of L_μ on $L^2(E;\mu)$ (cf. also Sect. 3 below). For concrete examples of E, H, K_0, b, and μ we refer to [AR 89], [AR 90 a], [AR 91] and Section 5 below.

In the following two sections we shall discuss some consequences of irreducibility. We close this section with the following proposition which is proved by standard means and essentially well–known. We include a short and simple proof for the convenience of the reader in the Appendix.

Proposition 2.3. *Let $\mu \in \mathcal{G}^b$. Then the following assertions are equivalent:*

(i) $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ *is irreducible.*

(ii) $(T_t^\mu)_{t\geq 0}$ *is irreducible.*

(iii) If $u \in L^2(E;\mu)$ such that $T_t^\mu u = u$ for all $t > 0$ then u is a constant.

(iv) $\int \left(T_t^\mu g - \int g\, d\mu\right)^2 d\mu \underset{t\to\infty}{\longrightarrow} 0$ *for all $g \in L^2(E;\mu)$. (In this case $(T_t^\mu)_{t>0}$ is also sometimes called* ergodic*).*

(v) If $u \in D(L_\mu)$ with $L_\mu u = 0$, then u is a constant.

We note that clearly $L_\mu 1 = 0$ (thus $T_t^\mu 1 = 1$ for all $t \geq 0$), hence the implication in Proposition 2.3 (v) is really an equivalence.

3. Irreducibility and shift ergodicity

Let E, K, b be as in Section 1 such that (1.4) holds, and let μ be a probability measure on $\mathcal{B}(E)$.

Definition 3.1. A $\mathcal{B}(E)$–measurable (real–valued) function f is called *K–(shift) invariant* if $f(z + tk) = f(z)$ for μ–a.e. $z \in E$ for all $t \in \mathbb{R}$ and all $k \in K$. $g \in L^2(E;\mu)$ is called *K–(shift) invariant* if there exists a $\mathcal{B}(E)$–measurable representative which is *K–(shift) invariant.*

Remark 3.2. We emphasize that for a K–invariant $g \in L^2(E;\mu)$ not every $\mathcal{B}(E)$–measurable representative is K–invariant unless μ is *K–quasi–invariant* (i.e., the images of μ under the maps $\tau_{tk} : E \to E, k \in K$, defined by $\tau_{tk}(z) = z + tk$, $z \in E$, are absolutely continuous w.r.t. μ for all $t \in \mathbb{R}$).

Definition 3.3. μ is called *K–ergodic* if every K–invariant $\mathcal{B}(E)$–measurable function f is constant μ–a.e..

The main result of this section is the following theorem including its converse (see Theorem 3.7 below). A special case thereof was first proved in[AH–K 77, Sect. 3].

Theorem 3.4. *Let $\mu \in \mathcal{G}^b$ and suppose that $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is irreducible. Then μ is K–ergodic.*

For the proof we need a result from [RZ 92] (cf. Proposition 3.5 below) which requires some preparations.

So, let $\mu \in \mathcal{G}^b$ and fix $k \in K$. We recall that for any closed subspace E_k of E such that $E = k\mathbb{R} \oplus E_k$ if $\pi_k : E \to E_k$ is the canonical projection, there exists a function $\rho_k : E_k \times \mathbb{R} \to [0, \infty[$ such that $\rho_k(x,s)ds$ is a subprobability kernel from $(E_k, \mathcal{B}(E_k))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and for all $u : E \to [0,\infty[, \mathcal{B}(E)$–measurable,

$$\int_E u(z)\mu(dz) = \int_{E_k} \int_{\mathbb{R}} u(x + sk)\rho_k(x,s)\, ds\, \nu_k(dx) \ .$$

Here $z = x + sk$ with $x \in E_k$, $s \in \mathbb{R}$ uniquely determined, $\nu_k := \mu \circ \pi_k^{-1}$, and ds denotes Lebesgue measure on $\mathbb{R}$. Furthermore, for ν_k–a.e. $x \in E_k$, $\rho_k(x,\cdot)$ satisfies Hamza's condition (H). We recall that a $\mathcal{B}(\mathbb{R})$–measurable function $\rho : \mathbb{R} \to [0,\infty[$ satisfies (H) if $\rho = 0\, ds$–a.e. on $\mathbb{R} \setminus R(\rho)$ where

$$R(\rho) := \left\{ t \in \mathbb{R} \mid \int_{t-\varepsilon}^{t+\varepsilon} \rho(s)^{-1}\, ds < \infty \text{ for some } \varepsilon > 0 \right\} \ ,$$

i.e., $R(\rho)$ is the largest open set $U \subset \mathbb{R}$ such that $\rho^{-1} \in L^1_{\text{loc}}(U; ds)$. (Here $\rho^{-1}(s) := +\infty$ if $\rho(s) = 0$.) Note that clearly $\rho > 0\, ds$–a.e. on $R(\rho)$. All this follows from the main result (i.e., Theorem 3.2, particularly part (ii)) in [AR 90 a] and also [AKR 90, Proposition 2.2.]).

We also note that correspondingly $L^2(E;\mu)$ can then be written as a direct integral of L^2–spaces over $\mathbb{R}$, i.e.,

$$L^2(E;\mu) = \int_{E_k}^{\oplus} L^2(\mathbb{R};\rho_k(x,s)ds)\nu_k(dx) \tag{3.1}$$

in the sense that each $u \in L^2(E;\mu)$ corresponds to a "field of vectors" $(u_x)_{x\in E_k}$ where $u_x := u(x+\cdot k)$, $x \in E_k$ (cf. [AR 90 a] for details and references).

Proposition 3.5. *Let $k \in K$ and $u \in L^2(E;\mu)$. Then $u \in D(\mathcal{E}_{\mu,k})$ if and only if for one (hence any) decomposition $E = k\mathbb{R} \oplus E_k$ as above for the corresponding element $(u_x)_{x\in E_k}$ in $\int_{E_k}^{\oplus} L^2(\mathbb{R};\rho_k(x,s)\,ds)\nu_k(dx)$ (see (3.1)) for ν_k–a.e. $x \in E_k$ there exists a locally absolutely continuous ds–version $\tilde{u}_x$ of u_x on $R(\rho_k(x,\cdot))$ and $\left(\frac{d\tilde{u}_x}{ds}\right)_{x\in E_k} \in \int_{E_k}^{\oplus} L^2(\mathbb{R};\rho_k(x,s)\,ds)\nu_k(dx)$. In this case*

$$\frac{\partial^\mu u}{\partial k} = \left(\frac{d\tilde{u}_x}{ds}\right)_{x\in E_k}$$

in the sense of (3.1). (See Remark 1.1 (ii) for the definition of $\frac{\partial^\mu u}{\partial k}$ for $u \in D(\mathcal{E}_{\mu,k})$).

Proof. [RZ 92, Theorem 5.1]. $\square$

Note that in Proposition 3.5 for ν_k–a.e. $x \in E_k$ the existence of the (ordinary) derivative $\frac{d\tilde{u}_x}{ds}$ on $R(\rho_k(x,\cdot))$ is sufficient since $\rho_k(x,\cdot)$ satisfies (H).

Corollary 3.6. Let $k \in K$ and let $f : E \to \mathbb{R}$ be $\mathcal{B}(E)$–measurable and k–invariant. Then (if f also denotes the corresponding $L^2(E;\mu)$–class) $f \in D(\mathcal{E}_{\mu,k})$ and $\frac{\partial^\mu f}{\partial k} = 0$.

Proof. By Fubini's theorem and the k–invariance of f we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}}\int |f(z+tk) - f(z)|\mu(dz)e^{-|t|}\,dt \\ &= \int_{E_k}\int_{\mathbb{R}}\int_{\mathbb{R}} |f(x+(s+t)k) - f(x+sk)|e^{-|t|}\,dt\rho_k(x,s)\,ds\ \nu_k(dx) \\ &= \int_{\mathbb{R}}\int_{E_k}\int_{\mathbb{R}} |f(x+tk) - f(x+sk)|e^{-|t-s|}\rho_k(x,s)\,ds\ \nu_k(dx)\,dt\ . \end{aligned}$$

Hence there exists $t \in \mathbb{R}$ such for ν_k–a.e. $x \in E_k$

$$f(x+sk) = f(x+tk)\,ds\text{–a.e. on } R(\rho_k(x,\cdot))\ ,$$

since $\rho_k(x,\cdot) > 0$ ds–a.e. on $R(\rho_k(x,\cdot))$. Therefore, for ν_k–a.e. $x \in E_k$, $s \mapsto f(x+sk)$ has a ds–version $\tilde{f}_x$ which is constant on $R(\rho_k(x,\cdot))$, hence $\frac{d\tilde{f}_x}{ds} = 0$ on $R(\rho_k(x,\cdot))$. Consequently,

$$\left(\frac{\partial\tilde{f}_x}{ds}\right)_{x\in E_k} = 0$$

and hence $f \in D(\mathcal{E}_{\mu,k})$ with $\frac{\partial^\mu f}{\partial k} = 0$ by Proposition 3.5. $\square$

Proof of Theorem 3.4. Let f be a K–invariant $\mathcal{B}(E)$–measurable function. Then by Corollary 3.6, $f \in D(\mathcal{E}_\mu)$ and $\mathcal{E}_\mu(f,f) = 0$. Hence by assumption f is constant (μ–a.e.). $\square$

We have the following converse to Theorem 3.4.

Theorem 3.7. *Let $\mu \in \mathcal{G}^b$ be such that the following condition holds:*

For every $k \in K$ there exists E_k as above such that for the corresponding ν_k, ρ_k one has that

$$R(\rho_k(x,\cdot)) = \mathbb{R} \text{ for } \nu_k\text{–a.e. } x \in E_k \; . \tag{3.2}$$

Then μ is K–quasi–invariant and, if μ is K–ergodic, then $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ is irreducible.

Proof. The K–quasi–invariance of μ follows by [AR 90 b, Proposition 2.6] , since $\rho_k(x,\cdot) > 0$ ds–a.e. on $R(\rho_k(x,\cdot))$ for ν_k–a.e. $x \in E_k$. The last part of the assertion follows by the following proposition. $\square$

Proposition 3.8. *Consider the situation of Theorem 3.7 but with μ not necesarrily K–ergodic. Let $u \in D(\mathcal{E}_\mu)$ such that $\mathcal{E}_\mu(u,u) = 0$. Then u is K–invariant.*

Proof. Let $k \in K$ and $t \in \mathbb{R}$. Then by Proposition 3.5 we have for ν_k–a.e. $x \in E_k$

$$u(x + (t+s)k) = u(x + sk) \text{ for } ds\text{–a.e. } s \in \mathbb{R} \; ,$$

since $R(\rho_k(x,\cdot)) = \mathbb{R}$ (and since Lebesgue measure is translation invariant). Hence

$$\begin{aligned} &\int |u(z+tk) - u(z)| \mu(dz) \\ &= \int_{E_k} \int_{\mathbb{R}} |u(x+(s+t)k) - u(x+sk)| \rho_k(x,s)\, ds \;\; \nu_k(dx) \\ &= 0 \; . \end{aligned}$$

$\square$

Remark 3.9. **(i)** It follows immediately by [AKR 90, Proposition 2.4] that

$$R(\rho_k(x,\cdot)) = \left\{ t \in \mathbb{R} \mid \inf_{s \in [t-\varepsilon, t+\varepsilon]} \rho_k(x,s) > 0 \text{ for some } \varepsilon > 0 \right\} \; . \tag{3.3}$$

In [AH–K 77] this equivalent description of $R(\rho_k(x,\cdot))$ was used. Note, however, that (3.3) only holds since $k \in K$. For a generalization of part of the above in case where instead of $k \in K$ it is merely assumed that $(\mathcal{E}_{\mu,k}, \widetilde{\mathcal{F}C}_b^\infty)$ is (well–defined and) closable on $L^2(E;\mu)$ we refer to [Ki 95].

(ii) It is well–known that condition (3.2) in Theorem 3.7 cannot be dropped. There is a corresponding example even in the case $E = \mathbb{R}$ (cf. [AH–K 77, Example 3.1]).

(iii) Again by [AKR 90, Proposition 2.4], it is known that condition (3.2) can be expressed entirely in terms of b. More precisely, (3.2) is equivalent to the following: for every $k \in K$ there exists E_k as above such that for the corresponding ν_k, ρ_k one has that for ν_k–a.e. $x \in E_k$

$$s \mapsto b_k(x + sk) \text{ is locally } ds\text{–integrable on } \mathbb{R} .$$

(iv) Under assumption (3.2) $\mathcal{G}^b$ can also be described by a Cameron–Martin type formula. In this case one can prove that $\mathcal{G}^b_{ext} \neq \emptyset$ provided that $\mathcal{G}^b \neq \emptyset$ and one also has an integral representation of an arbitrary element in $\mathcal{G}^b$ in terms of extreme ones. This generalizes a result in [SK 84] (proved by different means) in an essential way. For details we refer to [Ki 95].

4. Irreducibility and time ergodicity of the corresponding stochastic dynamics

Crucial for this section is the clear understanding of the subtle problem known by the name "Markov uniqueness problem". Here we use the in our framework simplest formulation of the problem and refer to [RZ 92], [RZ 94], [ARZ 93 a], [ARZ 93 b], [AKoR 93 b] for details, background literature and the origin of this terminology. Let E, K, b be as in Section 1 such that (1.4) holds and fix $\mu \in \mathcal{G}^b$. In this section for simplicity we assume that E is a separable Banach space. However, most of the results below also hold for other types of locally convex spaces (like e.g. conuclear spaces in the sense of [AR 89, Subsection 3 c)] provided $\int |\ell(z)|\mu(dz) < \infty$ for all $\ell \in E'$). Define $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ to be the closure of $(\mathcal{E}_\mu, \widetilde{\mathcal{F}C}{}^\infty_b)$ on $L^2(E;\mu)$. By [AR 90 a, Sect. 3] this is also a Dirichlet form such that $D(\mathcal{E}^0_\mu) \subset D(\mathcal{E}_\mu)$ and $\mathcal{E}^0_\mu(u,v) = \mathcal{E}_\mu(u,v)$ for all $u, v \in D(\mathcal{E}^0_\mu)$. Note, however, that unless $K = \{k\}$ it is not clear whether $D(\mathcal{E}^0_\mu) = D(\mathcal{E}_\mu)$, i.e., whether $\widetilde{\mathcal{F}C}{}^\infty_b$ is dense in $D(\mathcal{E}_\mu)$ w.r.t. the norm $\|u\|_{1,2,\mu} := (\mathcal{E}_\mu(u,u) + (u,u))^{1/2}, u \in D(\mathcal{E}_\mu)$. $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ resp. $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ are sometimes called the *minimal* resp. *maximal Dirichlet forms* associated with μ (cf. [AKR 90]).

Definition 4.1. We say that *Markov uniqueness holds* for μ if $(\mathcal{E}_\mu, D(\mathcal{E}_\mu)) = (\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$.

The main result of [RZ 94] (cf. Theorem 1.1 therein) says that Markov uniqueness always holds for μ if $\dim E < \infty$ and K consists of a linear basis of E. For sufficient conditions and examples in case $\dim E = \infty$ we refer to [RZ 92], [RZ 94], [ARZ 93 a], [ARZ 93 b], [AKoR 93 a], [AKoR 93 b]. However, so far, we do not have an example where $D(\mathcal{E}^0_\mu) \neq D(\mathcal{E}_\mu)$.

The importance of the smaller form $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ is that (in contrast to $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$) it is known to be *quasi–regular* (cf. below) which has several important consequences. In order to explain this in more detail and to recall the definition we need some preparations. We formulate everything for a fixed abstract Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ (cf. [MR 92]) on $L^2(E;\mu)$. For the simplicity the non–expert reader may think that $(\mathcal{E}, D(\mathcal{E}))$ stands for $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ or $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$.

Definition 4.2. (cf. [MR 92, Chap. III, Definitions 2.1 and 3.2])
(i) A sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of E is called an $\mathcal{E}$*–nest* if

$$\bigcup_{n\in\mathbb{N}} \{u \in D(\mathcal{E}) \mid u = 0 \ \mu\text{–a.e. on } E \setminus F_n\}$$

is dense in $D(\mathcal{E})$ w.r.t. $\mathcal{E}_1^{1/2}$, i.e., the norm determined by $\mathcal{E}_1 := \mathcal{E} + (\ ,\ )_{L^2(E;\mu)}$.
(ii) A set $N \subset E$ is called $\mathcal{E}$*–exceptional* if $N \subset E \setminus (\bigcup_{n\in\mathbb{N}} F_n)$ for some $\mathcal{E}$–nest $(F_n)_{n\in\mathbb{N}}$. We say that a property of points in E holds $\mathcal{E}$*–quasi–everywhere* (abbreviated $\mathcal{E}$*–q.e.*), if the property holds outside some $\mathcal{E}$–exceptional set.
(iii) A function $f : E \to \mathbb{R}$ is called $\mathcal{E}$*–quasi–continuous* if there exists an $\mathcal{E}$–nest $(F_n)_{n\in\mathbb{N}}$ such that the restriction $f|_{F_n}$ of f to F_n is continuous for all $n \in \mathbb{N}$.

Definition 4.3. (cf. [MR 92, Chap. IV, Definition 3.1])
$(\mathcal{E}, D(\mathcal{E}))$ is called *quasi–regular* if:

(i) There exists an $\mathcal{E}$–nest consisting of compact sets.

(ii) There exists an $\mathcal{E}_1^{1/2}$–dense subset of $D(\mathcal{E})$ whose elements have $\mathcal{E}$–quasi–continuous μ–versions.

(iii) There exist $u_n \in D(\mathcal{E})$, $n \in \mathbb{N}$, having $\mathcal{E}$–quasi–continuous μ–versions $\tilde{u}_n, n \in \mathbb{N}$, and an $\mathcal{E}$–exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

In Definition 4.3 and below μ–versions of elements in $L^2(E;\mu)$ are always assumed to be $\mathcal{B}(E)$–measurable.

By the main result in [AMR 93 a] the property of quasi–regularity analytically characterizes all Dirichlet forms associated with right–continuous strong Markov processes, hence all that have a probabilistic counterpart. It is known that this particularly applies to $(\mathcal{E}_\mu^0, D(\mathcal{E}_\mu^0))$, since it obviously has properties 4.3 (ii), (iii) (cf. [MR 92, Chap. IV, Subsection 4. b)]) and since it has been proved that it also satisfies 4.3 (iii) (cf. ibidem or [RS 91, Proposition 3.1]). This means we have the following

Theorem 4.4. *There exists a diffusion process* $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, (\theta_t)_{t\ge0}, (X_t)_{t\ge0}, (P_z)_{z\in E})$ *on* E *(cf. [Dy 65]) which is* properly associated *with* $(\mathcal{E}_\mu^0, D(\mathcal{E}_\mu^0))$*, i.e., for all ($\mu$–versions of)* $f \in L^2(E;\mu)$ *and all* $t \ge 0$ *the function*

$$z \mapsto p_t f(z) := \int_\Omega f(X_t)\, dP_z, z \in E \ , \tag{4.1}$$

is an $\mathcal{E}_\mu^0$*–quasi–continuous version of* $\overset{\circ}{T}{}_t^\mu f$. $\mathbb{M}$ *is up to m–equivalence unique (cf. [MR 92, Chap. IV, Sect. 6]). Here* $(\overset{\circ}{T}{}_t^\mu)_{t\ge0}$ *is the operator semigroup associated with* $(\mathcal{E}_\mu^0, D(\mathcal{E}_\mu^0))$*. In particular,* $\mathbb{M}$ *is* μ–symmetric, *i.e.,* $\int p_t f g \, d\mu = \int f p_t g \, d\mu$ *for all bounded* $\mathcal{B}(E)$*–measurable functions* $f, g : E \to \mathbb{R}, t > 0$*, as well as* conservative, *i.e.,* $p_t 1 = 1$ $\mathcal{E}_\mu^0$*–q.e. for all* $t \ge 0$*. Thus* μ *is an invariant measure for* $\mathbb{M}$*.*

The fact that $\mathbb{M}$ is a *diffusion*, i.e., has continuous sample paths follows by the main result in [AMR 93 b] (or cf. [MR 92, Chap. V, Sect. 1]). In our present case where

E is a separable Banach space Theorem 4.4 was first proved by B. Schmuland [S 90].

Remark 4.5. **(i)** In Theorem 4.4 $\mathbb{M}$ is canonical, i.e., $\Omega = C([0,\infty[\to E), X_t(\omega) = \omega(t), t \geq 0,\ \omega \in \Omega, (\mathcal{F}_t)_{t\geq 0}$ together with $\mathcal{F}$ is the corresponding minimum completed admissible family (cf. [F 80, Sect. 4.1]) and $\theta_t, t \geq 0$, are the corresponding natural time shifts.
(ii) Since $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ has properties 4.3 (i), (iii), so has $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$. Indeed, 4.3 (iii) obviously holds and 4.3 (i) is easily obtained as a consequence of [MR 92, Chap. III, Proposition 2.11 (with $h \equiv g \equiv 1$) and Exercise 2.10]. It is, however, not clear at all whether 4.3 (ii) is satisfied for $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$. This problem is still under investigation.

The fact that $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ is quasi–regular has two more important consequences (cf. Theorem 4.6 and Corollary 4.7 below). As usual we set

$$P_\mu := \int P_z\, \mu\,(dz)$$

and we recall that P_μ is called *(time) ergodic* if any bounded $\mathcal{F}$–measurable function $G : \Omega \to \mathbb{R}$ which is θ_t–invariant for all $t \geq 0$, is a constant (P_μ–a.e.). The following is now a consequence of Theorem 4.4 above and [F 82] (see also [Mü 92, Subsection 4.1]), resp. [F 83].

Theorem 4.6. *$(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ is irreducible if and only if P_μ is ergodic. In this case, if $(p_t)_{t>0}$ is defined by (4.1), then $\lim\limits_{t\to\infty} p_t f = \int f\, d\mu$ $\mathcal{E}^0_\mu$–q.e. for all bounded $\mathcal{B}(E)$–measurable functions $f : E \to \mathbb{R}$.*

The last part of Theorem 4.6 is known as "Fukushima's quasi–everywhere ergodic theorem". We note that the corresponding proof in [F 83] was only carried out for the smaller class of regular Dirichlet forms on locally compact separable metric state spaces E. But by the regularization method developed in [AMR 92] (see also [MR 92, Chap. VI]) this result immediately carries over to quasi–regular Dirichlet forms on general Hausdorff topological spaces E for which $\mathcal{B}(E)$ is generated by the continuous functions. Hence Theorem 4.6 indeed holds with $(p_t)_{t\geq 0}$ given by $\mathbb{M}$ associated with $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$.
We recall that a probability measure ν on $\mathcal{B}(E)$ is called $(p_t)_{t\geq 0}$–invariant if for all bounded $\mathcal{B}(E)$–measurable functions $f : E \to \mathbb{R}$

$$\int p_t f\, d\nu = \int f\, d\nu \text{ for all } t \geq 0\ .$$

Corollary 4.7. If $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ is irreducible then μ is the only $(p_t)_{t\geq 0}$–invariant probability measure on $\mathcal{B}(E)$ which does not charge $\mathcal{E}^0_\mu$–exceptional sets.

Proof. This an immediate consequence of Theorem 4.6 and the fact that μ does not charge $\mathcal{E}^0_\mu$–exceptional sets (cf. e.g. [MR 92, Exercise 2.6 (ii)]). $\square$

Remark 4.8. Concerning uniqueness of invariant measures, Corollary 4.7 is the best result one can hope for in our very general situation. The fact that we can prove uniqueness among all probability measures charging no $\mathcal{E}^0_\mu$–exceptional sets rather merely among all which are absolutely continuous w.r.t. μ, comes from the *quasi–strong–Feller property* of $(p_t)_{t>0}$ (cf. [BoR 93, Sect. 1]) (i.e., $p_t f$ is $\mathcal{E}^0_\mu$–quasi–continuous for all $t \geq 0$ and all $f : E \to \mathbb{R}$, bounded, $\mathcal{B}(E)$–measurable; cf. Theorem 4.4). On the other hand, in general, one cannot hope to prove uniqueness among all probability measures without further regularity assumptions on $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$.

Since trivially the irreducibility of $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ implies that of $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ we can summarize the results of this section and Section 2 as follows.

Theorem 4.9. *Let $\mu \in \mathcal{G}^b$. Consider the following assertions:*

(i) $\mu \in \mathcal{G}^b_{ext}$.

(ii) $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ *is irreducible.*

(iii) $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ *is irreducible.*

(iv) P_μ *is (time) ergodic.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv). In particular, if Markov uniqueness holds for μ, then (i) – (iv) are equivalent and all results of this section hold for $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$.

Remark 4.10. We note that if $\mu \in \mathcal{G}^b_{ext}$, hence $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ is irreducible or equivalently P_μ is (time) ergodic, the large deviations of the occupation density of $\mathbb{M}$ from ergodic behaviour are described by the rate function which is exactly the Dirichlet form $\mathcal{E}^0_\mu$. This is a result which in this generality is due to S. Mück (cf. [Mü 92], [Mü 94]). In particular, by the existence result on extreme elements in $\mathcal{G}^b$ described in Remark 3.9 (iv), we obtain a large variety of non–Gaussian examples in infinite dimensions to which S. Mück's result applies.

5. Applications in classical statistical physics

In this section we shall apply the results of the previous sections to the equivalence of extremality of Gibbs measures and ergodicity of the stochastic dynamics in some models of classical lattice systems with unbounded spins. Our considerations will also essentially use a general approach to the study of the essential self–adjointness of Dirichlet operators developed in [AKoR 92], [AKoR 93 a], [AKoR 93 b]. In order to avoid too many technical complications which usually arise in case the spin spaces are not bounded (cf. [COPP 78] and also [BeH–K 82]), we confine ourselves to models which are lattice approximations for Euclidean quantum fields of polynomial type ("$P(\varphi)_d$–type" for short). The main ideas will become clearer this way. An application to more general lattice models will be discussed in the forthcoming paper [AKoR 95].

5.1 Classical lattice systems and Gibbs measures.

We denote the d–dimensional integer lattice by $\mathbb{Z}^d \subset \mathbb{R}^d$ and the set of all real sequences $x = (x_k)_{k \in \mathbb{Z}^d}$ by $\mathbb{R}^{\mathbb{Z}^d}$; $\mathbb{R}^{\mathbb{Z}^d}_0 \subset \mathbb{R}^{\mathbb{Z}^d}$ denotes the subset of all such sequences with at most finitely many non–zero components. $\mathbb{R}^{\mathbb{Z}^d}$ is a polish space for the product topology since $\mathbb{Z}^d$ is countable. $\mathcal{B} := \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$ denotes the corresponding Borel σ–algebra on $\mathbb{R}^{\mathbb{Z}^d}$.

Let $\mathcal{F} := \mathcal{F}(\mathbb{Z}^d)$ be the set of all finite subsets of $\mathbb{Z}^d$. For $\Lambda \in \mathcal{F}$, let $\mathcal{B}_\Lambda$ be the σ–algebra generated by variables $(x_k)_{k\in\Lambda}$. If we identify $\mathbb{R}^{\mathbb{Z}^d}$ with $\mathbb{R}^\Lambda \times \mathbb{R}^{\Lambda^c} (\Lambda^c := \mathbb{Z}^d \setminus \Lambda)$ such that

$$\mathbb{R}^{\mathbb{Z}^d} \ni x \mapsto x_\Lambda \times x_{\Lambda^c} \in \mathbb{R}^\Lambda \times \mathbb{R}^{\Lambda_c} , \tag{5.1}$$

then a $\mathcal{B}_\Lambda$–measurable function is a function depending only on $x_\Lambda \in \mathbb{R}^\Lambda$. The symbol $\mathcal{M}(\mathbb{R}^{\mathbb{Z}^d})$ will denote the set of all probability measures on $(\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B})$.

We will also use some special subspaces in $\mathbb{R}^{\mathbb{Z}^d}$. First of all the role of the tangent space H will now be played by the space $\ell_2(\mathbb{Z}^d)$ of all square summable sequences over $\mathbb{Z}^d$, i.e.,

$$H := \ell_2(\mathbb{Z}^d) := \left\{ x \in \mathbb{R}^{\mathbb{Z}^d} \mid \|x\|_H^2 := \sum_{k\in\mathbb{Z}^d} x_k^2 < \infty \right\} .$$

For any $p \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ we introduce the Hilbert space

$$H_p := \left\{ x \in \mathbb{R}^{\mathbb{Z}^d} \mid \|x\|_{H_p}^2 := \sum_{k\in\mathbb{Z}^d} x_k^2 (1+|k|)^p < \infty \right\}$$

and its dual space (w.r.t. H)

$$H_{-p} := \left\{ x \in \mathbb{R}^{\mathbb{Z}^d} \mid \|x\|_{H_{-p}}^2 := \sum_{k\in\mathbb{Z}^d} x_k^2 (1+|k|)^{-p} < \infty \right\} .$$

Here $|k|$ denotes the Euclidean norm of $k \in \mathbb{Z}^d \subset \mathbb{R}^d$. The nuclear space of fastly decreasing sequences $\mathcal{S}(\mathbb{Z}^d)$ is defined by

$$\mathcal{S}(\mathbb{Z}^d) := pr \lim_{p\in\mathbb{Z}_+} H_p \left(= \bigcap_{p\in\mathbb{Z}_+} H_p \right) .$$

The corresponding dual space $\mathcal{S}'(\mathbb{Z}^d)$ (of slowly increasing sequences) has then the representation

$$\mathcal{S}'(\mathbb{Z}^d) = ind \lim_{p\in\mathbb{Z}_+} H_{-p} \left(= \bigcup_{p\in\mathbb{Z}_+} H_{-p} \right) .$$

Remark 5.1. Note that this definition of $\mathcal{S}'(\mathbb{Z}^d)$ is equivalent with that in [BeH–K 82]. Note also that there is an error in the appendix of [BeH–K 82]. In contrast to what is stated there, by [Kö 69, Statement (7) on p. 394], $\mathcal{S}'(\mathbb{Z}^d)$ equipped with the strong topology is *not* metrizible, since it is the dual of an infinite dimensional nuclear (hence not normable) space. Consequently, $\mathcal{S}'(\mathbb{Z}^d)$ equipped with the weak topology is *not* metrizable, too. Furthermore, also in contrast to what is erronously stated in [BeH–K 82] the weak and strong topologies on $\mathcal{S}'(\mathbb{Z}^d)$ do *not* coincide, but *only for sequences* (not for nets) weak and strong convergence is equivalent.

Any Hilbert space $H_{-p}, p \in \mathbb{Z}_+$, is a Borel subset of $\mathbb{R}^{\mathbb{Z}^d}$ and, therefore, $\mathcal{S}'(\mathbb{Z}^d) \in \mathcal{B}$ (see [BKo 88, Chapter 2]).

In the following we will use a subclass of measures from $\mathcal{M}(\mathbb{R}^{\mathbb{Z}^d})$.

Definition 5.2. (see [BeH–K 82]). A measure $\mu \in \mathcal{M}(\mathbb{R}^{\mathbb{Z}^d})$ is called *tempered* if its *mean* is in $\mathcal{S}'(\mathbb{R}^d)$, i.e.,

$$m^\mu := (m_k)_{k\in\mathbb{Z}^d} := (\mu(|x_k|))_{k\in\mathbb{Z}^d} \in \mathcal{S}'(\mathbb{Z}^d) \ .$$

Here (and further on) we use the notation

$$\mu(f) := \int_{\mathbb{R}^{\mathbb{Z}^d}} f(x)\, d\mu(x)$$

for any $f : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$ for which the integral makes sense. The set of all tempered measures will be denoted by $\mathcal{M}_t(\mathbb{R}^{\mathbb{Z}^d})$.

Note that for any $\mu \in \mathcal{M}_t(\mathbb{R}^{\mathbb{Z}^d})$ by [BeH–K 82, Proposition A.1] we have that $\mu(\mathcal{S}'(\mathbb{Z}^d)) = 1$. As an easy consequence of the uniform boundedness principle (cf. [AR 89, Proposition 3.7]) it follows that $\mu(H_{-p}) = 1$ for some $p \in \mathbb{Z}_+$ (see also Proposition 5.6 below). Therefore, any such measure can be considered as a probability measure on $(\mathcal{S}'(\mathbb{Z}^d), \mathcal{B}(\mathcal{S}'(\mathbb{Z}^d)))$.

The interaction for our lattice systems is given by a family $(\Phi_\Lambda)_{\Lambda\in\mathcal{F}}$ where

$$\begin{aligned} \Phi_\Lambda(x) &:= V(x_k), x \in \mathbb{Z}^d, \text{ if } \Lambda = \{k\} \\ \Phi_\Lambda(x) &:= W_{\{k,j\}}(x_k - x_j), x \in \mathbb{Z}^d, \text{ if } \Lambda = \{k,j\}, k \neq j \ , \\ \Phi_\Lambda &:\equiv 0 \text{ else } . \end{aligned} \tag{5.2}$$

Here $V, W_{\{k,j\}} : \mathbb{R} \to \mathbb{R}$ with $W_{\{k,j\}}$ even. This means that to any point $k \in \mathbb{Z}^d$ there corresponds one–particle potential $V(x_k)$ and to any two–point subset $\{k,j\} \subset \mathbb{Z}^d$ there corresponds a two particle potential $W_{\{k,j\}}(x_k - x_j)$ that describes the interaction between particles at sites k and j of the lattice.

In the said models of $P(\varphi)_d$–type the interaction Φ has a more concrete form. We will use as the one–particle potential V any polynomial of the form

$$V(q) = a_{2m}q^{2m} + \cdots + a_1 q + a_0, q \in \mathbb{R}, a_{2m} > 0 , \text{ with } m \geq 2 \ .$$

The interaction $W_{\{k,j\}}$ is a quadratic nearest neighbors interaction, i.e.,

$$W_{\{k,j\}}(x_k - x_j) = \begin{cases} (x_k - x_j)^2 & , \ |k-j| = 1 \\ 0 & , \ |k-j| > 1. \end{cases} \tag{5.3}$$

For the rest of Section 5 we fix Φ of the above type.

Definition 5.3. For each $\Lambda \in \mathcal{F}$ and given $\xi \in \mathcal{S}'(\mathbb{Z}^d)$ the *Hamiltonian* $H^{\Phi}_{\Lambda,\xi}$ *with boundary conditions* ξ_{Λ^c} outside the volume Λ is the function

$$H^{\Phi}_{\Lambda,\xi}(x_\Lambda) := \sum_{k\in\Lambda} V(x_k) + \sum_{\{k,j\}\subset\Lambda} W_{\{k,j\}}(x_k - x_j) + \sum_{\substack{\{k,j\}:\\ k\in\Lambda, j\in\Lambda^c}} W_{\{k,j\}}(x_k - \xi_j) \ . \tag{5.4}$$

Due to the assumptions about Φ this function is continuous w.r.t. $x_\Lambda \in \mathbb{R}^\Lambda$, $\xi \in \mathcal{S}'(\mathbb{Z}^d)$ and positive. In particular, the conditional partition function

$$Z^{\Phi}_{\Lambda}(\xi_{\Lambda^c}) := \int_{\mathbb{R}^\Lambda} e^{-H^{\Phi}_{\Lambda,\xi}(x_\Lambda)} \prod_{k\in\Lambda} dx_k \ , \ \xi \in \mathcal{S}'(\mathbb{Z}^d) \ ,$$

is well–defined.

Definition 5.4. For $\xi \in \mathcal{S}'(\mathbb{Z}^d)$ the probability measure $\pi^{\Phi}_{\Lambda}(\xi|\,\cdot)$ on $\mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$ defined by

$$\pi^{\Phi}_{\Lambda}(\xi|A) := \left[Z^{\Phi}_{\Lambda}(\xi_{\Lambda^c})\right]^{-1} \int \mathbb{1}_A(x_\Lambda \times \xi_{\Lambda^c}) e^{-H^{\Phi}_{\Lambda,\xi}(x_\Lambda)} \prod_{k\in\Lambda} dx_k \tag{5.5}$$

for $A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$ is called the *Gibbs distribution in the volume* Λ *with boundary condition* ξ *corresponding to the interaction* Φ.

Remark 5.5. If we consider the measures $\pi^{\Phi}_{\Lambda}(\xi|\ \cdot)$ on $\mathcal{B}(\mathcal{S}'(\mathbb{Z}^d))$, then it is easy to check that $(\pi^{\Phi}_{\Lambda})_{\Lambda\in\mathcal{F}}$ forms a specification in the sense of [P 76, Sect. 1].

Since $\mu(\mathcal{S}'(\mathbb{Z}^d)) = 1$ for all $\mu \in \mathcal{M}_t(\mathbb{R}^{\mathbb{Z}^d})$ we can now define the corresponding Gibbs measures in $\mathcal{M}_t(\mathbb{R}^{\mathbb{Z}^d})$.

Definition 5.6. A measure $\mu \in \mathcal{M}_t(\mathbb{R}^{\mathbb{Z}^d})$ is called a *tempered Gibbs measure for* Φ if and only if

$$\mu\pi^{\Phi}_{\Lambda} = \mu \text{ for all } \Lambda \in \mathcal{F}, \text{ i.e. } ,$$

$$\int d\mu(\xi)\ Z^{\Phi}_{\Lambda}(\xi_{\Lambda^c})^{-1} \int \mathbb{1}_A(x_\Lambda \times \xi_{\Lambda^c}) e^{-H^{\Phi}_{\Lambda,\xi}(x_\Lambda)} \prod_{k\in\Lambda} dx_k = \mu(A) \tag{5.6}$$

for all $\Lambda \in \mathcal{F}$, $A \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$.

(5.6) is called *Dobrushin–Lauford–Ruelle* (DLR) *equation.* The set of all tempered Gibbs measures for our interaction Φ will be denoted by $\mathcal{G}_t(\Phi)$. In the sequel we shall also use a subset $\mathcal{G}_{exp}(\Phi)$ of $\mathcal{G}_t(\Phi)$ which is characterized by the following condition:

$$\mu \in \mathcal{G}_{exp}(\Phi) :\Leftrightarrow \mu \in \mathcal{G}_t(\Phi) \text{ and there exists } C(\mu) \in]0,\infty[\text{ such that } \mu(e^{|x_k|}) \le C(\mu) \text{ for all } k \in \mathbb{Z}^d \ . \tag{5.7}$$

Due to [Si 82, Chap. 1, § 5], [PrF 91, Chap. 3, § 4] the set $\mathcal{G}_{exp}(\Phi)$ is a non–empty convex set which in general contains (strictly) more than one element. In fact in [Si 82] it is proven that $\mathcal{G}_t(\Phi) = \mathcal{G}_{exp}(\Phi)$. The latter is a direct consequence of the estimate (1.16) in [Si 82, Chap. 1, § 5].

Remark 5.7. By the definition of $\mathcal{G}_{exp}(\Phi)$ and since $\mathcal{G}_{exp}(\Phi) \neq \emptyset$, it is an immediate consequence of a famous result by Föllmer–Dynkin (cf. [Fö 75], [Dy 78] and, in particular [P 76, Theorem 2.2]) that any $\mu \in \mathcal{G}_{exp}(\Phi)$ can uniquely be represented as an integral over extreme elements in $\mathcal{G}_{exp}(\Phi)$. In particular, $(\mathcal{G}_{exp}(\Phi))_{ext} \neq \emptyset$.

Proposition 5.8. *Let $\mu \in \mathcal{G}_{exp}(\Phi)$. Then $\mu(H_{-p}) = 1$ provided $p > d$.*

Proof. The inequality $q^2 \leq 2e^{|q|}, q \in \mathbb{R}$, and (5.7) imply that for any $\mu \in \mathcal{G}_{exp}(\Phi)$ there exists $m_2 = m_2(\mu) \in]0, \infty[$ such that $\mu(x_k^2) \leq m_2$ for all $k \in \mathbb{Z}^d$. Let $p \in \mathbb{N}$ with $p > d$. Then

$$\sum_{k \in \mathbb{Z}^d} (1 + |k|)^{-p} < \infty ,$$

and therefore

$$\int \|x\|^2_{H_{-p}} \, d\mu(x) = \sum_{k \in \mathbb{Z}^d} \mu(x_k^2)(1 + |k|)^{-p} \leq$$
$$\leq m_2 \sum_{k \in \mathbb{Z}^d} (1 + |k|)^{-p} < \infty .$$

This shows that the function $\|x\|^2_{H_{-p}}$, $x \in \mathcal{S}'(\mathbb{Z}^d)$, is finite μ–a.s., i.e. $\mu(H_{-p}) = 1$. $\square$

So, any tempered Gibbs measure $\mu \in \mathcal{G}_t(\Phi)$ can be considered as a probability measure on any Hilbert space H_{-p} with $p > d$. In the following we fix some $p_0 > d$. Then in our considerations the space H_{-p_0} will play the role of the space E from Sections 1 – 4 and we have $E' = H_{p_0}$.

5.2 Characterization of the Gibbs measures by integration by parts.

As in Example 2.2 we will deal with the triple

$$E' := H_{p_0} \subset H := \ell_2(\mathbb{Z}^d) \subset H_{-p_0} =: E \tag{5.8}$$

with $p_0 > d$ fixed. Here we identified $\ell_2(\mathbb{Z}^d)$ with its dual so that j_H ist just the natural embedding of E' into H. Let K be the standard orthonormal basis $\{e_k | k \in \mathbb{Z}^d\}$ in H, i.e., $e_k = (\delta_{kj})_{j \in \mathbb{Z}^d}$, $k \in \mathbb{Z}^d$. Note that $K = K_0 \subset E'$ in the notation of Example 2.2.

For any $k \in \mathbb{Z}^d$ we consider the function

$$b_k^\Phi(x) := -V'(x_k) - 2 \sum_{j \in \mathbb{Z}^d, |j-k|=1} (x_k - x_j) \; , \; x \in E = H_{-p_0} \tag{5.9}$$

Let $\mu \in \mathcal{G}_{exp}(\Phi)$. We obviously have that $b_k^\Phi \in L^p(E; \mu)$ for all $p \geq 1$ and, moreover, that there exists $C_b(\mu) \in]0, \infty[$ such that

$$\|b_k^\Phi\|_{L^2(E;\mu)} \leq C_b(\mu) \text{ for all } k \in \mathbb{Z}^d . \tag{5.10}$$

Let $b := (b_k^\Phi)_{k \in \mathbb{Z}^d}$. We want to relate $\mathcal{G}_{exp}(\Phi)$ with $\mathcal{G}^{b^\Phi}$ as introduced in Section 1. To this end we define $\mathcal{M}_{exp} = \mathcal{M}_{exp}(\mathbb{R}^{\mathbb{Z}^d})$ to be the set of all probability measures on $\mathcal{B}(\mathbb{R}^{\mathbb{Z}^d})$ which satisfy the exponential bound (5.2).

Proposition 5.9. $\mathcal{G}_{exp}(\Phi) = \mathcal{G}^{b^\Phi} \cap \mathcal{M}_{exp}$.

Proof. From (5.4), (5.5) if follows immediately that

$$\mathcal{G}_{exp}(\Phi) \subset \mathcal{G}^{b^\Phi} \cap \mathcal{M}_{exp} \ .$$

The dual inclusion is proved analogously to [AAAK 93, Theorem 3.2]. $\square$

Remark 5.10. The fact that Gibbs measures can be characterized by an integration by parts formula instead of the DLR equations is well–known also in many other models. We refer e.g. to [Fr 74], [HSt 76], [Ro 77] and [RoeZ 93].

Let $(\mathcal{G}_{exp}(\Phi))_{ext}$ denote the extreme points of $\mathcal{G}_{exp}(\Phi)$.

Corollary 5.11. $(\mathcal{G}_{exp}(\Phi))_{ext} = \mathcal{G}^{b^\Phi}_{ext} \cap \mathcal{M}_{exp}$.

Proof. Obviously, by Proposition 5.9 the set on the right hand side is contained in the one on the left hand side. Conversely, if

$$\mu \in \left(\mathcal{G}_{exp}(\Phi)\right)_{ext}$$

then $\mu \in \mathcal{G}^{b^\Phi} \cap \mathcal{M}_{exp}$ by Proposition 5.9. Suppose $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$, $\alpha \in]0,1[$, $\mu_1, \mu_2 \in \mathcal{G}^{b^\Phi}(\Phi)$. Then $\mu_1, \mu_2 \in \mathcal{M}_{exp}$, hence $\mu_1, \mu_2 \in \mathcal{G}_{exp}(\Phi)$ by Proposition 5.9. Therefore, by assumption $\mu_1 = \mu_2$. $\square$

Remark 5.12. The reader should note that in Sections 1 – 4 it was not claimed that $\mathcal{G}^b_{ext} \neq \emptyset$ if $\mathcal{G}^b \neq \emptyset$. For general b this is at present on open question for us. But in the situation studied in this section $\mathcal{G}^{b^\Phi}_{ext} \neq \emptyset$ is ensured by Corollary 5.11 and Remark 5.7 above.

5.3 Markov uniqueness for $\mu \in \mathcal{G}_{exp}(\Phi)$.

We shall prove in this subsection that Markov uniqueness holds for all $\mu \in \mathcal{G}_{exp}(\Phi)$. So, let $\mu \in \mathcal{G}_{exp}(\Phi)$. Let $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ resp. $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ be the minimal resp. maximal Dirichlet forms associated with μ as introduced in Section 4 (with E, K as above). Note that $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$ $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ are defined since $\mu \in \mathcal{G}^{b^\Phi}$ by Proposition 5.9. Recall also that we are in the situation of Example 2.2 so that

$$\mathcal{E}^0_\mu(u,v) = \frac{1}{2}\int \langle \nabla u, \nabla v\rangle_H \, d\mu \text{ for all } u, v \in \mathcal{F}C_b^\infty$$

where $H = \ell_2(\mathbb{Z}^d)$. Let $(L^0_\mu, D(L^0_\mu))$ be the generator corresponding to $(\mathcal{E}^0_\mu, D(\mathcal{E}^0_\mu))$, i.e., the unique negative definite self–adjoint operator such that

$$\mathcal{E}^0_\mu(u,v) = \left(\sqrt{-L^0_\mu}u, \sqrt{-L^0_\mu}v\right)_{L^2(E,\mu)} \text{ for all } u, v \in D(\mathcal{E}^0_\mu) = D\left(\sqrt{-L^0_\mu}\right) \ .$$

Define the operator $\left(L_\Phi, \widetilde{\mathcal{F}C}_b^\infty(K)\right)$ by

$$L_\Phi u(x) = \frac{1}{2}\sum_{i=1}^N \partial_i^2 u_N\left(x_{k_1}, \dots, x_{k_N}\right) + \frac{1}{2}\sum_{i=1}^N b^\Phi_{k_i}(x)\partial_i u_N(x_{k_1}, \dots, x_{k_N}) \qquad (5.11)$$

for $u \in \mathcal{F}C_b^\infty(K)$, $u(x) = u_N(x_{k_1}, \dots, x_{k_N})$, $N \in \mathbb{N}$, $k_1, \dots, k_N \in \mathbb{Z}^d$, $u_N \in C_b^\infty(\mathbb{R}^N)$. By Example 2.2 both $(L^0_\mu, D(L^0_\mu))$ and $(L_\mu, D(L_\mu))$ extend $(L_\Phi, \widetilde{\mathcal{F}C}_b^\infty(K))$, in particular $(L_\Phi, \widetilde{\mathcal{F}C}_b^\infty(K))$ is well–defined as an operator on $L^2(E;\mu)$ by (5.11).

Theorem 5.13. *For any $\mu \in \mathcal{G}_{exp}(\Phi)$ the operator $(L_\Phi, \widetilde{\mathcal{F}C}_b^\infty(K))$ on $L^2(E;\mu)$ is* essentially self–adjoint *(i.e., the closure of $(L_\Phi, \widetilde{\mathcal{F}C}_b^\infty(K))$ on $L^2(E;\mu)$ is self–adjoint). In particular, Markov uniqueness holds for all $\mu \in \mathcal{G}_{exp}(\Phi)$.*

Proof. Let $\mu \in \mathcal{G}_{exp}(\Phi)$. The first part of the assertion is an immediate consequence of (the proof of) [AKoR 93 b, Theorem 5]. We emphasize that the assumption that there is only one Gibbs state made there, is not needed for the proof. It is easy to check that it is enough to have the exponential integrability property (5.7) that any $\mu \in \mathcal{G}_{exp}(\Phi)$ in our situation enjoys. The last part of the assertion is a simple consequence of the first. Since both $(L_\mu, D(L_\mu))$ and $(L_\mu^0, D(L_\mu^0))$ extend $(L_\Phi, \widetilde{\mathcal{F}C}_b^\infty(K))$ and hence its self–adjoint closure and since both these operators are self–adjoint they must both coincide with the closure. Thus $(L_\mu, D(L_\mu)) = (L_\mu^0, D(L_\mu^0))$, consequently $(\mathcal{E}_\mu, D(\mathcal{E}_\mu)) = (\mathcal{E}_\mu^0, D(\mathcal{E}_\mu^0))$. □

5.4 Characterizations of $(\mathcal{G}_{exp}(\Phi))_{ext}$.

Let $\mu \in \mathcal{G}_{exp}(\Phi)$ and $\mathbb{M} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\theta_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in E})$ be the diffusion process on E properly associated with $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ (cf. Theorem 4.4) and $P_\mu := \int P_x \,\mu\,(dx)$. Before we state and prove the main result of this section we note:

Remark 5.14. Since b_k^Φ is continuous on E for all $k \in \mathbb{Z}^d$, it follows by Remark 3.9 (iii) that condition (3.2) in Theorem 3.7 is fulfilled for any $\mu \in \mathcal{G}^{b^\Phi}$, hence by Proposition 5.9 by any $\mu \in \mathcal{G}_{exp}(\Phi)$. Consequently, Theorem 3.7 applies to any $\mu \in \mathcal{G}_{exp}(\Phi)$ and each is, in particular, K–quasi–invariant.

Theorem 5.15. *Let $\mu \in \mathcal{G}_{exp}(\Phi)$ and $\mathbb{M}$ as above. Then the following assertions are equivalent:*

(i) $\mu \in (\mathcal{G}_{exp}(\Phi))_{ext}$.

(ii) *μ is K–ergodic.*

(iii) *$(\mathcal{E}_\mu, D(\mathcal{E}_\mu))(= (\mathcal{E}_\mu^0, D(\mathcal{E}_\mu^0)))$ is irreducible.*

(iv) *$(T_t^\mu)_{t\geq 0}$ is irreducible (resp. ergodic).*

(v) *P_μ is (time) ergodic.*

In this case Fukushima's quasi–everywhere ergodic theorem applies.

Proof. Corollary 5.11, Theorems 3.4, 3.7, 4.9, 5.13, and Proposition 2.3 imply that (i) – (v) are all equivalent. The last part of the assertion follows by Theorem 4.6. □

A deficiency of the above theorem is that the diffusion process $\mathbb{M}$, in this case sometimes called the *stochastic dynamics associated with the interaction* Φ, is constructed from $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ hence depends on μ (i.e., all path space measures P_x, $x \in E$, already depend on μ). In our particular situation here we have enough regularity of b^Φ to give an alternative process construction using an appropriate stochastic differential equation instead of the Dirichlet form $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$. This process is then universal, i.e., independent of $\mu \in \mathcal{G}_{exp}(\Phi)$. This is possible due to known results and will be illustrated in the next subsection.

5.5 A universal construction of the associated stochastic dynamics.

The appropriate stochastic differential equation is a system of Langevin equations, namely:

$$\begin{aligned} d\xi_k^x(t) &= -\sum_{\substack{j\in\mathbb{Z}^d\\ |j-k|=1}} (\xi_k^x(t)-\xi_j^x(t))\,dt - \frac{1}{2}V'(\xi_k^x(t)\,dt + dw_k(t)\\ \xi_k^x(0) &= x_k\ , k\in\mathbb{Z}^d \end{aligned} \tag{5.12}$$

or in vector form

$$\begin{aligned} d\xi^x(t) &= \frac{1}{2}b^\Phi(\xi^x(t))\,dt + dw(t) \\ \xi^x(0) &= x\ . \end{aligned} \tag{5.14}$$

Here $w(t)=(w_k(t))_{k\in\mathbb{Z}^d}$, $t\geq 0$, is a family of independent standard one–dimensional Wiener processes on a probability space $(\widetilde{\Omega},\Sigma,P)$, $\xi^x(t)=(\xi_k^x(t))_{k\in\mathbb{Z}^d}\in\mathbb{R}^{\mathbb{Z}^d}$, $t\geq 0$, and $x\in\mathbb{R}^{\mathbb{Z}^d}$ is a given starting point.

As is well known, the system (5.12) has a unique strong solution

$$[0,+\infty[\ni t\mapsto \xi^x(t)\in E$$

for any $x\in E$ which defines a diffusion process with state space E (see e.g. [DRo 79, Theorems 3.2, 3.6], [LRi 85, Sections 3 and 5], and also [LRi 94] as well as references therein). Let us introduce the Markov semigroup which is generated by this process:

$$(p_t^\Phi f)(x) := E^P\left[f(\xi^x(t))\right], t\geq 0, x\in E\ ,$$

where $f\in C_b(E)$ (:= the space of bounded continuous functions on E) and E^P is the expectation on $(\widetilde{\Omega},\Sigma,P)$. Due to [DRo 79] we have the following estimate about the dependency of $\xi^x(t)$, $t\geq 0$, on the starting point $x\in E$: there exist $\lambda, C\in]0,\infty[$ such that for all $x',x''\in E$, $t\geq 0$,

$$\|\xi^{x'}(t)-\xi^{x''}(t)\|_E \leq C\,e^{\lambda t}\|x'-x''\|_E$$

P–a.s.. The latter, in particular, implies that each p_t^Φ is a *Feller* kernel, i.e., maps $C_b(E)$ into itself. Let $\mu\in\mathcal{G}_{exp}(\Phi)$. By [DRo 79, Theorem 4.10] we know that each p_t^Φ is μ–symmetric.

Hence (cf. e.g. [F 80, p. 27/28]) $(p_t^\Phi)_{t\geq 0}$ uniquely extends to a strongly continuous semigroup $(\widetilde{T}_t^\mu)_{t\geq 0}$ of symmetric contraction operators on $L^2(E;\mu)$. By the (finite dimensional) Itô–formula it easily follows that its generator coincides on $\mathcal{F}C_b^\infty(K)$ with $\frac{1}{2}L_\Phi$. Consequently, by Theorem 5.13 above $\widetilde{T}_t^\mu = T_t^\mu$ for all $t\geq 0$, where we recall that $(T_t^\mu)_{t\geq 0}$ is the operator semigroup associated with $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))(= (\mathcal{E}_\mu^0, D(\mathcal{E}_\mu^0)))$. Defining P_x to be the distribution on $\Omega = C([0,\infty[,E)$ of ξ^x under P for $x\in E$, we obtain a diffusion process $\mathbb{M}_0 = (\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},(\theta_t)_{t\geq 0},(X_t)_{t\geq 0}, (P_x)_{x\in E})$ on E (where $X_t:\omega\mapsto\omega(t), t\geq 0$, are the canonical evaluation maps) which (e.g. by [BoR 93, Proposition 1.9 and Remark 1.7, 1.]) is properly associated to $(\mathcal{E}_\mu^0, D(\mathcal{E}_\mu^0))$. Hence by the uniqueness part of Theorem 4.4 above for any $\mu \in \mathcal{G}_{exp}(\Phi)$ we can replace the μ–dependent diffusion process $\mathbb{M}$ in Theorem 5.15 above by the μ–independent diffusion process $\mathbb{M}_0$.

We emphasize, however, that the fact that a diffusion process associated to $(\mathcal{E}_\mu^0, D(\mathcal{E}_\mu^0))$ as $\mathbb{M}_0$ above, whose transition semigroup is Feller, exists, is in the general case discussed in Sections 1 – 4 not true, but only holds in the special type of lattice models studied in this section.

Furthermore, by Theorem 5.15 and Fukushima's quasi–everywhere ergodic theorem (cf. Theorem 4.6) we know that

$$\lim_{t\to\infty} p_t^\Phi f = \int f\,d\mu \quad \mathcal{E}_\mu^0\text{–q.e.} \tag{5.15}$$

for all bounded $\mathcal{B}(E)$–measurable functions $f : E \to \mathbb{R}$ and all $\mu \in (\mathcal{G}_{exp}(\Phi))_{ext}$. Since $\mathcal{B}(E)$ is countably generated, $(p_t^\Phi)_{t\geq 0}$ is independent of $\mu \in (\mathcal{G}_{exp}(\Phi))_{ext}$, and the definition of $\mathcal{E}_\mu^0$–exceptionality resp. $\mathcal{E}_\mu^0$–nests we hence deduce the following

Theorem 5.16. *Let $\mu_1, \mu_2 \in (\mathcal{G}_{exp}(\Phi))_{ext}$, $\mu_1 \neq \mu_2$. Then for $i \in \{1,2\}$ there exist $\mathcal{E}_{\mu_i}^0$–nests $(K_n^{(i)})_{n\in\mathbb{N}}$ consisting of compacts such that, if $M_i := \bigcup_{n\in\mathbb{N}} K_n^{(i)}$, then $M_1 \cap M_2 = \emptyset$.*

Since $E \setminus M_i$ is $\mathcal{E}_{\mu_i}^0$–exceptional and hence of μ_i–measure zero, $i = 1, 2$, Theorem 5.16 extends the classical result that two different extreme Gibbs measures are singular w.r.t. each other.

Remark 5.17. The process $\mathbb{M}_0$ (as well as the process $\mathbb{M}$) is also called the *stochastic quantization* of $\mu \in \mathcal{G}_{exp}(\Phi)$. Note that by (5.15) each $\mu \in (\mathcal{G}_{exp}(\Phi))_{ext}$ is obtained as a limit distribution of the one dimensional marginals of $\mathbb{M}_0$ as $t \to \infty$. Hence depending on the starting point on can "construct" any such μ from $\mathbb{M}_0$ (cf. (5.15) and Theorem 5.16). A more detailed study of this and of the structure of the set of starting points which lead to the same $\mu \in (\mathcal{G}_{exp}(\Phi))_{ext}$ as a limit in (5.15) will be carried out in a forthcoming paper.

Appendix

For simplicity of notation we drop the index μ.

Proof of Proposition 2.3. (i) $\Rightarrow$ (v): Let $u \in D(L)$ with $Lu = 0$ then $\mathcal{E}(u,u) = (-Lu, u)_{L^2(E;\mu)} = 0$, hence $u =$ const. and (v) is proved.

(v) $\Rightarrow$ (iv): By the spectral theorem we know that for all $u \in L^2(E;\mu)$

$$\lim_{t\to\infty} T_t u = \Pi_0 u \text{ in } L^2(E;\mu)\ ,$$

where Π_0 is the projection on $\ker L$. Hence by assumption $\Pi_0 u =$ const.. But by symmetry for all $t > 0$

$$\int T_t\, u\, d\mu = \int T_t 1\, u\, d\mu = \int u\, d\mu$$

since $T_t 1 = 1$. Hence $\Pi_0 u = \int u \, d\mu$ and (iv) is proved.

(iv) $\Rightarrow$ (iii): Let $u \in L^2(E;\mu)$ such that $T_t u = u$ for all $t > 0$, then $\lim_{t\to\infty} T_t u = u$. Hence by assumption $u = \int u \, d\mu$ and (iii) is shown.

(iii) $\Rightarrow$ (ii): Since $f \equiv 1 \in L^2(E;\mu) \cap L^\infty(E;\mu)$, this implication is trivial.

(ii) $\Rightarrow$ (i): Let $u \in L^2(E;\mu)$ such that $\mathcal{E}(u,u) = 0$. Let $n \in \mathbb{N}$. Then $u_n := (u \wedge n) \vee (-n) \in D(\mathcal{E})$, $\mathcal{E}(u_n, u_n) = 0$ (cf. e.g. [MR 92, Theorem 4.17]), and $u_n \underset{n\to\infty}{\longrightarrow} u$ in $L^2(E;\mu)$. Hence replacing u by $u_n + n + 1$ we may assume that $1 \le u \le c$ for some $c > 0$. For all $v \in D(L)$ and $w \in D(\mathcal{E})$ we have that $uv \in D(\mathcal{E})$ and

$$\begin{aligned}
\mathcal{E}(uv, w) &= \frac{1}{2} \sum_{k\in K} \int u \frac{\partial^\mu v}{\partial k} \frac{\partial^\mu w}{\partial k} \, d\mu \\
&= \mathcal{E}(v, uw) \\
&= \int (-uLv) \cdot w - d\mu
\end{aligned}$$

where we used the product rule for $\frac{\partial^\mu}{\partial k}$ defined in Remark 1.1 (ii). But $(-uLv) \in L^2(E;\mu)$. Hence (e.g. by [MR 92, Proposition 2.16]) $uv \in D(L)$ and $L(uv) = uLv$. Since $u \ge 1$ it follows that for all $v \in D(L)$

$$\frac{u^{-1} T_t(uv) - v}{t} = u^{-1} \frac{T_t(uv) - uv}{t} \to Lv$$

in $L^2(E;\mu)$ as $t \downarrow 0$. Since $T_t^u f := u^{-1} T_t(uf)$, $f \in L^2(E;\mu)$, $t \ge 0$, defines a strongly continuous semigroup, this means that its generator extends $(L, D(L))$. Since such generators cannot strictly extend each other, they must coincide. This implies $(T_t^u)_{t\ge 0} = (T_t)_{t \ge 0}$. It follows by assumption (ii) that $u = \text{const}$. $\square$

Remark. The proof of Proposition 2.3 obviously remains valid for any symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ of square field operator type such that $1 \in D(\mathcal{E})$ and $\mathcal{E}(1,1) = 0$.

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