THE GOODWILLIE TOWER OF THE IDENTITY IS A LOGARITHM

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Abstract. We offer the point of view that the Goodwillie tower of the identity functor is a formal inverse to the standard filtration of stable homotopy in the same sense as $\ln(x)$ is inverse to $\exp(x)$.

0. Introduction

The *Goodwillie tower of the identity* (see [2, 3, 4] for a general reference on the Goodwillie calculus) is a tower of functors and natural transformations, which starts with stable homotopy and converges to unstable homotopy. The functors in this tower are characterized by certain universal properties, and according to [4] they should be thought of as the Taylor polynomials of the identity. The study of these polynomial approximations was initiated in [5] and continued in [1]. In particular, an explicit description of the fibers in the tower was derived there. However, the nature of the tower and the reason that it converges remained a bit mysterious. It appears that the Goodwillie tower is a natural, canonical object, and therefore one expects to have a basic understanding of it in terms of some fundamental object in mathematics. Our goal in this short, simple note is to show that there is a formal analogy between the Goodwillie tower of the identity and the Taylor series of the logarithm function and to suggest the point of view that the Goodwillie tower is an inverse to stable homotopy in the same way as logarithm is an inverse to exponential.

Let $D_n(X)$ denote the n-th fiber in the Goodwillie tower of the identity (the n-th differential). It was shown in [1] (see also [5] for a different approach) that $D_1(X) =$

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$\Omega^\infty\Sigma^\infty(X)$ and that for $n > 1$

$$D_n(X) \simeq \Omega^\infty \left(\mathrm{Map}_*\left(SK_n, \Sigma^\infty X^{\wedge n}\right)_{h\Sigma_n}\right)$$

where K_n is the unreduced suspension of the Folkman complex (essentially the geometric realization) of the category of partitions of the set $\underline{n} = \{1, \ldots, n\}$, and SK_n is the suspension of K_n. SK_n is a complex with a natural action of the symmetric group Σ_n. Furthermore, it is well-known that non-equivariantly

$$SK_n \simeq \bigvee_{i=1}^{(n-1)!} S^{n-1}.$$

Thus, the coefficient of the n-th differential of the identity looks like $K_n/\Sigma_n \approx \frac{(n-1)!}{n!} = \frac{1}{n}$, which reminds one of the Taylor series of the function $g(x) = \ln(1+x)$. We will see that this is not a coincidence, that the fact that the Goodwillie tower converges is a jazzed up version of the fact that the Taylor series of logarithm is inverse to the Taylor series of exponential. More precisely, our point of view is that stable homotopy is analogous to the function e^{x-1} rather than to a linear function, and the Goodwillie tower is an infinite product, rather than an infinite sum, namely it is analogous to the product

$$e^{x-1} \cdot e^{-\frac{(x-1)^2}{2}} \cdot e^{\frac{(x-1)^3}{3}} \cdots = e^{\ln(1+(x-1))} = x.$$

It is clear that in the context of the Goodwillie tower it is not good enough to think of the coefficient of the n-th term in the Taylor expansion of the function $\ln(1+x)$ as $\frac{1}{n}$. Rather, it is $\frac{\underline{(n-1)!}}{n!}$, where $\underline{(n-1)!}$ is a fancy $(n-1)!$, as given by the reduced homology, equivalently by the reduced Euler characteristic, of the partition lattice, considered as a Σ_n-representation. As a biproduct we will obtain an elementary proof of the well-known fact that the reduced Euler characteristic of K_n is $(-1)^n(n-1)!$. We will see that in our context the partition lattice is the "correct" way to think of the umbral coefficient in the Taylor tower of $\ln(1+x)$. We believe it to be true in other situations too. We emphasize, however, that the combinatorics in this paper is elementary and the corresponding topological results about the Goodwillie tower of the identity, as well as their proofs, are contained in [1]. It is their juxtaposition which we hope will be of some interest.

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1. Combinatorics versus homotopy theory

In the category of topological spaces the categorical product (cartesian product) and the categorical sum (disjoint union) satisfy the distributivity relation

$$X \times \left(A \coprod B\right) \cong (X \times A) \coprod (X \times B),$$

which makes the category of spaces into a semiring with the sum and the product being the categorical ones. The empty set and the one-point space are the zero and the unit element of this semiring. More generally, suppose we are given a commutative square

$$\begin{array}{ccc} X_0 & \to & X_1 \\ \downarrow & & \downarrow \\ X_2 & \to & X_{12} \end{array} \tag{1}$$

If it is a homotopy pullback, we would like to interpret it heuristically as

$$X_{12} = \frac{X_1 \cdot X_2}{X_0}.$$

If it is a homotopy pushout, we interpret it as

$$X_{12} = X_1 + X_2 - X_0.$$

As an illustration of the distributivity law in this more general setting, consider the following well-known fact: Suppose we are given a pushout square as in (1). Let Y be a connected, pointed space and let $f : X_{12} \to Y$ be a map. Then the fiber of f is homotopy equivalent to the pushout of the obvious diagram of fibers of the maps $X_i \to Y$, where $i = 0, 1$ or 2. This fact has no formal dual in the category of spaces, and it is analogous to the fact that

$$\frac{X_1 + X_2 - X_0}{Y} = \frac{X_1}{Y} + \frac{X_2}{Y} - \frac{X_0}{Y}.$$

Thus, from our point of view, a tower of functors is analogous to an infinite product of functions, while according to the philosophy of calculus it is analogous to an infinite sum.

Remark 1.1. Of course, our analogy is very crude. For instance, it suggests that ΩX corresponds to $\frac{1}{X}$ and therefore $\Omega^2 X$ can be identified with X, which is not generally true in the category of spaces.

Now let $L : \text{Spaces}_* \to \text{Spaces}_*$ be a reduced homology theory, or equivalently, in the terminology of [2], a homogeneous linear functor. Thus $L(*) \simeq *$, where $*$ is the one-point space, and L takes homotopy pushout squares to homotopy pullback squares. According to our picture, L is analogous to a function f (say from reals to reals) which satisfies $f(1) = 1$ and $f(a + b - c) = \frac{f(a) \cdot f(b)}{f(c)}$. A typical function with these properties has the form $f(x) = a^{x-1}$. Thus, our first suggestion is that a reduced homology theory is analogous to a function of this form rather than to a linear function. Our next suggestion is that the counterpart of the functor $Q(X) = \Omega^\infty \Sigma^\infty(X)$ is the function

$$h\colon R \to R, \quad x \mapsto e^{x-1}.$$

We give two reasons for this. First, $Q(X)$ gives the best possible approximation of the identity functor by a reduced homology theory. More precisely, $Q(X)$ is the unique homology theory such that there is a natural map $X \to Q(X)$, which is $2k+1$ connected where k is the connectivity of X. This corresponds to the fact that e^{x-1} is the best approximation of x by a function of the form a^{x-1}. Notice also that with topological spaces the higher the connectivity of X is, the better the approximation, while with numbers, the closer x to 1, the better the approximation. Indeed, we have already said that a contractible space corresponds to 1, and therefore a highly connected space corresponds to a number close to 1. The second reason is that the standard (May-Milgram) filtration of $Q(X)$ formally corresponds to the Taylor expansion of e^{x-1}. We leave it to the reader to convince himself of this (keeping in mind that product corresponds to cartesian product, not smash product).

We denote by h^k the composition of k copies of h, for every $k \geq 1$. For $k = 0$ we define h^0 to be the identity mapping of R. More generally, for k positive integers $i_1 < i_2 < \cdots < i_k$ we define $h_{i_1,\ldots,i_k} = h^k$. In [1] the Goodwillie tower was studied by comparing it with the Bousfield-Kan tower converging to stable homotopy localization. To understand better this part of [1], we construct the counterpart in the realm of real valued functions. We would like to define a sequence of functions f_k corresponding to the Bousfield-Kan tower. In particular, we expect f_k to converge to the identity function. We proceed as follows. Let $f_0 = h$. Thus f_0 is the first approximation to the identity (corresponding to stable homotopy!). We define

$$f_{k+1}(x) = f_k(x) \cdot \frac{h(x)}{f_k(h(x))}. \tag{2}$$

Motivation: Clearly, $x = f_k(x) \cdot \frac{x}{f_k(x)}$. Therefore, it is reasonable to hope that if one replaces x with $h(x)$ in $\frac{x}{f_k(x)}$, one will get a better approximation to x than f_k, see Proposition 1.4 below.

Proposition 1.2. *For $k > 1$*

$$f_{k-1}(x) = \cfrac{\prod\limits_{1\leq i\leq k} h_i(x)}{\left(\cfrac{\prod\limits_{1\leq i_1<i_2\leq k} h_{i_1,i_2}(x)}{\left(\cfrac{\vdots}{\prod\limits_{1\leq i_1<\cdots<i_k\leq k} h_{i_1\ldots i_k}(x)}\right)}\right)}$$

Proof. The proof is an easy induction on k, observing that whenever $1 \leq i_1 < \cdots < i_k \leq k$, one can write

$$h_{i_1,\ldots,i_k}(h(x)) = h_{i_1,\ldots,i_k,k+1}(x).$$

□

Corollary 1.3.

$$f_{k-1}(x) = \frac{(h(x))^{\binom{k}{1}}(h^3(x))^{\binom{k}{3}}\cdots(h^k(x))^{(-1)^{k+1}\binom{k}{k}}}{(h^2(x))^{\binom{k}{2}}(h^4(x))^{\binom{k}{4}}\cdots}$$

and

$$\frac{f_k}{f_{k-1}} = \prod_{i=0}^{k}\left(h^{i+1}\right)^{(-1)^i\binom{k}{i}}.$$

The sequence (f_k) converges to the identity function. More precisely, we have the following proposition,

Proposition 1.4. *Let $\sum_{m=0}^{\infty} a_m(x-1)^m$ be the Taylor series of f_{k-1} near 1. Then $a_0 = a_1 = 1$ and $a_m = 0$ for $2 \leq m \leq k$.*

Proof. The claim is trivially true for $k = 1$. Assume the claim holds for $k = n$. Then we can write

$$f_{n-1}(x) = x + a_{n+1}(x-1)^{n+1} + O[(x-1)^{n+2}].$$

Using formula (2) we get

$$f_n(x) = \frac{e^{x-1}\left(x + a_{n+1}(x-1)^{n+1} + O[(x-1)^{n+2}]\right)}{e^{x-1} + a_{n+1}(e^{x-1}-1)^{n+1} + O[(e^{x-1}-1)^{n+2}]}.$$

Since

$$a_{n+1}(e^{x-1}-1)^{n+1} = a_{n+1}(x-1)^{n+1} + O[(x-1)^{n+2}]$$

and

$$O[(e^{x-1}-1)^{n+2}] = O[(x-1)^{n+2}],$$

it follows that in the above formula the inverse of the denominator equals to

$$\begin{aligned} & \textstyle\sum_{i=0}^{\infty}(1-e^{x-1})^i - a_{n+1}(e^{x-1}-1)^{n+1} + O[(e^{x-1}-1)^{n+2}] \\ = \; & e^{1-x} - a_{n+1}(x-1)^{n+1} + O[(x-1)^{n+2}]. \end{aligned}$$

Therefore

$$\begin{aligned} f_n(x) &= e^{x-1}\left(x + a_{n+1}(x-1)^{n+1} + O[(x-1)^{n+2}]\right) \\ &\quad \cdot \left(e^{1-x} - a_{n+1}(x-1)^{n+1} + O[(x-1)^{n+2}]\right) \\ &= \left(x + a_{n+1}(x-1)^{n+1} + O[(x-1)^{n+2}]\right) \\ &\quad \cdot \left(1 - a_{n+1}(x-1)^{n+1} + O[(x-1)^{n+2}]\right) \\ &= x - x a_{n+1}(x-1)^{n+1} + O[(x-1)^{n+2}] + a_{n+1}(x-1)^{n+1} \\ &= x + O[(x-1)^{n+2}], \end{aligned}$$

which completes the proof. □

To compare the sequence (f_k) with its homotopy-theoretic counterpart, the Bousfield-Kan stable homotopy localization tower, we rewrite Proposition 1.4 as follows. Let $g(x)$ be a function, such that g and $\ln(g)$ are analytic around 1. Let

$$\ln(g(x)) = a_0 + a_1(x-1) + a_2(x-1)^2 + \cdots .$$

Then

$$g(x) = e^{a_0} \cdot e^{a_1(x-1)} \cdot e^{a_2(x-2)^2} \cdots . \tag{3}$$

We call (3) the *logarithmic expansion* of $g(x)$. It is the point of this paper that the Goodwillie tower is the homotopy-theoretic analog of logarithmic expansion, rather than of Taylor series.

It follows from Proposition 1.4 that

$$\begin{aligned} \ln f_{n-1}(x) &= \ln(1 + (x-1) + O[(x-1)^{n+1}]) \\ &= (x-1) - \tfrac{1}{2}(x-1)^2 + \ldots + (-1)^{n+1}\tfrac{1}{n}(x-1)^n + O[(x-1)^{n+1}] \\ &= \ln(x) + O[(x-1)^{n+1}]. \end{aligned}$$

Thus Proposition 1.4 can be reformulated as follows:

Corollary 1.5. *The logarithmic expansion of f_{k-1} coincides with the logarithmic expansion of the identity up to the k-th term.*

Proposition 1.4 and Corollary 1.5 illustrate the convergence of the Bousfield-Kan tower (of course, they are far from proving it). By Bousfield-Kan tower we mean the tower of functors from based spaces to based spaces

$$\cdots \to F_k(-) \to \cdots$$

associated with the cosimplicial space

$$QX \rightrightarrows QQX \cdots .$$

In this tower $F_0(X) = Q(X)$ and fiber $(F_k \to F_{k-1})$ is given by the k-th loop space of the iterated fiber of a certain k-dimensional cubical diagram defined by $\chi(U) = Q^{|\underline{k}\setminus U|+1}(-)$ for $U \subseteq \underline{k}$. For instance

$$\text{fiber}\,(F_1 \to F_0) \simeq \Omega\,(\text{fiber}\,(QQ(X) \to Q(X)))$$

and

$$\text{fiber}\,(F_2 \to F_1) \simeq \Omega^2 \text{iterated fiber} \left(\begin{array}{ccc} QQQ(X) & \longrightarrow & QQ(X) \\ \downarrow & & \downarrow \\ QQ(X) & \longrightarrow & Q(X) \end{array} \right)$$

We leave it to the interested reader to convince himself that this formula is exactly parallel to the formula for $\frac{f_k}{f_{k-1}}$ in Corollary 1.3. The functors F_k correspond to the functions f_k. The tower $F_k(X)$ converges to the identity functor for reasonable spaces X. More precisely, the Goodwillie tower of $F_{k-1}(X)$ coincides with the Goodwillie tower of the identity up to the k-th degree approximation. This is the topological counterpart of Corollary 1.5.

Next, we would like to link the coefficients of the Taylor series of $\ln(f_k)$ with partitions. We denote the number of partitions of $\underline{n} = \{1, 2, \ldots, n\}$ with k components by $P_{k,n}$. These are the *Stirling numbers.* Let P_n^1 denote the total number of partitions of $\underline{n}$. Then $P_n^1 = P_{1,n} + P_{2,n} + \cdots + P_{n,n}$. More generally, define a partial ordering on the set of partitions of $\underline{n}$ by $\lambda_1 \leq \lambda_2$ if λ_1 is a refinement of λ_2. Let $\hat{1}$ and $\hat{n}$ be the obvious minimal and maximal objects in this poset. We define the set of *k-chains of partitions of $\underline{n}$* to be the set of sequences $(\hat{1} = \lambda_0, \lambda_1, \ldots, \lambda_k, \lambda_{k+1} = \hat{n})$ such that $\hat{1} \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \hat{n}$. We denote the number of k-chains by P_n^k, for $k \geq -1$ and $n \geq 1$. To avoid confusion, we note that $P_1^{-1} = 1$ and $P_n^{-1} = 0$ for $n > 1$. We have no doubt that the following two propositions can be found in the literature, but for completeness we include their proofs.

Proposition 1.6. *Let $k \geq 1$. Then*

$$P_n^k = \sum_{\sum_{l=1}^{n} l k_l = n} \frac{n!}{1!(2!)^{k_2} \cdots (n!)^{k_n} k_1! k_2! \cdots k_n!} \left(P_1^{k-1}\right)^{k_1} \left(P_2^{k-1}\right)^{k_2} \cdots \left(P_n^{k-1}\right)^{k_n}.$$

Proof. By abuse of notation, and only for the purposes of this proof, we let P_n^k denote the set of k-chains as well as its cardinality. Let $(\hat{1}, \lambda_1, \ldots, \lambda_k, \hat{n})$ be a k-chain. Consider the partition λ_k, the coarsest partition in the chain. Let m be the number of components of λ_k and let $i_1, \ldots, i_m$ be their cardinalities. Thus $i_1 + \cdots + i_m = n$. For $l = 1, 2, \ldots$, let k_l be the number of components of λ_k whose cardinality is l. Clearly, $k_l = 0$ for $l > n$. We call the sequence $(k_1, \ldots, k_n)$ the *type* of λ_k. The set P_n^1 has a natural action of Σ_n, and it is easy to see that two partitions are in the same orbit iff they have the same type. Given that λ_k has type $(k_1, \ldots, k_n)$, it is obvious that the number of ways to choose $\lambda_1, \ldots, \lambda_{k-1}$ such that $(\hat{1}, \lambda_1, \ldots, \lambda_k, \hat{n})$ is a chain is $\left(P_1^{k-1}\right)^{k_1} \left(P_2^{k-1}\right)^{k_2} \cdots \left(P_n^{k-1}\right)^{k_n}$. On the other hand, the number of partitions of a given type $(k_1, \ldots, k_n)$ equals the number of cosets of the corresponding isotropy subgroup of Σ_n, which is easily seen to be

$$\Sigma_{k_1} \wr \Sigma_1 \times \Sigma_{k_2} \wr \Sigma_2 \times \cdots \times \Sigma_{k_n} \wr \Sigma_n.$$

Thus the number of k-chains $(\hat{1}, \lambda_1, \ldots, \lambda_k, \hat{n})$ such that λ_k has a given type $(k_1, \ldots, k_n)$ is

$$\frac{n!}{1!(2!)^{k_2} \cdots (n!)^{k_n} k_1! k_2! \cdots k_n!} \left(P_1^{k-1}\right)^{k_1} \left(P_2^{k-1}\right)^{k_2} \cdots \left(P_n^{k-1}\right)^{k_n}.$$

This proves the proposition. $\square$

Proposition 1.7. *The coefficient of* $\frac{(x-1)^n}{n!}$ *in the Taylor series of* $h^{k+1}(x) - 1$ *is* P_n^k, *for* $n \geq 1$ *and* $k \geq -1$.

Proof. By induction on k. The claim is trivially verified for $k = -1$. We can write

$$h^k(x) - 1 = \sum_{i=1}^{\infty} a_i (x-1)^i.$$

Then

$$h^{k+1}(x) - 1 = e^{h^k(x)-1} - 1 = \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{i=1}^{\infty} a_i (x-1)^i \right)^m.$$

Therefore the coefficient of $(x-1)^n$ in the Taylor series of $h^{k+1}(x) - 1$ is

$$\sum_{m=1}^{n} \frac{1}{m!} \left(\sum_{\sum_j i_j = n, i_j \geq 1} a_{i_1} a_{i_2} \cdots a_{i_m} \right).$$

Fix an index m and a monomial $a_{i_1} a_{i_2} \cdots a_{i_m}$ in the above formula. For $l = 1, 2, \ldots$, let k_l be the number of the terms i_j which are equal to l. Thus $0 \leq k_l \leq n$ for every l, $k_l = 0$ for $l > n$ and $\sum_{l=1}^{n} l k_l = n$. The terms a_{i_j}, $1 \leq j \leq m$, can be arranged in $m!$ ways of which $\frac{m!}{k_1! k_2! \cdots k_n!}$ are different. Thus

$$\sum_{m=1}^{n} \frac{1}{m!} \sum_{\sum_j i_j = n, i_j \geq 1} a_{i_1} a_{i_2} \cdots a_{i_m} = \sum_{\sum_l l k_l = n} \frac{1}{k_1! k_2! \cdots k_n!} a_1^{k_1} a_2^{k_2} \cdots a_n^{k_n}.$$

By induction assumption $P_i^{k-1} = a_i i!$ for $1 \leq i \leq n$, and the sum above equals to

$$\sum_{\sum_l l k_l = n} \frac{1}{k_1! k_2! \cdots k_n!} \left(\frac{P_1^{k-1}}{1!} \right)^{k_1} \left(\frac{P_2^{k-1}}{2!} \right)^{k_2} \cdots \left(\frac{P_n^{k-1}}{n!} \right)^{k_n}.$$

Hence the coefficient of $\frac{(x-1)^n}{n!}$ in the Taylor series of $h^{k+1}(x)-1$ is exactly P_n^k by Proposition 1.6. □

P_n^k is the number of k-chains of partitions such that $\hat{1} \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \hat{n}$. For $k \geq 0$ and $n \geq 1$ we let C_n^k be the number of chains $(\hat{1}, \lambda_1, \ldots, \lambda_k, \hat{n})$ such that $\hat{1} < \lambda_1 < \cdots < \lambda_k < \hat{n}$. We call these the non-degenerate k-chains. To avoid confusion, we record that $C_1^{-1} = 1$, $C_n^{-1} = 0$ for $n > 1$, $C_1^0 = 0$ and $C_n^0 = 1$ for $n > 1$.

Let $(\hat{1} = \lambda_0, \lambda_1, \ldots, \lambda_k, \lambda_{k+1} = \hat{n})$ be a k-chain. Let j_0 be the largest index j such that $\lambda_j = \hat{1}$, let j_1 be the largest index j such that $\lambda_j = \lambda_{j_0+1}$, and so on. Let $m+2$ be the cardinality of the set $\{\hat{1}, \lambda_1, \ldots, \lambda_k, \hat{n}\}$. There are $\binom{k+1}{m+1}$ different ways in which the indices $j_0, \ldots, j_m$ can occur. Thus the number of k-chains $(\hat{1}, \lambda_1, \ldots, \lambda_k, \hat{n})$ with $m+2$ distinct elements is $\binom{k+1}{m+1} C_k^m$. This implies that

$$C_n^k = P_n^k - \sum_{m=-1}^{k-1} \binom{k+1}{m+1} C_n^m,$$

for $k \geq 0$ and $n \geq 1$. The following proposition is essentially the inclusion-exclusion principle.

Proposition 1.8. *For $n \geq 1$ and $k \geq 0$*

$$C_n^k = \sum_{l=-1}^{k} \binom{k+1}{l+1} (-1)^{k-l} P_n^l.$$

Proof. The formula is trivially true for $k = 0$. Let us assume it holds for all $0 \leq m \leq k-1$. Then

$$\begin{aligned}
C_n^k &= P_n^k - \textstyle\sum_{m=-1}^{k-1} \binom{k+1}{m+1} C_n^m \\
&= P_n^k - \textstyle\sum_{m=-1}^{k-1} \binom{k+1}{m+1} \left(\sum_{l=-1}^{m} \binom{m+1}{l+1} (-1)^{m-l} P_n^l \right) \\
&= P_n^k - \textstyle\sum_{s=-1}^{k-1} \sum_{m=s}^{k-1} \binom{k+1}{m+1} \binom{m+1}{s+1} (-1)^{m-s} P_n^s \\
&= P_n^k - \textstyle\sum_{s=-1}^{k-1} \binom{k+1}{s+1} \left(\sum_{m=s}^{k-1} \binom{k-s}{k-m} (-1)^{m-s} P_n^s \right) \\
&= P_n^k - \textstyle\sum_{s=-1}^{k-1} \binom{k+1}{s+1} \left(\sum_{j=0}^{k-s-1} \binom{k-s}{j} (-1)^j \right) P_n^s \\
&= P_n^k + \textstyle\sum_{s=-1}^{k-1} \binom{k+1}{s+1} (-1)^{k-s} P_n^s \\
&= \textstyle\sum_{s=-1}^{k} \binom{k+1}{s+1} (-1)^{k-s} P_n^s.
\end{aligned}$$

□

Now we return to the functions f_k. It is easy to see from Corollary 1.3 and the definition of h that

$$\frac{f_{k+1}(x)}{f_k(x)} = e^{\sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (h^j(x)-1)}$$

for $k \geq 0$. According to Proposition 1.7 the coefficient of $\frac{(x-1)^n}{n!}$, for $n \geq 1$ in the Taylor series of $h^{k+1}(x) - 1$ is P_n^k. Therefore the coefficient of $\frac{(x-1)^n}{n!}$, $n \geq 1$, in the Taylor series of $\ln f_{k+1}(x) - \ln f_k(x)$ is

$$\sum_{j=0}^{k+1} \binom{k+1}{j} (-1)^j P_n^{j-1} = \sum_{l=-1}^{k} \binom{k+1}{l+1} (-1)^{l+1} P_n^l = (-1)^{k+1} C_n^k.$$

Recall that K_n is a simplicial complex whose k-simplices are the non-degenerate $k+1$ chains. Let $\mathrm{e}(K_n)$ be the reduced Euler characteristic of K_n. Then

$$\mathrm{e}(K_n) = \sum_{k=0}^{n-2} (-1)^{k+1} C_n^k = \sum_{k=0}^{\infty} (-1)^{k+1} C_n^k$$

for $n \geq 2$. Therefore the reduced Euler characteristic of K_i equals to the coefficient of $\frac{(x-1)^i}{i!}$ for $i \leq n$ in the Taylor series of $\ln f_{n-1}(x) - \ln f_0(x)$. Since $\ln f_0(x) = x - 1$, its Taylor coefficients vanish for $n \geq 2$. On the other hand, Corollary 1.5 implies that for $i \leq n$ the coefficient of $\frac{(x-1)^i}{i!}$ in the Taylor series of $\ln f_{n-1}(x)$ is $(-1)^{i+1}(i-1)!$. Hence we have naturally obtained a proof of the following well-known fact:

Proposition 1.9. *The reduced Euler characteristic of K_n is $(-1)^{n+1}(n-1)!$ for $n \geq 2$.*

Another way to say the above is that the infinite product

$$\exp(x-1) \cdot \prod_{i=2}^{\infty} \exp\left(-\frac{\mathrm{e}(K_i)\,(x-1)^i}{i!}\right)$$

converges to the identity. This corresponds to the fact that the Goodwillie tower of the identity converges.

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