PAIRS OF SHORT ROOT SUBGROUPS IN A CHEVALLEY GROUP

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§ 1. Introduction

The purpose of this paper is to describe subgroups of a Chevalley group G of type B_l, C_l or F_4, generated by a pair of unipotent short root subgroups. In fact we do somewhat more: namely, we classify orbits of a Chevalley group acting by simultaneous conjugation on pairs of unipotent short root subgroups. In the sequel of this paper we consider analogous problems for the case of the Chevalley group of type G_2.

For unipotent long root subgroups such a description is very well known, see M.Aschbacher and G.M.Seitz [AS]. Namely, it turns out that any pair (X_1, X_2) of long root subgroups is simultaneously conjugate to a pair of elementary long roots subgroups (X_α, X_β). The orbits of G on pairs of root subgroups are essentially determined by the angle between α and β, so that generically there are five possible configurations, some of which may give two or three orbits, compare Theorem A below.

This result played an important role in understanding the irreducible subgroups generated by long root subgroups. Geometry of long root subgroups in Chevalley groups is now a well established field, see, for example, J.McLaughlin [M1], [M2], B.S.Stark [S1] – [S3], H.Pollatsek [P], A.Wagner [W], A.E.Zalesskii and V.N.Serezhkin [ZS], W.M.Kantor [K1], B.N.Cooperstein [C1] – [C4], R.Brown

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and S.P.Humphries [BH], Li Shang Zhi [L1] – [L3], F.G.Timmesfeld [T1] – [T5], M.W.Liebeck and G.M.Seitz [LS], H.Cuypers [Cu].

Irreducible subgroups of the classical groups generated by short root subgroups have been classified by B.S.Stark [S2] and Li Shang Zhi [L3]. Nevertheless the geometry of short root subgroups is far from being properly understood. To the best of our knowledge even for the classical cases the orbits of G on pairs of short root subgroups were not classified.

In [V1] – [V3] N.A.Vavilov calculated Bruhat decomposition of root elements. Using these results he classified the orbits of a Chevalley group on pairs of a long and a short root subgroups [V2]. In this case generically there are six possible configurations, compare Theorem B below. This is the starting point of our work.

In the present paper we make the next step and get a similar list for pairs of two short root subgroups. However our list is considerably longer, for odd characteristics there are 21 possible configurations (not all of them occur for each type), some of which split into infinitely many orbits, parametrised by a continuous parameter. This result is summarized in Theorem 1. For characteristic 2 short root subgroups behave exactly as the long root ones and the answer is much simpler, see Theorem 2.

One of the reasons why the answer is so much more complicated is that a short root subgroup (unlike a long root one) does not lie in the center of the unipotent radical U of a Borel subgroup, so that conjugation by elements of U leads to much more complicated configurations of roots. In fact, the mutual position of two long root subgroups depends only on two long roots and that of a pair consisting of a long and a short root subgroups on two or three roots (see [V2]). But the mutual position of two short root subgroups depends on two, three, four and in two cases on five roots.

The present paper is organized as follows. In § 2 we recall the known results about pairs of long – long (Theorem A) and long – short (Theorem B) subgroups and state our principal results about pairs of short – short subgroups (Theorems 1 and 2). The rest of the paper is devoted to the proofs of these results. Namely, in § 3 using the Bruhat decomposition of short root subgroups we reduce description of orbits to a combinatorial problem about roots. In § 4 for each type we draw a list of pairs, (X_1, X_2), which contains representatives of all orbits (Tables 5 – 7). In § 5 we identify the corresponding spans $X = \langle X_1, X_2\rangle$ and study long and short root subgroups contained in X. Finally, in § 6 (for characteristic not 2) and § 7 (for characteristic 2) we prove that pairs corresponding to different configurations are not conjugate and determine non-conjugate pairs within the same configuration.

One can find further details related to these problems, as well as some applications and many additional references, in the surveys [C3], [K], [K2], [T4], [V4], [V5], [Z] and in the fundamental works of B.Cooperstein [C2], W.M.Kantor [K1], M.W.Liebeck and G.M.Seitz [LS] and F.Timmesfeld [T2], [T3], [T5]. In the sequel of this paper we hope to be able to imitate one of the main results of [C2] and to classify some important classes of subgroups generated by triples of short root subgroups (compare also [DV], [V3]).

§ 2. Statement of results

All properties of Chevalley groups used in the sequel can be found in [C] or [S]. First we fix the notation. Let Φ be a reduced irreducible root system, $G = G(\Phi, K)$ be a Chevalley group of type Φ over a field K. Usually we may (and will) assume that G is simply connected. For each root $\alpha \in \Phi$ and each element $t \in K$ one denotes by $x_\alpha(t)$ the corresponding elementary root unipotent. Further $X_\alpha = \{x_\alpha(t) \mid x \in K\}$ is the elementary unipotent root subgroup corresponding to α. For $\alpha \in \Phi$ and $t \in K^*$ we set $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ and $h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$. Fix an order on Φ. Let Φ^+, Φ^- be the corresponding sets of positive and negative roots. Then the subgroups U, V and H of G are defined as follows

$$U = U(\Phi, K) = \langle x_\alpha(t),\ \alpha \in \Phi^+,\ t \in K\rangle,$$

$$V = V(\Phi, K) = \langle x_\alpha(t),\ \alpha \in \Phi^-,\ t \in K\rangle,$$

$$H = H(\Phi, K) = \langle h_\alpha(t),\ \alpha \in \Phi,\ t \in K^*\rangle.$$

Here for a subset X of G one denotes by $\langle X\rangle$ the subgroup of G generated by X. For a root $\alpha \in \Phi$ we set $H_\alpha = \langle h_\alpha(t),\ t \in K\rangle$. The Weyl group of the system Φ is denoted by $W = W(\Phi)$.

There are two dominant roots in a system of type B_l, C_l, F_4 or G_2. One of them is the maximal root δ and the other one is the dominant short root ρ. A subgroup X is called a unipotent long root subgroup if it is conjugate to X_δ and a unipotent short root subgroup if it is conjugate to X_ρ. The elements of X are called unipotent long or short root elements respectively. In the sequel we always omit the word "unipotent" and say simply "long/short root subgroup/element".

In the usual realization of root systems (see [B]) we have:

$$\mathrm{B}_l = \{\pm e_i, \pm e_i \pm e_j\}, \quad \rho = e_1, \quad \delta = e_1 + e_2,$$

$$\mathrm{C}_l = \{\pm 2e_i, \pm e_i \pm e_j\}, \quad \rho = e_1 + e_2, \quad \delta = 2e_1,$$

$$\mathrm{F}_4 = \{\pm e_i, \pm e_i \pm e_j, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4\}, \quad \rho = e_1, \quad \delta = e_1 + e_2.$$

Here $1 \leq i \neq j \leq l$ (with $l = 4$ for $\Phi = \mathrm{F}_4$). In the sequel, when no confusion may arise, we write simply $\pm i$, $\pm i \pm j$, $\pm 1 \pm 2 \pm 3 \pm 4$ to denote the corresponding roots of B_l, C_l or F_4.

Denote the set of all roots strictly larger than ρ by Σ. Then for B_l and F_4 one has $\Sigma = \{e_1 + e_2, \dots, e_1 + e_l\}$, where as usual $l = 4$ for $\Phi = \mathrm{F}_4$; for C_l one has $\Sigma = \{2e_1\}$. The elements of Σ will be denoted by $\rho_2, \dots, \rho_l$, where $m = l, 4, 2$ for

$\Phi = \mathrm{B}_l, \mathrm{F}_4, \mathrm{C}_l$, respectively. Namely, $\rho_i = e_1 + e_i$, $2 \le i \le l$, for $\Phi = \mathrm{B}_l$ and F_4, and $\rho_2 = 2e_1$ for $\Phi = \mathrm{C}_l$. The subset of long roots is denoted by Φ_l and that of short roots by Φ_s.

Let $\Lambda(G)$ and $\Omega(G)$ be the sets of short and long root subgroups respectively. For a subgroup $X \leqslant G(\Phi, K)$ of G we denote by

$$\Lambda(X) = \{\, Y \leqslant X \mid Y \in \Lambda(G)\},$$

$$\Omega(X) = \{\, Y \leqslant X \mid Y \in \Omega(G)\},$$

the sets of short and long root subgroups contained in X. As usual $|\Lambda(X)|$ and $|\Omega(X)|$ are the cardinalities of these sets. We write $K = \mathbb{F}_q$ for the finite field of q elements. Finally, $X \sim Y$ means that the subgroups X and Y are conjugate in G, whereas $X_1 \times X_2$, $X_1 \leftthreetimes X_2$ and $[X_1, X_2]$ denote the direct product, the semidirect product and the mutual commutator subgroup of the groups X_1 and X_2 respectively.

Before stating our main results, Theorems 1 and 2, we reproduce the known results about pairs of long root subgroups (see [AS], [K1] or [C2]) and pairs of a long and a short root subgroups (essentially contained in [V2], but we state it in a slightly more precise form). These results will be used in the statement of Theorem 1. It is also quite instructive to compare them with Theorems 1 and 2. For brevity sake we write Λ, Ω instead of $\Lambda(\langle X_1, X_2\rangle)$, $\Omega(\langle X_1, X_2\rangle)$.

Theorem A ([AS], [K1], [C2], [V1]). *Let X_1, X_2 be two distinct long root subgroups in a universal Chevalley group $G = G(\Phi, K)$. Then*

1) *X_1 and X_2 generate one of the groups* (L1) – (L5) *listed below;*
2) *pairs (X_1, X_2) of different cases are not conjugate in G and the number of orbits under conjugation action of G on pairs of the same case is equal to d (we omit $d = 1$).*

(L1) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Omega| = q + 1$ *or* ∞, $|\Lambda| = 0$, $d = 2$ *for* A_l, $\Phi \ne \mathrm{C}_l$;

(L2) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Omega| = 2$, $|\Lambda| = 0$, $d = 3$ *for* D_l, $\Phi \ne \mathrm{C}_l$, F_4, G_2;

(L3) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Omega| = 2$, $|\Lambda| = q - 1$ *or* ∞, $d = 2$ *for* F_4, $\Phi = \mathrm{B}_l$, C_l, F_4;

(L4) $\langle X_1, X_2\rangle = X_1 X_2 Z \simeq U(\mathrm{A}_2, K)$, $Z = [X_1, X_2] \in \Omega$, (X_i, Z) as (L1), $i = 1, 2$, $\Phi \ne \mathrm{C}_l$;

(L5) $\langle X_1, X_2\rangle = \mathrm{SL}_2(K)$.

Remarks. If (X_1, X_2) and (Y_1, Y_2) are such as in (L1), $\Phi = \mathrm{A}_l$, then there exists $g \in G$ such that $gX_1g^{-1} = Y_1$ or Y_2, $gX_2g^{-1} = Y_2$ or Y_1 respectively.

Two orbits for D_l in case (L2) are fused by an outer automorphism.

Theorem B ([V2]). *Let Φ be a root system of type B_l, C_l, F_4; X be a long, Y a short root subgroups in a universal Chevalley group $G(\Phi, K)$. Suppose* $\operatorname{char} K \ne 2$. *Then*

1) *X and Y generate one of the groups* (LS1) – (LS6) *listed below;*

2) *pairs* (X, Y) *are conjugate in* G *if and only if they belong to the same case.*

(LS1) $\langle X, Y\rangle = X \times Y$, $|\Lambda| = q$ *or* ∞, $|\Omega| = 1$

(LS2) $\langle X, Y\rangle = X \times Y$, $|\Lambda| = 1$, $|\Omega| = 1$

(LS3) $\langle X, Y\rangle = X \times Z$, $Z \in \Omega$, $|\Lambda| = q-1$ *or* ∞, $|\Omega| = 2$

(LS4) $\langle X, Y\rangle = XYZ$, $Z = [X, Y] \in \Omega$, $|\Lambda(\langle X, Z\rangle)| = 0$, $|\Omega(\langle X, Z\rangle)| = q+1$ *or* ∞, (Y, Z) *as* (LS1), $\Phi \neq \mathrm{C}_l$

(LS5) $\langle X, Y\rangle = U(\mathrm{C}_2, K)$

(LS6) $\langle X, Y\rangle = \mathrm{SL}_2(K) \times Z$, $Z \in \Omega$,

If $\operatorname{char} K = 2$ *then only* (LS1), (LS2), (LS6) *may occur.*

The goal of this paper is to obtain analogues of Theorems A and B for pairs of short roots subgroups. Namely, we prove the following results.

Theorem 1. *Let* X_1, X_2 *be two distinct short root subgroups in a universal Chevalley group* $G(\Phi, K)$ *of type* B_l, C_l *or* F_4. *Suppose* $\operatorname{char} K \neq 2$. *Then*

1) X_1 *and* X_2 *generate one of the groups* (S1) – (S21) *listed below;*
2) *pairs* (X_1, X_2) *of different cases are not conjugate in* G *and the number of orbits under conjugation action of* G *on pairs of the same case is equal to* d *(we omit* $d = 1$*).*

(S1) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q+1$ *or* ∞, $|\Omega| = 0$, $\Phi = \mathrm{C}_l$, F_4;

(S2) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q+1$ *or* ∞, $|\Omega| = 0$, $d = |\{\pm c \in K \mid c^2 - 4 \notin K^{*2}\}|$, $\Phi = \mathrm{B}_l$, C_l, F_4;

(S3) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q-1$ *or* ∞, $|\Omega| = 2$, $d = |\{\pm c \in K \mid c^2 - 4 \in K^{*2}\}|$, $\Phi = \mathrm{B}_l$, C_l, F_4;

(S4) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q$, *or* ∞, $|\Omega| = 1$, $\Phi = \mathrm{B}_l$, C_l, F_4;

(S5) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = 3$, $|\Omega| = 0$, $\Phi = \mathrm{C}_l$, F_4;

(S6) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = 2$, $|\Omega| = 0$, $d = 1$ *for* $\Phi = \mathrm{B}_l$ *and 2 for* $\Phi = \mathrm{C}_l$, F_4;

(S7) $\langle X_1, X_2\rangle = X_1X_2Z$, $Z = [X_1, X_2] \in \Omega$, $|\Lambda| = q^2 + q$ *or* ∞, $|\Omega| = 1$, (X_i, Z) *as* (LS1), $i = 1, 2$, $\Phi = \mathrm{B}_l$, C_l, F_4;

(S8) $\langle X_1, X_2\rangle = X_1X_2Z$, $Z = [X_1, X_2] \in \Omega$, $|\Lambda| = 2q$ *or* ∞, $|\Omega| = 1$, $d = (q-1)/2$ *or* ∞, (X_i, Z) *as* (LS1), $i = 1, 2$, $\Phi = \mathrm{B}_l$, F_4;

(S9) $\langle X_1, X_2\rangle = X_1X_2Z$, $Z = [X_1, X_2] \in \Omega$, $|\Lambda| = 2q$ *or* ∞, $|\Omega| = 1$, $d = 2$, (X_i, Z) *as* (LS1), $i = 1, 2$, $\Phi = \mathrm{F}_4$;

(S10) $\langle X_1, X_2\rangle = X_1X_2Y \simeq U(\widetilde{\mathrm{A}}_2, K)$, $Y = [X_1, X_2] \in \Lambda$,$(X_i, Y)$ *as* (S1), $i = 1, 2$, $\Phi = \mathrm{C}_l$, F_4;

(S11) $\langle X_1, X_2\rangle = Y_1Y_2Z_1Z_2 \simeq U(\mathrm{C}_2, K)$, Y_1, $Y_2 \in \Lambda$, Z_1, $Z_2 \in \Omega$, (Y_1, Y_2) *as* (S1), (Z_1, Z_2) *as* (L3), (Y_1, Z_1), (Y_2, Z_i) *as* (LS1), $i = 1, 2$, $\Phi = \mathrm{B}_l$, C_l, F_4;

(S12) $\langle X_1, X_2\rangle = X_1X_2YZ \simeq U(\mathrm{C}_2, K)$, $Y \in \Lambda$, $Z \in \Omega$, (X_1, Y) *as* (S4), (X_2, Y) *as* (S5), (X_2, Z), (Y, Z) *as* (LS1), (X_1, Z), *as* (LS2), $\Phi = \mathrm{C}_l$, F_4;

(S13) $\langle X_1, X_2\rangle = X_1X_2YZ$, $Y \in \Lambda$, $Z \in \Omega$, (X_2, Z), (Y, Z) *as* (LS1), (X_1, Z) *as* (LS3), (X_1, Y) *as* (S2), (X_2, Y) *as* (S5), $\Phi = \mathrm{B}_l$, F_4;

(S14) $\langle X_1, X_2\rangle = U(\mathrm{C}_2, K) \times Z$, $Z \in \Omega$, $\Phi = \mathrm{F}_4$;

(S15) $\langle X_1, X_2\rangle = \mathrm{SL}_2(K)$, $\Phi = \mathrm{B}_l$, C_l, F_4;

(S16) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(K)$, $d = |\{\pm c \in K \mid c^2 - 4 \in K^{*2}\}|$, $\Phi = \mathrm{B}_l$, C_l, F_4;

(S17) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(L)$, *$L$ is a quadratic extension of K*, $d = |\{\pm c \in K \mid c^2 - 4 \notin K^{*2}\}|$, $\Phi = \mathrm{B}_l$, C_l, F_4.

(S18) $\langle X_1, X_2\rangle \simeq U(\mathrm{C}_2, K)H_\rho w_\rho X_\rho$, $\Phi = \mathrm{B}_l$, C_l, F_4.

(S19) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(K) \times Z$, $Z \in \Omega$, *if* $K \neq \mathbb{F}_3$ *and* $\mathrm{SL}_2(K)$ *if* $K = \mathbb{F}_3$, $d = 1$ *for* $\Phi = \mathrm{B}_l$ $(l \geq 3)$, F_4 *and d is equal to index* $\nu = [K^* : K^{*2} \cup -K^{*2}]$ *for* $\Phi = \mathrm{C}_l$;

(S20) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(K) \times Z_1 \times Z_2$, $Z_1, Z_2 \in \Omega$, (Z_1, Z_2) *as* (L3), *if* $K \neq \mathbb{F}_3$, *and* $\mathrm{SL}_2(K) \times Z$, *if* $K = \mathbb{F}_3$, *Z is a cyclic group of order 3*, $\Phi = \mathrm{C}_l$, F_4;

(S21) $\langle X_1, X_2\rangle \simeq \mathrm{SL}_2(K) \rtimes (Z_1 \times Z_2)$, $Z_1, Z_2 \in \Omega$, (Z_1, Z_2) *as* (L3), $\Phi = \mathrm{F}_4$.

When $\operatorname{char} K = 2$ short root subgroups behave exactly as long root subgroups. In this case we have the following result.

Theorem 2. *Assume as in Theorem 1 but* $\operatorname{char} K = 2$. *Then one has the following cases:*

(T1) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = q + 1$ *or* ∞, $|\Omega| = 0$, $\Phi = \mathrm{C}_l$, F_4;

(T2) $\langle X_1, X_2\rangle = X_1 \times X_2$, $|\Lambda| = 2$, $|\Omega| = 0$, $d = 1$ *for* $\Phi = \mathrm{B}_l$, F_4 *and 2 for* C_l $(l \geq 4)$;

(T3) $\langle X_1, X_2\rangle = U(\widetilde{\mathrm{A}}_2, K)$, $\Phi = \mathrm{C}_l$, F_4;

(T4) $\langle X_1, X_2\rangle = \mathrm{SL}_2(K)$, $\Phi = \mathrm{B}_l$, C_l, F_4

Remarks. The index ν is equal to 1 if K is really or quadratically closed, or if $K = \mathbb{F}_q$ where $q \equiv 3 \pmod 4$, and ν is equal to 2 if $K = \mathbb{F}_q$ where $q \equiv 1 \pmod 4$.

In cases (S2), (S3), (S8), (S16), (S17) each orbit corresponds to an element of the field up to sign.

For an algebraically closed field K the pairs of cases (S1) – (S3), (S8), (S15) – (S17) are conjugate in the appropriate $\mathrm{GL}(n, K)$.

There exists no automorphism that identifies two orbits of cases (S6) and (S9).

For a finite field $\mathbb{F}_q$, $q \neq 2^n$ there exist $6 + (3q + 1)/2$ orbits for a root system of type B_l, $10 + \nu + q$ for C_l, $16 + (3q + 1)/2$ for F_4.

The length of the list in Theorem 1 alone suggests that it is much harder to work with the short root subgroups than with the long root ones. The presence of continuous parameters and arithmetic conditions in the description of orbits shows that, unlike the geometry of long root subgroups, geometry of short root subgroups essentially depends on the arithmetic nature of the ground field. This might explain why they have not attracted as much attention as the long root subgroups.

The rest of the paper is devoted to the proof of Theorems 1 and 2.

§ 3. Possible configurations of roots

The starting point of our work is the calculation of the Bruhat decomposition of a short root element [V2].

Lemma 1. *Let $\Phi = \mathrm{B}_l$, C_l, F_4 and x be a short root element of the Chevalley group $G(\Phi, K)$. If $\operatorname{char} K \neq 2$, then there exist such $u \in U$ and either such a short root γ, that $x = ux_\gamma(t)u^{-1}$, or a pair of orthogonal long roots α and β whose semi-difference is a root, that $x = ux_\alpha(s)x_\beta(r)u^{-1}$. Moreover u, γ, α, β and $r/s \in -K^{*2}$ do not depend on the choice of $x \neq e$ from a given root subgroup X.*

If $\operatorname{char} K = 2$ only the first of these cases may occur.

This result implies the following key lemma.

Lemma 2. *Let X_1, X_2 be short root subgroups in $G(\Phi, K)$ where $\Phi = \mathrm{B}_l$, C_l, F_4 and $\operatorname{char} K \neq 2$. Then there exists $g \in G(\Phi, K)$ such that*

$$gX_1g^{-1} = \langle x_\rho(t)x_{\rho_2}(c_2t)\cdots x_{\rho_m}(c_mt),\ t \in K\rangle,$$

and

$$gX_2g^{-1} = X_\gamma \qquad \text{or} \qquad gX_2g^{-1} = X_{\alpha,\beta} = \langle x_\alpha(s)x_\beta(-s),\ s \in K\rangle,$$

where γ, α, β have the same meaning as in Lemma 1; $c_2, \dots, c_m \in K$ depend only on X_1 and X_2; $m = l, 2, 4$ for $\Phi = \mathrm{B}_l$, C_l, F_4 respectively.

If $\operatorname{char} K = 2$ one has $gX_1g^{-1} = X_\rho$, $gX_2g^{-1} = X_\gamma$.

Proof. There is $z \in G$ such that $zX_1z^{-1} = X_\rho$. By Lemma 1 one may pick $u \in U$ such that $u^{-1}zX_2z^{-1}u$ coincides with X_γ or $\{x_\alpha(st)x_\beta(rt) \mid t \in K\}$. In the first case set $g = u^{-1}z$, in the second one set $g = h_\beta(k)u^{-1}z$, where $-k^2 = s/r$. Then gX_2g^{-1} will have the required form and $gX_1g^{-1} = u^{-1}X_\rho u$. Since $u \in U$ is a product of $x_\tau(t)$, $\tau \in \Phi^+$, the conjugate $u^{-1}x_\rho(r)u$ equals $\prod x_\sigma(*)$ with $\sigma \in \Sigma$.

This means that the elements of gX_1g^{-1} have the following form

$$x_\rho(t)x_{\rho_2}(c_2t)\cdots x_{\rho_m}(c_mt),$$

where $m = l, 2, 4$ for $\Phi = \mathrm{B}_l$, C_l, F_4 respectively. Now it follows from the Chevalley commutator formula that all c_i are divisible by 2 and so if $\operatorname{char} K = 2$ they are all equal to zero. This proves the lemma.

Note that since x_ρ, x_{ρ_i} commute we do not have to worry about their order.

Denote by R_J the set of those $\rho_i \in \Sigma$ which appear in the decomposition of elements of gX_1g^{-1} with non-zero coefficients c_i. By J we denote the set of the corresponing indices i. We set

$$X_{\rho,J} = \langle x_\rho(t) \prod_i x_{\rho_i}(c_i t),\ i \in J,\ c_i \neq 0,\ t \in K\rangle.$$

If $J = \emptyset$, $X_{\rho,J} = X_\rho$, and if $J = \{i\}$ we write $X_{\rho,c}$, where $c = c_i$.

Recall that the angle between short roots can take the values 0, $\pi/3$, $\pi/2$, $2\pi/3$, π for root systems of type C_l and F_4 and 0, $\pi/2$, π for root systems of type B_l. The angle between a short and a long root is $\pi/4$, $\pi/2$, $3\pi/4$ (see [B]).

Case 1. $gX_1g^{-1} = X_{\rho,J}, gX_2g^{-1} = X_\gamma$.
First suppose that $R_J = \emptyset$, i.e. $gX_1g^{-1} = X_\rho$. Denote by θ the angle between ρ and γ. The possibilities for γ are listed in Table 1 below. In the following tables we always assume that the indices i, j, k are pair-wise distinct in the same variant and strictly greater than 1 in the cases B_l, F_4 or 2 in the case C_l.

Table 1

	θ	γ	Φ
1	0	e_1	$\mathrm{B}_l, \mathrm{F}_4$
		$e_1 + e_2$	C_l
2	$\pi/3$	$\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$	F_4
		$e_{1,2} \pm e_j$	C_l
3	$\pi/2$	e_i	$\mathrm{B}_l, \mathrm{F}_4$
		$e_1 - e_2$	C_l
4	$\pi/2$	$\pm e_j \pm e_k$	C_l
5	$2\pi/3$	$\frac{1}{2}(-e_1 \pm e_2 \pm e_3 \pm e_4)$	F_4
		$-e_{1,2} \pm e_j$	C_l
6	π	$-e_1$	$\mathrm{B}_l, \mathrm{F}_4$
		$-e_1 - e_2$	C_l

Note that the angle $\pi/2$ corresponds to two variants for C_l: when ρ and γ generate a subsystem of type C_2 or $2\widetilde{\mathrm{A}}_1$.

Now suppose $R_J \neq \emptyset$. Introduce the notation (θ, ψ), where $\theta = \angle(\rho, \gamma)$ is the angle between ρ and γ and $\psi = \max\{\angle(\rho_i, \gamma)\}$, i runs over J. Since $\angle(\rho, \rho_i) = \pi/4$

for any $\rho_i \in \Sigma$, the configurations $(0, \pi/2)$, $(0, 3\pi/4)$, $(\pi/3, 3\pi/4)$, $(2\pi/3, \pi/4)$, $(\pi, \pi/4)$, $(\pi, \pi/2)$ are impossible and we have nine variants for C_l, F_4 and five variants for B_l. For each pair (θ, ψ) we fix an index i such that $\psi = \angle(\rho_i, \gamma)$. The corresponding roots γ are listed in Table 2.

TABLE 2

	(θ, ψ)	γ	Φ
7	$(0, \pi/4)$	e_1	$\mathrm{B}_l, \mathrm{F}_4$
		$e_1 + e_2$	C_l
8	$(\pi/3, \pi/4)$	$\frac{1}{2}(e_1 + e_i \pm e_j \pm e_k)$	F_4
		$e_1 \pm e_j$	C_l
9	$(\pi/3, \pi/2)$	$\frac{1}{2}(e_1 - e_i \pm e_j \pm e_k)$	F_4
		$e_2 \pm e_j$	C_l
10	$(\pi/2, \pi/4)$	e_i	$\mathrm{B}_l, \mathrm{F}_4$
		$e_1 - e_2$	C_l
11	$(\pi/2, \pi/2)$	$\pm e_j$	$\mathrm{B}_l, \mathrm{F}_4$
		$\pm e_j \pm e_k$	C_l
12	$(\pi/2, 3\pi/4)$	$-e_i$	$\mathrm{B}_l, \mathrm{F}_4$
		$-e_1 + e_2$	C_l
13	$(2\pi/3, \pi/2)$	$\frac{1}{2}(-e_1 + e_i \pm e_j \pm e_k)$	F_4
		$-e_2 \pm e_j$	C_l
14	$(2\pi/3, 3\pi/4)$	$\frac{1}{2}(-e_1 - e_i \pm e_j \pm e_k)$	F_4
		$-e_1 \pm e_j$	C_l
15	$(\pi, 3\pi/4)$	$-e_1$	$\mathrm{B}_l, \mathrm{F}_4$
		$-e_1 - e_2$	C_l

Case 2. $gX_1g^{-1} = X_{\rho,J}$, $gX_2g^{-1} = X_{\alpha,\beta}$.
The restrictions imposed upon the roots α, β imply that (α, β) is one of the following pairs: $\{(\pm e_i \pm e_j, \pm e_i \mp e_j),\ i \neq j\}$ for B_l; $\{(\pm 2e_i, \pm 2e_j),\ i \neq j\}$ for C_l; and $\{(\pm e_i \pm e_j, \pm e_i \mp e_j),\ i \neq j\}$ or $\{(\pm e_1 \pm e_i, \pm e_j \pm e_k),\ i \neq j \neq k \neq i\}$ for F_4. We set

$$(\theta_1, \theta_2) = (\angle(\rho, \alpha), \angle(\rho, \beta))$$
$$(\psi_1, \psi_2) = (\max\{\angle(\rho_i, \alpha)\}, \max\{\angle(\rho_i, \beta)\},\ i \in J), \quad \text{if } J \neq \emptyset$$

Interchanging the roots α and β we may assume that $\theta_1 \leq \theta_2$ and $\psi_1 \leq \psi_2$.

Suppose that $R_J = \emptyset$. Then the possible configurations of roots are determined by the angles (θ_1, θ_2). In Tables 3 and 4 the indices i, j, k are used according to the above conventions.

TABLE 3

	(θ_1, θ_2)	α	β	Φ
16	$(\pi/4, \pi/4)$	$e_1 \pm e_j$	$e_1 \mp e_j$	$\mathrm{B}_l, \mathrm{F}_4$
		$2e_1$	$2e_2$	C_l
17	$(\pi/4, \pi/2)$	$e_1 \pm e_j$	$\pm e_j \pm e_k$	F_4
		$2e_1$	$\pm 2e_j$	C_l
18	$(\pi/2, \pi/2)$	$\pm e_j \pm e_k$	$e_j \mp e_k$	$\mathrm{B}_l, \mathrm{F}_4$
		$\pm 2e_j$	$\pm 2e_k$	C_l
19	$(\pi/4, 3\pi/4)$	$e_1 \pm e_j$	$-e_1 \pm e_j$	$\mathrm{B}_l, \mathrm{F}_4$
		$2e_1$	$-2e_2$	C_l
20	$(\pi/2, 3\pi/4)$	$\pm e_j \pm e_k$	$-e_1 \pm e_i$	F_4
		$\pm 2e_j$	$-2e_2$	C_l
21	$(3\pi/4, 3\pi/4)$	$-e_1 \pm e_j$	$-e_1 \mp e_j$	$\mathrm{B}_l, \mathrm{F}_4$
		$-2e_2$	$-2e_1$	C_l

For the case $R_J \neq \emptyset$ we get 26 possible configurations. To describe these configurations we have to use both (θ_1, θ_2) and (ψ_1, ψ_2). In the following Table 4 the angles ψ_1, ψ_2 are usually formed by the same root ρ_i with α, β. However in Variants 24, 27, 30, 34, 38, 44, 47 the angle ψ_1 is formed by ρ_j and α, where $j \in J$, $j \neq i$. To stress this we append 'j' to the corresponding numbers.

TABLE 4

	(θ_1, θ_2), (ψ_1, ψ_2)	α	β	Φ
22	$(\pi/4, \pi/4), (\pi/3, \pi/3)$	$e_1 \pm e_j$	$e_1 \mp e_j$	$\mathrm{B}_l, \mathrm{F}_4$
23	$(\pi/4, \pi/4), (0, \pi/2)$	$e_1 + e_i$	$e_1 - e_i$	$\mathrm{B}_l, \mathrm{F}_4$
		$2e_1$	$2e_2$	C_l
24j	$(\pi/4, \pi/4), (\pi/3, \pi/2)$	$e_1 + e_i$	$e_1 - e_i$	$\mathrm{B}_l, \mathrm{F}_4$
25	$(\pi/4, \pi/2), (\pi/3, \pi/3)$	$e_1 \pm e_j$	$e_i \pm e_k$	F_4

26	$(\pi/4,\pi/2),(0,\pi/2)$	e_1+e_i	$\pm e_j\pm e_k$	F_4
		$2e_1$	$\pm 2e_j$	C_l
27j	$(\pi/4,\pi/2),(\pi/3,\pi/2)$	e_1+e_i	$e_j\pm e_k$	F_4
28	$(\pi/4,\pi/2),(\pi/2,\pi/2)$	e_1-e_i	$\pm e_j\pm e_k$	F_4
		$2e_2$	$\pm 2e_j$	C_l
29	$(\pi/4,\pi/2),(\pi/3,2\pi/3)$	$e_1\pm e_j$	$-e_i\pm e_k$	F_4
30j	$(\pi/4,\pi/2),(\pi/2,2\pi/3)$	e_1-e_j	$-e_i\pm e_k$	F_4
31	$(\pi/2,\pi/2),(\pi/3,\pi/3)$	$e_i\pm e_k$	$e_i\mp e_k$	$\mathrm{B}_l,\mathrm{F}_4$
32	$(\pi/2,\pi/2),(\pi/2,\pi/2)$	$\pm e_j\pm e_k$	$\pm e_j\mp e_k$	$\mathrm{B}_l,\mathrm{F}_4$
		$\pm 2e_j$	$\pm 2e_k$	C_l
33	$(\pi/2,\pi/2),(\pi/3,2\pi/3)$	$e_i\pm e_k$	$-e_i\pm e_k$	$\mathrm{B}_l,\mathrm{F}_4$
34j	$(\pi/2,\pi/2),(\pi/2,2\pi/3)$	$e_i\pm e_k$	$-e_i\pm e_k$	$\mathrm{B}_l,\mathrm{F}_4$
35	$(\pi/2,\pi/2),(2\pi/3,2\pi/3)$	$-e_i\pm e_j$	$-e_i\mp e_j$	$\mathrm{B}_l,\mathrm{F}_4$
36	$(\pi/4,3\pi/4),(0,\pi/2)$	e_1+e_i	$-e_1+e_i$	$\mathrm{B}_l,\mathrm{F}_4$
		$2e_1$	$-2e_2$	C_l
37	$(\pi/4,3\pi/4),(\pi/3,2\pi/3)$	$e_1\pm e_j$	$-e_1\pm e_j$	$\mathrm{B}_l,\mathrm{F}_4$
38j	$(\pi/4,3\pi/4),(\pi/2,2\pi/3)$	e_1-e_j	$-e_1+e_j$	$\mathrm{B}_l,\mathrm{F}_4$
39	$(\pi/4,3\pi/4),(\pi/2,\pi)$	e_1-e_i	$-e_1-e_i$	$\mathrm{B}_l,\mathrm{F}_4$
		$2e_2$	$-2e_1$	C_l
40	$(\pi/2,3\pi/4),(\pi/2,\pi/2)$	$\pm e_j\pm e_k$	$-e_1+e_i$	F_4
		$\pm 2e_j$	$-2e_2$	C_l
41	$(\pi/2,3\pi/4),(\pi/3,2\pi/3)$	$e_i\pm e_j$	$-e_1\pm e_k$	F_4
42	$(\pi/2,3\pi/4),(2\pi/3,2\pi/3)$	$-e_i\pm e_j$	$-e_1\pm e_k$	F_4
43	$(\pi/2,3\pi/4),(\pi/2,\pi)$	$\pm e_j\pm e_k$	$-e_1-e_i$	F_4
		$\pm 2e_j$	$-2e_1$	C_l
44j	$(\pi/2,3\pi/4),(2\pi/3,\pi)$	$-e_j\pm e_k$	$-e_1-e_i$	F_4
45	$(3\pi/4,3\pi/4),(2\pi/3,2\pi/3)$	$-e_1\pm e_j$	$-e_1\mp e_j$	$\mathrm{B}_l,\mathrm{F}_4$
46	$(3\pi/4,3\pi/4),(\pi/2,\pi)$	$-e_1+e_i$	$-e_1-e_i$	$\mathrm{B}_l,\mathrm{F}_4$
		$-2e_2$	$-2e_1$	C_l
47j	$(3\pi/4,3\pi/4),(\pi/2,\pi)$	$-e_1+e_i$	$-e_1-e_i$	$\mathrm{B}_l,\mathrm{F}_4$

Using Tables 1 – 4 we are in a position to construct a list of pairs (X_1, X_2) which contains representative(s) of every orbit of G on pairs of short roots subgroups. This will be done in the next section.

§ 4. Representatives of orbits

In Sections 4 – 6 we finish the proof of Theorem 1. In particular, we always assume that the characteristic of the ground field is distinct from 2. When char $K = 2$ only very few of the configurations listed in Tables 1 – 4 may occur, see Section 7.

It follows from the contents of Section 3 that each pair (X_1, X_2) in question can be reduced to one of the forms $(X_{\rho,J}, X_\gamma)$ or $(X_{\rho,J}, X_{\alpha,\beta})$ and thus corresponds to one of the variants listed above in Tables 1 – 4. From now on, we mean by '*Variant N*' a pair of subgroups which has the same configuration of roots as the one in Variant N of Tables 1 – 4.

Lemma 3. *Let X_1, X_2 be short root subgroups of $G(\Phi, K)$ where $\Phi = \mathrm{B}_l$, C_l, F_4. Then there exists $f \in G$ such that $fX_2f^{-1} = X_\gamma$ or $X_{\alpha,\beta}$ and one of the following holds*

(1) $fX_1f^{-1} = X_\rho$,
(2) $fX_1f^{-1} = X_{\rho,c} = \{x_\rho(t)x_{\rho_i}(c_it),\ t \in K\}$
(3) $fX_1f^{-1} = \{x_\rho(t)x_{\rho_i}(c_it)x_{\rho_j}(c_jt),\ t \in K\}$

for some $\rho_i, \rho_j \in R_J$, $c_i, c_j \in K^*$, $i \neq j$. Case (3) only occurs in Variants 29 and 42.

Proof. By Lemma 2 there exists $g \in G$ such that $gX_1g^{-1} = X_{\rho,J}$ and $gX_2g^{-1} = X_\gamma$ or $X_{\alpha,\beta}$. If $J = \emptyset$, we take $f = g$. It happens in Variants 1 – 6, 16 – 21 described in Tables 1 and 3. Suppose $J \neq \emptyset$. We try to find f of the form $\prod x_\tau(k_\tau)g$, where $\tau \in \Phi$, $k\tau \in K^*$, such that $\tau + \gamma$ or $\tau + \alpha$ and $\tau + \beta$ are not in Φ. Then one can kill some of the $x_{\rho_i}(c_it)$ of $X_{\rho,J}$ by varying k_τ.

In Variants 8, 10, 13, 22, 25, 26, 27, 31, 32, 36, 37, 40, 41, 45 and in Variant 11 (for C_l only), we set $\tau = \rho_i - \rho$, where ρ_i runs over R_J and $k_\tau = \mp c_i/2$. Then we have

$$x_{\rho_i-\rho}(\mp c_i/2)x_\rho(t)x_{\rho_i}(c_i)x_{\rho_i-\rho}(\pm c_i/2) = x_\rho(t)$$

Since for $\Phi = \mathrm{C}_l$ the set Σ consists of one root, this finishes the proof for this case.

Let $\Phi = \mathrm{B}_l$ or F_4. If R_J consists of one root, we are done. Thus for the remaining variants we may assume that R_J contains at least two elements. Fix a root ρ_i from R_J which forms the maximal angle with γ or β. Then in Variants 7, 9, 11 (for B_l, F_4), 12, 14, 15, 28, 29, 35, 39, 42, 43 we set $\tau = \rho_j - \rho_i$, $j \in J$, $j \neq i$ and $k_\tau = \mp c_j/c_i$ respectively. We have

$$x_{-i+j}(\mp c_j/c_i)x_{1+i}(c_it)x_{1+j}(c_jt)x_{-i+j}(\pm c_j/c_i) = x_{1+i}(c_it)$$

In Variants 24, 33, 34, 46, 47 we set $\tau = e_j$, $i \neq j \in J$ and $k_\tau = \mp c_j/2$ respectively.

In Variant 23 the set R_J consists of a single root ρ_i we are done. In Variant 38 we fix ρ_j such that the angle between ρ_j and α is maximal and conjugates by $x_k(\mp c_k/2)$, $k \in J$, $k \neq j$.

It remains to consider Variants 30 and 44. We may assume that R_J consists of three roots. Fix ρ_i which forms the maximal angle with β, and ρ_j which has the same property with respect to α. Then we may set $\tau = \rho_k - \rho_i$, $k \neq i, j$ and $k_\tau = \mp c_k/c_i$. This completes the proof of the lemma.

We call a *coppia*[1] of Variant N, $1 \le N \le 47$, a pair of sets of roots occuring in the description of X_2 and X_1 in Lemma 3, where (X_1, X_2) belongs to Variant N. It follows from Lemma 3 that a coppia may have one of the following five possible forms:

$$(\{\rho\}, \{\gamma\}),\ (\{\rho\}, \{\alpha, \beta\}),\ (\{\rho, \rho_i\}, \{\gamma\}),\ (\{\rho, \rho_i\}, \{\alpha, \beta\}),\ (\{\rho, \rho_i, \rho_j\}, \{\alpha, \beta\}).$$

The next lemma shows that a variant determines its coppia essentially uniquely.

Lemma 4. *All coppie belonging to the same variant of Tables 1 – 4 are conjugate by an element of the Weyl group.*

Proof. The proof in all cases proceeds as follows. First we observe that for any variant the subsystems generated by the corresponding coppie always have the same type. The Weyl group usually acts transitively on the subsystems of the same type, with the exceptions explicitly known (see [D]).

Variants 1 – 6, 8, 10, 11 (for C_l), 13 have coppie of the form $(\{\rho\}, \{\gamma\})$, where ρ and γ generate one of the subsystems $\widetilde{\mathrm{A}}_1$, $\widetilde{\mathrm{A}}_2$, $2\widetilde{\mathrm{A}}_1$ or $\widetilde{\mathrm{D}}_2$.

Coppie associated with Variants 7, 9, 11 (for $\mathrm{B}_l, \mathrm{F}_4$), 12, 14, 15 are $(\{\rho, \rho_i\}, \{\gamma\})$, where ρ_i and γ generate either B_2 or $\widetilde{\mathrm{A}}_1 + \mathrm{A}_1$. These root subsystems are conjugate in W by a transformation which leaves ρ invariant.

Variants 16 – 21, 22, 25 – 27, 31, 32, 36 – 38, 40, 41, 45 lead to $(\{\rho\}, \{\alpha, \beta\})$. Consider one of pairs (α, β), $(\alpha \pm \rho, \beta)$, $(\alpha, \beta \pm \rho)$, $(\alpha \pm \rho, \beta \pm \rho)$, where one sets $\alpha \pm \rho$ or $\beta \pm \rho$ is taken if and only if it is a root in given variant. Such pairs of roots are conjugate in the Weyl group.

Variants 23, 24, 28, 29, 33 – 35, 39, 42, 43, 46, 47 lead to $(\{\rho, \rho_i\}, \{\alpha, \beta\})$. We consider root subsystems generated by the triples of roots $(\rho_i - \rho, \alpha, \beta)$, $(\rho_i - \rho, \alpha \pm \rho, \beta)$, $(\rho_i - \rho, \alpha, \beta \pm \rho)$, $(\rho_i - \rho, \alpha \pm \rho, \beta \pm \rho)$ as above. These triples generate root systems $\widetilde{\mathrm{A}}_1$, $\widetilde{\mathrm{A}}_1 + \mathrm{A}_1$, B_2, B_3.

Finally, the sets $(\rho_i - \rho, \rho_j - \rho, \alpha - \rho, \beta)$, $(\rho_i - \rho, \rho_j - \rho, \alpha, \beta + \rho)$ in Variants 30, 44 generate B_3 and it is easily seen that they are conjugate in W.

Thus without loss of generality we may assume that $\rho_i = \delta$, $j = 3$ and $k = 4$, whenever these indices occur in Tables 1 – 4.

[1] Plural: *coppie*.

Lemma 5. *Let X_1, X_2 be as above. Then, apart from Variants 23, 33, 46 and Variant 39 (for* C_l*), there exists $g \in G$ such that*

$$gX_1g^{-1} = X_\rho \quad \text{or} \quad \{x_\rho(t)x_\delta(t)\} \quad \text{or} \quad \{x_\rho(t)x_{\rho_2}(t)x_{\rho_3}(t)\},$$

and

$$gX_2g^{-1} = X_\gamma \quad \text{or} \quad X_{\alpha,\beta}.$$

Proof. Straightforward calculations using the elements of H.

It follows from Lemmas 3 and 4 that some variants may be identified since the corresponding coppie have the same form. We write $M(N)$ to denote that Variant N is conjugate to Variant M and does not lead to a new orbit. Then

variants	coppie
1, 2(8), 3(10), 4(11,C_l), 5(13)	$(\{\rho\},\{\gamma\})$
7, 9, 11(B_l, F_4), 12, 14, 15	$(\{\rho,\rho_i\},\{\gamma\})$
16(22), 17(25–27), 18(31,32), 19(36–38), 20(40,41), 21(45)	$(\{\rho\},\{\alpha,\beta\})$
23(24), 28, 29, 33(34), 35, 39, 42, 43, 46(47)	$(\{\rho,\rho_i\},\{\alpha,\beta\})$
30, 44	$(\{\rho,\rho_i,\rho_j\},\{\alpha,\beta\})$

Lemma 6. *The subgroups of the following variants are conjugate in G: 7 and 23 with $c = \pm 2$; 9 and 17; 11 and 32 for* C_l*; 11 and 33 for* B_l*, F_4; 12 and 36; 14 and 40; 15 and 46 with $c = \pm 2$.*

Proof. Straightforward calculation.

Summarizing the above we are in a position to explicitly list representatives (possibly non-unique) of every orbit of G on pairs of short root subgroups. Set $c = c_i$, $x_\alpha = x_\alpha(t)$ and $x_\alpha(c) = x_\alpha(ct)$. We collect these representatives in the three following tables. Table 5 lists the representatives for types C_l.

TABLE 5: $\Phi = \mathrm{C}_l$

case	variants	representatives	
(S1)	2(8)	$\{x_{1+2}\}$	$\{x_{1+3}\}$
(S2), (S3)	16	$\{x_{1+2}\}$	$\{x_1 x_2(-1)\}$
(S2) – (S4)	23(7)	$\{x_{1+2}x_1(c)\}$	$\{x_1 x_{-2}(-1)\}$
(S5)	28	$\{x_{1+2}x_1\}$	$\{x_2 x_3(-1)\}$
$(\mathrm{S6})_1$	4(11,18,32)	$\{x_{1+2}\}$	$\{x_{3+4}\}$
$(\mathrm{S6})_2$	9(17,26)	$\{x_{1+2}x_1\}$	$\{x_{2+3}\}$
(S7)	3(10)	$\{x_{1+2}\}$	$\{x_{1-2}\}$
(S10)	5(13)	$\{x_{1+2}\}$	$\{x_{-1+3}\}$
(S11)	12(19,36)	$\{x_{1+2}x_1\}$	$\{x_{-1+2}\}$
(S12)	14(20,40)	$\{x_{1+2}x_1\}$	$\{x_{-1+3}\}$
(S15)	6	$\{x_{1+2}\}$	$\{x_{-1-2}\}$
(S16), (S17)	21	$\{x_{1+2}\}$	$\{x_{-2}x_{-1}(-1)\}$
(S16) – (S18)	46(15)	$\{x_{1+2}x_1(c)\}$	$\{x_{-2}x_{-1}(-1)\}$
(S19)	39	$\{x_{1+2}x_1(c)\}$	$\{x_2 x_{-1}(-1)\}$
(S20)	43	$\{x_{1+2}x_1\}$	$\{x_3 x_{-1}(-1)\}$

The representatives of orbits for type B_l are collected in Table 6. Type F_4 has the same orbits as B_4 plus the additional orbits listed in Table 7 below.

TABLE 6: $\Phi = \mathrm{B}_l$

case	variants	generators/representatives	
(S2), (S3)	16(22)	$\{x_1\}$	$\{x_{1+2}x_{1-2}(-1)\}$
(S2) – (S4)	23(7,24)	$\{x_1 x_{1+2}(c)\}$	$\{x_{1+2}x_{1-2}(-1)\}$
$(\mathrm{S6})_1$	18(31,32)	$\{x_1\}$	$\{x_{2+3}x_{2-3}(-1)\}$
(S7)	3(10)	$\{x_1\}$	$\{x_2\}$
(S8)	33(11,34)	$\{x_1 x_{1+2}(c)\}$	$\{x_{2+3}x_{-2+3}(-1)\}$
(S11)	12(19,36,37,38)	$\{x_1 x_{1+2}\}$	$\{x_{-2}\}$
(S13)	35	$\{x_1 x_{1+2}\}$	$\{x_{-2+3}x_{-2-3}(-1)\}$
(S15)	6	$\{x_1\}$	$\{x_{-1}\}$
(S16), (S17)	21(45)	$\{x_1\}$	$\{x_{-1+2}x_{-1-2}(-1)\}$
(S16) – (S18)	46(15)	$\{x_1 x_{1+2}(c)\}$	$\{x_{-1+2}x_{-1-2}(-1)\}$
(S19)*	39	$\{x_1 x_{1+2}\}$	$\{x_{1-2}x_{-1-2}(-1)\}$

* In this case $l \geq 3$

TABLE 7: $\Phi = \mathrm{F}_4$

case	variants	generators/representatives	
(S1)	2(8)	$\{x_1\}$	$\{x_{1+2+3+4}\}$
(S5)	28	$\{x_1 x_{1+2}\}$	$\{x_{1-2}x_{3+4}(-1)\}$
$(\mathrm{S6})_2$	9(17,25–27)	$\{x_1 x_{1+2}\}$	$\{x_{1-2+3+4}\}$
$(\mathrm{S9})_1$	29	$\{x_1 x_{1+2}\}$	$\{x_{1+3}x_{-2+4}\}$
$(\mathrm{S9})_2$	30	$\{x_1 x_{1+2} x_{1+3}\}$	$\{x_{1-3}x_{-2+4}\}$
(S10)	5(13)	$\{x_1\}$	$\{x_{-1+2+3+4}\}$
(S12)	14(20,40,41)	$\{x_1 x_{1+2}\}$	$\{x_{-1-2+3+4}\}$
(S14)	42	$\{x_1 x_{1+2}\}$	$\{x_{-2+3}x_{-1+4}\}$
(S20)	43	$\{x_1 x_{1+2}\}$	$\{x_{3+4}x_{-1-2}(-1)\}$
(S21)	44	$\{x_1 x_{1+2} x_{1+3}\}$	$\{x_{-3+4}x_{-1-2}\}$

The cases listed in Tables 5 – 7 correspond exactly to the cases of Theorem 1. In Sections 5 – 7 we prove that two pairs belonging to different cases are never conjugate. In Case (S19) two pairs of subgroups are conjugate if and only if parameters c and c' of their generators differ only by sign and/or square factor, $c = \pm d^2 c'$ for some $d \in K^*$. In Cases (S2) – (S4), (S8), (S16) – (S18) two pairs of subgroups are conjugate if and only if the corresponding parameters c and c' differ only by sign, $c = \pm c'$.

Notice also that Cases (S2) and (S16) arise if $c^2 - 4 \notin K^{*2}$, whereas Cases (S3) and (S17) arise if $c^2 - 4 \in K^{*2}$ and, finally, Cases (S4) and (S18) arise if $c = \pm 2$.

In Section 5 we study subgroups generated by each of the above pairs of short root subgroups. Namely, we determine these subgroups up to isomorphism and, moreover, describe root subgroups contained in them. It is clear, that if either the isomorphism class or the number of long or short root subgroups in $\langle X_1, X_2\rangle$ and $\langle Y_1, Y_2\rangle$ are distinct, then the pairs (X_1, X_2) and (Y_1, Y_2) are non-conjugate. The cases when the spans of pairs of short root subgroups coinside are considered in Section 6.

§ 5. Structure of spans

We start with some general remarks. For a representative (X_1, X_2) of an orbit of G on pairs of short root subgroups, listed in Tables 5 – 7 we denote by X their span $X \sim \langle X_1, X_2\rangle$. Note that sometimes we pass to a conjugate of X by an element of the Weyl group.

As a rule our calculations do not depend on the signs of the structure constants. When they do, it is always sufficient to calculate inside a Chevalley group of type B_4 or C_2. In these cases we use the following matrix representation (see [C]).

In the case B_4 let V be a 9-dimensional vector space and let Q be a quadratic form defined on V. We choose a base of V such that $Q(x,x) = x_1x_9 + x_2x_8 + x_3x_7 + x_4x_6 + x_5^2$, where $x = (x_1, \dots, x_9)^t$ is vector of V. Here, as usual, x^t denotes the transpose of x.

In the case $\Phi = C_2$ let V be a 4-dimensional vector space and let B be a skew-simmetric bilinear form defined on V. We choose a base of V such that

$$B(x,y) = x_1y_4 - x_4y_1 + x_2y_3 - x_3y_2,$$

where $x = (x_1, \dots, x_4)^t$, $y = (y_1, \dots, y_4)^t$.

Counting the root subgroups contained in $\langle X_1, X_2\rangle$ in all non-trivial cases we have to work inside $U(\Phi, K)$. We proceed as follows: we take an arbitrary one-parameter subgroup of X, say $Y = \{x(t),\ t \in K\}$, and try to solve the following equations in $u \in U$

$$\begin{aligned} ux(t)u^{-1} &= x_\lambda(s), \quad \lambda \in \Phi_l \cap \Phi^+, \\ ux(t)u^{-1} &= x_\gamma(s), \\ ux(t)u^{-1} &= x_\alpha(s)x_\beta(-s), \end{aligned}$$

where γ, α and β have the same meaning as in Lemma 1 and are positive. The first equation is obvious from the Bruhat decomposition of long root element (see [V1]) or can be checked directly.

Non-solvability of (1) or both (2) and (3) means that Y is not a long or not a short root subgroup respectively. As in Section 4 we tacitly assume that t runs over K and write simply $\{x(t)\}$ instead of $\{x(t),\ t \in K\}$.

Now we pass to the case by case analysis of orbits.

(S1). $\Phi = \mathrm{C}_l,\ F_4$. Here $X = \langle x_\rho(t), x_\gamma(s)\rangle$, where $\rho - \gamma \in \Phi$, $\rho + \gamma \notin \Phi$. The subgroups of the form $Y = \{x_\rho(t)x_\gamma(kt)\}$, $k \in K$ and X_γ are short root subgroups.

(S2) – (S4). $\Phi = \mathrm{B}_l,\ \mathrm{C}_l,\ \mathrm{F}_4$. Here

$$X = \langle x_\rho(t)x_\delta(ct), x_\delta(s)x_{2\rho-\delta}(-s)\rangle \sim X_1 \times X_2.$$

Let $s = kt$, $k \in K$ is fixed. Conjugating by $x_{\delta-\rho}(r)$, $r \in K^*$, we get

$$Y = \{x_\rho((1 + k + rk)t)x_\delta((2r + c + 2 + 2rk + kr^2)t)x_{2\rho-\delta}(-kt)\}.$$

Now if $1 + k + rk = 0$ or $2r + c + 2 + 2rk + kr^2 = 0$ but not both, then Y is a short root subgroup, otherwise Y is a long root subgroup. Specifically, setting $r = -(1 + k)/k$ the second equation becomes $k^2 - kc + 1 = 0$. Hence we fall into the case (S2) if $c^2 - 4 \notin K^{*2}$; (S3) if $c^2 - 4 \in K^{*2}$ and (S4) if $c = \pm 2$.

(S5). $\Phi = \mathrm{C}_l,\ \mathrm{F}_4$. Here

$$X = \langle x_\rho(t)x_\delta(t), x_{2\rho-\delta}(s)x_\beta(-s)\rangle \sim X_1 \times X_2,$$

where β is orthogonal to ρ and δ. The subgroups of X are $\{x_{2\rho-\delta}(t)x_\beta(-t)\}$ and $\{x_\rho(t)x_\delta(t)x_{2\rho-\delta}(kt)x_\beta(-kt)\}$ when k runs over K. Consider equations

$$ux_\rho(t)x_\delta(t)x_{2\rho-\delta}(kt)x_\beta(-kt)u^{-1} = x_\lambda(s),$$

λ is a positive long root. One immediately sees that it is impossible to get rid of the factors $x_{2\rho-\delta}(*)$ and $x_\beta(*)$, so that the set $\Omega(X)$ is empty. The same argument as above shows that the equality

$$ux_\rho(t)x_\delta(t)x_{2\rho-\delta}(kt)x_\beta(-kt)u^{-1} = x_\gamma(s),$$

where γ is a positive short root, is impossible. Finally, let

$$ux_\rho(t)x_\delta(t)x_{2\rho-\delta}(kt)x_\beta(-kt)u^{-1} = x_\alpha(s)x_b(-s),$$

where α, β are as in Lemma 1. Since $x_{2\rho-\delta}(*)$ and $x_\beta(*)$ cannot be eliminated, the coefficients in $x_\rho(*)$ and $x_\delta(*)$ have to be zeros. It is possible only when $u = x_{\delta-\rho}(r)$ and the following holds: $x_\rho((1+kr)t) = 1$ and $x_\delta((1+2r+r^2k)t) = 1$. Hence $k = 1$ and $r = -1$ and therefore $|\Lambda(X)| = 3$.

$(\mathbf{S6})_1$. First, let $\Phi = \mathrm{B}_l$, F_4. Here
$$X = \langle x_1(t), x_{2+3}(s)x_{2-3}(-s)\rangle \sim X_1 \times X_2.$$
The subgroups of X are $\{x_{2+3}(s)x_{2-3}(-s)\}$, $\{x_1(t)x_{2+3}(kt)x_{2-3}(-kt)\}$, where $k \in K$ is fixed. We have to consider the following equations
$$ux_1(t)x_{2+3}(kt)x_{2-3}(-kt)u^{-1} = x_\lambda(s), \quad \text{or} \quad x_\gamma(s), \quad \text{or} \quad x_\alpha(s)x_b(-s).$$
It is easy to see that one cannot get rid of $x_1(t)$ and $x_{2-3}(-kt)$. Thus $|\Lambda| = 2$, $|\Omega| = 0$.

If $\Phi = \mathrm{C}_l$, then $X = \langle x_{1+2}(t), x_{3+4}(s)\rangle$. The number of long and short root subgroups is as above.

$(\mathbf{S6})_2$. $\Phi = \mathrm{C}_l$, F_4. Here $X = \langle x_\rho(t)x_\delta(t), x_\gamma(s)\rangle$, where $\rho - \gamma \in \Phi$, δ and γ are orthogonal. Since $x_\rho(*)$ and $x_\gamma(*)$ are not killed by any $u \in U$, the group X has the required structure.

(S7). $\Phi = \mathrm{B}_l$, C_l, F_4. Here $X = \langle x_\rho(t), x_{\delta-\rho}(s)\rangle = X_\rho X_{\delta-\rho} X_\delta$. The one-parameter subgroups of X have the form
$$\{x_\rho(at)x_{\delta-\rho}(bt)x_\delta(kt - abt^2)\}, \quad a,\ b,\ k \text{ are fixed}$$
If $a, b = 0$, $k \neq 0$ we get a long root subgroup Y. If either one of the following conditions is satisfied we get a short root subgroup: $k = 0$ and exactly one of the elements a and b equals 0; or exactly one of a, b, k equals zero; or a, b and k are all non-zero. In particular, for a finite field $\mathbb{F}_q$ we get $1 + 1 + (q-1) + (q-1) + (q-1) + (q-1)^2 = q^2 + q$ short root subgroups.

(S8). $\Phi = \mathrm{B}_l$, F_4. Here
$$X = \langle x_1(t)x_{1+3}(ct), x_{2+3}(s)x_{2-3}(-s)\rangle \sim X_1X_2Z$$
The one-parameter subgroups of X are
$$\{x_1(at)x_{1+3}(act)x_{2+3}(bt)x_{2-3}(-bt)x_{1+2}(kt - \tfrac{1}{2}abt^2)\}.$$
If $a, b = 0$, $k \neq 0$ we have a long root subgroup Z. If exactly one of the elements a, b equals zero we get a short root subgroup. In particular, for a finite field $\mathbb{F}_q$ we get $2q$ short root subgroups. It is easy to check that if $a, b \neq 0$ we cannot get a root subgroup.

(S9). $\Phi = \mathrm{F}_4$. One has
$$X = \langle x_1(t)x_{1+4}(t)x_{1-3}(\varepsilon t), x_{1+3}(s)x_{2-4}(-s)\rangle \sim X_1X_2Z,$$
where $\varepsilon = 0$ or 1 for $(\mathrm{S9})_1$ or $(\mathrm{S9})_2$ respectively (see Table 7).

The one-parameter subgroups of X are
$$\{x_1(at)x_{1+4}(at)x_{1-3}(a\varepsilon t)x_{1+3}(bt)x_{2-4}(-bt)x_{1+2}(kt - \tfrac{1}{2}abt^2)\}.$$
The number of root subgroups is calculated exactly as above. Under the conjugation action of U either $x_1(*)$ and $x_{1+2}(*)$ are nontrivial if $\varepsilon = 0$, or $x_{1-3}(*)$ and $x_{1+2}(*)$ if $\varepsilon = 1$.

(S10). $\Phi = \mathrm{C}_l$, F_4. Here $X = \langle x_\rho(t), x_\gamma(s)\rangle \simeq U(\widetilde{A}_2, K)$, since $\rho+\gamma$ is a short root.

(S11). $\Phi = \mathrm{B}_l$, C_l, F_4. Here

$$X = \langle x_\rho(t)x_\delta(t), x_{\rho-\delta}(s)\rangle \simeq U(\mathrm{C}_2, K).$$

This is immediate.

(S12). $\Phi = \mathrm{C}_l$, F_4. Again

$$X = \langle x_\rho(t)x_\delta(t), x_\gamma(s)\rangle \simeq U(\mathrm{C}_2, K),$$

where ρ, γ, $\rho+\gamma \in \Phi_s$ and $\{\delta, \gamma, \delta+\gamma, \delta+2\gamma\} = \Phi^+(\mathrm{C}_2)$, and

$$\begin{aligned}
&[x_\rho(t)x_\delta(t), x_\gamma(s)] = x_{\rho+\gamma}(\pm ts)x_{\delta+\gamma}(\pm ts)x_{\delta+2\gamma}(\pm ts^2),\\
&[[x_\rho(t)x_\delta(t), x_\gamma(s)], x_\gamma(r)] = x_{\delta+2\gamma}(\pm 2tsr).
\end{aligned}$$

Thus the subgroup X is generated by the elements of the form $x_\gamma(t)$, $x_\rho(t)x_\delta(t)$, $x_{\rho+\gamma}(t)x_{\delta+\gamma}(t)$ and $x_{\delta+2\gamma}(t)$, which satisfy the same defining relations as the standard generators of $U(\mathrm{C}_2, K)$. In other words, the map defined by

$$\begin{aligned}
x_\gamma(t) &\mapsto x_\gamma(t),\\
x_\rho(t)x_\delta(t) &\mapsto x_\delta(t),\\
x_{\rho+\gamma}(t)x_{\delta+\gamma}(t) &\mapsto x_{\delta+\gamma}(t),\\
x_{\delta+2\gamma}(t) &\mapsto x_{\delta+2\gamma}(t)
\end{aligned}$$

is an isomorphism of X onto $U(\mathrm{C}_2, K)$ as abstract groups.

(S13). $\Phi = \mathrm{B}_l$, F_4. Here

$$X = \langle x_1(t)x_{1+2}(t), x_{-2+3}(s)x_{-2-3}(-s)\rangle \sim X_1X_2YZ.$$

It suffices to calculate the commutators:

$$\begin{aligned}
&[x_1(t)x_{1+2}(t), x_{-2+3}(s)x_{-2-3}(-s)] = x_{1+3}(-ts)x_{1-3}(ts)x_{1-2}(-ts^2),\\
&[x_{-2+3}(s)x_{-2-3}(-s), x_{1+3}(-r)x_{1-3}(r)x_{1-2}(rt)] = x_{1-2}(-2rs).
\end{aligned}$$

(S14). $\Phi = \mathrm{F}_4$. Here

$$X = \langle x_1(t)x_{1+2}(t), x_{-1+4}(s)x_{-2+3}(-s) \simeq U(\mathrm{C}_2, K) \times Z.$$

Calculate the commutators:

$$[x_1(t)x_{1+2}(t), x_{-1+4}(s)x_{-2+3}(-s)] = x_{2+4}(ts)x_{1+3}(-ts)x_4(ts)x_{1+4}(-t^2s)x_{3+4}(ts^2),$$

$$[x_{2+4}(rs)x_{1+3}(-rs)x_4(rs)x_{1+4}(-r^2s)x_{3+4}(rs^2), x_1(t)x_{1+2}(t)] = x_{1+4}(2rst),$$

$$[x_{2+4}(rs)x_{1+3}(-rs)x_4(rs)x_{1+4}(-r^2s)x_{3+4}(rs^2), x_{-1+4}(t)x_{-2+3}(-t)] = x_{3+4}(2rst).$$

The quotient group X/X_{3+4} is isomorphic to $U(\mathrm{C}_2, K)$. On the generators the isomorphism may be defined by:

$$\begin{aligned} x_1(t)x_{1+2}(t)X_{3+4} &\mapsto x_\alpha(t), \\ x_{-1+4}(t)x_{-2+3}(-t)X_{3+4} &\mapsto x_\beta(t), \\ x_{2+4}(t)x_{1+3}(-t)x_4(t)X_{3+4} &\mapsto x_{\alpha+\beta}(t), \\ x_{1+4}(t)X_{3+4} &\mapsto x_{2\alpha+\beta}(t), \end{aligned}$$

where α and β are the short and the long fundamental roots respectively.

(S15). $\Phi = \mathrm{B}_l$, C_l, F_4. Clearly $X = \langle x_\rho(t), x_{-\rho}(s)\rangle \simeq \mathrm{SL}_2(K)$.

(S16) – (S18). $\Phi = \mathrm{B}_l$, C_l, F_4. Here

$$X = \langle x_\rho(t)x_\delta(ct), x_{\delta-2\rho}(-s)x_{-\delta}(s)\rangle.$$

Obviously X may be considered as a subgroup of $\mathrm{Sp}(V)$, where $\dim(V) = 4$. Set $x(t,c) = x_\rho(t)x_\delta(ct)$, $y(s) = x_{\delta-2\rho}(-s)x_{-\delta}(s)$. Then

$$x(t,c) = \begin{pmatrix} 1 & 0 & t & ct \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad y(s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -s & 1 & 0 \\ s & 0 & 0 & 1 \end{pmatrix}.$$

Let $\overline{V}$ be a 3-dimensional projective space whose points are 1-dimensional subspaces of V. Denote by $\overline{x}(t,c)$, $\overline{y}(s)$ the projective transformations induced by $x(t,c)$, $y(s)$ respectively and let $\overline{X}$ be the group generated by $\overline{x}(t,c)$, $\overline{y}(s)$.

A point of $\overline{V}$ is determined by its homogeneuos coordinates $(p_1, p_2, p_3, p_4)^t$ and by duality the same holds for the planes of $\overline{V}$: a plane is determined by (p^1, p^2, p^3, p^4). The lines of $\overline{V}$ may be described by non-zero skew-symmetric 4×4 matrices P with $\det(P) = 0$.

Now we find the subspaces of $\overline{V}$ invariant under the action of $\overline{X}$. Let $p = (p_1, p_2, p_3, p_4)^t$ be a point, $q = (q^1, q^2, q^3, q^4)$ be a plane and $P = (p_{ij})$ be a line of $\overline{V}$, where $p_{ij} = -p_{ji}$ and $p_{12}p_{34} + p_{13}p_{24} - p_{14}p_{23} = 0$.

Then the action of $\overline{z} \in \mathrm{PSp}(V)$ is determined as follows

$$p \mapsto \overline{z}p, \qquad q \mapsto q\overline{z}^{-1}, \qquad P \mapsto \overline{z}^t P \overline{z}.$$

It is easy to verify that there are no invariant points and planes in $\overline{V}$ under $\overline{X}$. Moreover a line P in $\overline{V}$ is invariant under $\overline{X}$ if and only if

$$\overline{x}^t(t,c)P = aP\overline{x}(-t,c), \qquad a \in K^*$$

$$\overline{y}^t(s)P = bP\overline{y}(-s), \qquad b \in K^*$$

Thus P must have the form

$$\begin{pmatrix} 0 & 0 & p_{13} & cp_{13} + p_{23} \\ 0 & 0 & p_{23} & -p_{13} \\ -p_{13} & -p_{23} & 0 & 0 \\ -cp_{13} - p_{23} & p_{13} & 0 & 0 \end{pmatrix},$$

where $p_{13}^2 + cp_{13}p_{23} + p_{23}^2 = 0$. The solvability of this last equation is equivalent to that of $t^2 + ct + 1 = 0$. This leads to three separate cases.

(S16). The equation $t^2 + ct + 1 = 0$ has two distinct roots t_1 and t_2 in K, i.e. $\sqrt{c^2 - 4}$ belongs to K^*. Then the subgroup X has two 2-dimensional invariant subspaces in V, namely the planes spaned by $(1, t_i, 0, 0)$ and $(0, 0, -t_i, 1)$, $i = 1, 2$. Thus X is conjugate to $\langle X_\rho, X_{-\rho}\rangle$ by an element of $\mathrm{GL}_4(K)$ whose determinant is $(t_2 - t_1)^2$. It follows from the definition of the symplectic group that this element cannot possibly belong to $\mathrm{Sp}_4(K)$.

(S17). The equation $t^2 + ct + 1 = 0$ has no roots in K, i.e. $\sqrt{c^2 - 4}$ does not belong to K. Then we pass to the quadratic extension $L = K(\sqrt{c^2 - 4})$ and use the same argument as in Case (S16). Observe that this result agrees with [L3].

Remark. If we do not pass to a quadratic extension of the ground field, it may be proved that X is a primitive subgroup containing no long root subgroups and leaving invariant a regulus in a projective space (see [M]).

If $K = \mathbb{F}_3$, then $X \simeq G_{960}$, the semidirect product of the elementary abelian group of order 16 by the alternating group A_5 (see [S2]).

(S18). The equation $t^2 + ct + 1 = 0$ has a unique root t_0 in K, i.e. $c = \pm 2$. Then the subgroup X is conjugate to $\langle x_\rho(t)x_\delta(t), x_{-\rho}(s)\rangle$. Each element of this group may be written as yu, where $y \in \langle X_\rho, X_{-\rho}\rangle \simeq \mathrm{SL}_2(K)$, $u \in U(\mathrm{C}_2, K)$.

Consider the map ϕ from $X \longrightarrow \mathrm{SL}_2(K)$ defined by $yu \mapsto y$. It is a surjective homomorphism. We have to calculate the kernel $U_0 = \ker\phi$ of this homomorphism.

It is well known (see [S]) that the defining relations of $\mathrm{SL}_2(K)$ are

$$x_\alpha(t)x_\alpha(s) = x_\alpha(t+s),$$
$$w_\alpha(t)x_\alpha(s)w_\alpha(-t) = x_{-\alpha}(-t^2 s),$$
$$h_\alpha(t)h_\alpha(s) = h_\alpha(ts),$$

where $\alpha \in \Phi(\mathrm{A}_1)$, t, s run over K.

Let $x(t) = x_\rho(t)x_\delta(t)$, $y(s) = x_{-\rho}(s)$. We set

$$w(t) = x(t)y(-t^{-1})x(t), \qquad h(t) = w(t)w(-1).$$

Then

$$U_0 = \langle w(t)x(s)w(-t)y(t^2 s), h(t)h(s)h(-t^{-1}s^{-1})\rangle$$

and a straightforward calculations yields

$$U_0 = \langle x_\delta(-f(t^{-1}, s^{-1}))x_{\delta-2\rho}(f(t,s)), x_{\delta-\rho}(-t^{-1}s)x_{\delta-2\rho}(-t^{-2}s + t^{-3}s^2)\rangle,$$

where $f(t,s) = t^2s^2 - ts^2 - ts + s^2 - s + 1$. Setting $s = t$ we conclude that $\langle x_{\delta-\rho}(1)\rangle$ is contained in U_0. Set $s = kt$, where k is an element of the prime subfield of K. Then $x_{\delta-2\rho}(kt^{-1}(k-1))$ lies in U_0 for any $t \in K^*$ and therefore U_0 is generated by $X_{\delta-2\rho}$, $X_{\delta-\rho}$, X_δ.

(S19). $\Phi = \mathrm{B}_l$, C_l, F_4. Here

$$X = \langle x_\rho(t)x_\delta(ct), x_{2\rho-\delta}(s)x_{-\delta}(-s)\rangle \simeq \mathrm{SL}_2(K)\times Z \quad \text{or} \quad \mathrm{SL}_2(K).$$

Conjugating generators by $x_{\rho-\delta}(-1/c)$ we get

$$\langle x_{2\rho-\delta}(-t/c)x_\delta(ct), x_{2\rho-\delta}(s)x_{-\delta}(-s)\rangle.$$

First suppose that $K \neq \mathbb{F}_3$. Then the commutator subgroup $[X,X]$ is isomorphic to $\mathrm{SL}_2(K)$ and we are done.

Let $K = \mathbb{F}_3$. This group is a subgroup of $\mathrm{Sp}_4(K)$. We consider this subgroup in the same manner as in Case (S18). Then calculations with generators show that $\ker\phi$ is trivial.

(S20). $\Phi = \mathrm{C}_l$, F_4. Here

$$X = \langle x_\rho(t)x_\delta(t), x_{-\delta}(s)x_\beta(-s)\rangle,$$

where β is orthogonal to ρ and δ. Conjugating by $x_{\rho-\delta}(-1)$ we get

$$\langle x_{2\rho-\delta}(t)x_\delta(t), x_{-\delta}(s)x_\beta(-s)\rangle.$$

Again our statement for $K \neq \mathbb{F}_3$ follows from the fact that $[X,X] \simeq \mathrm{SL}_2(K)$.

Let $K = \mathbb{F}_3$. Proceeding as in (S18) and (S19) we deduce that $U_0 = \ker\phi = \langle x_{2\rho-\delta}(t)x_\beta(t)\rangle$ is the cyclic group of order 3. Note that for F_4 the subgroup U_0 is a short root subgroup.

(S21). $\Phi = \mathrm{F}_4$. Here

$$X = \langle x_1(t)x_{1-2}(t)x_{1-3}(t), x_{3-4}(s)x_{-1+2}(-s)\rangle.$$

Now X may be considered as a subgroup of $\Omega_9(K, Q)$, the commutator subgroup of the orthogonal group with respect to the form Q mentioned at the beginning of this section. Let $x(t)$ stand for the first generator and $y(t)$ for the second one. It is clear that X is isomorphic to the semi-direct product $\mathrm{SL}_2(K) \leftthreetimes U_0$, where U_0 is a subgroup of $U(\mathrm{B}_4, K)$. To determine U_0 we proceed as above (compare (S18)). Calculations with $x(t)$ and $y(t)$ (in the matrix representation) show that U_0 is generated by

$$w(t)x(s)w(-t)y(t^{-2}s) = x_{1+2}(s)x_{2-4}(t^{-2}s), x_{3-4}(-t^{-2}s),$$
$$h(t)h(s)h(t^{-1}s^{-1}) = x_{1+2}(2f(t,s))x_{2-4}(-f(t^{-1},s^{-1}))x_{3-4}(f(t^{-1},s^{-1})),$$

where $f(t,s) = t^{-1}s^{-1}+s+t-3$. Varying t and s we conclude that U_0 is generated by long root subgroups X_{1+2} and $\{x_{2-4}(t)x_{3-4}(t)\}$.

§ 6. Completion of proof of Theorem 1

To finish the proof of Theorem 1 it remains only to show that subgroups falling into different Cases (S1) – (S21) are non-conjugate and to calculate the number of non-conjugate subgroups of the same case.

It is clear, that may restrict our attention only to the cases when the structure of subgroups $\langle X_1, X_2\rangle$ in the sense of Section 5 (i.e. the isomorphism class plus the numbers of short and long root subgroups contained in $\langle X_1, X_2\rangle$) is the same for distinct representatives of Tables 5 – 7.

We start with some terminology. The roots α and β are called *strictly orthogonal* if $\alpha \pm \beta$ are not roots. For a subset of roots Δ in Φ we denote by $\Delta^{\perp}$ (resp. $\Delta_0^{\perp}$) the set of roots orthogonal (resp. strictly orthogonal) to Δ. Let $\overline{\Delta}$ be the closure of the set Δ, i.e. the minimal closed set of roots, containing Δ.

For $u \in U$ we write $u = \prod x_\sigma(c_\sigma)$, $\sigma \in \Phi^+$. Then the set

$$\Gamma(u) = \{\, \sigma \in \Phi^+ \mid c_\sigma \neq 0 \,\},$$

is called the *support* of u. Recall also that the reduced Bruhat decomposition asserts that any element $g \in G$ can be uniquely written in the form $g = uhn_wv$, where $u \in U$, $h \in H$, $v \in U \cap w^{-1}Vw$ and $n_w \in N$. Since N/H is isomorphic to W, we will ignore the diagonal factor and write simply w instead of n_w.

We start with the Cases (S2) – (S4), (S8), (S16) – (S18) and (S19) (the latter for $\Phi = \mathrm{C}_l$). These pairs are parameterized by an element of the field K. Thus, we have to study when two pairs of the form $(X_{\rho,c}, X_{\alpha,\beta})$ and $(X_{\rho,c'}, X_{\alpha,\beta})$ are conjugate. We may assume that $c \neq 0$ and c' is an arbitrary element of K.

First consider the effect of conjugation by elements of $H(\Phi)$. Without loss of generality we may assume that $\Phi = \mathrm{F}_4$ or C_2. An element of $H(\mathrm{F}_4)$ may be decomposed as

$$h_{1-2-3-4}(k_1)h_{2-3}(k_2)h_{3-4}(k_3)h_4(k_4), \qquad k_i \in K^*, \quad 1 \leq i \leq 4,$$

and an element of $H(\mathrm{C}_2)$ as $h_{1-2}(k_1)h_2(k_2)$, $k_i \in K^*$, $i = 1, 2$. Thus the condition

$$hx_\rho(t)x_\delta(ct)h^{-1} = x_\rho(t')x_\delta(c't')$$

amounts to the following system of equations:

$$(1) \qquad \begin{cases} k_1 t = t', \\ k_2 ct = c't', \end{cases} \quad (\text{for } \mathrm{F}_4), \qquad \text{and} \qquad \begin{cases} k_2 t = t', \\ k_1^2 ct = c't', \end{cases} \quad (\text{for } \mathrm{C}_2).$$

In the same manner condition

$$hx_\alpha(s)x_\beta(-s)h^{-1} = x_\alpha(s')x_\beta(-s')$$

furnishes us the second system of equations. Now we analyse these two systems case by case.

(S2) – (S4), $\Phi = \mathrm{B}_l$, C_l, F_4. We have

$$(2) \qquad \begin{cases} k_2 s = s', \\ k_1^2 k_2^{-1} s = s', \end{cases} \quad (\text{for } \mathrm{F}_4) \qquad \text{and} \qquad \begin{cases} k_1^2 s = s', \\ k_1^{-2} k_2^2 s = s', \end{cases} \quad (\text{for } \mathrm{C}_2).$$

Comparing (1) and (2) we conclude that $c = \pm c'$.

(S8), $\Phi = \mathrm{B}_l$, F_4. Then

$$(3) \qquad \begin{cases} k_1^2 k_3 s = s', \\ k_2^{-2} k_3 s = s'. \end{cases}$$

Again comparing (1) and (3) we conclude that $c = \pm c'$.

(S16) – (S18), $\Phi = \mathrm{B}_l$, C_l, F_4. Then

$$(4) \qquad \begin{cases} k_1^{-2} k_2 s = s', \\ k_2^{-1} s = s', \end{cases} \quad (\text{for } \mathrm{F}_4) \qquad \text{and} \qquad \begin{cases} k_1^2 k_2^{-2} s = s', \\ k_1^{-2} s = s', \end{cases} \quad (\text{for } \mathrm{C}_2).$$

Again we have $c = \pm c'$.

(S19), $\Phi = \mathrm{C}_l$. Then

$$\begin{cases} k_1^{-2}k_2^2 s = s', \\ k_1^{-2} s = s'. \end{cases} \tag{5}$$

Thus $c = \pm k^2 c'$ for any $k \in K$.

This finishes the analysis of the conjugation by the elements of H and we pass to the conjugation by an arbitrary element.

(S2) – (S4). The fact that our representatives are conjugate is equivalent to the solvability of the following system:

$$hwvx_\rho(t)x_\delta(ct)v^{-1}w^{-1}h^{-1} = u^{-1}x_\rho(t')x_\delta(c't')u, \tag{6}$$

$$hwvx_\alpha(s)x_\beta(s)v^{-1}w^{-1}h^{-1} = u^{-1}x_\alpha(s')x_\beta(-s')u, \tag{7}$$

for some $g = uhwv$, $(\alpha, \beta) = (\delta, 2\rho - \delta)$. We may rewrite (6) in the form

$$hwx_\rho(t)x_\delta((c+2c_2)t)\prod_{i>2} x_{\rho_i}(c_i t)w^{-1}h^{-1} = x_\rho(t')x_\delta((c'+2c_2')t)\prod_{j>2} x_{\rho_j}(c_j' t'), \tag{8}$$

where $v = \prod x_\sigma(c_\sigma)$, $u = \prod x_\sigma(c_\sigma')$.

The last equation implies that $w\rho = \rho$ and thus $w \in W(\{\rho\}^\perp)$. If $c + 2c_2 \neq 0$ or $c_i \neq 0$ for some $i > 2$, then $c' + 2c_2' \neq 0$, $c_j \neq 0$, $j > 2$. Thus $w(\rho_i - \rho)$ is a positive root and $\rho_i - \rho \notin \Gamma(v)$. This means that, $c_i = 0$, $i > 2$, and either $c_2 = 0$ or $c + 2c_2 = 0$, $c_2 \neq 0$.

Suppose $c_2 = 0$ and $c' + 2c_2' \neq 0$. Then $w\delta = \delta$ and we have

$$g = u_1 x_{\delta-\rho}(c_2')hw_1v_1, \tag{I}$$

where $w_1 \in W(\{\rho, \delta\}^\perp)$, $\Gamma(v_1) \subset \Gamma_1 = \{\rho, \delta\}_0^\perp$, $\Gamma(u_1) \subset \Gamma_2 = \{\sigma \in \Phi^+ \mid \sigma + \rho, \sigma + \delta \notin \Phi\}$,

If $c_2 = 0$, $c' + 2c_2' = 0$, $c_j' \neq 0$ for some $j > 2$ we conclude that $w\delta = \rho_j$ and

$$g = u_1 x_{\rho_j - \rho}(c_j')x_{\delta-\rho}(-c'/2)hw_{\delta-\rho_j}w_1v_2, \tag{II}$$

where u_1, w_1 are the same as above, $\Gamma(v_2) \subset \Gamma_1 \cup \{\delta - \rho_3, \dots, \delta - \rho_j\}$.

Finally, suppose that $c + 2c_2 = 0$ so $c_j' = 0$, $j > 2$ and $c' + 2c_2' = 0$. The element g has the form

$$g = u_1 x_{\delta-\rho}(-c'/2)hw_\lambda w_1 x_{\delta-\rho}(-c/2)v_3, \tag{III}$$

where u_1, w_1 are as above, $\lambda = \delta + \rho_j - 2\rho$ for some $j > 2$ or $\lambda = \delta - \rho$, $\Gamma(v_3) \subset \Gamma_1 \cup \{\rho_j - \rho, \rho_j - \rho_k, k > j, \rho_j + \rho_k - 2\rho, 2 \leq k \neq j\}$.

If g is of type (I), (II) or (III) with $j \neq 2$, it is easily seen that (7) is impossible. If g is of type (III) and $j = 2$ we get

$$hx_\delta(s)x_\rho(-cs)x_{2\rho-\delta}(s - c^2 s)h^{-1} = x_\delta(s' - c'^2 s')x_\rho(-c's')x_{2\rho-\delta}(s').$$

The above analysis of the conjugation by h implies that $c^2 = c'^2$.

(S16) – (S18). The arguments are analogous to those used for cases (S2) – (S4). Equation (7) becomes

$$hw_{\delta-\rho}x_{-\delta}(s)x_{-\rho}(-cs)x_{\delta-2\rho}(s - c^2 s)w_{\delta-\rho}h^{-1} = x_{-\delta}(s')x_{-\rho}(-c's')x_{\delta-2\rho}(s' - c'^2 s').$$

Again one deduces that $c^2 = c'^2$.

(S8). Here we have to consider the system

$$hwvx_1(t)x_{1+3}(ct)v^{-1}w^{-1}h^{-1} = u^{-1}x_1(t')x_{1+3}(c't')u, \tag{9}$$

$$hwvx_{2+3}(s)x_{2-3}(s)v^{-1}w^{-1}h^{-1} = u^{-1}x_{2+3}(s')x_{2-3}(-s')u, \tag{10}$$

for some $g = uhwv$.

Proceeding as in cases (S2) – (S4) we deduce from (9) that g has one of the following forms:

$$g = u_1x_3(c_3')hw_1v_1, \tag{I}$$

where $w_1 \in W(\{\rho, \rho_3\}^\perp)$, $\Gamma(v_1) \subset \Gamma_1 = \{\rho, \rho_3\}_0^\perp$, $\Gamma(u_1) \subset \Gamma_2 = \{\sigma \in \Phi^+ \mid \sigma + \rho, \sigma + \rho_3) \notin \Phi\}$;

$$g = u_1x_3(-c_3')x_2(c_2')x_{2-3}(-c_2'/2)x_j(c_j')hw_{3-j}w_1v_2, \tag{II}$$

where $c_j' \neq 0$, $j > 3$. u_1, w_1 are as above, $\Gamma(v_2) \subset \Gamma_1 \cup \{\rho_3 - \rho_4, \ldots, \rho_3 - \rho_j\}$; or

(III)
$$g = u_1x_3(-c'/2)x_2(c_2')x_{2-3}(-c_2'/2)hw_{3-j}w_{2+k}w_1x_3(-c/2)x_2(c_2)x_{2-3}(-c_2/2)v_3,$$

where $k \neq j > 3$, u_1, w_1 are as above, and $v_3 \in U$.

Now (10) reduces to the action of $h \in H$ on $\{x_{2+3}(*)x_2(*)x_{2-3}(*)\}$. It follows from the above that $c^2 = c'^2$.

(S19), $\Phi = \mathrm{C}_l$. Here we deal with the following system

$$hwvx_{1+2}(t)x_2(ct)v^{-1}w^{-1}h^{-1} = u^{-1}x_{1+2}(t')x_2(c't')u,$$

$$hwvx_1(s)x_{-2}(-s)v^{-1}w^{-1}h^{-1} = u^{-1}x_1(s')x_{-2}(-s')u,$$

for some $g = uhwv$. The proof for this case is analogous to the above.

Now we prove that the Cases (S6) and (S9) lead to two orbits.

(S6). For $\Phi = \mathrm{F}_4$ consider the following pairs (see Tables 5 – 7)

$$(X_\rho, \{x_{2+3}x_{2-3}(-1)\}) \quad \text{and} \quad (X_\rho, \{x_{1+3}x_{2+4}(-1)\})$$

Let $N(X_\rho)$ be the normalizer of X_ρ and $g = uhwv \in N(X_\rho)$. Then $w \in W(\{\rho\}^\perp)$, $\Gamma(v) \subset \{\rho\}_0^\perp$, $\Gamma(u) \subset \{\sigma \in \Phi^+ \mid \sigma + \rho \notin \Phi\}$. For such an element g the equation

$$hwvx_{2+3}x_{2-3}(-1)v^{-1}w^{-1}h^{-1} = u^{-1}x_{1+3}x_{2+4}(-1)u$$

is nonsolvable.

For $\Phi = \mathrm{C}_l$ we have

$$(X_\rho, \{x_{3+4}\}) \quad \text{and} \quad (X_\rho, \{x_{1+3}x_{2+3}\}),$$

and the same argument shows that these pairs are not conjugate.

(S9). Here we consider the pairs

$$(x_1x_{1+2}, \{x_{1+3}x_{-2+4}(-1)\}) \quad \text{and} \quad (x_1x_{1+2}x_{1+3}, \{x_{1-3}x_{-2+4}(-1)\}).$$

If $g = uhwv$ transforms x_1x_{1+2} to $x_1x_{1+2}x_{1+3}$, then $w \in \langle w_{2-3}, w_{3-4}, w_4\rangle$ and $\Gamma(v) \subset \{2\pm 3, 2\pm 4, 3\pm 4, 2, 3\}$. If these pairs are conjugate, then X_{1+4} transforms to itself. Thus $w = w_{2-3}$, $\Gamma(v) \subset \{2-3\}$, $\{e_3\} \in \Gamma(u)$. Returning to the conditions which guaranteed that $\{x_{1-3}x_{-2+4}(-1)\}$ was carried to $\{x_{1-3}x_{-2+4}(-1)\}$, we see that this is impossible.

It remains only to consider different cases for which $\langle X_1, X_2\rangle$ has the same description in the sense of § 5. These are Cases (S1) and (S2), (S8) and $(\mathrm{S}9)_{1,2}$, (S15) and (S16).

Cases (S1) and (S2). Here we have

$$(X_\rho, X_\gamma) \quad \text{and} \quad (\{x_\rho x_\delta(c)\}, \{x_\delta x_{2\rho-\delta}(-1)\}),$$

where c such that $c^2 - 4 \notin K^{*2}$. If these pairs were conjugate, this would also hold over an algebraically closed field, which is absurd.

Cases (S8) and (S9), $\Phi = \mathrm{F}_4$. Here

$$(x_1x_{1+2}(c), \{x_{2+3}x_{-2+3}(-1)\}) \quad \text{and} \quad (x_1x_{1+2}x_{1+3}(\varepsilon), \{x_{1-3}x_{-2+4}(-1)\}),$$

where $c \in K^*$, $\varepsilon = 0$ or 1. These pairs are not conjugate due to the same reason, as in Case (S6).

In cases (S15) and (S16) the assertion follows from the discussion of Cases (S16) – (S18) in Section 5.

This completes the proof of Theorem 1.

Notice that the existence of two orbits in Cases (S6) and (S9) is explained by the presence in the corresponding X of non-conjugate one-parameter subgroups which are not root subgroups.

§ 7. Proof of Theorem 2

It follows from Lemma 2 that now we have to consider only Variants 1 – 6. For the sake of convience we extract from Tables 5 – 7 the representatives of those orbits which may arise in this case.

Table 8

case	variants	$\Phi = \mathrm{C}_l$	$\Phi = \mathrm{B}_l$, F_4
(T1)	2	$\{x_{1+2}, x_{1+3}\}$	$\{x_1, x_{1+2+3+4}\}$
$(\mathrm{T2})_1$	3	$\{x_{1+2}, x_{1-2}\}$	$\{x_1, x_2\}$
$(\mathrm{T2})_2$	4	$\{x_{1+2}, x_{3+4}\}$	—
(T3)	5	$\{x_{1+2}, x_{-1+3}\}$	$\{x_1, x_{-1+2+3+4}\}$
(T4)	6	$\{x_{1+2}, x_{-1-2}\}$	$\{x_1, x_{-1}\}$

Cases (T1), $(\mathrm{T2})_2$, (T3), (T4) have been already discussed as Cases (S1), (S4), (S9) and (S12) respectively in the proof of Theorem 1. Characteristic plays no role here. The only new case $(\mathrm{T2})_1$ arises when the subgroup $\langle X_1, X_2\rangle$ is generated by $x_\rho(t)$ and $x_\gamma(s)$, where $\rho+\gamma$ is a long root. Then $\langle X_1, X_2\rangle = X_1 \times X_2$ since the corresponding coefficient in the Chevalley commutator formula vanishes in characteristic 2. This proves Theorem 2.

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