Martingale Decomposition of Dirichlet Processes on the Banach space $C_0[0,1]$

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Abstract

We prove that for a given symmetric Dirichlet form of type $\mathcal{E}(u,v) = \int_E \langle A(z)\nabla u(z), \nabla v(z)\rangle_H \mu(dz)$ with $E = C_0[0,1]$ and $H =$ classical Cameron-Martin space the corresponding diffusion process (under P_μ) can be decomposed into a forward and a backward E-valued martingale. Applications to prove tightness of laws of diffusions of the above kind are given.

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1 Introduction and framework

Let $E = C_0[0,1]$ be the Banach space of all continuous functions on $[0,1]$ with initial value zero. E is endowed with the usual maximum norm denoted by $\| \quad \|$. Let μ be a probability measure on $(E, \mathcal{B}(E))$ with $\mathrm{supp}[\mu] = E$. Let $L^2(E;\mu)$ $(:= L^2(E \to \mathbb{R};\mu))$ denote the corresponding L^2-space. Let H denote the classical Cameron-Martin space, i.e.,

$$H := \{h \in C_0[0,1] \mid h \text{ is absolutely continuous on } [0,1] \text{ and } \int_0^1 h'^2(s)ds < +\infty\}, \tag{1.1}$$

where ds denotes Lebesgue measure on $\mathbb{R}^1$. Then H is a Hilbert space with inner product

$$\langle h_1, h_2 \rangle_H := \int_0^1 h_1'(s)h_2'(s)ds.$$

Define the linear space

$$\begin{aligned} \mathcal{F}C_b^\infty \quad := \quad & \{u : E \to \mathbb{R} \mid u(z) = f(l_1, \ldots, l_m),\ m \in \mathbb{N}, \\ & f \in C_b^\infty(\mathbb{R}^m) \text{ and } l_1, \ldots, l_m \in E'\}, \end{aligned} \tag{1.2}$$

where $C_b^\infty(\mathbb{R}^m)$ is the set of all infinitely differentiable functions on $\mathbb{R}^m$ with all partial derivatives bounded and E' is the dual of E. Identifying H with its dual E' we have

$$E' \subset H \subset E \quad \text{densely and continuously,} \tag{1.3}$$

where E' is equipped with the usual operator norm. More precisely an element m in E', i.e., a signed measure of finite total variation on $\mathcal{B}([0,1])$, is identified with $\int_0^\cdot m([s,1])ds \in H$. Let $k \in E$ and define for $u \in \mathcal{F}C_b^\infty$

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds}u(z+sk)|_{s=0} \ , z \in E. \tag{1.4}$$

We denote by $\nabla u(z)$ the unique element in H representing the continuous linear map $k \mapsto \frac{\partial u}{\partial k}u(z), k \in H$.

Let $L^\infty(H)$ denote the collection of all continuous linear operators from H to H. Suppose we are given a strongly measurable map $A : E \to L^\infty(H)$, satisfying

(i) $A(z)$ is self-adjoint for each $z \in E$.

(ii) There exists a positive constant $\delta > 0$ such that

$$\delta I_H \leq A(z) \leq \frac{1}{\delta} I_H \quad \text{in the sense of quadratic forms,} \tag{1.5}$$

where I_H is the identity operator on H.

We introduce the following symmetric form $(\mathcal{E}, \mathcal{F}C_b^\infty)$ on $L^2(E;\mu)$,

$$\mathcal{E}(u,v) = \frac{1}{2}\int_E \langle A(z)\nabla u(z), \nabla v(z)\rangle_H \mu(dz);\ \ u,v \in \mathcal{F}C_b^\infty. \tag{1.6}$$

We assume that $(\mathcal{E}, \mathcal{F}C_b^\infty)$ is closable on $L^2(E;\mu)$, and we denote its closure by $(\mathcal{E}, D(\mathcal{E}))$. ($(\mathcal{E}, D(\mathcal{E}))$ is then a *Dirichlet form* (cf. [AR90, Section 3] and [F80] or [MR92] for the terminology here and below). Sufficient conditions for the closability of the above symmetric form are given in [AR90, Theorem 3.2] and [AR91, Sect. 3]. For instance, if an integration by parts formula holds for μ, then the form in (1.6) is closable on $L^2(E;\mu)$ (cf. [AKR90, Proposition 2.2]). Under the above assumptions, it follows from [S90] (see also [MR92, Chap. IV, Subsection 4b]) that there exists a diffusion process $\mathcal{M} = \{\Omega, \mathcal{F}, (\mathcal{F}_t), (X_t), (P_z)_{z\in E}\}$ associated with $(\mathcal{E}, D(\mathcal{E}))$, that is for every bounded $\mathcal{B}(E)$-measurable function u on E and all $t > 0$

$$\int_\Omega u(X_t)dP_z = \left(e^{tL}u\right)(z) \quad \text{for } \mu\text{-a.e. } z \in E, \tag{1.7}$$

where L is the *generator* of $(\mathcal{E}, D(\mathcal{E}))$ (cf. e.g. [MR92, Chap.I, Sect.2]). Define a probability measure P_μ on $(\Omega, \mathcal{F})$ by

$$P_\mu = \int_E P_z \ \mu(dz). \tag{1.8}$$

The aim of this paper is to prove that the *Dirichlet process* $\mathcal{M}$ can be decomposed into a forward and a backward martingale on the infinite dimensional space E under the probability measure P_μ (cf. Theorem 2.1 below). This extends the corresponding componentwise decomposition given in [LZhe88]. Our approach uses a modification of Levy's method to construct Brownian motion. The forward and backward decomposition incorporates important cancellations and implies useful estimates for the Dirichlet process. An important application of the decomposition is to prove tightness of a sequence of diffusion processes on the Banach space E. This is carried out in Section 3. The main result is proved in Section 2.

2 Martingale Decomposition

Consider our process on a fixed time intervall $[0,T]$. For simplicity let $T := 1$. Define

$$\mathcal{F}_t := \sigma(X_s, s \le t) \quad \text{and} \quad \bar{\mathcal{F}}_t := \sigma(X_{1-s}, s \le t), \quad t \ge 0.$$

Since in this paper E is the classical Wiener space, we denote elements of E by w instead of z.

Our main result is the following:

Theorem 2.1 *Assume that $E' \subset L^2(E;\mu)$. Then under P_μ there exists an E- valued continuous $(\mathcal{F}_t)$-martingale M and an E-valued continuous $(\bar{\mathcal{F}}_t)$-martingale $\bar{M}$ such that*

(i) $X_t - X_0 = \frac{1}{2}M_t - \frac{1}{2}(\bar{M}_1 - \bar{M}_{1-t}),\ 0 \le t \le 1, \quad P_\mu$*-a.e*

(ii) $M_t, \bar{M}_t \in L^p(\Omega \to E; P_\mu)$ *for any* $p \ge 1$ *and* $t \ge 0$.

Proof. Since $E' \subset L^2(E;\mu)$ by assumption, it follows that $E' \subset D(\mathcal{E})$. Hence by [LZhe88] resp. [LZ91] for any $l \in E'$ the following decomposition holds

$$l(X_t) - l(X_0) = \frac{1}{2}M_t^l - \frac{1}{2}(\bar{M}_t^l - \bar{M}_{1-t}^l),\ 0 \le t \le 1, \quad P_\mu - \text{a.e.}, \tag{2.1}$$

where M^l is a continuous square integrable $(\mathcal{F}_t)$-martingale and $\bar{M}^l$ is a continuous square integrable $(\bar{\mathcal{F}}_t)$-martingale. Moreover, the brackets of M^l and $\bar{M}^l$ are given as

$$\begin{aligned} \langle M^l \rangle &= \int_0^\cdot \langle A(X_s)l, l\rangle_H ds, \\ \langle \bar{M}^l \rangle &= \int_0^\cdot \langle A(X_{1-s})l, l\rangle_H ds \end{aligned} \tag{2.2}$$

(cf. [AR91, Proposition 4.5] and [LZ91]). The proof will be finished if we find a continuous E-valued $(\mathcal{F}_t)$-adapted martingale M and a continuous E-valued $(\bar{\mathcal{F}}_t)$-martingale $\bar{M}$ satisfying (ii) such that for every $l \in E'$ and each $0 \le t \le 1$

$$l(M_t) = M_t^l, \quad l(\bar{M}_t) = \bar{M}_t^l \quad P_\mu\text{-a.e.} \tag{2.3}$$

This will be done by the following two lemmas.

Lemma 2.2 *There exists a continuous E-valued $(\mathcal{F}_t)$-adapted process M and a continuous E-valued $(\bar{\mathcal{F}}_t)$-adapted process $\bar{M}$ such that for $l \in E'$ and $0 \le t \le 1$*

$$l(M_t) = M_t^l, \quad l(\bar{M}_t) = \bar{M}_t^l, \quad P_\mu\text{-a.e.}$$

Proof. We shall give a direct construction of M and $\bar{M}$ by using a procedure similar to Levy's construction of Brownian motion. Let us first recall the definition of the Haar functions. For $n \ge 1$ and odd $k < 2^n$, the Haar function $f_{\frac{k}{2^n}}$ is defined by

$$f_{\frac{k}{2^n}}(t) := 2^{\frac{n-1}{2}} \left(1_{[\frac{k-1}{2^n}, \frac{k}{2^n}[} - 1_{[\frac{k}{2^n}, \frac{k+1}{2^n}[} \right) \tag{2.4}$$

Set $f_0(t) \equiv 1$. Then $\{f_{\frac{k}{2^n}}, f_0, n \ge 1, \mathrm{odd}\, k < 2^n\}$ forms an orthonormal basis of $L^2([0,1]; ds)$. For $n \ge 1$ and odd $k < 2^n$, we define an element $h^*_{k,n} \in E'$ by

$$h^*_{k,n}(w) := -2^{\frac{n-1}{2}} \left(w(\frac{k+1}{2^n}) - 2\ w(\frac{k}{2^n}) + w(\frac{k-1}{2^n}) \right) \text{ for } w \in E. \tag{2.5}$$

Set $h^*_0(w) := w(1)$ for $w \in E$. Clearly, the linear span of $\{h^*_0, h^*_{k,n}, \mathrm{odd}\ k < 2^n, n \ge 1\}$ is dense in E' and the following is easy to check:

$$\begin{aligned} h^*_{k,n}(w) &= \langle h_{k,n}, w \rangle_H, \\ h^*_0(w) &= \langle h_0, w \rangle_H, \quad w \in H, \end{aligned} \tag{2.6}$$

where $h_{k,n} = \int_0^\cdot f_{\frac{k}{2^n}}(s)ds$, $h_0 = \int_0^\cdot f_0(s)ds \in H$.

Let $(M_t(s))$ be a random field on $[0,1] \times [0,1]$ defined by

$$M_t(s) := M_t^{h^*_0} \int_0^s f_0(s')ds' + \sum_{n=1}^\infty \sum_{\substack{k\ odd \\ k<2^n}} M_t^{h^*_{k,n}} \int_0^s f_{\frac{k}{2^n}}(s')ds'. \tag{2.7}$$

Since $\sum_{\substack{k\ odd \\ k<2^n}} M_t^{h^*_{k,n}} \int_0^s f_{\frac{k}{2^n}}(s')ds'$ is jointly continuous with respect to t and s, to prove that $(M_t(s))$ is well defined (i.e., that the sum in (2.7) converges) and that it is continuous on $[0,1] \times [0,1]$ P_μ-a.s., it sufficies to show that the sum in (2.7) converges uniformly on $[0,1] \times [0,1]$ for P_μ-a.e. $w \in \Omega$. Set

$$\alpha_n := \sup_{0 \le s,t \le 1} \left| \sum_{\substack{k\ odd \\ k<2^n}} M_t^{h^*_{k,n}} \int_0^s f_{\frac{k}{2^n}}(s')ds' \right|. \tag{2.8}$$

It is clear that

$$\begin{aligned}\alpha_n &\le \max_{\substack{k\,odd\\k<2^n}} \sup_{0\le t\le 1} |M_t^{h^*_{k,n}}| \sup_{0\le s\le 1} |\sum_{\substack{k\,odd\\k<2^n}} \int_0^s f_{\frac{k}{2^n}}(s')ds'| \\ &= 2^{-\frac{(n+1)}{2}} \max_{\substack{k\,odd\\k<2^n}} \sup_{0\le t\le 1} |M_t^{h^*_{k,n}}| \,. \end{aligned} \tag{2.9}$$

Thus for $\vartheta > 0$

$$\begin{aligned} P[\alpha_n > \vartheta(2^{-n}\ln 2^n)^{\frac{1}{2}}] &\le P\left[\max_{\substack{k\,odd\\k<2^n}} \sup_{0\le t\le 1} |M_t^{h^*_{k,n}}| > \vartheta\sqrt{2n\ln 2}\right] \\ &\le \sum_{\substack{k\,odd\\k<2^n}} P\left[\sup_{0\le t\le 1} |M_t^{h^*_{k,n}}| > \vartheta\sqrt{2n\ln 2}\right]. \end{aligned} \tag{2.10}$$

Note that $M_t^{h^*_{k,n}}$ is a martingale with bracket

$$\begin{aligned} \langle M_t^{h^*_{k,n}}\rangle &= \int_0^\cdot \langle A(X_s)\nabla h^*_{k,n}, \nabla h^*_{k,n}\rangle_H ds \\ &= \int_0^\cdot \langle A(X_s)h_{k,n}, h_{k,n}\rangle_H ds \end{aligned} \tag{2.11}$$

(cf. [AR91, Proposition 4.5]). By standard theory (since $\langle M^{h^*_{k,n}}\rangle_t \ge \delta\, t$ by (1.5)) there exists a Brownian motion (B_t) such that $M_t^{h^*_{k,n}} = B_{\langle M^{h^*_{k,n}}\rangle_t}$, $t\ge 0$. Using the fact that by (1.5) $\langle M^{h^*_{k,n}}\rangle_t \le \frac{1}{\delta}\, t$, we have that

$$\begin{aligned} P\left[\sup_{0\le t\le 1} |M_t^{h^*_{k,n}}| > \vartheta\sqrt{2n\ln 2}\right] &= P\left[\sup_{0\le t\le 1} |B_{\langle M^{h^*_{k,n}}\rangle_t}| > \vartheta\sqrt{2n\ln 2}\right] \\ &\le P\left[\sup_{0\le t\le \frac{1}{\delta}} |B_t| > \vartheta\sqrt{2n\ln 2}\right] \\ &= 4P\left[B_1 > \sqrt{\delta}\vartheta\sqrt{2n\ln 2}\right] \\ &= 4\int_{\sqrt{\delta}\vartheta\sqrt{2n\ln 2}}^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &\le 4\left(\vartheta\sqrt{\delta}\sqrt{2n\ln 2}\right)^{-1} \frac{1}{\sqrt{2\pi}} \exp\left(-\left(\sqrt{\delta}\vartheta\right)^2 n\ln 2\right) . \end{aligned} \tag{2.12}$$

By (2.10) we now conclude that

$$
\begin{aligned}
P\left[\alpha_n > \vartheta(2^{-n}\ln 2^n)^{1/2}\right] &\leq \frac{1}{\sqrt{2\pi}} 2^{n-1} 4(\vartheta\sqrt{\delta})(\sqrt{2n\ln 2})^{-1} 2^{-(\vartheta\sqrt{\delta})^2 n} \\
&\leq M\vartheta\sqrt{\delta}\, 2^{n(1-(\vartheta\sqrt{\delta})^2)} n^{-\frac{1}{2}} \qquad (2.13)
\end{aligned}
$$

with M a numerical constant. Choosing $\vartheta > \frac{1}{\sqrt{\delta}}$, from (2.13) we obtain that

$$
\sum_{n=1}^{\infty} P\left[\alpha_n > \vartheta\left(2^{-n}\ln 2^n\right)^{1/2}\right] < +\infty\ . \qquad (2.14)
$$

By the Borel-Cantelli lemma, we conclude that

$$
P\left[\bigcap_n \bigcup_{m\geq n} \left\{l_m > \vartheta\left(2^{-m}\ln 2^m\right)^{1/2}\right\}\right] = 0\ . \qquad (2.15)
$$

This implies

$$
\sum_n \alpha_n < +\infty \quad P_\mu\text{-a.e.},
$$

which proves the path continuity of $(M_t(s))$ as a random field on $[0,1]\times[0,1]$. Consequently, (M_t) is a continuous E-valued process. Its $(\mathcal{F}_t)$-adaptiveness follows from the $(\mathcal{F}_t)$-adaptiveness of $M^{h^*_{k,n}}$. By the same proof, one can show that $\bar{M}_t$ is given by

$$
\bar{M}_t(s) = \bar{M}_t^{h_0^*}\int_0^s f_0(s')ds' + \sum_{n=1}^{\infty}\sum_{\substack{k\,odd\\ k<2^n}} \bar{M}_t^{h^*_{k,n}}\int_0^s f_{\frac{k}{2^n}}(s')ds'\ . \qquad (2.16)
$$

Now it remains to show that for every $l\in E'$, $0\leq t\leq 1$,

$$
l(M_t) = M_t^l\,,\ l(\bar{M}_t) = \bar{M}_t^l \quad P_\mu - a.s. \qquad (2.17)
$$

By the definition, (2.17) holds for any l in the linear span of $\{h_0^*, h^*_{k,n}, n\geq 1, k\,\text{odd}, k<2^n\}$. Combining this with the fact that $M_t^{l_n}\to M_t^l$ in $L^2(\Omega;P_\mu)$ if $l_n\to l$ in H (which follows from (2.2)), we have (2.17) for any $l\in E'$. $\square$

To state our next lemma, we recall the definition of the space $W^{\alpha,2m}$. Let $0<\alpha<\frac{1}{2}$, $m\in\mathbb{N}$. Define

$$
W^{\alpha,2m} := \{w\in L^{2m}([0,1];ds)\mid \|w\|_{\alpha,2m} < +\infty\}, \qquad (2.18)
$$

where

$$\|w\|_{\alpha,2m} := \left[\int_0^1 \int_0^1 \frac{|w(t)-w(s)|^{2m}}{|t-s|^{1+2m\alpha}} ds\, dt\right]^{\frac{1}{2m}} . \tag{2.19}$$

If $2m\alpha > 1$ and $m \geq 2$, it is well-known that the Banach space $W_0^{\alpha,2m} := \{w \in W^{\alpha,2m} | w(0) = 0\}$ with norm $\| \quad \|_{\alpha,2m}$ is continuously embedded into E and that the embedding is compact (see e.g. [Tr78, Section 4.5, in particular, Theorem 4.6.1].)

Let $M, \bar{M}$ be the processes in Lemma 2.2.

Lemma 2.3 *(i)* $M_t, \bar{M}_t \in L^p(\Omega \to E; P_\mu)$ *for any* $p \geq 1,\ t \geq 0$.

(ii) (M_t) *is an* $(\mathcal{F}_t)$*-martingale and* $(\bar{M}_t)$ *is an* $(\bar{\mathcal{F}}_t)$*-martingale.*

Proof. (i): We actually will prove a stronger result, that is, for any $m \geq 1$, $0 < \alpha < \frac{1}{2}$, and $t \geq 0$,

$$\int_\Omega \|M_t\|_{\alpha,2m}^{2m} dP_\mu < +\infty \tag{2.20}$$

$$\int_\Omega \|\bar{M}_t\|_{\alpha,2m}^{2m} dP_\mu < +\infty \tag{2.21}$$

We only prove (2.20); (2.21) is proved similarly. Now fix $t \geq 0$. For any $s \in [0,1]$ define $\delta_s \in E'$ by

$$\delta_s(w) = w(s) . \tag{2.22}$$

Then by the explicit form of the embedding $E' \subset H$ (cf . the sentence after (1.3))

$$\nabla \delta_s(w) = \int_0^\cdot 1_{[0,s]}\, ds' \quad \text{for all } w \in E.$$

Thus, if $s_1, s_2 \in [0,1], s_1 \leq s_2$, the bracket of the $(\mathcal{F}_t)$-martingale $M_t^{\delta_{s_2}-\delta_{s_1}}$ is

$$\begin{aligned} \langle M^{\delta_{s_2}-\delta_{s_1}}\rangle_t &= \int_0^t \langle A(X_{s'})\nabla(\delta_{s_2}-\delta_{s_1})(X_{s'}), \nabla(\delta_{s_2}-\delta_{s_1})(X'_s)\rangle_H ds' \\ &\leq \frac{1}{\delta}\int_0^t \langle \nabla(\delta_{s_2}-\delta_{s_1})(X_{s'}), \nabla(\delta_{s_2}-\delta_{s_1})(X_{s'})\rangle_H ds' \\ &= \frac{t}{\delta}(s_2-s_1)\,,\ t\geq 0. \end{aligned} \tag{2.23}$$

This implies that for any $m \geq 1$

$$\begin{aligned}
\int_\Omega |M_t(s_2) - M_t(s_1)|^{2m} dP_\mu &= \int_\Omega |M_t^{\delta_{s_2}} - M_t^{\delta_{s_1}}|^{2m}\ dP_\mu \\
&= \int_\Omega |M_t^{\delta_{s_2}-\delta_{s_1}}|^{2m}\ dP_\mu \\
&\leq C \int_\Omega \langle M^{\delta_{s_2}-\delta_{s_1}} \rangle_t^m dP_\mu \\
&\text{(where } C \text{ is a constant independent of } M^{\delta_{s_2}-\delta_{s_1}}) \\
&\leq C \left(\frac{t}{\delta}\right)^m (s_2 - s_1)^m \ . \qquad (2.24)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_\Omega \|M_t\|_{\alpha,2m}^{2m} dP_\mu &= \int_\Omega \left(\int_0^1 \int_0^1 \frac{|M_t(s_2) - M_t(s_1)|^{2m}}{|s_2 - s_1|^{1+2m\alpha}} ds_1 ds_2 \right) dP_\mu \\
&= \int_0^1 \int_0^1 \frac{ds_1 ds_2}{|s_2 - s_1|^{1+2m\alpha}} \int_\Omega |M_t(s_2) - M_t(s_1)|^{2m} dP_\mu \\
&\leq C \left(\frac{t}{\delta}\right)^m \int_0^1 \int_0^1 \frac{|s_2 - s_1|^m}{|s_2 - s_1|^{1+2m\alpha}} ds_1 ds_2 \\
&= C \left(\frac{t}{\delta}\right)^m \int_0^1 \int_0^1 \frac{ds_1 ds_2}{|s_2 - s_1|^{1+2m\alpha-m}} \\
&< +\infty, \quad \text{since } 1 + 2m\alpha - m < 1.
\end{aligned}$$

This proves (i).

(ii): To show that (M_t) is an $(\mathcal{F}_t)$-martingale, we need to prove that for any $A \in \mathcal{F}_s$, $s \leq t$

$$\int_\Omega M_t 1_A dP_\mu = \int_\Omega M_s 1_A dP_\mu \ . \qquad (2.25)$$

But this is a consequence of

$$\int_\Omega M_t^l 1_A dP_\mu = \int_\Omega M_s^l 1_A dP_\mu \quad \text{for all } l \in E'. \qquad (2.26)$$

By the same reason, $(\bar{M}_t)$ is also an $(\bar{\mathcal{F}}_t)$-martingale. $\square$

3 Tightness for $C_0[0,1]$-valued diffusion processes

In this section we apply the martingale decomposition to the study of tightness of diffusion processes with state space $E := C_0[0,1]$. (For an implementation of this method in the case $E = \mathbb{R}^d$, see [T89].) Let $A_n(z), n \geq 1$, be strongly measurable maps from E to $L^\infty(H)$, which satisfy the conditions (i) and (ii) in (1.5) with a constant δ independent of n. Suppose we are given a probability measure μ_n on $(E, \mathcal{B}(E))$ with $\mathrm{supp}[\mu_n] = E$ for each $n \geq 1$. Define for $n \geq 1$ the following symmetric form $(\mathcal{E}_n, \mathcal{F}C_b^\infty)$ on $L^2(E;\mu_n)$,

$$\mathcal{E}_n(u,v) := \int_E \langle A_n(z)\nabla u(z), \nabla v(z)\rangle_H \, \mu_n(dz); \; u,v \in \mathcal{F}C_b^\infty \; . \tag{3.1}$$

Again we impose the minimum condition that $(\mathcal{E}_n, \mathcal{F}C_b^\infty))$ is closable on $L^2(E;\mu_n)$ and we denote its closure by $(\mathcal{E}_n, D(\mathcal{E}_n))$.

Let $\mathcal{M}^n = \{\Omega, \mathcal{F}, (\mathcal{F}_t), (X_t), (P_z^n)_{z\in E}\}$ be the diffusion process associated with $(\mathcal{E}_n, D(\mathcal{E}_n))$. Here $\Omega := C([0,\infty) \to E)$ is the path space equipped with the topology of locally uniform convergence, $X_t(w) := w(t), w \in \Omega$, is the coordinate process, $(\mathcal{F}_t)$ the corresponding filtration, and $\mathcal{F} := \sigma\{X_t | t \geq 0\}$. For $n \geq 1$ define a probability measure P_n on $(\Omega, \mathcal{F})$ by

$$P_n = \int_E P_z^n \, \mu_n(dz) \; . \tag{3.2}$$

Theorem 3.1 *Assume $\{\mu_n \,|n \geq 1\}$ is tight on $(E, \mathcal{B}(E))$. Then $\{P_n \,|n \geq 1\}$ is tight on the path space Ω.*

Proof. Let $T > 0$ and $\Omega_T := C([0,T] \to E)$. It sufficies to prove the tightness of (the restrictions of) $\{P_n \,|\, n \in \mathbb{N}\}$ on Ω_T. For simplicity again we set $T := 1$. First we are going to prove that there exists a constant $c_m > 0$, which is independent of n, such that

$$E_n \left[\|X_t - X_s\|^{2m} \right] \leq c_m \, (t-s)^m, \quad \text{for all } 0 \leq s,t \leq 1, \tag{3.3}$$

where E_n denotes the expectation with respect to P_n. By Theorem 2.1 we have that under P_n

$$X_t - X_0 = \frac{1}{2} M_t^n - \frac{1}{2}(\bar{M}_1^n - \bar{M}_{1-t}^n) \, , \; 0 \leq t \leq 1. \tag{3.4}$$

Here M_t^n is an $(\mathcal{F}_t)$-martingale, and $\bar{M}_t^n$ is an $(\bar{\mathcal{F}}_t)$-martingale. Thus to prove (3.3) it sufficies to show the following for $0 < \alpha < \frac{1}{2}$ (cf. Section 2): there exist constants $c_m^{(1)}, c_m^{(2)} > 0$ such that for all $0 \le s, t \le 1$

$$E_n\left[\|M_t^n - M_s^n\|_{\alpha,2m}^{2m}\right] \le c_m^{(1)} (t-s)^m , \tag{3.5}$$

$$E_n\left[\|\bar{M}_t^n - \bar{M}_s^n\|_{\alpha,2m}^{2m}\right] \le c_m^{(2)} (t-s)^m , \tag{3.6}$$

where $\| \quad \|_{\alpha,2m}$ is defined as in Section 2. We only prove (3.5); (3.6) is proved analogously.

For $s_1 \le s_2$, $s \le t$ we have that

$$\begin{aligned}
& \int_\Omega |\,(M_t^n - M_s^n)\,(s_2) - (M_t^n - M_s^n)\,(s_1)|^{2m}\, dP_n \\
= & \int_\Omega |M_t^n(s_2) - M_t^n(s_1) + M_s^n(s_1) - M_s^n(s_2)|^{2m}\, dP_n \\
= & \int_\Omega |(M^n)_t^{\delta_{s_2}-\delta_{s_1}} - (M^n)_s^{\delta_{s_2}-\delta_{s_1}}|^{2m}\, dP_n
\end{aligned}$$

(where $\delta_{s_1}, \delta_{s_2} \in E'$ is defined as in (2.22))

$$\le c(m) \int_\Omega \left(\langle (M^n)^{\delta_{s_2}-\delta_{s_1}}\rangle_t - \langle (M^n)^{\delta_{s_2}-\delta_{s_1}}\rangle_s\right)^m\, dP_n$$

(where $c(m) > 0$ is a constant independent of t, s, s_1, s_2, n)

$$= c(m) \int_\Omega \left(\int_s^t \langle A_n(X_{s'})\nabla(\delta_{s_2}-\delta_{s_1})(X_{s'}), \nabla(\delta_{s_2}-\delta_{s_1})(X_{s'})\rangle_H ds'\right)^m\, dP_n$$

(by (2.2) applied to M^n)

$$\begin{aligned}
\le & \; c(m)\,(t-s)^{m-1} \int_\Omega \int_s^t \left(\frac{1}{\delta}\right)^m \langle \nabla(\delta_{s_2}-\delta_{s_1})(X_{s'}), \nabla(\delta_{s_2}-\delta_{s_1})(X_{s'})\rangle_H^m\, ds' dP_n \\
= & \; c(m) \left(\frac{1}{\delta}\right)^m (t-s)^m\,(s_2-s_1)^m
\end{aligned} \tag{3.7}$$

(cf. (2.23)).

By (3.7) we obtain that

$$\begin{aligned}
& E_n\left[\|M_t^n - M_s^n\|_{\alpha,2m}^{2m}\right] \\
= & \int_\Omega \left(\int_0^1\int_0^1 \frac{|(M_t^n - M_s^n)(s_2) - (M_t^n - M_s^n)(s_1)|^{2m}}{|s_2-s_1|^{1+2m\alpha}}\, ds_1\, ds_2\right) dP_n \\
= & \int_0^1\int_0^1 \frac{ds_1\, ds_2}{|s_2-s_1|^{1+2m\alpha}} \int_\Omega |(M_t^n - M_s^n)(s_2) - (M_t^n - M_s^n)(s_1)|^{2m} dP_n
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^1 \int_0^1 \frac{ds_1\, ds_2}{|s_2 - s_1|^{1+2m\alpha}} \;\; c(m) \left(\frac{1}{\delta}\right)^m (t-s)^m\, |s_2 - s_1|^m \\
&= c_m^{(1)}\, (t-s)^m
\end{aligned}
$$

where $c_m^{(1)} := (\frac{1}{\delta})^m \int_0^1 \int_0^1 \frac{ds_1\, ds_2}{|s_2-s_1|^{1+2m\alpha-m}} < \infty$. This proves (3.5), and hence (3.3). □

Note that for any $t \ge 0$, $\nu_t^n := P_n \circ X_t^{-1} = \mu_n$. In particular, $\{\nu_t^n\, | n \ge 1\}$ is tight. Combining this with (3.3) the assertion follows by [EK86, Chap.3, Theorem 7.2]

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