On the second real eigenvalue of nonnegative and Z–matrices *

Shmuel Friedland
University of Illinois at Chicago
Department of Mathematics
851 S. Morgan St., Chicago, Illinois 60607-7045
USA

Reinhard Nabben†
Fakultät für Mathematik
Universität Bielefeld
Postfach 10 01 31
33 501 Bielefeld
Germany

Abstract

We give bounds for the second real eigenvalue of nonnegative matrices and Z–matrices. Furthermore, we establish upper bounds for the maximal spectral radii of principal submatrices of nonnegative matrices. Using this bounds we prove that our inequality for the second real eigenvalue of the incidence matrix of a connected regular graph improves a well known bound for the second eigenvalue using Cheeger's inequality.

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1 Introduction

Let $\mathbb{R}^{n,n}$ be the set of $n \times n$ real valued matrices. Assume that $A = [a_{ij}] \in \mathbb{R}^{n,n}$ and arrange the eigenvalues of A as follows

$$Re(\lambda_n(A)) \geq Re(\lambda_{n-1}(A)) \geq \ldots \geq Re(\lambda_1(A)).$$

Suppose that A is nonnegative ($A \geq 0$). Then $\lambda_n(A) = \rho(A)$ is the spectral radius of A. In many applications one needs to estimate the separation between $\rho(A)$ and $Re(\lambda_{n-1})$. See for example [F2] and the references therein. For symmetric nonnegative irreducible matrices the known inequalities are matrix variations of Cheeger's inequality [C]. For general nonnegative irreducible matrices a matrix version of Cheeger's inequality is given in [FG].

$$Re(\lambda_{n-1}(A)) \leq \rho(A) - \frac{1}{2}(\rho(A) - \max_{1\leq i\leq n} a_{ii})\epsilon_{\lfloor \frac{n}{2} \rfloor}(A)^2,$$
$$\epsilon_{\lfloor \frac{n}{2} \rfloor}(A) = \inf_{U\subset <n>, |U|\leq \lfloor \frac{n}{2} \rfloor} \frac{\sum_{i\in U, j\in <n>\backslash U} a_{ij}v_iu_j + a_{ji}v_ju_i}{\sum_{i\in U} 2(\rho(A) - a_{ii})v_iu_i},$$

where $u, v \in \mathbb{R}^{n,n}$ are the right and left positive eigenvectors corresponding to $\rho(A)$.

The aim of this paper is to give a different type of estimate on the real eigenvalue of A different from $\rho(A)$. This estimate is given in terms of spectral radii of principle submatrices of A. Namely, for $U \subset < n >$ let $A(U)$ be the principle submatrix of A whose rows and columns are in U. Set

$$\rho_m(A) = \max_{U\subset <n>, |U|=m} \rho(A(U)).$$

We then show that any real eigenvalue $\lambda, \lambda \neq \rho(A)$ satisfies $\lambda \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A)$. In the case that A is a symmetric nonnegative matrix, or more generally A is diagonally similar to a nonnegative symmetric matrix (in particular time reversible Markov chains), we have the inequality $\lambda_{n-1}(A) \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A)$. For any nonnegative irreducible matrix we show

$$\rho_{\lfloor \frac{n}{2} \rfloor}(A) \leq \rho(A) - \frac{1}{2}(\rho(A) - \max_{1\leq i\leq n} a_{ii})\epsilon_{\lfloor \frac{n}{2} \rfloor}(A)^2. \tag{1.1}$$

It then follows that our inequality $\lambda_{n-1}(A) \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A)$ for the incidence matrix A of a connected regular graph improves a well known bound for the second eigenvalue using Cheeger's inequality, e.g. [F2].

For Z–matrices we can get a similar bound for the second smallest real eigenvalue. If we define

$$\tau_m(A) = \min_{U \subset <n>, |U|=m} \lambda_1(A(U))$$

then we show that any real eigenvalue $\lambda \neq \lambda_1(A)$ of a Z–matrix satisfies $\lambda \geq \tau_{\lfloor \frac{n}{2} \rfloor}(A)$.

The paper is organized as follows. In Section 2 we establish our bounds on the second real eigenvalue of nonnegative and Z–matrices. In Section 3 we give upper bounds for $\rho_s(A)$, $s = 1, \ldots, n$. We conclude with an example that shows that our bound is best asymptotically.

2 Eigenvalue bounds

For $n \in \mathbb{N}$ set $< n > = \{1, \ldots, n\}$ and let $U = \{u_1, \ldots, u_m\} \subset < n >$ be a nontrivial subset of $< n >$ with $1 \leq u_1 < u_2 < \ldots < u_m \leq n$. Thus $|U| = m$.

Assume that $A = [a_{ij}] \in \mathbb{R}^{n,n}$. Denote by $A(U)$ the principal submatrix $[a_{u_\alpha u_\beta}]_{\alpha,\beta=1}^m$ of A which rows and columns are indexed by the entries of U. Set

$$\phi_m(A) = \max_{U \subset <n>, |U|=m} Re(\lambda_m(A(U))).$$

Assume that $A \geq 0$. Then $\phi_m(A) = \rho_m(A)$ is the maximum of the spectral radii of all $m \times m$ principal submatrices of A. In particular, $\rho_n(A) = \rho(A)$. Since A is a nonnegative matrix we have

$$\rho_1(A) \leq \rho_2(A) \leq \ldots \leq \rho_n(A).$$

If A is a positive matrix ($A > 0$) the inequalities are strict. Also, it is well-known that if $A \geq 0$ is irreducible, then $\rho_{n-1}(A) < \rho_n(A)$. However, the other inequalities are not necessarily strict.

A real matrix $A \in \mathbb{R}^{n,n}$ is called a Z–matrix if $A = tI - B$ where $t \in \mathbb{R}$ and $B \geq 0$, i.e. all off-diagonal elements of A are nonpositive. If $t \geq \rho(B)$ then A is called an M–matrix. Fiedler and Markham [FM] used the values $\rho_m(A)$ where A is nonnegative, to establish a classification of Z–matrices, see also [N].

For a Z–matrix A we have parallel inequalities.

$$\tau_m(A) = \min_{U \subset <n>, |U|=m} \lambda_1(A(U)).$$

We have

$$\tau_n(A) \leq \tau_{n-1}(A) \leq \ldots \leq \tau_1(A).$$

If A is irreducible then $\tau_n(A) < \tau_{n-1}(A)$. Furthermore, if all off-diagonal entries of A are negative, all inequalities are strict. We will call such a Z–matrix an *essentially negative matrix.*

Before we state our theorem we need some Lemmata which can be obtained from well-known Lemmata for nonnegative matrices (see e.g. [HJ]).

Lemma 2.1 *Let $A \in \mathrm{I\!R}^{n,n}$ be a Z–matrix. Let $x \in \mathrm{I\!R}^n$, $x \geq 0$ and $x \neq 0$. If $Ax \leq \alpha x$ for some $\alpha \in \mathrm{I\!R}$ then $\lambda_1(A) \leq \alpha$. If A is irreducible and in addition $Ax \neq \alpha x$ then $\lambda_1(A) < \alpha$.*

Lemma 2.2 *Let $A \in \mathrm{I\!R}^{n,n}$ be an irreducible Z–matrix. Let $u = [u_i] \in \mathrm{I\!R}^n$ such that $Au = \lambda u$ for some $\lambda \in \mathrm{I\!R}$ with $\lambda \neq \rho(A)$. Then, there exist two indices i, j, $1 \leq i < j \leq n$ such that $u_i u_j < 0$.*

We now state and prove our main result of this section for Z–matrices.

Theorem 2.3 *Let $A \in \mathrm{I\!R}^{n,n}$ be a Z–matrix. Then any real eigenvalue λ of A different from $\lambda_1(A)$ satisfies the inequality*

$$\lambda \geq \tau_{\lfloor \frac{n}{2} \rfloor}(A). \tag{2.2}$$

If A is essentially negative the inequality (2.2) *is strict.*

Proof. First we assume that A is irreducible. Let λ be a real eigenvalue different from $\rho(A)$, and let $u \in \mathrm{I\!R}^n$ be the corresponding eigenvector, i.e. $Au = \lambda u$. Without loss of generality we can assume that u is given in the form

$$u = \begin{bmatrix} v \\ -w \end{bmatrix}$$

where $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^{n-m}$ satisfy $u \geq 0$ and $w > 0$. Lemma 2.2 yields that $m > 0$. Partition A as u to obtain

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} v \\ -w \end{bmatrix} = \lambda \begin{bmatrix} v \\ -w \end{bmatrix};$$

or

$$\begin{aligned} A_{11}v - A_{12}w &= \lambda v \\ A_{22}w - A_{21}v &= \lambda w. \end{aligned}$$

Hence

$$\begin{aligned} A_{11}v &\leq \lambda v \\ A_{22}w &\leq \lambda w. \end{aligned}$$

Lemma 2.1 implies $\lambda \geq \lambda_1(A_{11})$ and $\lambda \geq \lambda_1(A_{22})$. Moreover, strict inequalities hold if A is essentially negative. Since either m or $n-m$ is less or equal to $\lfloor \frac{n}{2} \rfloor$ we obtain $\lambda \geq \tau_{\lfloor \frac{n}{2} \rfloor}(A)$, or $\lambda > \tau_{\lfloor \frac{n}{2} \rfloor}(A)$ if A is essentially negative.

If A is reducible (2.2) follows immediately from the irreducible case considering the reducible normal form.

$\square$

Theorem 2.3 improves a result given in Lemma 1.1 of [F2] which is stated for diagonally symmetrizable irreducible matrices, i.e. matrices A for which a positive diagonal matrix D exists such that DA is symmetric.

Restating Theorem 2.3 for $-A$ we get

Theorem 2.4 *Let $A \in \mathbb{R}^{n,n}$ be essentially nonnegative, i.e. $-A$ is a Z-matrix. Then any real eigenvalue λ of A different from $\lambda_n(A)$ satisfies the inequality*

$$\lambda \leq \phi_{\lfloor \frac{n}{2} \rfloor}(A) \tag{2.3}$$

Assume that $A \geq 0$. Then

$$\lambda \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A) \tag{2.4}$$

If A is positive the inequality (2.4) *is strict.*

The inequality (2.4) improves some known results. It was proved by Koteljanskii that any real eigenvalue λ, different from $\rho(A)$, of a positive matrix satisfies $\lambda < \rho_{n-2}(A)$ [K]. The weak inequality $\lambda \leq \rho_{n-2}(A)$ is proved by R.L. Smith for nonnegative matrices [S].

In general one can not get better results as (2.4). One can easily find reducible matrices A for which the second largest real eigenvalue is equal to $\rho_{\lfloor \frac{n}{2} \rfloor}(A)$. In the next section we give an example that shows that inequality (2.4) is best possible asymptotically for irreducible matrices.

The proof of the Theorems 2.4 and 2.3 uses essentially the minima of $|\alpha|$ and $|\beta|$, where $\alpha = \{i \in < n > \mid u_i > 0\}$ and $\beta = \{i \in < n > \mid u_i < 0\}$ and $u = [u_i] \in \mathbb{R}^{n,n}$ satisfies $Au = \lambda u$ with $\lambda \neq \lambda_1(A)$ or $\lambda \neq \rho(A)$ respectively. Thus we have

Theorem 2.5 *Let* $A \in \mathbb{R}^{n,n}$, $A \geq 0$, $\lambda \in \mathbb{R}$ *and* $u \in \mathbb{R}^n$ *such that* $Au = \lambda u$ *with* $\lambda \neq \rho(A)$. *Set* $\alpha = \{i \in < n > \mid u_i > 0\}$, $\beta = \{i \in < n > \mid u_i < 0\}$ *and let* $t = \min\{|\alpha|, |\beta|\}$. *Then*

$$\lambda \leq \rho_t(A). \tag{2.5}$$

If A is positive the inequality (2.5) *is strict.*

A similar result can be stated for Z–matrices.

Next we will show that one cannot expect inequalities like (2.4) for arbitrary matrices even if they are symmetric.

Assume that A is an $n \times n$ symmetric (not necessary nonnegative) matrix. Let $< n >= U_n \supset \cdots \supset U_1, |U_i| = i, i = 1, ..., n$. Denote by $\lambda_{i,i} \geq \cdots \lambda_{i,1}$ the eigenvalues of $A(U_i)$ for $i = 1, ..., n$. The only conditions on the eigenvalues $\lambda_{i,j}, i = 1, ..., n, j = 1, ..., i$, are the Cauchy interlacing inequalities $\lambda_{i,j} \leq \lambda_{i-1,j} \leq \lambda_{i,j+1}, i = 2, ..., n, j = 1, ..., i-1$, see [F1]. Assume that A is a nonnegative symmetric matrix. Then Theorem 2.4 implies that $\phi_{\lfloor \frac{n}{2} \rfloor}(A) \geq \lambda_{n-1}(A)$. The following example show that the above inequality fails for general symmetric matrices at least for $n = 3$. Let

$$A = \begin{bmatrix} 0 & 1 & r \\ 1 & 0 & 1 \\ r & 1 & 0 \end{bmatrix}$$

Clearly, $\lambda_{1,1} = 0$ for any $U_1, |U_1| = 1$. The Cauchy interlacing inequalities applied for $U = \{1, 2\}$ yield that $\lambda_3(A) \geq 1, \lambda_1(A) \leq -1$. Moreover $det(A) = 2r$. Thus if $r < 0$ we have that $\lambda_2(A) > 0 = \phi_1(A)$. We suspect that our theorem fails for any $n > 3$ for general symmetric matrices.

Our estimate for the second real eigenvalues or for the second largest eigenvalue in the symmetric case halfs the size of the problem. However in any case the right hand side of (2.4) has to be estimated and the principle submatrix has to be found which belongs to the maximal spectral radius. To do so one can use comparison theorems for the spectral radius of nonnegative matrices and every approximation of their spectral radii. As an example we just mention the following approximation which is based on Gershgorin disks.

Define for $A \in \mathbb{R}^{n,n}$ and $t \in \{1, \ldots, n\}$

$$\omega_t(A) = \max_{V \subset <n>, |V| = t} \max_{i \in V} \sum_{j \in V} |a_{ij}|.$$

Then the Gershgorin disk theorem gives

$$\rho_t(A) \leq \omega_t(A).$$

Thus, given a nonnegative matrix A the maximal sum of $\lfloor \frac{n}{2} \rfloor$ entries in a row or column gives an upper bound for $\rho_{\lfloor \frac{n}{2} \rfloor}(A)$ and hence for the second real eigenvalue. This strategy is useful only when the matrix is not sparse.

From Theorem 2.4 we obtain eigenvalue bounds for M–matrices.

Theorem 2.6 *Let $A \in \mathbb{R}^{n,n}$ be an M–matrix of the form $A = sI - B$ with $B \geq 0$ and $s \geq \rho(B)$. Then every real eigenvalue μ of A which is not equal the smallest one, satisfies*

$$\mu \geq s - \rho_{\lfloor \frac{n}{2} \rfloor}(B).$$

Proof. The proof follows immediately from Theorem 2.4. □

More applications of our estimate of the second real eigenvalue of a nonnegative matrix are given in the next section and in [N].

3 Upper estimates of $\rho_s(A)$

Let $A \in \mathbb{R}^{n,n}$ be an irreducible nonnegative matrix. Denote by u and v be the left and right eigenvectors of A corresponding to $\rho(A)$, i.e.

$$Au = \rho(A)u, \qquad A^T v = \rho(A)v, \qquad 0 < u, v \in \mathbb{R}^n$$

Set

$$\epsilon(A,U) = \min_{\emptyset \neq V \subset U} \frac{\sum_{i\in V, j\in <n>\setminus V} a_{ij}v_iu_j + a_{ji}v_ju_i}{\sum_{i\in V} 2(\rho(A)-a_{ii})v_iu_i}, \emptyset \neq U \subset <n>,$$
$$\epsilon_s(A) = \min_{\emptyset \neq V \subset <n>, |V|\leq s} \frac{\sum_{i\in V, j\in <n>\setminus V} a_{ij}v_iu_j + a_{ji}v_ju_i}{\sum_{i\in V} 2(\rho(A)-a_{ii})v_iu_i} = \min_{U\subset <n>, |U|=s} \epsilon(A,U).$$

The main result of this section is inspired by the main result of [FG].

Theorem 3.1 *Let $A = [a_{ij}]$ be a nonnegative irreducible matrix with the positive left and right eigenvectors u and v. Then*

$$\rho_s(A) \leq \rho(A) - \frac{1}{2}(\rho(A) - \max_i a_{ii})\epsilon_s(A)^2$$

for $s = 1, \ldots, n-1$

Proof. Assume first that $A = A^T$, $u = v$. Consider the singular M–matrix $H = \rho(A)I - A$. It was shown in [F2] that

$$H = \tilde{B}^T W \tilde{B}$$

where $\tilde{B}$ and W are constructed as follows. Let G be the loopless connected graph induced by the matrix A. Direct the edges in G in an arbitrary manner to obtain a set of edges $E \in <n> \times <n>$. Let $|E| = m$ be the number of edges in G. Then $\tilde{B}$ is an the following $m \times n$ (modified edge-vertex incidence) matrix. If there is an edge $e = (i,j)$ directed from i to j then the eth row of $\tilde{B}$ has exactly two nonzero entries: $1/u_i$ at the ith column and $-1/u_j$ at the j column. W is a diagonal matrix $diag\{w_1, ..., w_m\}$, where the weight w_e of the edge $e = (i,j)$ is $a_{ij}u_iu_j$. It then follows that the total weight (degree) of the vertex i, δ_i, which is denoted by $|E(\{i\})|$ in [F2], is equal to

$$\delta_i = \sum_{1\leq j\leq n, j\neq i} a_{ij}u_iu_j = (\rho(A) - a_{ii})u_i^2.$$

Let $Q = diag\{u_1, ..., u_n\}$. Then the standard edge-vertex incidence matrix B given in [F2] is $\tilde{B}Q$. We now apply Theorem 2.6 in [F2] to the matrix

$$C = D^{-1}B^TWB = Q^{-1}HQ, \quad D = Q^2.$$

Let $U \subset < n >$. Note that $C(U)$ and $H(U)$ are diagonally similar. The inequality (2.7) of [F2] yields.

$$\lambda_1(H(U)) = \lambda_1(C(U)) \geq \frac{1}{2}(\rho(A) - \max_{i \in U} a_{ii})\epsilon(A, U)^2.$$

Let $U_s^* \subset \{1, \ldots, n\}$ such that $\rho_s(A) = \rho(A(U_s^*))$. Use the above inequality and the definition of $\epsilon_s(A)$ to deduce the theorem in the symmetric case.
Assume that A is not symmetric. Let $\tilde{A} = DAD^{-1}$, such that $Du = D^{-1}v = w$. Note that the spectrum of A and $\tilde{A}$ are the same. Moreover $\epsilon(A, U) = \epsilon(\tilde{A}, U)$ for any $U \subset < n >$.

Next consider $\hat{A} = (\tilde{A} + \tilde{A})/2$. We have $\rho(\hat{A}) = \rho(\tilde{A})$ and $\epsilon(\hat{A}, U) = \epsilon(\tilde{A}, U)$ for all $U \subset < n >$. The maximal characterization of $\rho(\hat{A}(U))$ implies the inequality $\rho(\hat{A}(U)) \geq \rho(\tilde{A}(U))$. In particular, $\rho_s(\hat{A}) \geq \rho_s(A)$. Apply the theorem to the symmetric matrix $\hat{A}$ to deduce the theorem for a nonsymmetric irreducible A. $\square$

Let G be a connected $k - regular$ graph. Denote by A its incidence matrix. Observe that A is symmetric, $\rho(A) = k$ and $Au = ku, u = (1, ..., 1)^T$. As A is symmetric we can apply Theorem 2.4 to estimate from above $\lambda_{n-1}(A)$. We can estimate $\rho_{\lfloor \frac{n}{2} \rfloor}(A)$ using Theorem 3.1. It then follows

$$\lambda_{n-1}(A) \leq \rho_{\lfloor \frac{n}{2} \rfloor}(A) \leq k - \frac{1}{2}k\epsilon_{\lfloor \frac{n}{2} \rfloor}(A)^2. \tag{3.6}$$

The above inequality shows that the upper bound given by Theorem 2.4 is better then the standard Cheeger's upper estimate for $\lambda_{n-1}(A)$ as given by the right-hand (3.6). Consult with [F2] for the references on the Cheeger's inequality.

In the following example we compare our bound (2.4) with Cheeger's bound - the right part of (3.6).

Example 3.2 For $n > 2$ let

$$A_n = \begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 1 & & & 1 & 0 \end{bmatrix} \in \mathrm{I\!R}^{\mathrm{n,n}}.$$

A_n is the incidence matrix of a connected 2-regular graph which is a cycle. Note that $A = P + P^T$ where P is a cyclic permutation. It then follows that the eigenvalues of this matrix are

$$\lambda_j = 2cos\frac{2\pi j}{n} \quad j = 0, \ldots, n-1.$$

Hence we have $\rho(A_n) = 2$. Let $V \subset< n >, |V| \leq \lfloor \frac{n}{2} \rfloor$. Then $A_n(V)$ is the incidence matrix of a union of a finite number of chains. It then follows that $\rho_{\lfloor \frac{n}{2} \rfloor}(A_n) = \rho(A_n(U^*)$ for any U^* consists of $\lfloor \frac{n}{2} \rfloor$ consecutive integers. Then $A_n(U^*)$ is a tridiagonal matrix with 0's on the diagonal and 1's on the upper and lower subdiagonals. It is known that $\rho_{\lfloor \frac{n}{2} \rfloor}(A_n) = 2cos\frac{\pi}{\lfloor \frac{n}{2} \rfloor + 1}$. For large n we get

$$\lambda_n = 2 \quad \lambda_{n-1} \approx 2 - \frac{4\pi^2}{n^2}$$

and

$$\rho_{\lfloor \frac{n}{2} \rfloor}(A_n) \approx 2 - \frac{\pi^2}{(\lfloor \frac{n}{2} \rfloor + 1)^2} \approx 2 - \frac{4\pi^2}{n^2}$$

Thus our upper bound is best possible asymptotically for reducible matrices.
It is easy to show that

$$\epsilon_{\lfloor \frac{n}{2} \rfloor}(A_n) = \epsilon(A_n, U^*) = \frac{1}{\lfloor \frac{n}{2} \rfloor}. \tag{3.7}$$

Hence, the bound (3.6) gives

$$\lambda_{n-1} = 2cos\frac{2\pi}{n} \leq 2 - \frac{4}{n^2}.$$

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