Co-Multiplication and Suspension

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1 Introduction

It has been known for a long time that an $(n-1)$-connected CW-complex of dimension $\leq 2n-1$ has the homotopy type of a suspension. In 1963 Bernstein and Hilton proved that an $(n-1)$-connected based CW-complex X of dimension $\leq 3n-3$ has the homotopy type of a suspension provided X admits a co-multiplication $X \to X \vee X$ with homotopy co-unit [1] (Bernstein and Hilton made the additional technical requirement that the homology groups of X are finitely generated). In 1970 Ganea extended this result: An $(n-1)$-connected co-H-space X of dimension $\leq 4n-5$, $n \geq 2$, has the homotopy type of a suspension ΣY if and only if it is homotopy co-associative. Moreover, the homotopy equivalence $\Sigma Y \to X$ is a homomorphism up to homotopy [5]. Bernstein-Hilton and Ganea conjectured that a suitable theory of co-A_n-spaces, dual to Stasheff's theory of A_n-H-spaces [12], would lead to extensions of their results. Indeed, in the A_n-space situation the following result holds [13], [16].

1.1 Proposition: Let X be an $(n-1)$-connected well-pointed A_{k-1}-space, $k \leq 3$, admitting a homotopy inverse, such that $\pi_i X = 0$ for $i > (k+1)n-4$. Then there is an A_{k-1}-homomorphisms $f : X \to \Omega_M Y$ which is a weak equivalence. Here $\Omega_M Y$ is the Moore loop space on Y with its natural monoid structure.

Recall that an A_k-homomorphism is a homomorphism up to homotopy satisfying coherence conditions up to $(k-1)$-dimensional homotopies. The purpose of this paper is to prove the following dual result.

1.2 Theorem: Let X be an $(n-1)$-connected based CW-complex with $\dim X \leq k(n-2)+3$, $n \geq 2$. Then X is of the homotopy type of a suspension if and only if X is a co-A_{k-1}-space. Moreover, the homotopy equivalence $f : \Sigma Y \to X$ is a co-A_{k-1}-homomorphism.

1.3 Corollary: A 2-connected space X is of the weak homotopy type of a suspension if and only if it is a co-A_∞-space.

We then use our approach to prove a dual version of Segal's delooping result [11]

1.4 Theorem: Let $X : \triangle \to \mathcal{T}op^*$ be a co-simplicial space such that X_0 is contractible, X_1 is 2-connected and $(\pi_1, \ldots, \pi_n) : \vee_{k=1}^{n} X_1 \to X_n$ is a homotopy equivalence where $\pi_k : [1] \to [n]$ maps 0 to $i-1$ and 1 to i. Then there is a weak homotopy equivalence $\Sigma RX \to X_1$, where RX is a functorial, thickened version of the topological realization of X.

A result of this type has been formulated by Hopkins [7] with the weaker condition that X_1 is only 1-connected, but remarks by Bousfield [3, (4.9)] indicate that additional assumptions are needed. Instead of the maps π_k Hopkins uses the equivalence $(\iota_1, \ldots, \iota_n) : \vee_{k=1}^{n} X_1 \xrightarrow{\simeq} X_n$ where $\iota_k : [1] \to [n]$ maps 0 to 0 and 1 to k. A cosimplicial space in his sense induces one in our sense by a base change.

The approach to desupension which we develop here is motivated by potential applications. A fibrant variant of the main result of this paper (see [8], [9]) will be used by the first-named author in a subsequent paper to prove a Haefliger-type embedding theorem in the category of Poincaré duality spaces. Such embedding and desuspension results also play a fundamental role in the original Poincaré surgery program as formulated by Browder et. al. It is hoped that the results of this paper can be adapted so as to complete the Browder program.

Our proofs will make extensive use of homotopy limits and co-limits of cube diagrams and their analysis by Goodwillie [6]. They are in spirit quite different from the proofs of Bernstein-Hilton and Ganea. To stay away from point set topological difficulties we work in the category $\mathcal{T}op$ of compactly generated spaces in the sense of [14]. Based spaces are assumed to be well-pointed unless stated otherwise.

2 A_k-spaces and co-A_k-spaces

We recall the definition of an A_k-space using the terminology of [2].

Let $\mathcal{G}$ be the category with $ob\ \mathcal{G} = \mathbb{N} = \{0, 1, \ldots\}$, exactly one morphism $\lambda_n : n \to 1$ for $n \geq 1$, and $\mathcal{G}(n,k)$ consisting of all formal expressions

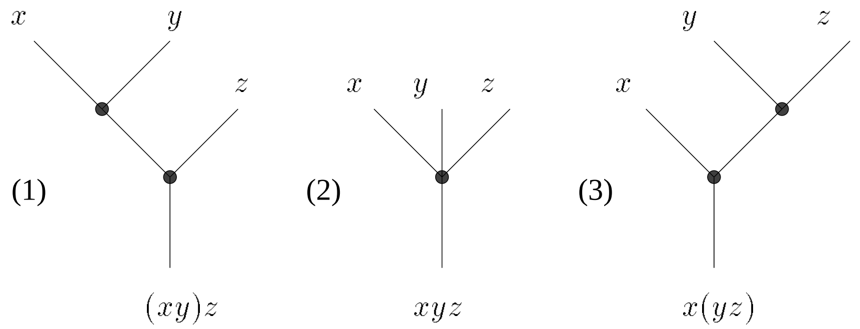
$$\lambda_{i_1} \oplus \ldots \oplus \lambda_{i_k}, \quad i_1 + \ldots + i_k = n$$

$\mathcal{G}(0,0) = \{id_0\}$ and $\mathcal{G}(0,n) = \emptyset = \mathcal{G}(n,0)$ for $n > 0$. Composition is defined by the requirement that $\oplus$ is a bifunctor.

The category $\mathcal{G}$ describes semigroups: A semigroup G determines a functor $\mathcal{G} \to \mathcal{T}op$, $n \mapsto G^n$, transforming $\oplus$ into $\times$. Conversely, if $G : \mathcal{G} \to \mathcal{T}op$ is a functor such that $G(n) = G(1)^n$ and $G(f \oplus g) = G(f) \times G(g)$ then $G(1)$ is a semigroup with multiplication $G(\lambda_2)$.

We symbolize λ_n, $n \geq 2$, by a box with n inputs $x_1, \ldots, x_n$ and one output $x_1 \ldots x_n$. Composite operations are obtained by wiring boxes together to a tree shaped circuit, e.g.

2.1



represent all possible non-trivial composite operations $3 \to 1$. In $\mathcal{G}$ they coincide but for homotopy associative multiplications they only coincide up to homotopy. To account for this we give each connection (between boxes) a length $t \in I = [0,1]$. Given a tree T the various lengths of its connections make up the points of a cube $C(T)$ whose dimension is the number of connections of T. We thus obtain a new category $\mathcal{TG}$ with $ob\,\mathcal{TG} = ob\,\mathcal{G}$ and $\mathcal{TG}(n,1) = \coprod C(T)$, T running through all trees with n inputs, $\mathcal{TG}(n,k) = \coprod \mathcal{TG}(i_1,1) \times \ldots \times \mathcal{TG}(i_k,1)$ with $i_1 + \ldots + i_k = n$. Let T be a tree with lengths in $\mathcal{TG}(k,1)$ and let $(T_1,\ldots,T_k) \in \mathcal{TG}(n,k)$ be a k-tuple of trees then $T \circ (T_1,\ldots,T_k)$ is the tree obtained from T by sticking T_i onto its i-th input and giving the newly created connections the lengths 1. The trivial tree serves as id_1. We have a continuous pairing $\oplus : \mathcal{TG} \times \mathcal{TG} \to \mathcal{TG}$ defined by $(T_1,\ldots,T_k) \oplus (T_{k+1},\ldots,T_{k+l}) = (T_1,\ldots,T_{k+l})$. Composition then is determined by the requirement, that $\oplus$ is a bifunctor. It is continuous.

Let $\mathcal{WG}$ be the quotient category of $\mathcal{TG}$ obtained by imposing the relation that a connection of length 0 may be shrunk by amalgamating the two boxes at its ends to a single one. Observe that the trees (2.1.1) and (2.1.3) represent unit intervals in $\mathcal{WG}(3,1)$ which are identified at 0 with the tree (2.1.2). Hence the trees of (2.1) codify an associating homotopy.

2.2 Definition: Let $\mathcal{W}_k\mathcal{G}$ be the full subcategory of $\mathcal{WG}$ of objects $0, 1, \ldots, k$. An *A_k-space* is a continuous functor $X : \mathcal{W}_k\mathcal{G} \to \mathcal{T}op$ mapping $\oplus$ to $\times$ (in particular $X(n) = X(1)^n$) such that the multiplication $X(\lambda_2) : X(1)^2 \to X(1)$ admits a homotopy unit. (Note that we do not require any coherence for the homotopy unit.) In abuse of notation we often denote $X(1)$ by X.

2.3 Remark: The map $\varepsilon : \mathcal{WG} \to \mathcal{G}$ which maps a tree with n inputs to λ_n is a continuous functor admitting a non-functorial section $\eta : \mathcal{G} \to \mathcal{WG}$ mapping λ_n to the tree with a single vertex and n inputs if $n > 1$ and to the trivial tree for $n = 1$. As maps of morphism spaces $\eta \circ \varepsilon \simeq id$ (the homotopy shrinks the lengths of connections by the factor t at time t).
It is easy to show that $\mathcal{WG}(n,1)$ is a subdivision of the Stasheff cell K_n of [12] into cubes. So our definition coincides with the one of Stasheff with the exception that he makes the stronger requirement that $X(\lambda_2)$ admits a strict unit e. But if $\{e\} \subset X$ is a closed co-fibration both definition are equivalent [2, Chapt. I].

co-A_k-spaces are defined dually:

2.4 Definition: A *co-A_k-space* is a continuous functor $X : (\mathcal{W}_k\mathcal{G})^{op} \to \mathcal{T}op^*$ into the category of well-based topological spaces mapping $\oplus$ to the wedge sum $\vee$ such that the comultiplication $X(\lambda_2) : X(1) \to X(1) \vee X(1)$ admits a homotopy co-unit (i.e. $p_i \circ X(\lambda_2) \simeq id$, where $p_i : X(1) \vee X(1) \to X(1)$ is the i-th projection, $i = 1, 2$). Again we often write X for $X(1)$.

2.5 Examples: (1) Let C be an operad without permutations in the sense of [10]. We associate a category $\mathcal{C}$ with $ob\,\mathcal{C} = \mathbb{N}$, $\mathcal{C}(n,k) = \coprod C(i_1) \times \ldots \times C(i_k)$ with $i_1 + \ldots + i_k = n$ and all $i_j > 0$. $\mathcal{C}(0,0) = \{id_0\}$ and $\mathcal{C}(0,n) = \mathcal{C}(n,0) = \emptyset$ if $n > 0$. Taking products defines a pairing $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Composition

$$\mathcal{C}(k,1) \times \mathcal{C}(n,k) \longrightarrow \mathcal{C}(n,1)$$

is defined by the structure map γ of the operad. General composition then is determined by the requirement that $\oplus$ is a bifunctor.

Call C an *A_k-operad* if $C(j) \simeq *$ for $j \leq k$. If C is an A_k-operad, there is a continuous $\oplus$-preserving functor $F_C : \mathcal{W}_k\mathcal{G} \to \mathcal{C}$ by [2, 3.17]. If F'_C is another such functor then there is a homotopy through continuous $\oplus$-preserving functors from F_C to F'_C.

A *co-C-space* is a collection of maps

$$\mu_n : X \wedge C(n)^+ \longrightarrow X \vee \ldots \vee X = X^{\vee n}$$

(n-fold wedge) where $C(n)^+ = C(n) \sqcup \{+\}$, such that $\mu_1(x,1) = x$ and

$$\begin{tikzcd}
X \wedge C(k)^+ \wedge C(i_1)^+ \wedge \ldots \wedge C(i_k)^+ \arrow[r, "\mu_k \wedge id"] \arrow[d, "\cong"] & X^{\vee k} \wedge C(i_1)^+ \wedge \ldots \wedge C(i_k)^+ \arrow[d, "proj"] \\
X \wedge (C(k) \times C(i_1) \times \ldots \times C(i_k))^+ \arrow[d, "id \wedge \gamma"] & (X \wedge C(i_1)^+) \vee \ldots \vee (X \wedge C(i_k)^+) \arrow[d, "\mu_{i_1} \vee \ldots \vee \mu_{i_k}"] \\
X \wedge C(n)^+ \arrow[r, "\mu_n"] & X^{\vee n}
\end{tikzcd}$$

commutes. It determines a *co-$\mathcal{C}$-space*, i.e. a continous functor $\mathcal{C}^{op} \to \mathcal{T}op^*$ mapping $\oplus$ to $\vee$.

If C is an A_k-operad, $k \geq 2$, then any co-C-space has a homotopy co-unit. Hence it is a co-A_k-space.

(2) Let $\mathcal{Q}$ be the "little 1-cubes" category [2, 2.53] describing loop space structures. A point in $\mathcal{Q}(n,1)$ is a tuple $(x_1, y_1, \ldots, x_n, y_n)$ of points in I such that $0 \leq x_1 < y_1 \leq x_2 < y_2 \leq \ldots \leq x_n < y_n \leq 1$, and should be considered as an embedding of n intervals $[x_i, y_i]$ into I such that the images have disjoint interiors.
Let Z be a based space. Then ΣZ is a co-$\mathcal{Q}$-space and hence a co-A_∞-space with

$$\mu_n : \Sigma Z \wedge \mathcal{Q}(n,1)^+ \longrightarrow \Sigma Z \vee \ldots \vee \Sigma Z$$

$$\mu_n((z,t),(x_1,y_1,\ldots,x_n,y_n) = \begin{cases} \left(z, \dfrac{t-x_k}{y_k-x_k}\right) \in & k\text{-the summand if } t \in [x_k, y_k] \\ \qquad * & \text{otherwise} \end{cases}$$

3 Cubical diagrams of co-A_k-spaces

Let $\mathcal{D}$ be a small category. Homotopy coherent $\mathcal{D}$-diagrams are codified by the topological category $\mathcal{WD}$ [2, Chapt. VII] with $ob\,\mathcal{WD} = ob\,\mathcal{D}$ and

$$\mathcal{WD}(A,B) = \left(\coprod_n \mathcal{D}_{n+1}(A,B) \times I^n\right) / \sim$$

where $\mathcal{D}_{n+1}(A,B)$ is the set of $(n+1)$-tuples $(f_n, \ldots, f_0)$ of morphisms in $\mathcal{D}$ such that $f_n \circ \ldots \circ f_0 : A \to B$ is defined, with the relations

$$(f_n, t_n, \ldots, f_1, t_1, f_0) = \begin{cases} (f_{n-1}, t_{n-1}, \ldots, f_0) & \text{if } f_n = id \\ (f_n, \ldots, f_{i+1}, \max(t_{i+1}, t_i), f_{i-1} \ldots, f_0) & \text{if } f_i = id \\ (f_n, t_n, \ldots, t_2, f_1) & \text{if } f_0 = id \\ (f_n, \ldots, f_{i+1} \circ f_i, t_i \ldots, f_0) & \text{if } t_{i+1} = 0. \end{cases}$$

Composition is defined by

$$(f_n, t_n, \ldots, f_0) \circ (g_p, u_p, \ldots, g_0) = (f_n, t_n, \ldots, f_0, 1, g_p, u_p \ldots, g_0)$$

$\mathcal{WD}$ can be viewed as obtained from $\mathcal{D}$ by taking the free category over the diagram $\mathcal{D}$ and putting back the relations up to coherent homotopies.

3.1 Lemma: The augmentation $\varepsilon : \mathcal{WD} \to \mathcal{D},\ (f_n, t_n, \ldots, f_0) \mapsto f_n \circ \ldots \circ f_0$ is a homotopy equivalence of morphism spaces with non-functorial homotopy inverse $\eta : \mathcal{D} \to \mathcal{WD},\ f \mapsto (f)$. [2, Prop. 3.15]

3.2 Definition: [2] A *homotopy coherent $\mathcal{D}$-diagram* is a continuous functor $D : \mathcal{WD} \to \mathcal{T}op^*$.

3.3 Notation: $\mathcal{Q}_n$ denotes the poset of all subsets of $\underline{n} = \{1, \ldots, n\}$,
$\mathcal{J}_n$ denotes the poset of all subsets of $\underline{n}$ different from $\underline{n}$
$\mathcal{T}_n$ denotes the poset of all non-empty subsets of $\underline{n}$

We are going to associate to each co-A_{k-1}-space a homotopy coherent $\mathcal{T}_k$ diagram $k \geq 2$.

Let $Inj(n, k)$ denote the set of ordered injections $\underline{n} \to \underline{k}$. We also allow $\underline{0} = \emptyset$. We enlarge $\mathcal{WG}$ by set operations coming from injections (compare [2, p. 57 ff]) to a category $\mathcal{W}^*\mathcal{G}$ as follows: $ob\, \mathcal{W}^*\mathcal{G} = ob\, \mathcal{WG}$ and

$$\mathcal{W}^*\mathcal{G}(n, k) = \coprod_{n} \mathcal{WG}(l, k) \times Inj(l, n)$$

In particular, $\mathcal{W}^*\mathcal{G}(n, 0)$ consists of a single element $(id_0, \emptyset \to n)$.
Composition of $(f, \alpha) \in \mathcal{WG}(l, k) \times Inj(l, n)$ with $(g_1 \oplus \ldots \oplus g_n, \beta) \in \mathcal{WG}(m, n) \times Inj(m, p)$ is defined by

$$(f, \alpha) \circ (g_1 \oplus \ldots \oplus g_n, \beta) = (f \circ (g_{\alpha(1)} \oplus \ldots \oplus g_{\alpha(l)}), \beta \circ \alpha(m_1, \ldots, m_n))$$

Here $g_i \in \mathcal{WG}(m_i, 1),\ m = m_1 + \ldots + m_n$, and

$$\alpha(m_1, \ldots, m_n) : \underline{m_{\alpha(1)} + \ldots + m_{\alpha(l)}} \to \underline{m}$$

is the injection sending $m_{\alpha(1)} + \ldots + m_{\alpha(i-1)} + j$ to $m_1 + m_2 + \ldots + m_{\alpha(i)-1} + j$ for $1 \leq j \leq m_{\alpha(i)}$.

The bifunctor $\oplus$ extends to $\mathcal{W}^*\mathcal{G}$ by

$$(f, \alpha) \oplus (g, \beta) = (f \oplus g, \alpha \sqcup \beta)$$

where $\alpha \sqcup \beta$ is the ordered disjoint union.

Let $\mathcal{W}_k^*\mathcal{G}$ again denote the full subcategory of $\mathcal{W}^*\mathcal{G}$ consisting of all objects $j \leq k$. Given a co-A_k-space $X : (\mathcal{W}_k\mathcal{G})^{op} \to \mathcal{T}op^*$ it extends to a continuous functor

$$X^* : (\mathcal{W}_k^*\mathcal{G})^{op} \longrightarrow \mathcal{T}op^*$$

mapping $\oplus$ to $\vee$ by

$$X^* : (f, \alpha) : X^{\vee k} \xrightarrow{f} X^{\vee l} \xrightarrow{\alpha_*} X^{\vee n}$$

for $(f, \alpha) \in \mathcal{W}\mathcal{G}(l, k) \times Inj(l, n)$, where α_* maps the i-th summand identically onto the $\alpha(i)$-th one (the checking of functoriality is left to the reader).

In an analogous way we can adjoin the injections to $\mathcal{G}$ to obtain a category $\mathcal{G}^*$. We have the following fact.

3.4 The augmentation functor $\varepsilon : \mathcal{W}\mathcal{G} \to \mathcal{G}$ and its section $\eta : \mathcal{G} \to \mathcal{W}\mathcal{G}$ extend to a functor $\varepsilon^* : \mathcal{W}^*\mathcal{G} \to \mathcal{G}^*$ and a non-functorial map $\eta^* : \mathcal{G}^* \to \mathcal{W}^*\mathcal{G}$. Again ε^* is a homotopy equivalence on morphism spaces with homotopy inverse η^*.

3.5 Let $\triangle^{inj}$ denote the category of ordered sets $[n] = \{0, 1, \ldots, n\}$ and order preserving injections and $\triangle_k^{inj}$ the full subcategory of objects $[j], j \leq k$. Define a functor

$$\vartheta_k : \mathcal{T}_k \longrightarrow \triangle_{k-1}^{inj}$$

by $\vartheta_k(\{i_1, \ldots, i_p\}) = [p-1]$. Identify $i_1 < \ldots < i_p$ in order with $0 < 1 < \ldots < p-1$. Then an inclusion $\alpha : \{i_1, \ldots, i_p\} \subset \{j_i, \ldots, j_q\}$ defines a unique order preserving map $\vartheta_k(\alpha) : [p-1] \to [q-1]$. (We always list the elements of a subset of $\underline{n}$ in increasing order.)

3.6
$$\varphi_k : \triangle_k^{inj} \longrightarrow (\mathcal{G}_k^*)^{op}$$

is defined by sending $[j]$ to j and $\alpha : [p] \to [q]$ to $\left(\bigoplus_{i=1}^{p} \lambda_{\alpha(i)-\alpha(i-1)}, \sigma_\alpha\right)$, where $\sigma_\alpha : \underline{\alpha(p) - \alpha(0)} \to \underline{q}$ is the ordered injection missing the elements $\{1, \ldots, \alpha(0), \alpha(p)+1, \ldots, q\}$

3.7 Let $X : (\mathcal{W}_{k-1}\mathcal{G})^{op} \to \mathcal{T}op^*$ be a co-A_{k-1}-space. To simplify our future argument we explicitly construct the 2-faces of a homotopy $\mathcal{T}_k$-diagram

$$\psi_k : \mathcal{W}\mathcal{T}_k \xrightarrow{\theta_k} (\mathcal{W}_{k-1}^*\mathcal{G})^{op} \xrightarrow{X^*} \mathcal{T}op^*$$

before we apply the universal property of $\mathcal{WT}_k$. Since ψ_k factors through X^* it is functorial in X. On objects θ_k is the composite $\varphi_{k-1} \circ \vartheta_k$, and on morphisms $\eta(f)$ it is given by $\eta^* \circ \varphi_{k-1} \circ \vartheta_k(f)$, where $f \in \mathcal{T}_k$ and $\eta : \mathcal{T}_k \to \mathcal{WT}_k$, $\eta^* : (\mathcal{G}^*_{k-1})^{op} \to (\mathcal{W}^*_{k-1}\mathcal{G}^*)^{op}$ are the sections. Finally we specify the "homotopies" of 2-dimensional faces defined by $\{i_1, \ldots, i_p\} \subset \{j_1, \ldots, j_{p-2}\}$ in $\mathcal{T}_k$. Let $\iota_r : 1 \to 2$ denote the injection missing the r-th element.
If $j_1 < j_2 < i_1$ the face is mapped to the commutative square

(3.7.1)

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "\lambda_2"] \\ 1 \arrow[r, "\iota_1"] & 2 \end{tikzcd} \qquad \oplus\, id_p$$

If $j_1 < i_1 < i_r < j_q < i_{r+1}$ the face is mapped to the commutative square

(3.7.2)

$$\begin{tikzcd} r \arrow[r, "\iota_1 \oplus id_{r-1}"] \arrow[d, "id_{r-1} \oplus \lambda_2"] & r+1 \arrow[d, "id_r \oplus \lambda_2"] \\ r+1 \arrow[r, "\iota_1 \oplus id_r"] & r+2 \end{tikzcd} \qquad \oplus\, id_{p-r-1}$$

If $i_r < j_q < i_{r+1} < i_s < j_t < i_{s+1}$ the face is mapped to the commutative square

(3.7.3)

$$id_{r-1} \oplus \qquad \begin{tikzcd} s-r+1 \arrow[r, "\lambda_2 \oplus id_{s-r}"] \arrow[d, "id_{s-r} \oplus \lambda_2"] & s-r+2 \arrow[d, "id_{s-r+1} \oplus \lambda_2"] \\ s-r+2 \arrow[r, "\lambda_2 \oplus id_{s-r+1}"'] & s-r+3 \end{tikzcd} \qquad \oplus\, id_{p-s-1}$$

If $i_r < j_q < j_{q+1} < i_{r+1}$ the face is mapped to the diagram

(3.7.4)

$$id_{r-1} \oplus \quad \begin{array}{ccc} 1 & \xrightarrow{\lambda_2} & 2 \\ {\scriptstyle\lambda_2}\downarrow & \searrow^{\lambda_3} & \downarrow{\scriptstyle\lambda_2 \oplus id} \\ 2 & \xrightarrow[id \oplus \lambda_2]{} & 3 \end{array} \quad \oplus\, id_{p-r-1}$$

and the associating homotopies in $\mathcal{W}^*_{k-1}\mathcal{G}$.

The faces with $i_r < i_p < j_{p+1}$ and $i_p < j_{p+1} < j_{p+2}$ are mapped analogously to (3.7.2) and (3.7.1) with ι_1 replaced by ι_2 and the orders with respect to $\oplus$ reserved.

Having specified θ_k on the 2-dimensional faces we extend it over all of $\mathcal{WT}_k$ by applying the relative lifting theorem [2, 3.17].

3.8 Lemma: Let X be a 1-connected co-A_{k-1}-space, $k \geq 3$. Then for each $(k-1)$-dimensional face $\mathcal{S}$ of $\mathcal{T}_k$ the homotopy coherent $\mathcal{S}$-diagram

$$\psi_k|\mathcal{WS} : \mathcal{WS} \longrightarrow \mathcal{T}op^*$$

is strongly (homotopy) co-Cartesian in the sense of [6, 2.1].

Proof: We have to show that the specified faces of the functor θ_k are homotopy pushouts after composition with X^*. This is trivially true for (3.7.2) and (3.7.3): Given maps $f : A \to B$ and $g : C \to D$ in $\mathcal{T}op^*$, the square

$$\begin{array}{ccc} A \vee C & \xrightarrow{f \vee id} & B \vee C \\ \downarrow{\scriptstyle id \vee g} & & \downarrow{\scriptstyle id \vee g} \\ A \vee D & \xrightarrow{f \vee id} & B \vee D \end{array}$$

is always a homotopy pushout. For (3.7.1) and the corresponding one with ι_1 replaced ι_2 the result holds by [18, Lemma 3.3]. For (3.7.4) we use the fact that a 1-connected homotopy associative co-H-space which is a CW-complex has a 2-sided inverse [5, Prop.3.6]. $\square$

4 Proof of Theorem 1.2

4.1 Lemma: The results of Ellis-Steiner [4] and Goodwillie listed in [6, Thm. 2.3,..., Thm. 2.6], which are formulated for commutative cube diagrams, also apply to coherently homotopy commutative cube diagrams. (A stronger version of [6, Thm. 2.4] for homotopy commutative cube diagrams can be found in [17, Prop. 5.5]).

Proof: By [15, Prop. 5.4] each coherently homotopy commutative cube is equivalent to a strictly commutative one, and the equivalence induces an equivalence of their homotopy limits and colimits. $\square$

4.2 Construction: Let $\psi_l : \mathcal{WT}_l \to \mathcal{T}op^*$, $2 \le l \le k$ be the cube diagrams determined by the co-A_{k-1}-space X. Let $\mathcal{S}_{k-1} \subset \mathcal{T}_k$ be the face defined by all subsets of $\underline{k}$ containing 1, and $\mathcal{TS}_{k-1}$ obtained from it by deleting the initial vertex. Let P_l denote the homotopy limit (in the sense of [15]) of the coherently homotopy commutative $\mathcal{T}_l$-diagram ψ_l and R_{k-1} the one of the subdiagram $\mathcal{WTS}_{k-1} \to \mathcal{T}op^*$. Observe that P_l and R_{k-1} are well-pointed by [15, Prop. 6.9]. Then P_k is the homotopy limit of the induced diagram

$$* = \psi_k\{1\} \longrightarrow R_{k-1} \longleftarrow P_{k-1}$$

(compare [17, Prop. 5.4]). In particular, we have a sequence of fibrations

$$P_k \xrightarrow{p_k} P_{k-1} \xrightarrow{p_{k-1}} \cdots \xrightarrow{p_3} P_2 = \Omega X$$

with fiber$(p_k) = \Omega R_{k-1}$. Since $\psi_k | \mathcal{WS}_{k-1}$ is strongly co-Cartesian and each map $\psi_k(\underline{k}\backslash\{i\} \to \underline{k}) : X^{\vee(k-2)} \to X^{\vee(k-1)}$ is $(n-1)$-connected the induced map $* \to R_{k-1}$ is $(1-(k-1)+(k-1)(n-1)) = (1+(k-1)(n-2))$-connected. Since $n \ge 2$, R_{k-1} is at least 1-connected and hence each P_k connected.

P_k together with the universal map $P_k \to \psi_k$ extends ψ_k to a $\mathcal{WQ}_k$-diagram $\Theta_k : \mathcal{WQ}_k \to \mathcal{T}op^*$, i.e. to a homotopy coherent k-dimensional cube diagram. Let $\tilde{\mathcal{T}} \subset \mathcal{Q}_k$ be an l-dimensional terminal face, $l < k$. By (3.8) it is strongly co-Cartesian, and hence by [4] $(1 + l(n-2))$-Cartesian. Since Θ_k is infinitely Cartesian it is $(k - 1 + 2 + k(n-2))$-co-Cartesian by [6, Thm. 2.6]. Recall that $\mathcal{J}_l \subset \mathcal{Q}_k$ is the subcategory of all subsets of $\underline{l} \subset \underline{k}$, $l \le k$, except of $\underline{l}$

itself. Let $\mathcal{I}_{l-1} \subset \mathcal{J}_l$ denote the subcategory of all subsets containing 1 and $\mathcal{J}\mathcal{I}_{l-1}$ the full subcategory of $\mathcal{I}_{l-1}$ containing all but the terminal element.

Let M_l denote the homotopy colimit of $\Theta_k|\mathcal{W}\mathcal{J}_l$. By (3.8) the homotopy colimit of $\Theta_k|\mathcal{W}\mathcal{J}\mathcal{I}_{l-1}$ is $X^{\vee(l-1)}$ and the induced map to the terminal vertex of $\mathcal{I}_{l-1}$ is a homotopy equivalence. By [17, Prop. 5.4] we have homotopy pushout diagrams

4.3

$$\begin{tikzcd} M_{l-1} \arrow[r, "r_{l-1}"] \arrow[d, "q_l"'] & X^{\vee(l-2)} \arrow[d] \\ X^{\vee(l-1)} \arrow[r, "g_l"] & M_l \end{tikzcd}$$

for $3 \leq l \leq k$, where $r_l : M_l \longrightarrow X^{\vee(l-1)}$ is the induced map into the terminal vertex of $\Theta_k|\mathcal{W}\mathcal{Q}_l$. Then, as noted above, $r_l \circ g_l$ is a homotopy equivalence. Since Θ_k is $(k(n-2)+k+1)$-co-Cartesian, r_k is $(k+k(n-2))$-connected, hence g_k is $(k(n-2)+k)$-connected. Since (4.3) is a homotopy pushout, r_{k-1} is homology-$(k(n-2)+k)$-connected. By downwards induction we obtain

$$r_2 : \Sigma P_k = M_2 \longrightarrow X$$

is homology $(k(n-2)+3)$-connected. Since ΣP_k and X are both 1-connected, this implies that r_2 is $(k(n-2)+3)$-connected.

The final diagram 4.3 is obtained from the cube

$$\begin{tikzcd} P_k \arrow[rr] \arrow[dr] \arrow[dd] & & * \arrow[dr] \arrow[dd] & \\ & * \arrow[rr] \arrow[dd] & & X \arrow[dd, "\mu"'] \\ * \arrow[rr] \arrow[dr] & & X \arrow[dr] & \\ & X \arrow[rr] & & X \vee X \end{tikzcd}$$

By definition, q_l factors through the standard pinch map and a wedge so that we arrive at a homotopy commutative diagram

$$
\begin{tikzcd}
\Sigma P_k \arrow[r, "r_2"] \arrow[d, "pinch"] & X \arrow[dd] \arrow[ddr, "\mu"] & \\
\Sigma P_k \vee \Sigma P_k \arrow[d, "f \vee g"] & & \\
X \vee X \arrow[r, "g_3"] & M_3 \arrow[r, "r_3"] & X \vee X
\end{tikzcd}
$$

Since $r_3 \circ g_3 \simeq id$ and the pinch map and μ have co-units we deduce $f \simeq r_2 \simeq g$. Hence r_2 is a homomorphism of co-H-spaces up to homotopy (to obtain higher coherence one has to include the higher dimensional cubes into the argument; we leave this to the reader). We summarize

4.4 Proposition: Let X be an $(n-1)$-connected co-A_{k-1}-space, $n \geq 2$, $k \geq 3$. Let P_k be the homotopy limit of its associated coherently homotopy commutative $\mathcal{T}_k$-diagram. Then the induced map

$$r : \Sigma P_k \longrightarrow X$$

is a $(k \cdot (n-2)+3)$-connected co-A_{k-1}-homomorphism.

Proof of the Theorem 1.2: Let K be the CW-approximation of P_k, and suppose that X is a CW-complex with $\dim X \leq k(n-2)+3$. Since $H_{k(n-2)+3}X$ is free abelian and r induces a map $q : \Sigma K \to X$ which is a homology isomorphism in dimensions less than $k(n-2)+3$ and a homology epimorphism in dimensions $k(n-2)+3$, there is a CW-complex Y having the same $(k(n-2)+1)$-skeleton as K by [1, Thm. 2.1] and a homology isomorphism $\Sigma Y \to X$. Since ΣY and X are 1-connected this is a homotopy equivalence.

5 Proof of Theorem 1.4

Let $X : \triangle \to \mathcal{T}op^*$ be a based cosimplicial space such that X_0 is contractible, X_1 is 2-connected, and $(\pi_1, \ldots, \pi_n) : \bigvee_{k=1}^{n} X_1 \to X_n$ is a homotopy equivalence. Define

$$\mu : X \xrightarrow{d^1} X_2 \longrightarrow X \vee X$$

where the second map is a homotopy inverse of $(\pi_1, \pi_2) = (d^2, d^0)$. (In abuse of notation we write X for X_1). Then (X, μ) is a co-H-space because

$$\begin{tikzcd}
X_2 \arrow[d, "s^1"'] & X \arrow[l, "d^1"'] \arrow[r, "d^1"] \arrow[d, "\mu"'] & X_2 \arrow[d, "s^0"] \\
X & X \vee X \arrow[l, "proj_1"] \arrow[r, "proj_1"'] \arrow[ul, "{(d^2, d^0)}"'] \arrow[ur, "{(d^2, d^0)}"] & X
\end{tikzcd}$$

commutes up to homotopy since X_0 is contractible. In a similar way one verifies that (X, μ) is homotopy associative.

5.1 Lemma: Each $(k-1)$-dimensional face of $X \circ \vartheta_k : \mathcal{T}_k \to \triangle_{k-1}^{inj} \to \mathcal{T}op^*$ is strongly homotopy co-Cartesian.

Proof: The 2-dimensional face determined by $\alpha : \{i_1, \ldots, i_p\} \subset \{j_1, \ldots, j_{p+2}\}$ is mapped to

$$\begin{tikzcd}
X_{p-1} \arrow[r, "d^q"] \arrow[d, "d^{r-1}"] & X_p \arrow[d, "d^r"] \\
X_p \arrow[r, "d^q"] & X_{p+1}
\end{tikzcd}$$

if α misses $j_{q+1} < j_{r+1}$. For $0 < i < p+1$ consider

$$\begin{tikzcd}
 & X_{i-1} \vee X \vee X_{p-i} \arrow[r, "\simeq"] \arrow[dl, "id \vee \mu \vee id"'] \arrow[d, "id \vee d^1 \vee id"] & X_p \arrow[d, "d^i"] \\
X_{i-1} \vee (X \vee X) \vee X_{p-i} \arrow[r, "id \vee (d^2, d^0) \vee id"] & X_{i-1} \vee X_2 \vee X_{p-i} \arrow[r, "\simeq"] & X_{p+1}
\end{tikzcd}$$

where the horizontal equivalences are induces by the inclusions of $[i-1]$, the map π_i respectively (π_i, π_{i+1}) and the injections of $[p-i]$ onto the terminal parts of $[p]$ respectively $[p+1]$. The square commutes while the triangle commutes up to homotopy. For d^0 and d^{p+1} the corresponding squares are

$$\begin{tikzcd}
X \vee X_p \arrow[d, "(\pi_1, d^0)"', "\simeq"] & X_p \arrow[l, "incl."'] \arrow[r, "incl."] \arrow[d, equal] & X_p \vee X \arrow[d, "\simeq"', "(d^{p+1}, \pi_{p+1})"] \\
X_{p+1} & X_p \arrow[l, "d^0"'] \arrow[r, "d^{p+1}"] & X_{p+1}
\end{tikzcd}$$

Hence, up to identities on wedge summands, we arrive at the same diagrams as in the proof of Theorem 1.2. □

Proof of Corollary 1.3 and Theorem 1.4: Let P_k denote the homotopy limit of $X \circ \vartheta_k$. As in the proof of Theorem 1.2 we obtain a sequence of fibrations

$$\cdots \longrightarrow P_4 \xrightarrow{p_4} P_3 \xrightarrow{p_3} P_2 \simeq \Omega X$$

such that $p_k : P_k \to P_{k-1}$ is k-connected because X is 2-connected, and a $(k+3)$-connected map $r_k : \Sigma P_k \to X$. Let P be the homotopy limit of the P_k. Since $\lim^1 \pi_i(P_k) = 0$ we have

$$\pi_k(P) \cong \lim \pi_k(P_i) \cong \pi_k(P_k).$$

In particular, the natural projection $q_k : P \to P_k$ is k-connected. The maps $r : \Sigma P_k \to X$ are compatible with the p_k up to homotopy and induce a map

$$\Sigma P \longrightarrow X.$$

Since $\Sigma P \to \Sigma P_k \to X$ is $(k+1)$-connected, this map is a weak equivalence.

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