MULTIDIMENSIONAL VERSION OF THE RESULTS OF KOMLÓS, MAJOR AND TUSNÁDY FOR VECTORS WITH FINITE EXPONENTIAL MOMENTS

A. Yu. Zaitsev

St. Petersburg Branch of Steklov Mathematical Institute

Revised version, July 1995

Abstract. A multidimensional version of the results of Komlós, Major and Tusnády for the Gaussian approximation of the sequence of successive sums of independent random vectors with finite exponential moments is obtained.

0. Introduction

The statement of the problem is well known. It is required to construct on a probability space a sequence of independent random vectors $X_1, \dots, X_n$ and a corresponding sequence of independent Gaussian random vectors $Y_1, \dots, Y_n$ (this means that Y_i has the same mean and the same covariance operator as X_i, $i = 1, \dots, n$) so that the quantity

$$\Delta(X, Y) = \max_{1 \leqslant k \leqslant n} \left| \sum_{i=1}^{k} X_i - \sum_{i=1}^{k} Y_i \right| \tag{1}$$

would be so small as possible with large probability. Here

$$|x| = \max_{1 \leqslant j \leqslant d} |x_j|, \qquad x = (x_1, \dots, x_d) \in \mathbb{R}^d. \tag{2}$$

1991 *Mathematics Subject Classification.* Primary 62F17; secondary 60F15.

Key words and phrases. multidimensional invariance principle, strong approximation, sums of independent random vectors.

Research supported by the Alexander von Humboldt Foundation, by the Russian Foundation of Fundamental Researches, grant 93-11-01454, and by the International Science Foundation, grant R3600.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Sometimes we shall use other notations for summands and, by analogy with [21], write

$$\Delta(\widetilde{X},\widetilde{Y}) = \max_{1\leqslant k\leqslant n} \left| \sum_{i=1}^{k} \widetilde{X}_i - \sum_{i=1}^{k} \widetilde{Y}_i \right| \tag{3}$$

and so on.

In some sense this problem is one of the most important in probability approximations because many well-known probability theorems can be considered as consequences of results about strong approximation of sequences of sums by corresponding Gaussian sequences. This is related to the law of iterated logarithm, to several theorems about large deviations, to the estimates for the rate of convergence of the Prokhorov distance in the invariance principles [4, 15, 24] as well as to the Strassen-type approximations [26] (see, for example, [6]).

The rate of approximation in the invariance principle was investigated by many authors (see, e.g., [4, 15, 24] and the bibliography in [5, 6, 16–18]). Skorokhod [24] worked out a powerful method known now as Skorokhod embedding. For a long time it seemed that it is impossible to propose a method which could be better. However, in 1975, Komlós, Major and Tusnády (KMT) worked out a new method of dyadic approximation [10, 11]. With the help of this method they obtained an exhaustive solution of the problem mentioned above for sequences of independent identically distributed random variables. Sakhanenko [21] generalized and essentially improved KMT results for the case of non-identically distributed random variables.

The first attempts to extend the KMT approximations to the multidimensional case (see [2, 3, 7, 8, 14]) had a partial success only. Comparatively recently U. Einmahl [9] obtained multidimensional analogs of KMT results which are close to optimal. However, in his results (obtained in general case for the vectors with finite exponential moments) there is an unnecessary logarithmic factor. In this paper we improve the result of Einmahl [9] and obtain multivariate analogs of KMT results.

For formulations of results we need some notations. Writing $z \in \mathbb{R}^d$ (resp. $\mathbb{C}^d$), we always keep in mind that $z = (z_1, \ldots, z_d) = z_1 e_1 + \cdots + z_d e_d$, where $z_j \in \mathbb{R}^1$ (resp. $\mathbb{C}^1$) and the e_j are the basis vectors. The scalar product of vectors $x, y \in \mathbb{R}^d$ (resp. $\mathbb{C}^d$) is denoted by $\langle x, y\rangle = x_1\overline{y}_1 + \cdots + x_d\overline{y}_d$. For $z \in \mathbb{R}^d$ (or $\mathbb{C}^d$) we shall use the Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$ and the maximum norm $|z| = \max\limits_{1\leqslant j\leqslant k} |z_j|$. For $b > 0$ we denote $\log^+ b = \max\{1, \log b\}$. The distribution and the corresponding covariance operator of a random vector ξ will be denoted by $\mathcal{L}(\xi)$ and $\operatorname{cov}\xi$ (or $\operatorname{cov} F$, if $F = \mathcal{L}(\xi)$). The symbol I will be used for the identity operator. The formulas are numbered in each section separately. When we refer to the formula from another section, we indicate in addition the number of this section. For example, (1.2) is the formula (2) from Section 1.

In [30] the author has introduced a new class $\mathcal{A}_d(\tau)$ of d-dimensional probability distributions depending on a parameter $\tau \geqslant 0$. The class $\mathcal{A}_d(\tau)$ consists of d-dimen-

sional distributions F for which the function

$$\varphi(z) = \varphi(F, z) = \log \int_{\mathbb{R}^d} e^{\langle z, x\rangle} F\{dx\} \qquad (\varphi(0) = 0) \tag{4}$$

is defined and analytic for $\|z\|\,\tau < 1,\ z \in \mathbb{C}^d,$ and

$$\bigl|d_u d_v^2\,\varphi(z)\bigr| \leqslant \|u\|\,\tau\,\bigl\langle Dv, v\bigr\rangle \tag{5}$$

for all $u,\ v \in \mathbb{R}^d$ and $\|z\|\,\tau < 1,$ where $D = \operatorname{cov} F,$ and the derivative d_u of φ in the direction u is given by the formula

$$d_u\varphi(z) = \lim_{\alpha\to 0}\ \frac{\varphi(z+\alpha\,u) - \varphi(z)}{\alpha}\ . \tag{6}$$

In Section 1 we consider the properties of classes $\mathcal{A}_d(\tau)$ and the connection of $\mathcal{A}_d(\tau)$ with other classes of probability distributions.

The following theorem is the main result of the paper.

Theorem 1. *Suppose that $\tau \geqslant 1,\ \alpha > 0$ and $\xi_1, \dots, \xi_n$ are random vectors with distributions $\mathcal{L}(\xi_k) \in \mathcal{A}_d(\tau),\ \mathbf{E}\,\xi_k = 0,\ \operatorname{cov}\xi_k = I,\ \ k = 1, \dots, n.$ Then one can construct on a probability space a sequence of independent random vectors $X_1, \dots, X_n$ and a corresponding sequence of independent Gaussian random vectors $Y_1, \dots, Y_n$ so that $\mathcal{L}(X_k) = \mathcal{L}(\xi_k),\ k = 1, \dots, n,$ and*

$$\mathbf{E}\,\exp\Bigl(\frac{c_1(\alpha)\,\Delta(X,Y)}{\tau\, d^3\,\log^+ d}\Bigr) \leqslant \exp\bigl(c_2(\alpha)\,d^{9/4+\alpha}\log^+ n\bigr), \tag{7}$$

where $c_1(\alpha)$, $c_2(\alpha)$ are quantities depending only on α.

Corollary 1. *In the conditions of Theorem 1 for all $x \geqslant 0$ the following inequality is valid*

$$\mathbf{P}\Bigl\{\,\Delta(X,Y) \geqslant \frac{c_2(\alpha)\,\tau\, d^{21/4+\alpha}\log^+ d\,\log^+ n}{c_1(\alpha)} + x\Bigr\} \leqslant \exp\Bigl(-\frac{c_1(\alpha)\,x}{\tau\, d^3\,\log^+ d}\Bigr). \tag{8}$$

It is evident that Corollary 1 can be easily derived from Theorem 1 with the help of the Chebyshev inequality. Corollary 1 is more usual version of the formulation of KMT-type results. However, Theorem 1 is more compact and more natural statement than Corollary 1, because the inequality (7) does not include the additional parameter x. Therefore, in Theorems 2–5 below, we formulate only the analogs of the inequality (7) and omit the analogs of the inequality (8). Note that such an approach was earlier used by Sakhanenko [21]. The formulation of his main result is also expressed in terms of some parameter entering in the conditions imposed on

the distributions of summands. In the particular case, when $d = 1$ and all summands have a common variance, Theorem 1 is equivalent to the main result of [21] (see (1.35), (1.36), Lemma 1.5)[1]. Einmahl [9] considered multidimensional vectors with identical covariance operators, satisfying multidimensional analogs of Sakhanenko's conditions (see (16)).

One can easily verify that if a vector ξ has finite exponential moments $\mathbf{E}\, e^{\langle h,\xi\rangle}$, for $h \in V$, where $V \subset \mathbb{R}^d$ is some neighborhood of zero, then $F = \mathcal{L}(\xi) \in \mathcal{A}_d(c(F))$. Therefore, Theorem 1 can be considered as a generalization and refinement of the main result of KMT [10, 11]. In particular, from Theorem 1 one can easily derive the following result, obtained by KMT [10, 11] in the one-dimensional case.

Corollary 2. *Suppose that a random vector ξ has finite exponential moments $\mathbf{E}\, e^{\langle h,\xi\rangle}$, for $h \in V$, where $V \subset \mathbb{R}^d$ is some neighborhood of zero. Then one can construct on a probability space a sequence of independent random vectors $X_1, X_2, \ldots$ and a corresponding sequence of independent Gaussian random vectors $Y_1, Y_2, \ldots$ so that $\mathcal{L}(X_k) = \mathcal{L}(\xi)$, $k = 1, 2 \ldots$, and*

$$\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} Y_j = O(\log n) \qquad a.s. \tag{9}$$

An analog of this result was obtained by Einmahl [9] under additional smoothness-type restrictions on the distributions $\mathcal{L}(\xi)$. Einmahl has also proved an analog of Theorem 1 (see [9, Theorem 10]), but only for sufficiently smooth distributions $\mathcal{L}(\xi)$. As it is noted in [10], from the results of Bártfai [1] it follows that the rate of convergence in (9) is the best possible.

The following Theorems 2 and 3 will be used in the proof of Theorems 1 and 4. However, they are of independent interest because sometimes they can give a more exact dependence of constants on the characteristics of distributions of summands. The complication of the formulation of Theorem 3 in comparison with that of Theorem 2 is necessary to make more precise the dependence of the denominators of the fractions in the left-hand sides of inequalities (7), (15) on τ.

Theorem 2. *Suppose that $\tau \geqslant 1$, $\lambda \geqslant 1$, $\alpha > 0$ and $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n$ are random vectors with distributions $\mathcal{L}(\widetilde{\xi}_k) \in \mathcal{A}_d(\tau)$, $\mathbf{E}\,\widetilde{\xi}_k = 0$, $\operatorname{cov}\widetilde{\xi}_k = I$, $k = 1, \ldots, n$. Assume that there exist random vectors $\widetilde{\eta}_k$ which have the same moments of the first three orders as the vectors $\widetilde{\xi}_k$, $\mathcal{L}(\widetilde{\eta}_k) \in \mathcal{A}_d(\tau)$ and $\mathbf{P}\left\{|\widetilde{\eta}_k| \leqslant \lambda\right\} = 1$, $k = 1, \ldots, n$. Then one can construct on a probability space a sequence of independent random*

[1] One should take into account that, using the Hölder inequality, we can transfer the factors under exponential sign in the right-hand side of (7) to the denominator of the fraction in the left-hand side of (7). Naturally, this makes the result weaker.

vectors $\widetilde{X}_1, \dots, \widetilde{X}_n$ *and a corresponding sequence of independent Gaussian random vectors* $Y_1, \dots, Y_n$ *so that* $\mathcal{L}(\widetilde{\xi}_k) = \mathcal{L}(\widetilde{X}_k)$, $k = 1, \dots, n$, *and*

$$\mathbf{E} \exp\Big(\frac{c_3(\alpha)\, \Delta(\widetilde{X}, Y)}{\lambda d^3 \tau^2} \Big) \leqslant \exp\Big(c_4(\alpha)\, (\lambda d^{3/2} \tau)^{3/2+\alpha} \log^+ n \Big), \tag{10}$$

where $c_3(\alpha)$, $c_4(\alpha)$ *are quantities depending only on* α.

Theorem 3. *Suppose that* $\tau_1 \geqslant \tau_2 \geqslant 1$, $\tau_3 \geqslant \tau_2$, $\lambda \geqslant 1$, $\alpha > 0$ *and* $\xi_1, \dots, \xi_n$ *are random vectors with distributions* $\mathcal{L}(\xi_k) \in \mathcal{A}_d(\tau_1)$, $\mathbf{E}\, \xi_k = 0$, $\operatorname{cov} \xi_k = I$, $k = 1, \dots, n$. *Let* $n = 2^N \geqslant \tau_1^2 \tau_2^{-2}$ *for some integer* N *and let the integer* $M \leqslant N$ *be defined by the relation*

$$\tau_1^2 \leqslant 2^M \tau_2^2 < 2\tau_1^2. \tag{11}$$

Denote

$$\widetilde{\xi}_m = 2^{-M/2} \big(\xi_{(m-1)\cdot 2^M + 1} + \dots + \xi_{m \cdot 2^M} \big), \qquad m = 1, \dots, 2^{N-M}. \tag{12}$$

Assume that there exist random vectors $\widetilde{\eta}_m$, $m = 1, \dots, 2^{N-M}$, *which have the same moments of the first three orders as* $\widetilde{\xi}_m$, $\mathcal{L}(\widetilde{\eta}_m) \in \mathcal{A}_d(\tau_3)$, $\mathbf{P}\Big\{ \big|\widetilde{\eta}_m\big| \leqslant \lambda \Big\} = 1$. *Then one can construct on a probability space a sequence of independent random vectors* $X_1, \dots, X_n$ *and a corresponding sequence of independent Gaussian random vectors* $Y_1, \dots, Y_n$ *so that* $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$, $k = 1, \dots, n$, *and*

$$\mathbf{E} \exp\Big(\frac{c_5(\alpha)\, \tau_2\, \Delta(X, Y)}{\lambda d^3\, \tau_1\, \tau_3^2} \Big) \leqslant \exp\Big(c_6(\alpha)\, (\lambda d^{3/2} \tau_3)^{3/2+\alpha} \log^+ n \Big), \tag{13}$$

where $c_5(\alpha)$, $c_6(\alpha)$ *are quantities depending only on* α.

The following Theorem 4 shows that if all moments of the third order of the vectors $\xi_1, \dots, \xi_n$ in Theorem 1 are equal to zero, then the dependence of constants on the dimension can be slightly sharpened.

Theorem 4. *Suppose that* $\tau \geqslant 1$, $\alpha > 0$ *and* $\xi_1, \dots, \xi_n$ *are random vectors with distributions* $\mathcal{L}(\xi_k) \in \mathcal{A}_d(\tau)$, $\mathbf{E}\, \xi_k = 0$, $\operatorname{cov} \xi_k = I$, $k = 1, \dots, n$. *Assume that for all* $u, v, w \in \mathbb{R}^d$ *we have*

$$\mathbf{E} \big\langle \xi_k, u \big\rangle \big\langle \xi_k, v \big\rangle \big\langle \xi_k, w \big\rangle = 0, \qquad k = 1, \dots, n. \tag{14}$$

Then one can construct on a probability space a sequence of independent random vectors $X_1, \dots, X_n$ *and a corresponding sequence of independent Gaussian random vectors* $Y_1, \dots, Y_n$ *so that* $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$, $k = 1, \dots, n$, *and*

$$\mathbf{E} \exp\Big(\frac{c_7(\alpha)\, \Delta(X, Y)}{d^3\, \tau} \Big) \leqslant \exp\Big(c_8(\alpha)\, d^{9/4+\alpha} \log^+ n \Big), \tag{15}$$

where $c_7(\alpha)$, $c_8(\alpha)$ *are quantities depending only on* α.

Remark 0.1. It is evident that the condition (14) is automatically fulfilled if the vectors $\xi_1, \dots, \xi_n$ have symmetric distributions.

Theorem 5 below was formulated without proof in [33] (with change of (18) by corresponding analog of (8)). Theorem 5 can be considered as a refinement of the main result of Einmahl [9] (in [9] the character of dependence of constants on the dimension d is not clarified). One can verify that Einmahl's [9] multidimensional version of one-dimensional Sakhanenko's conditions [21] (see (1.36))

$$\mathbf{E}\,\xi_j = 0, \quad \operatorname{cov}\xi_j = I, \quad \beta\,\mathbf{E}\,\|\xi_j\|^3 \exp\bigl(\beta\,\|\xi_j\|\bigr) \leqslant 1, \quad \beta < {}^1\!/_2, \quad j = 1,\dots,n, \tag{16}$$

(see [9, (4.1)–(4.3)]) imply that, for all $u, v \in \mathbb{R}^d$, $m = 3, 4\dots$,

$$\begin{aligned}\bigl|\,\mathbf{E}\,\bigl\langle\xi_j, v\bigr\rangle^2\bigl\langle\xi_j, u\bigr\rangle^{m-2}\bigr| &\leqslant \frac{\|v\|^2\,\|u\|^{m-2}}{\beta^{m-2}}\,\frac{(m-3)!\,\mathbf{E}\,\beta\,\|\xi_j\|^3\,(\beta\,\|\xi_j\|)^{m-3}}{(m-3)!}\\ &\leqslant (m-3)!\,\frac{\|v\|^2\,\|u\|^{m-2}}{\beta^{m-2}}\,\beta\,\mathbf{E}\,\|\xi_j\|^3\exp\bigl(\beta\,\|\xi_j\|\bigr)\\ &\leqslant \frac{1}{2}\,m!\,\tau^{m-2}\,\|u\|^{m-2}\,\mathbf{E}\,\bigl\langle\xi_j, v\bigr\rangle^2, \quad j = 1,\dots,n,\end{aligned} \tag{17}$$

with $\tau = \beta^{-1} > 2$. According to Lemma 1.5 (see Section 1), this implies that $\mathcal{L}(\xi_j) \in \mathcal{A}_d(c_9\,\tau)$ with an absolute positive constant c_9. Therefore, Theorem 11 from the paper by Einmahl [9] is a simple consequence of Theorem 5 of this paper. Note that Theorem 5 (as well as the corresponding version of Theorem 11 from [9]) has the factor $\log^+ n$ in the denominator of the fraction in the left-hand sides of inequality (18). This factor is absent in the corresponding inequality (7) of Theorem 1. However, Theorem 5 is of independent interest because the inequality (18) can be stronger than (7), e.g., if $\log^+ n < d^{3/2}\log^+ d$.

Theorem 5. *Suppose that $\tau \geqslant 1$ and $\widetilde{\xi}_1,\dots,\widetilde{\xi}_n$ are independent random vectors such that $\mathcal{L}(\widetilde{\xi}_j) \in \mathcal{A}_d(\tau)$, $\mathbf{E}\,\widetilde{\xi}_j = 0$, $\operatorname{cov}\widetilde{\xi}_j = I$, $j = 1,\dots,n$. Then one can construct on a probability space a sequence of independent random vectors $\widetilde{X}_1,\dots,\widetilde{X}_n$ and a corresponding sequence of independent Gaussian random vectors $\widetilde{Y}_1,\dots,\widetilde{Y}_n$ so that $\mathcal{L}(\widetilde{X}_j) = \mathcal{L}(\widetilde{\xi}_j)$, $j = 1,\dots,n$, and*

$$\mathbf{E}\,\exp\Bigl(\frac{c_{10}\,\Delta(\widetilde{X},\widetilde{Y})}{d^{3/2}\,\tau\,\log^+ n}\Bigr) \leqslant \exp\bigl(c_{11}\,\log^+ d\,\log^+ n\bigr), \tag{18}$$

where c_{10}, c_{11} are absolute positive constants.

In Theorems 1–5 the random vectors $\xi_1,\dots,\xi_n$ or $\widetilde{\xi}_1,\dots,\widetilde{\xi}_n$ are, generally speaking, non-identically distributed. However, they have the same covariance operator I. Therefore, the problem of obtaining the adequate multidimensional generalization of the main result of Sakhanenko [21] remains open. By the author's opinion, the proof of Theorem 1 can be transformed in the proof of an analogous result for vectors with non-identical covariance operators. The conditions $\operatorname{cov}\xi_j = I$, $\tau \geqslant 1$ can be apparently changed by $\sigma_j^2 \leqslant \tau^2$, $\sigma_j^2 \leqslant L\,s_j^2$, where σ_j^2, s_j^2 are respectively the maximal

and the minimal eigenvalues of $\operatorname{cov}\xi_j$, $j = 1, \dots, n$. The constants will depend, naturally, on L, and n should be changed in the right-hand side of (7) by the maximal eigenvalue of the covariance operator of the last sum $\xi_1 + \dots + \xi_n$. The author hopes to consider this situation in a separate paper. Besides, it is well known that the results similar to Theorems 1–5 imply many useful consequences. For example, one can derive estimates in the invariance principle in the case when the summands X_j satisfy less restrictive moment conditions (see, e.g., [16] and the bibliography in [16]). The author is going to devote to the consequences of Theorems 1–5 also a separate publication.

Note that Massart [12] has extended the KMT-type approach to the estimation of the rate of approximation in the invariance principle for set-indexed partial sums of random variables. Rio [17, 18] has used the quantiles estimates of Einmahl [9] to investigate the corresponding multidimensional situation.

The proof of Theorems 1–5 consists of several steps, many of which are close to corresponding steps from the proofs of the main results of [9, 10, 11, 21]. In Section 1 we introduce the necessary notations and formulate auxiliary results. In particular (see (1.10), (1.11)), we define the classes $\mathcal{R}_d\big(\tau, \varepsilon\big) \subset \mathcal{A}_d(\tau) \times \mathcal{A}_d(\tau)$ of pairs of distributions with close cumulants of the second and third orders and discuss the properties of classes $\mathcal{A}_d(\tau)$ and $\mathcal{R}_d\big(\tau, \varepsilon\big)$. We introduce also the classes $\mathcal{A}_d^*(\tau, \delta, \rho) \subset \mathcal{A}_d(\tau)$ of distributions satisfying some smoothness conditions which guarantee the existence of smooth densities (see (1.44), (1.45)).

It is known (see [30]) that if the eigenvalues of the covariance operator of a distribution $F \in \mathcal{A}_d(\tau)$ are sufficiently larger than τ^2, then this distribution is close to the corresponding Gaussian law. The parameter τ may be used for quantitative estimation of this closeness (see (1.33), (1.34)). In Sections 2 we obtain the uniform estimate of the closeness of densities in the central limit theorem (CLT) for smooth distributions from $\mathcal{A}_d(\tau)$. In Section 3 we prove a more precise bound for the closeness of densities of smooth distributions from $\mathcal{R}_d\big(\tau, \varepsilon\big)$. It is not very surprising that the additional information about the closeness of cumulants help us to obtain the better estimates. If $\varepsilon = 0$, the condition $(F_1, F_2) \in \mathcal{R}_d\big(\tau, \varepsilon\big)$ means that the distributions F_1, F_2 have identical cumulants of the second and third orders. Roughly speaking, if our distributions $(F_1, F_2) \in \mathcal{R}_d\big(\tau, 0\big)$ are distributions of sums of n i.i.d. random vectors, then we obtain in Section 3 the estimates of order $O(n^{-1})$. The estimates from Section 2 gives in this situation bounds of order $O(n^{-1/2})$.

In Section 4, using the result of Section 2, we obtain a non-uniform estimate for large deviations in the local CLT for smooth distributions from $\mathcal{A}_d(\tau)$ (see (4.2)). Besides, we prove that if $x \in \mathbb{R}^d$ belongs to some set separated from zero and from infinity, then the derivative of the logarithm of density in direction x at point x is negative. We give also an estimate of this derivative (see (4.3)). In Section 5, using the result of Section 3, we prove a non-uniform estimate for the closeness of densities of smooth distributions from $\mathcal{R}_d\big(\tau, \varepsilon\big)$ (see (5.10)). If ε is small, the inequality (5.10)

provides the more precise bound than the inequality (4.2) from Section 4.

In Section 6, using the result of Section 4, we obtain a non-uniform estimate for large deviations in the local CLT for conditional distributions of the last coordinates of smooth distributions from $\mathcal{A}_d(\tau)$ under condition that the first $d-1$ coordinates are fixed (see (6.4)). Analogously, in Section 7, using the result of Section 5, we prove a non-uniform estimate for the closeness of conditional densities of smooth distributions from $\mathcal{R}_d\big(\tau,0\big)$ (see (7.5)). Besides, using (4.3), we obtain an analog of (4.3) for conditional densities (see (7.6)).

In Sections 4–7 we use many times that the so-called conjugate distributions (defined by (1.1)) satisfy the conditions of previous results. For example, in Sections 5, 7 we use that, if $(F_1,F_2)\in\mathcal{R}_d\big(\tau,\varepsilon\big)$, then the pairs of conjugate distributions belong to $\mathcal{R}_d\big(\widetilde{\tau},\widetilde{\varepsilon}\big)$ (with other values of parameters, see Lemma 1.4, item b)). For the proof of Lemma 7.1 it is very important that Lemma 5.1 is proved not only for $\varepsilon=0$ because the coincidence of second and third cumulants of the distributions $(F_1,F_2)\in\mathcal{R}_d\big(\tau,0\big)$, does not imply the coincidence of corresponding cumulants of conjugate distributions.

In Sections 8, 9 we use the results of Sections 6, 7 and obtain the corresponding estimates of quantiles of conditional distributions. As usually, in Section 9, where we consider the pairs of smooth distributions $(F_1,F_2)\in\mathcal{R}_d\big(\tau,0\big)$, we prove more precise estimates than in the CLT for $F\in\mathcal{A}_d(\tau)$ in Section 8. Note that in Section 9 we essentially use the estimates for the derivatives of conditional densities.

One should note that the arguments from Sections 2, 4, 6, 8 are similar to those from [8, 9, 29–31]. On the other hand, in [8, 9, 29–31] the authors have used some ideas from a paper by Yurinskii [27].

In Section 10 we describe the KMT scheme of dyadic approximation. This scheme was essentially used by Sakhanenko [21] and Einmahl [9]. Note, however, that the formulations and the proofs of [10, inequality (2.5)], [21, Lemma 1, p. 34] and [9, inequality (4.11)] are close but slightly different. Our approach practically coincides with Einmahl's one. As in [9, 10, 21], we suppose that some independent random vectors with smooth distributions are already constructed and define independent random vectors with given absolutely continuous distributions by successive constructing of sums of blocks of summands, containing $2^N, 2^{N-1}, 2^{N-2}, \ldots, 4, 2, 1$ summands. For this we apply the so-called Rosenblatt quantile transformation [19] which was already used by Einmahl [9]. This transformation is a natural generalization of the usual one-dimensional quantile transformation which have been applied by KMT [10] and Sakhanenko [21].

The scheme of the proof of Theorem 1 is very close to that of the main result of Sakhanenko [21]. At first we suppose that the Gaussian vectors $Y_1,\ldots,Y_n$, $n=2^N$, are already constructed and construct the independent vectors which are bounded with probability one, have sufficiently smooth distributions and the same moments of the first, second and third orders as the needed independent random vec-

tors $X_1, \dots, X_n$. Then we construct the vectors $X_1, \dots, X_n$ in several steps. After each step the number of X_j which are not constructed becomes smaller in 2^p times, where p is a suitably chosen positive integer. In each step we begin with already constructed vectors which are bounded with probability one and have sufficiently smooth distributions and the needed moments up to the third order. Then we construct the vectors such that, in each block of 2^p summands, only the first vector has the same bounded smooth distribution. The rest $2^p - 1$ vectors have the needed distributions $\mathcal{L}(\xi_j)$. These $2^p - 1$ vectors from each block will be chosen as X_j and will be not involved in the next steps of the procedure. The coincidence of third moments will allow us to use more precise estimates of the closeness of quantiles of conditional distributions obtained in Section 9.

In Section 11 we realize the first step of the procedure just described. We estimate the rate of approximation in the case when we construct on a probability space the independent Gaussian vectors Y_j and the independent vectors X_j with smooth distributions and the needed moments up to the third order (see the conditions A1 between the formulas (11.14), (11.15)). Simultaneously, we consider the situation from [9] when the new vectors X_j have the needed distribution for even j and have the the same Gaussian distributions as $\mathcal{L}(Y_j)$ for odd j (see the conditions A2). In Section 14 this will help us to prove Theorem 5. The main result of Section 11 is formulated in Theorem 11.1.

Sections 12, 13 are devoted to the estimation of the rate of approximation in the case when we begin with the vectors $\widetilde{Y}_1, \dots, \widetilde{Y}_n$, $n = 2^N$, having smooth distribution with bounded supports and with the same moments up to the third order as the needed vectors $\widetilde{X}_1, \dots, \widetilde{X}_n$ and construct the vectors $\widetilde{X}_1, \dots, \widetilde{X}_n$ in several steps, diminishing after each step the number of $\widetilde{X}_j$ which are not constructed in 2^p times. The scheme of the proof is a modified version of that from [21]. We use the induction with respect to the number of steps of the procedure and prove that this procedure provides the good estimate for

$$\delta_j(N) = \left| \sum_{k=1}^{j} \widetilde{Y}_k - \sum_{k=1}^{j} \widetilde{X}_k \right| \tag{19}$$

(see (13.28), (13.94), (13.97)). It is very important that in (19) there is no supremum over $j = 1, \dots, 2^N$. This supremum is taken in the last moment only (see (13.98)). At the end of Section 13 we give the proof of Theorem 2.

Theorems 1, 3–5 are proved in Section 14. Theorem 3 is in fact a simple consequence of Theorem 2. However, its formulation is more convenient than that of Theorem 2 when we investigate the character of dependence of constants of the parameters. This is taken into account in the proofs of Theorems 1 and 4. For the proof of Theorem 1 we need also Lemma 14.1 which shows that there exist absolute positive constants c_{12}, c_{13} such that for any distribution $F \in \mathcal{A}_d(1)$ with mean zero

and $\operatorname{cov} F = I$ there exist a distribution $G \in \mathcal{A}_d(c_{12})$ with the same moments up to the third order and such that $G\big\{\{x : |x| \leqslant c_{13} \log^+ d\}\big\} = 1$. It can be verified that this statement is valid without $\log^+ d$ if all the moments of the third order of the distribution F are equal to zero. Therefore, the factor $\log^+ d$ from (7) is absent in the denominator of the fraction in the left-hand side of the inequality (15) of Theorem 4. At the end of Section 14 we give the proof of Theorem 5.

1. Notations and auxiliary results

At first we introduce some additional notations. Let $\mathfrak{F}_d$ be the set of all d-dimensional probability distributions defined on the σ-field $\mathfrak{B}_d$ of Borel subsets of the Euclidean space $\mathbb{R}^d$. Let $\mathfrak{G}_d$ be the collection of all d-dimensional Gaussian distributions.

The symbols $c, c_1, c_2, \ldots$ will be used for absolute positive constants. The letter c can denote different constants when we do not need to fix their numerical values. The numbering of constants starts in each new section again, except Sections 11–14, where we use the common enumeration. Below θ (with indices or without them) symbolizes different quantities not exceeding one in absolute value, E_a is the distribution concentrated at a point a. By $\widehat{F}(t)$, $t \in \mathbb{R}^d$, we denote the characteristic function of a distribution $F \in \mathfrak{F}_d$. The product of measures is understood as their convolution: $FG = F * G$. By $\mathcal{N}(a, M) \in \mathfrak{G}_d$ we denote the Gaussian distribution with mean a and covariance operator M. The notation $\Phi(F)$ will be used for the Gaussian distribution whose mean and covariance operator are the same as those of a distribution $F \in \mathfrak{F}_d$. In one-dimensional case we denote by $\Phi_\sigma(\,\cdot\,)$ the distribution function of the normal law with mean zero and variance σ^2 and by $\varphi_\sigma(\,\cdot\,)$ the corresponding density.

Let $\tau \geqslant 0$, $F = \mathcal{L}(\xi) \in \mathcal{A}_d(\tau)$, $\|h\|\,\tau < 1$, $h \in \mathbb{R}^d$. Then the conjugate distribution $\overline{F} = \overline{F}(h)$ is defined by

$$\overline{F}\{dx\} = \Big(\mathbf{E}\, e^{\langle h,\xi\rangle}\Big)^{-1} e^{\langle h,x\rangle} F\{dx\} \tag{1}$$

(sometimes it is called the Cramér transform). Denote by $\overline{\xi}(h)$ a random vector with $\mathcal{L}\big(\overline{\xi}(h)\big) = \overline{F}(h)$. Naturally, if the distribution F has a density $p(\cdot)$, then the distribution $\overline{F}(h)$ also has some density $\overline{p}(\,\cdot\,)$,

$$\overline{p}(x) = \Big(\mathbf{E}\, e^{\langle h,\xi\rangle}\Big)^{-1} e^{\langle h,x\rangle} p(x), \qquad x \in \mathbb{R}^d. \tag{2}$$

Denote by $D(h)$ the covariance operator of $\overline{\xi}(h)$.

It is clear that $\overline{F}(0) = F$. From (1) it follows that if $f(\cdot)$ is a function with finite $\mathbf{E}\,\big|\, f(\xi)\, e^{\langle h,\xi\rangle}\,\big| < \infty$, then

$$\mathbf{E}\, f(\overline{\xi}) = \Big(\mathbf{E}\, e^{\langle h,\xi\rangle}\Big)^{-1} \mathbf{E}\, f(\xi)\, e^{\langle h,\xi\rangle}. \tag{3}$$

It is easy to see that

$$\text{if} \quad U_1, \dots, U_n \in \mathcal{A}_d(\tau), \quad \|h\|\,\tau < 1, \quad U = \prod_{j=1}^{n} U_j, \quad \text{then} \quad \overline{U}(h) = \prod_{j=1}^{n} \overline{U}_j(h). \tag{4}$$

In the sequel we shall use the following simple fact, which can be easily verified with the help of (1) or (3).

Remark 1.1. For $h_1, h_2 \in \mathbb{R}^d$, $\|h_1\|\,\tau < 1/2$, $\|h_2\|\,\tau \leqslant 1/2$, we have

$$\overline{\big(\overline{F}(h_1)\big)}(h_2) = \overline{F}(h_1 + h_2), \qquad \|h_1 + h_2\|\,\tau < 1. \tag{5}$$

Let $\Phi = \mathcal{L}(\eta) \in \mathfrak{G}_d$ be a Gaussian distribution with zero mean and a covariance operator D. It is well known that

$$\mathbf{E}\,\big\langle z, \eta\big\rangle^2 = \big\langle Dz, \overline{z}\big\rangle, \qquad \mathbf{E}\,e^{\langle z, \eta\rangle} = \exp\Big(\frac{1}{2}\big\langle Dz, \overline{z}\big\rangle\Big) \tag{6}$$

for all $z \in \mathbb{C}^d$. The conjugate distributions $\overline{\Phi}(h)$ is also Gaussian for all $h \in \mathbb{R}^d$, with mean $D\,h$ and covariance operator D: for all $t \in \mathbb{R}^d$

$$\mathbf{E}\,\big\langle \overline{\eta}(h), t\big\rangle = \big\langle Dh, t\big\rangle, \qquad \mathrm{Var}\big\langle \overline{\eta}(h), t\big\rangle = \mathbf{E}\,\big\langle \eta, t\big\rangle^2 = \big\langle Dt, t\big\rangle = \big\|\,D^{1/2}t\,\big\|^2, \tag{7}$$

where $D^{1/2}$ is the square root of the non-negative-definite operator D.

Let $F = \mathcal{L}(\xi) \in \mathcal{A}_d(\tau)$, $\|h\|\,\tau < 1$, $h \in \mathbb{R}^d$. In the sequel the following relations will be very useful:

$$\mathbf{E}\,\big\langle \overline{\xi}(h), u\big\rangle = d_u\,\varphi(F, z)\Big|_{z=h}, \tag{8}$$

and

$$\big\langle D(h)\,u,\, v\big\rangle = d_u d_v\,\varphi(F, z)\Big|_{z=h} \qquad \text{for all} \quad u, v \in \mathbb{R}^d, \tag{9}$$

where $\varphi(F, z)$ is defined by (0.4).

For $\tau, \varepsilon \geqslant 0$ denote by $\mathcal{R}_d\big(\tau, \varepsilon\big)$ the collection of pairs of d-dimensional probability distributions (F_1, F_2) such that $F_k \in \mathcal{A}_d(\tau)$, $k = 1, 2$, and

$$\big|\,d_u d_v^2\,g_1(0) - d_u d_v^2\,g_2(0)\,\big| \leqslant \varepsilon\,\|u\|\,\tau\,\big\langle Dv, v\big\rangle, \tag{10}$$

$$\big|\,d_v^2\,g_1(0) - d_v^2\,g_2(0)\,\big| \leqslant \varepsilon^2\big\langle Dv, v\big\rangle, \tag{11}$$

for all $u, v \in \mathbb{R}^d$, where

$$D = D_1 + D_2, \quad g_k(z) = \varphi(F_k, z), \quad D_k = \mathrm{cov}\,F_k, \qquad k = 1, 2. \tag{12}$$

Remark 1.2. If $\varepsilon = 0$, then the relation $(F_1, F_2) \in \mathcal{R}_d\big(\tau, \varepsilon\big)$ means that the distributions $F_1, F_2 \in \mathcal{A}_d(\tau)$ have identical cumulants of second and third orders.

Taking into account the relations $\overline{F}(0) = F$, $D_k(0) = D_k$ and (9), it is easy to see that if $\varepsilon < 1$, then for $k = 1, 2$ and for all $v \in \mathbb{R}^d$

$$\begin{aligned} \big\langle D_1 v, v\big\rangle &\leqslant \frac{1+\varepsilon^2}{1-\varepsilon^2}\,\big\langle D_2 v, v\big\rangle, \\ \big\langle D_2 v, v\big\rangle &\leqslant \frac{1+\varepsilon^2}{1-\varepsilon^2}\,\big\langle D_1 v, v\big\rangle, \end{aligned} \qquad \text{and, hence,} \quad \big\langle Dv, v\big\rangle \leqslant \frac{2}{1-\varepsilon^2}\,\big\langle D_k v, v\big\rangle. \tag{13}$$

The classes $\mathcal{A}_d(\tau)$ and $\mathcal{R}_d\big(\tau, \varepsilon\big)$ have the properties which are very convenient for applying. Some of them was considered in [30, 32, 33]. In particular, the following lemmas are valid.

Lemma 1.1 [30, Lemmas 2.1, 3.1]**.** *Suppose that* $\tau \geqslant 0$, $F \in \mathcal{A}_d(\tau)$, $h \in \mathbb{R}^d$, $\|h\|\,\tau < 1$,

$$F = \mathcal{L}(\xi), \quad D = \operatorname{cov}\xi, \quad D(h) = \operatorname{cov}\overline{\xi}(h). \tag{14}$$

Let $\mathbf{E}\,\xi = 0$. *Denote by* σ^2 *and* $\sigma^2(h)$ *the minimal eigenvalues of operators* D *and* $D(h)$ *respectively. Then*

a) *for all* $u, v \in \mathbb{R}^d$ *the following relations are valid*:

$$\big\langle D(h)u, u\big\rangle = \big\langle Du, u\big\rangle\big(1 + \theta\,\|h\|\,\tau\big) \tag{15}$$

$$\big\langle D(h)u, v\big\rangle = \big\langle Du, v\big\rangle + \theta\,\|h\|\,\tau\big\langle Du, u\big\rangle^{1/2}\big\langle Dv, v\big\rangle^{1/2}, \tag{16}$$

$$\mathbf{E}\,\big\langle\overline{\xi}(h), u\big\rangle = \big\langle Dh, u\big\rangle + \frac{1}{2}\,\theta\,\|h\|\,\tau\big\langle Du, u\big\rangle^{1/2}\big\langle Dh, h\big\rangle^{1/2}, \tag{17}$$

$$\log\,\mathbf{E}\,e^{\langle h,\xi\rangle} = \frac{1}{2}\,\big\langle Dh, h\big\rangle\Big(1 + \frac{1}{3}\,\theta\,\|h\|\,\tau\Big), \tag{18}$$

$$\log\,\mathbf{E}\,e^{i\langle h,\xi\rangle} = -\frac{1}{2}\,\big\langle Dh, h\big\rangle\Big(1 + \frac{1}{3}\,\theta\,\|h\|\,\tau\Big); \tag{19}$$

b) *If* $\|h\|\,\tau \leqslant 1/2$, *then* $\overline{F}(h) \in \mathcal{A}_d(2\tau)$;

c) *There exists an absolute positive constant* c *such that*

$$\big(\det D(h)\big)^{1/2} = \big(\det D\big)^{1/2}\,\exp\big(c\theta d\,\|h\|\,\tau\big), \tag{20}$$

$$\sigma(h) = \sigma\,\exp\big(c\theta\,\|h\|\,\tau\big). \tag{21}$$

Lemma 1.2 [30, Lemma 3.2]. *In the conditions of Lemma* 1.1 *for* $x \in \Pi$,

$$\Pi = \Big\{ x \in \mathbb{R}^d : 4.8\,\tau\sigma^{-1} \big\| D^{-1/2} x \big\| \leqslant 1 \Big\},$$

there exists a parameter $\widetilde{h} = \widetilde{h}(x) \in \mathbb{R}^d$ *such that*

$$\mathbf{E}\,\overline{\xi}(\widetilde{h}) = x, \tag{22}$$

$$\|\widetilde{h}\|\,\tau \leqslant 1/2, \tag{23}$$

$$\sigma\,\|\widetilde{h}\| \leqslant \big\| D^{1/2}\widetilde{h} \big\| \leqslant 2.4\,\big\| D^{-1/2} x \big\|, \tag{24}$$

$$\big\| D^{1/2}\widetilde{h} - D^{-1/2} x \big\| \leqslant 2.88\,\tau\sigma^{-1} \big\| D^{-1/2} x \big\|^2, \tag{25}$$

$$\mathbf{E}\,\exp\Big(\langle \xi, \widetilde{h} \rangle - \langle x, \widetilde{h} \rangle \Big) = \exp\Big(-\frac{1}{2} \big\| D^{-1/2} x \big\|^2 + 10.08\,\theta\,\tau\sigma^{-1} \big\| D^{-1/2} x \big\|^3 \Big). \tag{26}$$

Lemma 1.3 [33, Lemma 1]. *Let* $\tau \geqslant 0$, $F \in \mathcal{A}_d(\tau)$. *Then for all* $u, v, w \in \mathbb{R}^d$ *and for all* $z \in \mathbb{C}^d$, $\|z\|\,\tau < 1/2$, *the inequality*

$$\big| d_w d_u d_v^2\, \varphi(F, z) \big| < 2\,\|w\|\,\tau\,\|u\|\,\tau \langle Dv, v \rangle \tag{27}$$

is valid.

Lemma 1.4 [33, Lemma 2]. *Suppose that* $\tau, \varepsilon \geqslant 0$, $(F_1, F_2) \in \mathcal{R}_d\big(\tau, \varepsilon\big)$, $h \in \mathbb{R}^d$, $\|h\|\,\tau < 1/2$,

$$F_j = \mathcal{L}(\xi_j), \quad D_j = \operatorname{cov} \xi_j, \quad D_j(h) = \operatorname{cov} \overline{\xi}_j(h), \qquad j = 1, 2.$$

Let $\mathbf{E}\,\xi_1 = \mathbf{E}\,\xi_2$. *Denote* $D = D_1 + D_2$. *Then*

a) *for all* $u, v \in \mathbb{R}^d$ *the following inequalities are valid*:

$$\big| \langle D_1(h)\,u,\, u \rangle - \langle D_2(h)\,u,\, u \rangle \big| \leqslant 2\,\big(\varepsilon + \|h\|\,\tau\big)^2 \langle Du, u \rangle, \tag{28}$$

$$\big| \langle D_1(h)\,u,\, v \rangle - \langle D_2(h)\,u,\, v \rangle \big| \leqslant 2\,\big(\varepsilon + \|h\|\,\tau\big)^2 \langle Du, u \rangle^{1/2} \langle Dv, v \rangle^{1/2}, \tag{29}$$

$$\big| \mathbf{E}\,\langle \overline{\xi}_1(h), u \rangle - \mathbf{E}\,\langle \overline{\xi}_2(h), u \rangle \big| \leqslant 2\,\big(\varepsilon + \|h\|\,\tau\big)^2 \langle Du, u \rangle^{1/2} \langle Dh, h \rangle^{1/2}, \tag{30}$$

$$\Big| \log \mathbf{E}\, e^{\langle h, \xi_1 \rangle} - \log \mathbf{E}\, e^{\langle h, \xi_2 \rangle} \Big| \leqslant \frac{1}{2}\,\big(\varepsilon + \|h\|\,\tau\big)^2 \langle Dh, h \rangle, \tag{31}$$

$$\Big| \log \mathbf{E}\, e^{i\langle h, \xi_1 \rangle} - \log \mathbf{E}\, e^{i\langle h, \xi_2 \rangle} \Big| \leqslant \frac{1}{2}\,\big(\varepsilon + \|h\|\,\tau\big)^2 \langle Dh, h \rangle; \tag{32}$$

b) $\big(\overline{F}_1(h), \overline{F}_2(h) \big) \in \mathcal{R}_d\big(2\tau,\, 2\,(\varepsilon + \|h\|\,\tau) \big)$.

It is evident that if $\tau_1 < \tau_2$, then $\mathcal{A}_d(\tau_1) \subset \mathcal{A}_d(\tau_2)$. Besides, it is easy to see that, for fixed τ, the class $\mathcal{A}_d(\tau)$ is closed with respect to convolution: if $F_1, F_2 \ldots, F_n \in \mathcal{A}_d(\tau)$, then $F_1 F_2 \ldots F_n \in \mathcal{A}_d(\tau)$. The class $\mathcal{A}_d(0)$ coincides with the class $\mathfrak{G}_d$ of all d-dimensional Gaussian distributions. The following inequality was proved in [30] and can be considered as an estimate of stability of this characterization: if $F \in \mathcal{A}_d(\tau)$, $\tau > 0$, then

$$\pi\big(F,\, \Phi(F)\big) \leqslant c d^2 \tau \, \log^+ \tau, \tag{33}$$

where $\pi(\cdot, \cdot)$ is the Prokhorov distance. Moreover, in [30] it was established that for all $X \in \mathfrak{B}_d$ and all $\lambda > 0$

$$\begin{aligned} F\big\{X\big\} &\leqslant \Phi(F)\big\{X^\lambda\big\} + c d^2 \exp\Big(- \frac{\lambda}{c\, d^2\, \tau} \Big), \\ \Phi(F)\big\{X\big\} &\leqslant F\big\{X^\lambda\big\} + c d^2 \exp\Big(- \frac{\lambda}{c\, d^2\, \tau} \Big), \end{aligned} \tag{34}$$

where $X^\lambda = \Big\{ y \in \mathbb{R}^d : \inf\limits_{x \in X} \|x - y\| < \lambda \Big\}$ is the λ-neighborhood of the set X. It is very essential (and important) that the inequalities (33), (34) are proved for all $\tau > 0$ and for arbitrary $\operatorname{cov} F$, in contrast to Theorems 1–5, where $\tau \geqslant 1$ and all covariance operators are of the form $n I$.

The classes $\mathcal{A}_d(\tau)$ are closely connected with other naturally defined classes of multidimensional distributions. From the definition of $\mathcal{A}_d(\tau)$ it follows that if $F = \mathcal{L}(\xi) \in \mathcal{A}_d(\tau)$ then the vector ξ has finite exponential moments $\mathbf{E}\, e^{\langle h, \xi \rangle}$, $h \in \mathbb{R}^d$, $\|h\| \tau < 1$. This leads to exponential estimates for the tail of distributions (see, e.g., Lemma 1.9).

The condition $F = \mathcal{L}(\xi) \in \mathcal{A}_1(c\tau)$ is equivalent to Statulevičius condition [25]

$$|\gamma_m| \leqslant c m!\, \tau^{m-2}, \qquad m = 3, 4, \ldots,$$

on the rate of increasing of cumulants γ_m of the random variable ξ. However, the condition $F = \mathcal{L}(\xi) \in \mathcal{A}_d(\tau)$ differs from multidimensional analogs of Statulevičius conditions, considered by Rudzkis [20] and Saulis [22].

An analog of the inequalities (33), (34) was proved in [29, 31] for the convolutions of distributions from the class

$$\begin{aligned} \mathcal{B}_d(\tau) = \Big\{ F = \mathcal{L}(\xi) \in \mathfrak{F}_d :\, & \mathbf{E}\, \xi = 0, \ \Big| \mathbf{E} \big\langle \xi, v \big\rangle^2 \big\langle \xi, u \big\rangle^{m-2} \Big| \\ & \leqslant \frac{1}{2}\, m!\, \tau^{m-2} \, \|u\|^{m-2}\, \mathbf{E} \big\langle \xi, v \big\rangle^2 \ \text{ for all } u, v \in \mathbb{R}^d,\ m = 3, 4 \ldots \Big\}. \end{aligned} \tag{35}$$

It can be easily verified that all distributions with zero mean, which are concentrated on the ball $\big\{ x \in \mathbb{R}^d : \|x\| \leqslant \tau \big\}$, belong to $\mathcal{B}_d(\tau)$. In the one-dimensional case

the condition $F \in \mathcal{B}_1(\tau)$ coincides with the conditions of the well-known Bernstein inequality. As is shown in [29], these conditions are equivalent to the conditions introduced by Sakhanenko [21] in the one-dimensional case:

$$\mathbf{E}\,\xi = 0, \qquad \mathbf{E}\,|\xi|^3 \exp\bigl(\tau^{-1}|\xi|\bigr) \leqslant \tau\,\mathbf{E}\,|\xi|^2. \tag{36}$$

This equivalence means that if one of these conditions is satisfied with parameter τ, then the second is valid with parameter $c\tau$.

The following Lemma 1.5 shows the relation between classes $\mathcal{A}_d(\tau)$ and $\mathcal{B}_d(\tau)$.

Lemma 1.5 [30, 32]. *Let $\sigma^2(F)$ be the maximal eigenvalue of the covariance operator of a distribution F. Then the following assertions are valid.*

a) *If $F = \mathcal{L}(\xi) \in \mathcal{B}_d(\tau)$, then $\sigma^2(F) \leqslant 12\,\tau^2$, $\mathbf{E}\,\xi = 0$ and $F \in \mathcal{A}_d(c\tau)$.*
b) *If $F = \mathcal{L}(\xi) \in \mathcal{A}_d(\tau)$, $\sigma^2(F) \leqslant \tau^2$ and $\mathbf{E}\,\xi = 0$, then $F \in \mathcal{B}_d(c\tau)$.*

Denote now by $\widetilde{\mathcal{A}}_d(\tau)$, $\tau \geqslant 0$, the class of distributions which is defined similarly to $\mathcal{A}_d(\tau)$, with the change of (0.5) by

$$\bigl|d_v^2\,\varphi(z)\bigr| \leqslant 2\,\bigl\langle Dv, v\bigr\rangle$$

for all $v \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$, $\|z\|\,\tau < 1$. One can show that $\mathcal{A}_d(\tau) \subset \widetilde{\mathcal{A}}_d(c\tau)$, $\widetilde{\mathcal{A}}_d(\tau) \subset \mathcal{A}_d(c\tau)$. This fact will be not used in this paper. Its proof is elementary (see the proof of [33, Lemma 1] and the relation (9)) and, therefore, is omitted.

As examples of distributions from $\mathcal{A}_d(c\tau)$ we can consider infinitely divisible distributions with spectral measures concentrated on the ball $\bigl\{x \in \mathbb{R}^d : \|x\| \leqslant \tau\bigr\}$ (see [30]). One can find in [30] a more general condition which ensure belonging of an infinitely divisible distribution to $\mathcal{A}_d(c\tau)$ (see [30, condition (1.8)]).

Assume, in particular, that $H = \mathcal{L}(\zeta) \in \mathcal{A}_d(\tau)$, $\mathbf{E}\,\zeta = 0$, $\operatorname{cov}\zeta = D$, σ_1^2 is the maximal eigenvalue of the operator D. Suppose that $\lambda > 0$ and an infinitely divisible distribution F is defined by its characteristic function

$$\widehat{F}(t) = \exp\biggl(\lambda \int_{\mathbb{R}^d} \bigl(e^{i\langle t,x\rangle} - 1\bigr)\,H\{dx\}\biggr), \qquad t \in \mathbb{R}^d.$$

Then it can be verified that $F \in \mathcal{A}_d(c\tau_0)$, where $\tau_0 = \max\bigl\{\tau, \sigma_1\bigr\}$.

Lemma 1.6. *Suppose that $\tau \geqslant 0$, $F = \mathcal{L}(\xi) \in \mathcal{A}_d(\tau)$, $y \in \mathbb{R}^m$, $\alpha \in \mathbb{R}^1$, and $A : \mathbb{C}^d \to \mathbb{C}^m$ is a linear operator such that $A(\mathbb{R}^d) \in \mathbb{R}^m$. Let $\widetilde{\xi} \in \mathbb{R}^k$ be the vector consisting of a subset of coordinates of the vector ξ. Then*

$$\mathcal{L}(A\xi + y) \in \mathcal{A}_m\bigl(\|A\|\,\tau\bigr), \qquad \textit{where} \quad \|A\| = \sup_{x \in \mathbb{R}^d,\ \|x\| \leqslant 1} \|Ax\|,$$

$$\mathcal{L}(\alpha\,\xi) \in \mathcal{A}_d(|\alpha|\,\tau), \qquad \mathcal{L}(\widetilde{\xi}) \in \mathcal{A}_k(\tau).$$

Proof. Put

$$\rho(z) = \log \mathbf{E}\, e^{\langle z, A\xi + y\rangle} = \log \mathbf{E}\, e^{\langle A^* z,\, \xi\rangle} + \big\langle z, y\big\rangle, \tag{37}$$

where $A^* : \mathbb{C}^m \to \mathbb{C}^d$ is the conjugate operator for A, $z \in \mathbb{C}^m$. Let $D = \operatorname{cov}\xi$, $M = \operatorname{cov} A\xi$. By virtue of (0.4), (0.5), (37), for all $u, v \in \mathbb{R}^m$, $z \in \mathbb{C}^m$, $\|A^* z\|\,\tau < 1$, the following relations are valid:

$$\begin{aligned}
\big|\, d_u d_v^2\, \rho(z)\,\big| &= \Big|\, d_{A^* u} d_{A^* v}^2\, \varphi(F, w)\,\big|_{w = A^* z}\,\Big| \\
&\leqslant \|A^* u\|\,\tau\,\big\langle D\, A^* v,\, A^* v\big\rangle \leqslant \|A^*\|\,\|u\|\,\tau\,\mathbf{E}\,\big\langle \xi - \mathbf{E}\,\xi,\, A^* v\big\rangle^2 \\
&= \|A\|\,\|u\|\,\tau\,\mathbf{E}\,\big\langle A\xi - \mathbf{E}\, A\xi,\, v\big\rangle^2 = \|u\| \cdot \|A\|\,\tau \cdot \big\langle M\, v,\, v\big\rangle.
\end{aligned} \tag{38}$$

If $\|z\| \cdot \|A\|\,\tau < 1$ then $\|A^* z\|\,\tau \leqslant \|A^*\|\,\|z\|\,\tau = \|z\| \cdot \|A\|\,\tau < 1$ and the inequality (38) is also valid. From (37), (38) and from definition of classes $\mathcal{A}_d(\tau)$ it follows that $\mathcal{L}(A\xi + y) \in \mathcal{A}_m\big(\|A\|\,\tau\big)$. The relations $\mathcal{L}(\alpha\,\xi) \in \mathcal{A}_d(|\alpha|\,\tau)$, $\mathcal{L}(\widetilde{\xi}) \in \mathcal{A}_k(\tau)$ follow directly from the statement just proved. It suffices to consider the operator $\alpha\, I$, where I is the identity operator, and the operator K, which associates to a vector $z \in \mathbb{C}^d$ the vector $\widetilde{z} \in \mathbb{C}^k$ constructed from coordinates of the vector z in the same way as the vector $\widetilde{\xi}$ was constructed from the coordinates of ξ. It is clear that $\|\alpha\, I\| = |\alpha|$, $\|K\| = 1$.

Corollary 1.1. *Let the conditions of Lemma* 1.6 *be satisfied,* $(F_1, F_2) \in \mathcal{R}_d\big(\tau, 0\big)$, $\alpha \in \mathbb{R}^1$, $F_j = \mathcal{L}(\xi_j)$, $y_j \in \mathbb{R}^m$, $j = 1, 2$, *and let* $\widetilde{\xi}_1, \widetilde{\xi}_2 \in \mathbb{R}^k$ *be the vectors consisting of identical subsets of coordinates of the vectors* ξ_1, ξ_2 *respectively. Then*

$$\begin{gathered}
\big(\mathcal{L}(A\xi_1 + y_1),\, \mathcal{L}(A\xi_2 + y_2)\big) \in \mathcal{R}_m\big(\|A\|\,\tau,\, 0\big), \\
\big(\mathcal{L}(\alpha\,\xi_1),\, \mathcal{L}(\alpha\,\xi_2)\big) \in \mathcal{R}_d\big(|\alpha|\,\tau, 0\big), \qquad \big(\mathcal{L}(\widetilde{\xi}_1),\, \mathcal{L}(\widetilde{\xi}_2)\big) \in \mathcal{R}_k(\tau, 0).
\end{gathered} \tag{39}$$

Proof. It is sufficient to use Lemma 1.6, the definition of classes $\mathcal{R}_d\big(\tau, \varepsilon\big)$ and the corresponding analogs of the relations (37), (38), taking into account Remark 1.2.

Lemma 1.7. *Suppose that* $\tau \geqslant 0$, $F_k = \mathcal{L}\big(\xi^{(k)}\big) \in \mathcal{A}_{d_k}(\tau)$, *and the vectors* $\xi^{(k)}$, $k = 1, 2$, *are independent. Let* $\xi \in \mathbb{R}^{d_1 + d_2}$ *be the vector with the first* d_1 *coordinates coinciding with those of* $\xi^{(1)}$ *and with the last* d_2 *coordinates coinciding with those of* $\xi^{(2)}$. *Then* $F = \mathcal{L}(\xi) \in \mathcal{A}_{d_1 + d_2}(\tau)$.

Proof. Let $D = \operatorname{cov}\xi$, $D^{(k)} = \operatorname{cov}\xi^{(k)}$, $k = 1, 2$. Using natural analogous nota-

tions, we easily derive that for all $u,\ v \in \mathbb{R}^{d_1+d_2}$ and $z \in \mathbb{C}^{d_1+d_2},\ \|z\|\,\tau < 1$,

$$\begin{aligned}\left|d_u d_v^2\,\varphi(F,z)\right| &= \left|d_u d_v^2\left(\varphi(F_1,z^{(1)})+\varphi(F_2,z^{(2)})\right)\right| \\ &= \left|d_{u^{(1)}} d_{v^{(1)}}^2\,\varphi(F_1,z^{(1)})\right| + \left|d_{u^{(2)}} d_{v^{(2)}}^2\,\varphi(F_2,z^{(2)})\right| \\ &\leqslant \|u^{(1)}\|\,\tau\left\langle D^{(1)}v^{(1)},v^{(1)}\right\rangle + \|u^{(2)}\|\,\tau\left\langle D^{(2)}v^{(2)},v^{(2)}\right\rangle \\ &\leqslant \|u\|\tau\left(\left\langle D^{(1)}v^{(1)},v^{(1)}\right\rangle + \left\langle D^{(2)}v^{(2)},v^{(2)}\right\rangle\right) \\ &= \|u\|\tau\left\langle Dv,v\right\rangle.\end{aligned}$$

It remains to use the definition of classes $\mathcal{A}_d(\tau)$.

Lemma 1.8. *Suppose that* $\tau \geqslant 0,\ \left(\mathcal{L}(\xi_1^{(k)}),\,\mathcal{L}(\xi_2^{(k)})\right) \in \mathcal{R}_{d_k}(\tau,0),\ k=1,2,$ *and the pairs of vectors* $\xi_j^{(1)},\xi_j^{(2)},\ j=1,2,$ *are the pairs of independent vectors. Let* $\xi_j \in \mathbb{R}^{d_1+d_2},\ j=1,2,$ *be the vectors with the first* d_1 *coordinates coinciding with those of* $\xi_j^{(1)}$ *and with the last* d_2 *coordinates coinciding with those of* $\xi_j^{(2)}$. *Then* $\left(\mathcal{L}(\xi_1),\,\mathcal{L}(\xi_2)\right) \in \mathcal{R}_{d_1+d_2}(\tau,0)$.

Proof. It suffices to apply Lemma 1.7, taking into account Remark 1.2.

Remark 1.3. Naturally, Lemmas 1.7, 1.8 can be evidently extended on the case of any finite number of independent vectors.

Remark 1.4. Lemma 1.7 can be easily derived from Lemma 1.6, using the completeness of classes $\mathcal{A}_d(\tau)$ with respect to convolution.

Lemma 1.9 below gives an analog of the well known Bernstein inequality for distributions from $\mathcal{A}_1(\tau)$.

Lemma 1.9. *Suppose that* $\tau > 0,\ \mathcal{L}(\xi) \in \mathcal{A}_1(\tau),\ \mathbf{E}\,\xi = 0$ *and* $\mathbf{E}\,\xi^2 = \sigma^2$. *Then*

$$\mathbf{P}\left\{|\xi| \geqslant x\right\} \leqslant 2\max\left\{\exp\left(-x^2/4\sigma^2\right),\,\exp\left(-x/4\tau\right)\right\}, \qquad x \geqslant 0. \tag{40}$$

Proof. Using Lemma 1.1, inequality (18), we see that, for $0 \leqslant t\tau < 1$,

$$\mathbf{E}\,e^{t\xi} \leqslant \exp\left(\frac{1}{2}\sigma^2t^2\left(1+\frac{1}{3}t\tau\right)\right) \leqslant e^{\sigma^2t^2}$$

and, by the exponential Chebyshev inequality,

$$\mathbf{P}\left\{\xi \geqslant x\right\} \leqslant \exp\left(\sigma^2t^2 - tx\right). \tag{41}$$

If $0 \leqslant x \leqslant \sigma^2/\tau$, we choose in (41) $t = x/2\sigma^2 < \tau^{-1}$. Then we have

$$\mathbf{P}\left\{\xi \geqslant x\right\} \leqslant \exp\left(-\frac{x^2}{4\sigma^2}\right). \tag{42}$$

In the case when $x \geqslant \sigma^2/\tau$, we take in (41) $t = (2\tau)^{-1}$ and obtain

$$\mathbf{P}\left\{\xi \geqslant x\right\} \leqslant \exp\Big(-\frac{x}{4\tau}\Big). \tag{43}$$

It is clear that we can analogously prove the estimates (42), (43) for $\mathbf{P}\Big\{\xi \leqslant -x\Big\}$. This completes the proof of the lemma.

Lemma 1.10. *Let* $\tau \geqslant 0$, $\mathcal{L}(\zeta_k) \in \mathcal{A}_1(\tau)$, $\mathbf{E}\,\zeta_k = 0$, $k = 1, \ldots, r$. *Let* $h \in \mathbb{R}^1$, $h\tau < 1$, $\zeta = \max\limits_{1\leqslant k\leqslant r} \big|\zeta_k\big|$, $\sigma^2 = \max\limits_{1\leqslant k\leqslant r} \mathbf{E}\,\zeta_k^2$. *Then* $\mathbf{E}\,e^{h\zeta} \leqslant 2r\,e^{h^2\sigma^2}$.

Proof. It is evident that the statement of the lemma can be easily derived if we take into account Lemma 1.6 and use the relation (18) of Lemma 1.1:

$$\mathbf{E}\,e^{h\zeta} \leqslant \sum_{k=1}^{r} \mathbf{E}\,e^{h\zeta_k} + \sum_{k=1}^{r} \mathbf{E}\,e^{-h\zeta_k} \leqslant 2r\,e^{h^2\sigma^2}.$$

Below we shall also use the following classes of distributions satisfying some special smoothness-type restrictions. Let $\tau \geqslant 0$, $\delta \geqslant 0$, $\rho \geqslant 0$. Denote by $\mathcal{A}_d^*(\tau, \delta, \rho)$ the class of distributions $F \in \mathcal{A}_d(\tau)$ such that for $h \in \mathbb{R}^d$, $\|h\|\,\tau < 1$, and for all $v \in \mathbb{R}^d$ the following inequalities hold:

$$(2\pi)^{-d} \int\limits_{\rho\|t\|\tau\, d\geqslant 1} \big|\widehat{F}_h(t)\big|\,dt \leqslant \frac{\tau^2\, d^2}{B^2\,(2\pi)^{d/2}\,(\det D)^{1/2}}, \tag{44}$$

$$(2\pi)^{-d} \int\limits_{\rho\|t\|\tau\, d\geqslant 1} \big|\big\langle t, v\big\rangle\, \widehat{F}_h(t)\big|\,dt \leqslant \frac{\big\langle D^{-1}v, v\big\rangle^{1/2}}{\delta\,(2\pi)^{d/2}\,(\det D)^{1/2}}, \tag{45}$$

where $F_h = \overline{F}(h)$, $D = \operatorname{cov} F$ and $B^2 > 0$ is the minimal eigenvalue of D.

According to (3), we have (if $F = \mathcal{L}(\xi)$)

$$\widehat{F}_h(t) = \mathbf{E}\,e^{\langle it, \overline{\xi}(h)\rangle} = \Big(\mathbf{E}\,e^{\langle h, \xi\rangle}\Big)^{-1} \mathbf{E}\,e^{\langle h+it, \xi\rangle}. \tag{46}$$

Therefore, the conditions (44), (45) play in this paper the role which is similar to that of [21, inequality (49), p. 9] or [9, inequality (1.5)]. Note, however, that the condition (45) will be needed when we shall estimate the derivatives of densities. Such estimates are necessary for obtaining more precise bounds for the closeness of conditional distributions when the compared distribution have the coinciding third moments.

2. Uniform estimate for the closeness of densities for $F \in \mathcal{A}_d(\tau)$

Lemma 2.1. *Let $\tau \geqslant 0$, $F \in \mathcal{A}_d(\tau)$. Suppose that*

$$(2\pi)^{-d} \int\limits_{\|t\|\,\tau\, d \geqslant 1} \bigl|\widehat{F}(t)\bigr|\, dt \leqslant \frac{\tau\, d^{3/2}}{\sigma\,(2\pi)^{d/2}\,(\det D)^{1/2}}\,, \tag{1}$$

where D is the common covariance operator of F and $\Phi = \Phi(F)$ and $\sigma^2 > 0$ is the minimal eigenvalue of D. Let $p(x)$, $q(x)$, $x \in \mathbb{R}^d$, are the probability densities of distributions F and Φ respectively. Then there exist absolute positive constants c_1, c_2 such that

$$\sup_{x \in \mathbb{R}^d} \bigl|p(x) - q(x)\bigr| \leqslant \frac{c_1\, \tau\, d^{3/2}}{\sigma\,(2\pi)^{d/2}\,(\det D)^{1/2}}\,, \tag{2}$$

if $\tau\, d^{3/2}/\sigma \leqslant c_2$. Besides, if we assume, in addition, that

$$(2\pi)^{-d} \int\limits_{\|t\|\,\tau\, d \geqslant 1} \bigl|\langle t, v\rangle\, \widehat{F}(t)\bigr|\, dt \leqslant \frac{\langle D^{-1} v, v\rangle^{1/2}}{(2\pi)^{d/2}\,(\det D)^{1/2}} \qquad \text{for some} \quad v \in \mathbb{R}^d, \tag{3}$$

then

$$\sup_{x \in \mathbb{R}^d} \bigl|d_v p(x)\bigr| \leqslant \frac{4\,\langle D^{-1} v, v\rangle^{1/2}}{(2\pi)^{d/2}\,(\det D)^{1/2}}\,. \tag{4}$$

Proof. By the inversion formula for densities we have

$$p(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle t, x\rangle}\, \widehat{F}(t)\, dt \tag{5}$$

and

$$\bigl|p(x) - q(x)\bigr| = (2\pi)^{-d} \left| \int_{\mathbb{R}^d} e^{-i\langle t, x\rangle} \Bigl(\widehat{F}(t) - \widehat{\Phi}(t)\Bigr)\, dt \right|. \tag{6}$$

Hence,

$$\bigl|d_v p(x)\bigr| = (2\pi)^{-d} \left| \int_{\mathbb{R}^d} \langle t, v\rangle\, e^{-i\langle t, x\rangle}\, \widehat{F}(t)\, dt \right|. \tag{7}$$

Denote

$$T_1 = \Bigl\{\, t \in \mathbb{R}^d : \|t\|\,\tau < d^{-1} \Bigr\}, \qquad T_2 = \mathbb{R}^d \backslash T_1. \tag{8}$$

Using the relation (1.19) of Lemma 1.1, we find for $t \in T_1$ that

$$\bigl|\widehat{F}(t)\bigr| = \left| \exp\Bigl(-\frac{1}{2}\,\langle Dt, t\rangle \Bigl(1 + \frac{1}{3}\theta\,\|t\|\,\tau\Bigr)\Bigr) \right| \leqslant \exp\Bigl(-\frac{1}{2}\,\langle Dt, t\rangle \Bigl(1 - \frac{1}{3d}\Bigr)\Bigr). \tag{9}$$

Using the inequality

$$\big|\,e^{z_1}-e^{z_2}\,\big| \leqslant \sqrt{2}\,|z_1-z_2|\,\max\big\{\,|e^{z_1}|,\,|e^{z_2}|\,\big\}, \tag{10}$$

which can be easily verified for all $z_1, z_2 \in \mathbb{C}^1$, and the relation (9), we find for $t \in T_1$ that

$$\begin{aligned}\Big|\,\widehat{F}(t)-\widehat{\Phi}(t)\,\Big| &\leqslant \Big|\;\frac{\sqrt{2}}{6}\;\|t\|\,\tau\,\big\langle Dt,t\big\rangle\exp\Big(-\frac{1}{2}\;\big\langle Dt,t\big\rangle\Big(1+\;\frac{1}{3}\,\theta\;\|t\|\,\tau\Big)\Big)\;\Big|\\ &\leqslant\;\frac{\sqrt{2}}{6}\;\|t\|\,\tau\,\big\langle Dt,t\big\rangle\exp\Big(-\frac{1}{2}\;\big\langle Dt,t\big\rangle\Big(1-\;\frac{1}{3d}\;\Big)\Big).\end{aligned} \tag{11}$$

From (11) it follows that, for $t \in T_1$,

$$\Big|\,\widehat{F}(t)-\widehat{\Phi}(t)\,\Big| \leqslant c\;\frac{\tau\,d^{3/2}}{\sigma}\;\exp\Big(-\frac{1}{2}\;\big\langle Dt,t\big\rangle\Big(1-\;\frac{1}{2\,d}\;\Big)\Big). \tag{12}$$

Fix now $x \in \mathbb{R}^d$ and introduce the notations

$$I_1 = (2\,\pi)^{-d}\int_{T_1}\Big|\,\widehat{F}(t)-\widehat{\Phi}(t)\,\Big|\,dt, \tag{13}$$

$$I_2 = (2\,\pi)^{-d}\int_{T_2}\Big(\,\big|\,\widehat{F}(t)\,\big|+\big|\,\widehat{\Phi}(t)\,\big|\,\Big)\,dt, \tag{14}$$

$$I_3 = (2\,\pi)^{-d}\int_{T_1}\Big|\,\big\langle t,v\big\rangle\,\widehat{F}(t)\,\Big|\,dt, \tag{15}$$

$$I_4 = (2\,\pi)^{-d}\int_{T_2}\Big|\,\big\langle t,v\big\rangle\,\widehat{F}(t)\,\Big|\,dt. \tag{16}$$

From (6), (13), (14) it follows that

$$\big|\,p(x)-q(x)\,\big| \leqslant I_1+I_2. \tag{17}$$

Using (7), (15), (16), we see that

$$\big|\,d_v p(x)\,\big| \leqslant I_3+I_4. \tag{18}$$

Taking into account the relations (12), (13) and

$$\int_{\mathbb{R}^d}\exp\Big(-\frac{1}{2}\;\big\langle Dt,t\big\rangle\Big)\,dt = (2\,\pi)^{d/2}\,(\det D)^{-1/2} \tag{19}$$

(valid for any non-degenerate covariance operator D), we conclude that

$$I_1 \leqslant\;\frac{c\,\tau\,d^{3/2}}{\sigma\,(2\,\pi)^{d/2}\,(\det D)^{1/2}}\,. \tag{20}$$

We used that, evidently, $\Big(1-\frac{1}{2\,d}\Big)^{d/2} \geqslant c$. Using (8), (19), we obtain the inequality

$$(2\,\pi)^{-d}\int_{T_2}\big|\,\widehat{\Phi}(t)\,\big|\,dt \leqslant (2\,\pi)^{-d}\,\exp\Big(-\frac{\sigma^2}{4\,d^2\,\tau^2}\Big)\int_{T_2}\exp\Big(-\frac{1}{4}\,\big\langle Dt,t\big\rangle\Big)\,dt$$
$$= \frac{2^{d/2}}{(2\,\pi)^{d/2}\,(\det D)^{1/2}}\,\exp\Big(-\frac{\sigma^2}{4\,d^2\,\tau^2}\Big)$$
$$\leqslant \frac{1}{(2\,\pi)^{d/2}\,(\det D)^{1/2}}\,\exp\Big(-\frac{\sigma^2}{8\,d^2\,\tau^2}\Big)$$
$$\leqslant \frac{c\,\tau\,d}{\sigma\,(2\,\pi)^{d/2}\,(\det D)^{1/2}}\,, \tag{21}$$

if c_2 is sufficiently small. The inequality (2) follows now from (17), (20), (21).

With the help of the relations (9), (15), (19) and

$$\int_{\mathbb{R}^d}\big\langle t,v\big\rangle^2\exp\Big(-\frac{1}{2}\,\big\langle Dt,t\big\rangle\Big)\,dt = (2\,\pi)^{d/2}\,(\det D)^{-1/2}\big\langle D^{-1}v,v\big\rangle \tag{22}$$

(valid for any non-degenerate covariance operator D and $v\in\mathbb{R}^d$), we obtain that

$$I_3 \leqslant (2\,\pi)^{-d}\Big(\int_{T_1}\big|\,\widehat{F}(t)\,\big|\;dt\;\int_{T_1}\big\langle t,v\big\rangle^2\,\big|\,\widehat{F}(t)\,\big|\,dt\Big)^{1/2} \leqslant \frac{e^{1/3}\,\sqrt{3}\,\big\langle D^{-1}v,v\big\rangle^{1/2}}{\sqrt{2}\,(2\,\pi)^{d/2}\,(\det D)^{1/2}} \tag{23}$$

(we also used that $e^{-2a}\leqslant 1-a$ for $0\leqslant a\leqslant {}^1\!/_2$). The inequality (4) is a simple consequence of the relations (3), (8), (16), (18), (23).

3. Uniform estimate for the closeness of densities for $(F_1,F_2)\in\mathcal{R}_d\big(\tau,\varepsilon\big)$

Lemma 3.1. *Suppose that* $\tau\geqslant 0$, $0\leqslant\varepsilon\leqslant 1/\sqrt{2}$, $(F_1,F_2)\in\mathcal{R}_d\big(\tau,\varepsilon\big)$. *Let* $\sigma_k^2>0$, $k=1,2$, *be the minimal eigenvalues of* $D_k=\operatorname{cov}F_k$. *Let* $p_k(x)$, $x\in\mathbb{R}^d$, *be the probability densities of distributions* F_k. *Let* $\mathbf{E}\,\xi_1=\mathbf{E}\,\xi_2$. *Denote* $\sigma^2=\max\limits_{k=1,2}\sigma_k^2$, $D=D_1+D_2$. *Assume that*

$$(2\,\pi)^{-d}\int\limits_{\|t\|\,\tau\,d\geqslant 1}\big|\widehat{F}_k(t)\big|\,dt \leqslant \frac{\tau^2\,d^2}{\sigma_k^2\,(2\,\pi)^{d/2}\,(\det D_k)^{1/2}}\,,\qquad k=1,2. \tag{1}$$

Then there exist absolute positive constants c_1,c_2 *such that the following estimate for the closeness of densities is valid*:

$$\sup_{x\in\mathbb{R}^d}\big|\,p_1(x)-p_2(x)\,\big| \leqslant c_1 d\,\Big(\varepsilon^2+\frac{d\,\tau^2}{\sigma^2}\Big)\max_{k=1,2}\;\frac{1}{(2\,\pi)^{d/2}\,(\det D_k)^{1/2}}\,, \tag{2}$$

if $\tau\,d^{3/2}\,\sigma^{-1}\leqslant c_2$.

Proof. Assume that the sets T_1, T_2 are defined by the relations (2.8). By the inversion formula for densities we have

$$\big|p_1(x) - p_2(x)\big| = (2\pi)^{-d} \left| \int_{\mathbb{R}^d} e^{-i\langle t,x\rangle} \big(\widehat{F}_1(t) - \widehat{F}_2(t)\big)\, dt \right|. \tag{3}$$

Note that if we take $0 \leqslant \varepsilon \leqslant 1/\sqrt{2}$, then from the definition of $\mathcal{R}_d\big(\tau,\varepsilon\big)$ it follows (see (1.13)) that, for $k, m = 1, 2$ and for all $v \in \mathbb{R}^d$,

$$\begin{gathered} \big\langle D_k v, v\big\rangle \leqslant \frac{1+\varepsilon^2}{1-\varepsilon^2}\, \big\langle D_m v, v\big\rangle \leqslant 3\big\langle D_m v, v\big\rangle, \qquad \sigma_k^2 = \inf_{\|v\|=1} \big\langle D_k v, v\big\rangle \leqslant 3\sigma_m^2, \\ \big\langle Dv, v\big\rangle \leqslant \frac{2}{1-\varepsilon^2}\, \big\langle D_k v, v\big\rangle \leqslant 4\big\langle D_k v, v\big\rangle. \end{gathered} \tag{4}$$

Using also the inequality (2.10), the relation (1.19) of Lemma 1.1 and the relation (1.32) of Lemma 1.4, we find for $t \in T_1$ that

$$\begin{aligned} \big|\widehat{F}_1(t) - \widehat{F}_2(t)\big| &\leqslant \left| \frac{\sqrt{2}}{2} \big(\varepsilon + \|t\|\,\tau\big)^2 \big\langle Dt, t\big\rangle \max_{k=1,2} \exp\!\left(-\frac{1}{2}\, \big\langle D_k t, t\big\rangle \Big(1 + \frac{1}{3}\,\theta_k\, \|t\|\,\tau\Big)\right)\right| \\ &\leqslant 4\sqrt{2}\, \big(\varepsilon^2 + \|t\|^2\tau^2\big) \max_{k=1,2} \big\langle D_k t, t\big\rangle \exp\!\left(-\frac{1}{2}\, \big\langle D_k t, t\big\rangle \Big(1 - \frac{1}{3\,d}\Big)\right). \end{aligned} \tag{5}$$

Using the conditions of the lemma, we derive from (4), (5) that, for $t \in T_1$,

$$\big|\widehat{F}_1(t) - \widehat{F}_2(t)\big| \leqslant cd\Big(\varepsilon^2 + \frac{d\,\tau^2}{\sigma^2}\Big) \max_{k=1,2} \exp\!\left(-\frac{1}{2}\, \big\langle D_k t, t\big\rangle \Big(1 - \frac{1}{2\,d}\Big)\right). \tag{6}$$

Fix now $x \in \mathbb{R}^d$ and introduce the notations

$$I_1 = (2\pi)^{-d} \int_{T_1} \big|\widehat{F}_1(t) - \widehat{F}_2(t)\big|\, dt, \tag{7}$$

$$I_2 = (2\pi)^{-d} \int_{T_2} \Big(\big|\widehat{F}_1(t)\big| + \big|\widehat{F}_2(t)\big|\Big)\, dt. \tag{8}$$

Using (3), (7), (8), we obtain the inequality

$$\big|p_1(x) - p_2(x)\big| \leqslant I_1 + I_2. \tag{9}$$

Taking into account the relations (6), (7) and (2.19), we conclude that

$$I_1 \leqslant cd\Big(\varepsilon^2 + \frac{d\,\tau^2}{\sigma^2}\Big) \max_{k=1,2} \frac{1}{(2\pi)^{d/2}\, (\det D_k)^{1/2}}\,. \tag{10}$$

We used again that, evidently, $\Big(1 - \frac{1}{2\,d}\Big)^{d/2} \geqslant c$. The inequality (2) follows now from (2.8), (1), (4), (8)–(10).

4. Non-uniform estimate for the closeness of densities for $F \in \mathcal{A}_d(\tau)$

Lemma 4.1. *Suppose that* $\tau \geqslant 0$,

$$F = \mathcal{L}(\xi) \in \mathcal{A}_d^*(\tau, 2, 2), \qquad \mathbf{E}\,\xi = 0, \tag{1}$$

$\sigma^2 > 0$ *is the minimal eigenvalue of* $D = \operatorname{cov}\xi$. *Let* $p(x)$, $q(x)$, $x \in \mathbb{R}^d$, *be the probability densities of distributions* F *and* $\Phi = \Phi(F)$ *respectively. Then there exist absolute positive constants* c_1, c_2, c_3 *such that for the density* $p(x)$ *and for the derivative of its logarithm* $d_x \log p(y)\big|_{y=x}$ *in the direction* x *at the point* x *the following relations are valid*:

$$\begin{aligned} p(x) &= q(x)\,\exp\Bigl(\theta\, c_1 \tau \sigma^{-1}\,\bigl(d^{3/2} + \bigl\|D^{-1/2}x\bigr\|^3\bigr)\Bigr) \\ &= \frac{1}{(2\,\pi)^{d/2}\,(\det D)^{1/2}}\,\exp\Bigl\{-\frac{1}{2}\bigl\|D^{-1/2}x\bigr\|^2 + \theta\, c_1 \tau \sigma^{-1}\,\bigl(d^{3/2} + \bigl\|D^{-1/2}x\bigr\|^3\bigr)\Bigr\}, \end{aligned} \tag{2}$$

$$d_x \log p(y)\Big|_{y=x} \leqslant -\frac{1}{7}\bigl\|D^{-1/2}x\bigr\|^2 \qquad \textit{for} \quad \bigl\|D^{-1/2}x\bigr\| > 7, \tag{3}$$

if $\tau\, d^{3/2}/\sigma \leqslant c_2$, $\tau\sigma^{-1}\bigl\|D^{-1/2}x\bigr\| \leqslant c_3\, d^{-1}$.

Proof. It is not hard to see that by choosing $c_3 \leqslant 5/24$ we can ensure that $x \in \Pi$ for $\tau\sigma^{-1}\bigl\|D^{-1/2}x\bigr\| \leqslant c_3\, d^{-1}$, where Π is the set from the formulation of Lemma 1.2. By this lemma, there exists a parameter $\widetilde{h} = \widetilde{h}(x) \in \mathbb{R}^d$, satisfying the relations (1.22)–(1.26). Choosing $c_3 \leqslant 5c_4/12$, we can ensure with the help of (1.24) the fulfilment of the relation

$$\bigl\|\widetilde{h}\,\bigr\|\tau \leqslant 2.4\,\tau\sigma^{-1}\bigl\|D^{-1/2}x\bigr\| \leqslant c_4\, d^{-1}, \tag{4}$$

for any constant c_4 (which we shall also choose sufficiently small). By (1), (1.23), the conditions of Lemma 1.1 are satisfied for $h = \widetilde{h}$. According to this lemma,

$$\overline{F}\bigl(\widetilde{h}\bigr) \in \mathcal{A}_d(2\tau) \tag{5}$$

and the relations (1.15)–(1.21) are valid (after the necessary change of notations) for $h = \widetilde{h}$.

Denote by $\widetilde{p}(\,\cdot\,)$, $\widetilde{r}(\,\cdot\,)$ the densities of distributions $\overline{F}\bigl(\widetilde{h}\bigr)$, $\Phi\Bigl(\overline{F}\bigl(\widetilde{h}\bigr)\Bigr)$ respectively. Then from (1.20), (1.22), (4) it follows that

$$\widetilde{r}(x) = \frac{1}{(2\,\pi)^{d/2}\,\bigl(\det D\bigl(\widetilde{h}\bigr)\bigr)^{1/2}} = \frac{\exp\bigl(c\theta\, d\,\|\widetilde{h}\|\,\tau\bigr)}{(2\,\pi)^{d/2}\,(\det D)^{1/2}} \geqslant \frac{c}{(2\,\pi)^{d/2}\,(\det D)^{1/2}}\,. \tag{6}$$

Taking into account (4), (6), (1.15) and choosing c_4 to be small enough, we derive from (2.22) (applied to the operators D and $D(\widetilde{h})$) that, for all $v \in \mathbb{R}^d$,

$$\bigl\langle D^{-1}(\widetilde{h})\, v, v\bigr\rangle = \bigl\langle D^{-1}v, v\bigr\rangle \exp\bigl(c\theta\, d\, \|\widetilde{h}\|\, \tau\bigr) = \bigl\| D^{-1/2}v \bigr\|^2 \exp\bigl(c\theta\, d\, \|\widetilde{h}\|\, \tau\bigr). \tag{7}$$

One can also verify that if the constants c_2, c_4 are sufficiently small, then, by virtue of (1), (4), (5), (7), (1.20), (1.21), Lemma 2.1 with parameter 2τ is applicable to the densities $\widetilde{p}(\,\cdot\,)$, $\widetilde{r}(\,\cdot\,)$. From this lemma and from (4), (7), (1.20), (1.21) it follows that

$$\sup_{y\in\mathbb{R}^d} \bigl|\widetilde{p}(y) - \widetilde{r}(y)\bigr| \leqslant \frac{c\,\tau\, d^{3/2}}{\sigma\,(2\,\pi)^{d/2}\,(\det D)^{1/2}} \tag{8}$$

and

$$\sup_{y\in\mathbb{R}^d} \bigl|d_v\widetilde{p}(y)\bigr| \leqslant \frac{4\,\bigl\langle D^{-1}v, v\bigr\rangle^{1/2} \exp\bigl(c\,d\,\|\widetilde{h}\|\,\tau\bigr)}{(2\,\pi)^{d/2}\,(\det D)^{1/2}} \qquad \text{for all} \quad v \in \mathbb{R}^d. \tag{9}$$

Choosing c_2 to be sufficiently small, we can ensure with the help of (6), (8) the validity of the inequality

$$\frac{\bigl|\widetilde{p}(x) - \widetilde{r}(x)\bigr|}{\widetilde{r}(x)} \leqslant c\,\frac{\tau\, d^{3/2}}{\sigma} \leqslant \frac{1}{2}\,. \tag{10}$$

From (4), (6), (10), (1.24) we obtain that

$$\begin{aligned}\widetilde{p}(x) &= \widetilde{r}(x)\left(1 + \frac{\widetilde{p}(x) - \widetilde{r}(x)}{\widetilde{r}(x)}\right)\\ &= \frac{1}{(2\,\pi)^{d/2}\,(\det D)^{1/2}}\,\exp\left(\theta\, c\left(\frac{\tau\, d^{3/2}}{\sigma} + \frac{d\,\tau}{\sigma}\,\bigl\|D^{-1/2}x\bigr\|\right)\right).\end{aligned} \tag{11}$$

From (1.2) it follows that

$$p(y) = \widetilde{p}(y)\ \mathbf{E}\exp\Bigl\{\bigl\langle\xi, \widetilde{h}\bigr\rangle - \bigl\langle y, \widetilde{h}\bigr\rangle\Bigr\} \qquad \text{for all} \quad y \in \mathbb{R}^d. \tag{12}$$

Besides,

$$q(x) = \frac{1}{(2\,\pi)^{d/2}\,(\det D)^{1/2}}\,\exp\Bigl\{-\frac{1}{2}\,\bigl\|D^{-1/2}x\bigr\|\Bigr\}. \tag{13}$$

The relation (2) follows now from (11)–(13) and (1.26). By (12), we have

$$d_x \log p(y)\Big|_{y=x} = d_x \log \widetilde{p}(y)\Big|_{y=x} - \bigl\langle x, \widetilde{h}\bigr\rangle. \tag{14}$$

Using (4), (9), (11) and choosing c_2, c_3, c_4 to be small enough, we see that

$$d_x \log \widetilde{p}(y)\Big|_{y=x} \leqslant 5\,\bigl\langle D^{-1}x, x\bigr\rangle^{1/2}. \tag{15}$$

From (1.25) it follows that

$$\Bigl|\bigl\langle D^{-1/2}x,\, D^{1/2}\widetilde{h} - D^{-1/2}x\bigr\rangle\Bigr| \leqslant 2.88\,\tau\sigma^{-1}\bigl\|D^{-1/2}x\bigr\|^3. \tag{16}$$

It is evident that

$$\bigl\langle x, \widetilde{h}\bigr\rangle = \bigl\langle D^{-1/2}x,\, D^{1/2}\widetilde{h}\bigr\rangle = \bigl\|D^{-1/2}x\bigr\|^2 + \bigl\langle D^{-1/2}x,\, D^{1/2}\widetilde{h} - D^{-1/2}x\bigr\rangle. \tag{17}$$

Choosing c_2, c_3, c_4 to be sufficiently small, we derive from (7) (with $v = x$), (14)–(17) the relation (3).

5. Non-uniform estimate for the closeness of densities for $(F_1, F_2) \in \mathcal{R}_d\big(\tau, \varepsilon\big)$

Lemma 5.1. *Suppose that* $\tau, \varepsilon \geqslant 0$, $(F_1, F_2) \in \mathcal{R}_d\big(\tau, \varepsilon\big)$, $\sigma_k^2 > 0$ *are the minimal eigenvalues of* $D_k = \operatorname{cov} F_k$, $F_k = \mathcal{L}(\xi_k)$, $k = 1, 2$. *Denote* $\sigma^2 = \max\limits_{k=1,2} \sigma_k^2$, $D = D_1 + D_2$. *Let* $p_k(x)$, $x \in \mathbb{R}^d$, *be the densities of distributions* F_k. *Suppose that* $\mathbf{E}\,\xi_1 = \mathbf{E}\,\xi_2 = 0$. *Then there exist absolute positive constants* $c_1, \ldots, c_8$ *such that if* $\tau d^{3/2} \sigma_k^{-1} \leqslant c_1$, $\tau \sigma_k^{-1} \big\| D_k^{-1/2} x_{(k)} \big\| \leqslant c_2$, $x_{(k)} \in \mathbb{R}^d$, $\varepsilon \leqslant c_3 d^{-1/2}$, *then for* $k = 1, 2$ *there exist parameters* $\widetilde{h}_k = \widetilde{h}_k(x_{(k)}) \in \mathbb{R}^d$ *satisfying the relations*

$$\mathbf{E}\,\overline{\xi}_k(\widetilde{h}_k) = x_{(k)}, \tag{1}$$

$$\big\| \widetilde{h}_k \big\| \tau \leqslant {}^1\!/_2, \tag{2}$$

$$\sigma_k \big\| \widetilde{h}_k \big\| \leqslant \big\| D_k^{1/2} \widetilde{h}_k \big\| \leqslant 2.4 \big\| D_k^{-1/2} x_{(k)} \big\|, \tag{3}$$

$$\big\| D_k^{1/2} \widetilde{h}_k - D_k^{-1/2} x_{(k)} \big\| \leqslant 2.88\, \tau \sigma_k^{-1} \big\| D_k^{-1/2} x_{(k)} \big\|^2, \tag{4}$$

$$\begin{aligned} &\mathbf{E}\, \exp\big(\big\langle \xi_k, \widetilde{h}_k \big\rangle - \big\langle x_{(k)}, \widetilde{h}_k \big\rangle \big) \\ &\qquad = \exp\Big\{ -\frac{1}{2} \big\| D_k^{-1/2} x_{(k)} \big\|^2 + 10.08\, \theta\, \tau \sigma_k^{-1} \big\| D_k^{-1/2} x_{(k)} \big\|^3 \Big\}, \end{aligned} \tag{5}$$

$$\begin{aligned} \sigma_l \big\| \widetilde{h}_1 - \widetilde{h}_2 \big\| &\leqslant \big\| D_l^{1/2} \widetilde{h}_1 - D_l^{1/2} \widetilde{h}_2 \big\| \leqslant \big\| D_l^{-1/2} x_{(1)} - D_l^{-1/2} x_{(2)} \big\| \\ &\quad + c_4 \max_{k=1,2} \big\| D_k^{-1/2} x_{(k)} \big\| \Big(\tau \sigma^{-1} \max_{k=1,2} \big\| D_k^{-1/2} x_{(k)} \big\| + \varepsilon^2 \Big), \quad l = 1, 2, \end{aligned} \tag{6}$$

$$\Big(\overline{F}_1\big(\widetilde{h}_1\big), \overline{F}_2\big(\widetilde{h}_2\big) \Big) \in \mathcal{R}_d\Big(2\tau,\, c_5 \big(\varepsilon + \tau \sigma^{-1} \max_{k=1,2} \big\| D_k^{-1/2} x_{(k)} \big\| + \sqrt{\tau \sigma^{-1} \Lambda} \big) \Big), \tag{7}$$

where

$$\Lambda = \big\| D_1^{-1/2} x_{(1)} - D_1^{-1/2} x_{(2)} \big\|. \tag{8}$$

Besides, under the additional suppositions $x_{(1)} = x_{(2)} = x$, $\tau \sigma_k^{-1} \big\| D_k^{-1/2} x \big\| \leqslant c_6 d^{-1}$, $\varepsilon \leqslant c_7 d^{-1/2}$ *and*

$$F_k \in \mathcal{A}_d^*(\tau, 2, 2), \qquad k = 1, 2, \tag{9}$$

the following representation is valid for the density $p_1(x)$*:*

$$p_1(x) = p_2(x)\, \exp\Bigg(\theta\, c_8 \Big(\varepsilon \Big(\sqrt{d} + \min_{k=1,2} \big\| D_k^{-1/2} x \big\| \Big) + \frac{\tau}{\sigma} \Big(d + \min_{k=1,2} \big\| D_k^{-1/2} x \big\|^2 \Big) \Big)^2 \Bigg). \tag{10}$$

Proof. From the definition of $\mathcal{R}_d\big(\tau,\varepsilon\big)$ (see (1.13)), taking into account (2.19), (2.22) and choosing c_3 to be small enough, we obtain, for $k,l=1,2$ and for all $v\in\mathbb{R}^d$, that

$$\begin{aligned}
&\big\langle D_k v,v\big\rangle \leqslant \tfrac{1+\varepsilon^2}{1-\varepsilon^2}\,\big\langle D_l v,v\big\rangle \leqslant 2\big\langle D_l v,v\big\rangle, \qquad \sigma_k^2=\inf_{\|v\|=1}\big\langle D_k v,v\big\rangle \leqslant 2\sigma_l^2,\\
&\big\langle Dv,v\big\rangle \leqslant \tfrac{2}{1-\varepsilon^2}\,\big\langle D_k v,v\big\rangle \leqslant 3\big\langle D_k v,v\big\rangle, \qquad \big\langle D_k^{-1}v,v\big\rangle \leqslant 2\big\langle D_l^{-1}v,v\big\rangle.
\end{aligned} \tag{11}$$

Fix $k=1$ or $k=2$. It is not hard to see that by choosing $c_2\leqslant 5/24$ we can ensure that $x_{(k)}\in\Pi_k$ for $\tau\sigma_k^{-1}\big\|\,D_k^{-1/2}x_{(k)}\,\big\|\leqslant c_2$, where

$$\Pi_k=\Big\{x\in\mathbb{R}^d : 4.8\,\tau\sigma_k^{-1}\big\|\,D_k^{-1/2}x\,\big\|\leqslant 1\Big\}.$$

Thus, the conditions of Lemma 1.2 (with $\sigma=\sigma_k$) are fulfilled for the random vector ξ_k. By this lemma, there exists a parameter $\widetilde{h}_k=\widetilde{h}_k(x_{(k)})\in\mathbb{R}^d$, satisfying the relations (1)–(5). Choosing $c_2\leqslant 5c_9/12$, we can ensure with the help of (3) the fulfilment of the relation

$$\big\|\,\widetilde{h}_k\,\big\|\tau\leqslant 2.4\,\tau\sigma_k^{-1}\big\|\,D_k^{-1/2}x_{(k)}\,\big\|\leqslant c_9, \tag{12}$$

for any constant c_9 (which will be also chosen sufficiently small). Using (2) and the inclusion $(F_1,F_2)\in\mathcal{R}_d\big(\tau,\varepsilon\big)$, we see that the conditions of Lemmas 1.1 and 1.4 are satisfied for $h=\widetilde{h}_k$. According to these lemmas,

$$\begin{aligned}
&\Big(\overline{F}_1\big(\widetilde{h}_k\big),\overline{F}_2\big(\widetilde{h}_k\big)\Big)\in\mathcal{R}_d\big(2\tau,\,2\big(\varepsilon+\|\widetilde{h}_k\|\,\tau\big)\big),\\
&\overline{F}_1\big(\widetilde{h}_k\big),\ \overline{F}_2\big(\widetilde{h}_k\big)\in\mathcal{A}_d(2\tau), \qquad k=1,2,
\end{aligned} \tag{13}$$

and the relations (1.15)–(1.21), (1.28)–(1.32) are valid (after the necessary change of notations) for $h=\widetilde{h}_k$.

Let us estimate $\big\|\,\widetilde{h}_1-\widetilde{h}_2\,\big\|$. Applying the inequality (1.29) (with $h=0$), we have for all $u,v\in\mathbb{R}^d$:

$$\Big|\big\langle D_1u,v\big\rangle-\big\langle D_2u,v\big\rangle\Big|\leqslant 2\varepsilon^2\big\langle Du,u\big\rangle^{1/2}\big\langle Dv,v\big\rangle^{1/2}. \tag{14}$$

By virtue of (1.17) for all $u\in\mathbb{R}^d$ we have:

$$\mathbf{E}\,\big\langle\overline{\xi}_k(\widetilde{h}_k),\,u\big\rangle=\big\langle D_k\widetilde{h}_k,u\big\rangle+\tfrac{1}{2}\theta\,\|\widetilde{h}_k\|\,\tau\big\langle D_ku,u\big\rangle^{1/2}\big\langle D_k\widetilde{h}_k,\widetilde{h}_k\big\rangle^{1/2} \tag{15}$$

and, taking into account (1),

$$\left|\left\langle D_1\widetilde{h}_1,u\right\rangle-\left\langle D_2\widetilde{h}_2,u\right\rangle\right|\leqslant\left|\left\langle x_{(1)}-x_{(2)},\,u\right\rangle\right|+\frac{1}{2}\sum_{k=1}^{2}\|\widetilde{h}_k\|\,\tau\left\langle D_ku,u\right\rangle^{1/2}\left\langle D_k\widetilde{h}_k,\widetilde{h}_k\right\rangle^{1/2}. \tag{16}$$

Now for all $u\in\mathbb{R}^d$ from (8), (11), (14), (16) it follows that

$$\begin{aligned}\left|\left\langle D_1\widetilde{h}_1,u\right\rangle-\left\langle D_1\widetilde{h}_2,u\right\rangle\right|&\leqslant\left|\left\langle D_1\widetilde{h}_1,u\right\rangle-\left\langle D_2\widetilde{h}_2,u\right\rangle\right|+\left|\left\langle D_2\widetilde{h}_2,u\right\rangle-\left\langle D_1\widetilde{h}_2,u\right\rangle\right|\\&\leqslant\left|\left\langle x_{(1)}-x_{(2)},\,u\right\rangle\right|+\frac{1}{2}\sum_{k=1}^{2}\|\widetilde{h}_k\|\,\tau\left\langle D_ku,u\right\rangle^{1/2}\left\langle D_k\widetilde{h}_k,\widetilde{h}_k\right\rangle^{1/2}\\&\quad+2\varepsilon^2\left\langle Du,u\right\rangle^{1/2}\left\langle D\widetilde{h}_2,\widetilde{h}_2\right\rangle^{1/2}\leqslant\Lambda\left\langle D_1u,u\right\rangle^{1/2}\\&\quad+\sum_{k=1}^{2}\|\widetilde{h}_k\|\,\tau\left\langle D_1u,u\right\rangle^{1/2}\left\langle D_k\widetilde{h}_k,\widetilde{h}_k\right\rangle^{1/2}+6\varepsilon^2\left\langle D_1u,u\right\rangle^{1/2}\left\langle D_2\widetilde{h}_2,\widetilde{h}_2\right\rangle^{1/2}.\end{aligned} \tag{17}$$

Setting here $u=D_1^{-1/2}v$, we obtain that for all $v\in\mathbb{R}^d$

$$\begin{aligned}&\left|\left\langle D_1^{1/2}\widetilde{h}_1,\,v\right\rangle-\left\langle D_1^{1/2}\widetilde{h}_2,\,v\right\rangle\right|\\&\qquad\leqslant\|v\|\left(\Lambda+\sum_{k=1}^{2}\|\widetilde{h}_k\|\,\tau\left\langle D_k\widetilde{h}_k,\widetilde{h}_k\right\rangle^{1/2}+6\varepsilon^2\left\langle D_2\widetilde{h}_2,\widetilde{h}_2\right\rangle^{1/2}\right).\end{aligned} \tag{18}$$

The minimal eigenvalue of the operator $D_1^{1/2}$ is equal to σ_1. Therefore, from (3), (11), (18) it follows that

$$\begin{aligned}\sigma_1\left\|\widetilde{h}_1-\widetilde{h}_2\right\|&\leqslant\left\|D_1^{1/2}\widetilde{h}_1-D_1^{1/2}\widetilde{h}_2\right\|\\&\leqslant\Lambda+\sum_{k=1}^{2}\|\widetilde{h}_k\|\,\tau\left\langle D_k\widetilde{h}_k,\widetilde{h}_k\right\rangle^{1/2}+6\varepsilon^2\left\langle D_2\widetilde{h}_2,\widetilde{h}_2\right\rangle^{1/2}\\&\leqslant\Lambda+c\left(\|\widetilde{h}_1\|\,\tau+\|\widetilde{h}_2\|\,\tau+\varepsilon^2\right)\sum_{k=1}^{2}\left\|D_k^{-1/2}x_{(k)}\right\|.\end{aligned} \tag{19}$$

The inequality (6) can be easily derived from (3), (8), (11), (19) and from the symmerty arguments.

Choosing c_2,c_3,c_9 to be sufficiently small and taking into account (8), we can ensure the fulfilment of the inequality

$$\begin{aligned}\varepsilon_0^2&=\tau\sigma^{-1}\left(\Lambda+\left(\|\widetilde{h}_1\|\,\tau+\|\widetilde{h}_2\|\,\tau+\varepsilon^2\right)\sum_{k=1}^{2}\left\|D_k^{-1/2}x_{(k)}\right\|\right)\\&\leqslant\left(\tau\sigma^{-1}\sum_{k=1}^{2}\left\|D_k^{-1/2}x_{(k)}\right\|+\|\widetilde{h}_1\|\,\tau+\|\widetilde{h}_2\|\,\tau+\varepsilon^2\right)^2+\tau\sigma^{-1}\Lambda\leqslant1.\end{aligned} \tag{20}$$

Introduce the functions

$$g_k(z) = \varphi(F_k, z), \qquad \overline{g}_{kl}(z) = \varphi\big(\overline{F}_k\big(\widetilde{h}_l\big), z\big), \qquad k, l = 1, 2, \tag{21}$$

(see (0.4)). Then, using (21), (1.3), (1.27), and taking into account (12), we obtain that

$$\begin{aligned}\big|\, d_u d_v^2\, \overline{g}_{11}(0) - d_u d_v^2\, \overline{g}_{12}(0)\,\big| &= \big|\, d_u d_v^2\, g_1\big(\widetilde{h}_1\big) - d_u d_v^2\, g_1\big(\widetilde{h}_2\big)\,\big| \\ &= \left|\, \int_0^1 d_{\widetilde{h}_2 - \widetilde{h}_1}\, d_u d_v^2\, g_1(z)\Big|_{z = \widetilde{h}_1 + \alpha(\widetilde{h}_2 - \widetilde{h}_1)} d\alpha \,\right| \\ &\leqslant 2\, \|\widetilde{h}_1 - \widetilde{h}_2\|\, \tau\, \|u\|\, \tau \big\langle D_1 v, v\big\rangle. \end{aligned} \tag{22}$$

From (11), (19), (20), (22) and from the definition of σ^2 it follows that

$$\big|\, d_u d_v^2\, \overline{g}_{11}(0) - d_u d_v^2\, \overline{g}_{12}(0)\,\big| \leqslant c\varepsilon_0^2\, \|u\|\, \tau \big\langle D_1 v, v\big\rangle. \tag{23}$$

Analogously, using (2), (21), (0.5), we have

$$\begin{aligned}\big|\, d_v^2\, \overline{g}_{11}(0) - d_v^2\, \overline{g}_{12}(0)\,\big| &= \big|\, d_v^2\, g_1\big(\widetilde{h}_1\big) - d_v^2\, g_1\big(\widetilde{h}_2\big)\,\big| \\ &= \left|\, \int_0^1 d_{\widetilde{h}_2 - \widetilde{h}_1}\, d_v^2\, g_1(z)\Big|_{z = \widetilde{h}_1 + \alpha(\widetilde{h}_2 - \widetilde{h}_1)} d\alpha \,\right| \\ &\leqslant \|\widetilde{h}_1 - \widetilde{h}_2\|\, \tau \big\langle D_1 v, v\big\rangle \end{aligned} \tag{24}$$

and, consequently,

$$\big|\, d_v^2\, \overline{g}_{11}(0) - d_v^2\, \overline{g}_{12}(0)\,\big| \leqslant c\varepsilon_0^2 \big\langle D_1 v, v\big\rangle. \tag{25}$$

It follows from (4), (11)–(13), (20), (21), (23), (25), (1.10), (1.11), (1.15) that

$$\begin{aligned}&\big|\, d_u d_v^2\, \overline{g}_{11}(0) - d_u d_v^2\, \overline{g}_{22}(0)\,\big| \\ &\quad\leqslant \big|\, d_u d_v^2\, \overline{g}_{11}(0) - d_u d_v^2\, \overline{g}_{12}(0)\,\big| + \big|\, d_u d_v^2\, \overline{g}_{12}(0) - d_u d_v^2\, \overline{g}_{22}(0)\,\big| \\ &\quad\leqslant c\, \Big(\varepsilon_0^2 + \varepsilon + \max_{k=1,2}\, \|\widetilde{h}_k\|\, \tau\Big)\, \|u\|\, \tau \big\langle D_1 v, v\big\rangle \\ &\quad\leqslant c\, \Big(\varepsilon + \tau\sigma^{-1} \max_{k=1,2}\, \big\|\, D_k^{-1/2} x_{(k)}\,\big\| + \sqrt{\tau\sigma^{-1}\Lambda}\Big)\, \|u\|\, \tau \big\langle D_1\big(\widetilde{h}_k\big)\, v,\, v\,\big\rangle \end{aligned} \tag{26}$$

and, analogously,

$$\begin{aligned}&\big|\, d_v^2\, \overline{g}_{11}(0) - d_v^2\, \overline{g}_{22}(0)\,\big| \leqslant \big|\, d_v^2\, \overline{g}_{11}(0) - d_v^2\, \overline{g}_{12}(0)\,\big| + \big|\, d_v^2\, \overline{g}_{12}(0) - d_v^2\, \overline{g}_{22}(0)\,\big| \\ &\quad\leqslant c\, \Big(\varepsilon_0 + \varepsilon + \max_{k=1,2}\, \|\widetilde{h}_k\|\, \tau\Big)^2 \big\langle D_1 v, v\big\rangle \\ &\quad\leqslant c\, \Big(\varepsilon + \tau\sigma^{-1} \max_{k=1,2}\, \big\|\, D_k^{-1/2} x_{(k)}\,\big\| + \sqrt{\tau\sigma^{-1}\Lambda}\Big)^2 \big\langle D_1\big(\widetilde{h}_k\big)\, v,\, v\,\big\rangle, \end{aligned} \tag{27}$$

for all $u, v \in \mathbb{R}^d$. The relation (7) follows now from (13), (21), (26), (27) and from the definition of classes $\mathcal{R}_d\bigl(\tau, \varepsilon\bigr)$.

Let us pass to the proof of the representation (10). Now we have

$$x_{(1)} = x_{(2)} = x, \quad \text{and, hence,} \quad \Lambda = 0 \tag{28}$$

(see (8)). Let $k = 1$ or $k = 2$. Denote by $\widetilde{p}_k(\,\cdot\,)$ and $\widetilde{r}_k(\,\cdot\,)$ the probability densities of distributions $\overline{F}_k\bigl(\widetilde{h}_k\bigr)$ and $\Phi\Bigl(\overline{F}_k\bigl(\widetilde{h}_k\bigr)\Bigr)$ respectively. Choosing $c_6 \leqslant 5c_{10}/12$, we can ensure with the help of (3) the fulfilment of the relation

$$\|\widetilde{h}_k\|\,\tau \leqslant 2.4\,\tau\,\sigma_k^{-1}\bigl\|\,D_k^{-1/2}x\,\bigr\| \leqslant c_{10}\,d^{-1}, \tag{29}$$

for any constant c_{10} (which can be also chosen sufficiently small). Then from (1.20), (1), (28), (29) it follows that

$$\widetilde{r}_k(x) = \frac{1}{(2\,\pi)^{d/2}\,\bigl(\det D_k\bigl(\widetilde{h}_k\bigr)\bigr)^{1/2}} = \frac{\exp\bigl(c\,\theta\,d\,\|\widetilde{h}_k\|\,\tau\bigr)}{(2\,\pi)^{d/2}\,(\det D_k)^{1/2}} = \frac{e^{c\,\theta}}{(2\,\pi)^{d/2}\,(\det D_k)^{1/2}}\,. \tag{30}$$

It can be verified that, if the constants c_1, c_{10} are sufficiently small, then, by virtue of (9), (13), (29), (1.15), (1.20), (1.21), (2.22), Lemma 2.1 with parameter $2\,\tau$ is applicable to the densities $\widetilde{p}_k(\,\cdot\,)$, $\widetilde{r}_k(\,\cdot\,)$. From this lemma and from (11), (29), (1.20), (1.21) it follows that

$$\bigl|\,\widetilde{p}_k(x) - \widetilde{r}_k(x)\,\bigr| \leqslant \frac{c\,\tau\,d^{3/2}}{\sigma\,(2\,\pi)^{d/2}\,(\det D_k)^{1/2}}\,. \tag{31}$$

Choosing c_1 to be sufficiently small, we can ensure with the help of (30), (31) the validity of the inequality

$$\frac{\bigl|\,\widetilde{p}_k(x) - \widetilde{r}_k(x)\,\bigr|}{\widetilde{r}_k(x)} \leqslant c\,\frac{\tau\,d^{3/2}}{\sigma} \leqslant \frac{1}{2}\,. \tag{32}$$

From (30), (32), we obtain that

$$\widetilde{p}_k(x) = \widetilde{r}_k(x)\,\Bigl(1 + \frac{\widetilde{p}_k(x) - \widetilde{r}_k(x)}{\widetilde{r}_k(x)}\Bigr) = \frac{e^{c\theta}}{(2\,\pi)^{d/2}\,(\det D_k)^{1/2}}\,. \tag{33}$$

Choosing c_7 to be sufficiently small and using (11), (2.19), we find that

$$\det D_1 = e^{c\theta\,d\varepsilon^2}\,\det D_2 = e^{c\theta}\,\det D_2. \tag{34}$$

From (1.2) it follows that

$$p_k(x) = \widetilde{p}_k(x)\;\mathbf{E}\,\exp\Bigl(\bigl\langle\widetilde{h}_k,\,\xi_k\bigr\rangle - \bigl\langle\widetilde{h}_k,\,x\bigr\rangle\Bigr). \tag{35}$$

Let us compare $\widetilde{p}_1(x)$ and $\widetilde{p}_2(x)$. Taking c_6, c_7, c_{10} to be sufficiently small and using (7), (9), (11), (28), (29), (1.15), (1.20), (1.21), we see that the distributions $\overline{F}_1\big(\widetilde{h}_1\big)$, $\overline{F}_2\big(\widetilde{h}_2\big)$ satisfy the conditions of Lemma 3.1 with parameters

$$\widetilde{\tau} = 2\tau, \qquad \widetilde{\varepsilon} = c_5\big(\varepsilon + \tau\sigma^{-1} \min_{k=1,2} \big\| D_k^{-1/2} x \big\|\big). \tag{36}$$

Using (2), (1.20), (1.21) and Lemma 3.1, we obtain that

$$\big|\widetilde{p}_1(x) - \widetilde{p}_2(x)\big| \leqslant cd\Big(\widetilde{\varepsilon}^2 + \frac{d\widetilde{\tau}^2}{\sigma^2}\Big) \max_{k=1,2} \frac{1}{(2\pi)^{d/2}\,(\det D_k)^{1/2}}\,. \tag{37}$$

From (33), (34), (36), (37) one can easily derive the inequality

$$\frac{\big|\widetilde{p}_1(x) - \widetilde{p}_2(x)\big|}{\widetilde{p}_2(x)} \leqslant cd\Big(\widetilde{\varepsilon}^2 + \frac{d\widetilde{\tau}^2}{\sigma^2}\Big) \leqslant \frac{1}{2}\,, \tag{38}$$

if c_1, c_6, c_7 are sufficiently small. Therefore, from (36), (38) it follows that

$$\begin{aligned}\widetilde{p}_1(x) &= \widetilde{p}_2(x)\Big(1 + \frac{\widetilde{p}_1(x) - \widetilde{p}_2(x)}{\widetilde{p}_2(x)}\Big) \\ &= \widetilde{p}_2(x)\,\exp\bigg(\theta cd\,\Big(\varepsilon + \frac{d^{1/2}\tau}{\sigma} + \frac{\tau}{\sigma} \min_{k=1,2} \big\| D_k^{-1/2} x \big\|\Big)^2\bigg).\end{aligned} \tag{39}$$

Consider the functions

$$\psi_k(z) = \varphi(F_k, z) - \big\langle z, x\big\rangle = \log \mathbf{E}\,\exp\big(\big\langle z, \xi_k\big\rangle - \big\langle z, x\big\rangle\big), \qquad k = 1, 2, \tag{40}$$

for $z \in \mathbb{R}^d$, $\|z\|\,\tau < 1$. By (1.8), (1),

$$d_u\,\psi_k(z)\Big|_{z=\widetilde{h}_k} = \mathbf{E}\,\big\langle \overline{\xi}_k(\widetilde{h}_k),\, u\big\rangle - \big\langle x, u\big\rangle = 0 \tag{41}$$

and, by virtue of (1.9),

$$d_u^2\,\psi_k(z)\Big|_{z=h} = \big\langle D_k\big(h\big)\,u,\, u\big\rangle > 0, \tag{42}$$

for all $u, h \in \mathbb{R}^d$, $u \neq 0$, $\|h\|\,\tau < 1$. From (41), (42) it can be easily derived that the functions $\psi_k(z)$ take at points $z = \widetilde{h}_k$ their minimal values for $z \in \mathbb{R}^d$, $\|z\|\,\tau < 1$. In particular, taking into account (2), we see that

$$\psi_1(\widetilde{h}_1) < \psi_1(\widetilde{h}_2), \qquad \psi_2(\widetilde{h}_2) < \psi_2(\widetilde{h}_1) \tag{43}$$

and, consequently,

$$\big|\psi_1(\widetilde{h}_1) - \psi_2(\widetilde{h}_2)\big| \leqslant \max\Big\{\big|\psi_1(\widetilde{h}_1) - \psi_2(\widetilde{h}_1)\big|,\ \big|\psi_1(\widetilde{h}_2) - \psi_2(\widetilde{h}_2)\big|\Big\}. \tag{44}$$

By (1.31), (3), (11), (40), for $l = 1, 2$ we have:

$$\begin{aligned}\big|\psi_1(\widetilde{h}_l) - \psi_2(\widetilde{h}_l)\big| &= \Big|\log \mathbf{E}\,\exp\big(\langle \widetilde{h}_l, \xi_1\rangle\big) - \log \mathbf{E}\,\exp\big(\langle \widetilde{h}_l, \xi_2\rangle\big)\Big| \\ &\leqslant \frac{1}{2}\,\big(\varepsilon + \|\widetilde{h}_l\|\,\tau\big)^2 \langle D\widetilde{h}_l, \widetilde{h}_l\rangle \\ &\leqslant c \min_{k=1,2}\,\Big(\varepsilon + \tau\sigma^{-1}\big\|D_k^{-1/2}x\big\|\Big)^2 \big\|D_k^{-1/2}x\big\|^2. \end{aligned} \tag{45}$$

According to (40), the formula (35) can be rewritten in the form

$$p_k(x) = \widetilde{p}_k(x)\, e^{\psi_k(\widetilde{h}_k)}, \qquad k = 1, 2. \tag{46}$$

Using (44), (45), we see that

$$e^{\psi_1(\widetilde{h}_1)} = e^{\psi_2(\widetilde{h}_2)}\,\exp\Big(c\theta \min_{k=1,2}\,\Big(\varepsilon + \tau\sigma^{-1}\big\|D_k^{-1/2}x\big\|\Big)^2 \big\|D_k^{-1/2}x\big\|^2\Big). \tag{47}$$

The relation (10) follows now from (39), (46), (47).

6. Non-uniform estimate for the closeness of conditional densities for $F \in \mathcal{A}_d(\tau)$

Below for $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $d \geqslant 2$, we shall denote by $x' = (x_1, \ldots, x_{d-1})$ the vector consisting of the first $d-1$ coordinates of x. Analogously, we shall use a prime to denote the matrix D' of size $(d-1)\times(d-1)$ consisting of the first $d-1$ rows and first $d-1$ columns of a matrix D of size $d \times d$.

Lemma 6.1. *Suppose that* $\tau \geqslant 0$,

$$F = \mathcal{L}(\xi) \in \mathcal{A}_d^*(\tau, 4, 4) \qquad \textit{and, if } d > 1,\ \ F' = \mathcal{L}(\xi') \in \mathcal{A}_{d-1}^*(\tau, 4, 4). \tag{1}$$

Assume that $\mathbf{E}\,\xi = 0$ *and the last coordinate* ξ_d *of the vector* ξ *is not correlated with the previous ones. Let* $\sigma^2 = \langle De_d, e_d\rangle = \mathbf{E}\,\xi_d^2 > 0$ *be the minimal eigenvalue of* $D = \operatorname{cov} F$. *Denote by* $p(x)$, $x \in \mathbb{R}^d$, *the probability density of distribution* F. *Let* $p'(x')$, $x' \in \mathbb{R}^{d-1}$, *be the density of the vector* ξ'. *Denote by*

$$p^{(d)}(x_d \,|\, x') = p(x)/p'(x'), \qquad x_d \in \mathbb{R}^1, \tag{2}$$

the conditional density of the d^{th} *coordinate* ξ_d *of the vector* $\xi \in \mathbb{R}^d$ *for a fixed value of* $\xi' = x'$. *For* $d = 1$ *we define* $p^{(1)}(x_1 \,|\, x') = p(x)$. *Denote by* $K_d = K_{d,x'}$ *the corresponding one-dimensional conditional distribution.*

Then there exist absolute positive constants $c_1, \dots, c_6$ *such that*[2] *the following statements are true for* $\tau d^{3/2}/\sigma \leqslant c_1$, $\left| \left(D'\right)^{-1/2} x' \right| \leqslant c_2 \sigma / d^{3/2} \tau$.

For $d \geqslant 2$ *there exists a parameter* $h' = h'(x') \in \mathbb{R}^{d-1}$ *which gives the solution of the equation* $\mathbf{E}\,\overline{\xi}'(h') = x'$. *Introduce the parameter* $h \in \mathbb{R}^d$ *which has the first* $d-1$ *coordinates coinciding with corresponding coordinates of* h' *and the last* d^{th} *coordinate* $h_d = 0$. *If* $d = 1$, *we take* $h = 0$. *Denote* $y = (0, 0, \dots, 0, y_d) \in \mathbb{R}^d$, *where* $y_d = \mathbf{E}\left\langle \overline{\xi}(h), e_d \right\rangle$. *Then*

$$|y_d| \leqslant 2.88\,\tau \left\| \left(D'\right)^{-1/2} x' \right\|^2 \leqslant 2.88\,\tau\, \frac{\|x'\|^2}{\sigma^2}\,, \tag{3}$$

and, if $|x_d| \leqslant \dfrac{c_3\,\sigma^2}{d\,\tau}$, *then for the density* $v(x_d)$ *of the distribution* $H_d = K_d E_{-y_d}$ *the following representation is valid*:

$$v(x_d) = \varphi_\sigma(x_d)\, \exp\left(\theta\, c_4\, \frac{\tau}{\sigma} \left(d^{3/2} + d \left\| \left(D'\right)^{-1/2} x' \right\| \left(1 + \frac{x_d^2}{\sigma^2} \right) + \frac{|x_d|^3}{\sigma^3} \right) \right), \tag{4}$$

and

$$\frac{1}{2\sqrt{2\,\pi}\,\sigma}\, \exp\left(-\frac{x_d^2}{\sigma^2} \right) \leqslant v(x_d) \leqslant \frac{2}{\sqrt{2\,\pi}\,\sigma}\, \exp\left(-\frac{x_d^2}{4\,\sigma^2} \right). \tag{5}$$

Besides, for all $x_d \in \mathbb{R}^1$,

$$v(x_d) \leqslant \frac{c_5}{\sigma}\, \exp\left(-\min\left\{ \frac{x_d^2}{4\,\sigma^2}\,,\, \frac{c_6\,|x_d|}{d\,\tau} \right\} \right). \tag{6}$$

Proof. At first we shall prove the relations (3)–(5). For $d = 1$ they can be easily derived from Lemma 4.1.

Let $d \geqslant 2$. Fix $x' \in \mathbb{R}^{d-1}$, $\left| \left(D'\right)^{-1/2} x' \right| \leqslant c_2 \sigma / d^{3/2} \tau$, where c_2 is so small as it is necessary for the following below arguments. Note that in the proof of (6) we shall use the evident possibility to choose c_2, c_3 so that $4 c_2 = c_3$. Taking now c_2 to be sufficiently small, we ensure the fulfilment of the inequality

$$\frac{12\,\tau}{5\,\sigma} \left\| \left(D'\right)^{-1/2} x' \right\| \leqslant \frac{12\,\tau\, d^{1/2}}{5\,\sigma} \left| \left(D'\right)^{-1/2} x' \right| \leqslant \frac{12\,c_2}{5\,d} \leqslant \frac{1}{2}\,. \tag{7}$$

The distribution $F' = \mathcal{L}(\xi')$ satisfies the $(d-1)$-dimensional version of Lemma 1.2. By this lemma, there exists a parameter $h' \in \mathbb{R}^{d-1}$ satisfying the relations

$$\begin{gathered} \mathbf{E}\,\overline{\xi}'(h') = x', \qquad \left\| h' \right\| \tau \leqslant {}^1\!/_2, \\ \sigma \left\| h' \right\| \leqslant \left\| (D')^{1/2} h' \right\| \leqslant 2.4 \left\| (D')^{-1/2} x' \right\| \end{gathered} \tag{8}$$

(it is evident that the minimal eigenvalue of the operator D' is at least σ^2).

[2]For $d = 1$ we use the natural agreement $\left| \left(D'\right)^{-1/2} x' \right| = \left\| \left(D'\right)^{-1/2} x' \right\| = |x'| = \|x'\| = 0$.

Now we can define the parameter $h \in \mathbb{R}^d$ which has the first $d-1$ coordinates coinciding with corresponding coordinates of h' and the d^{th} coordinate $h_d = 0$. It is clear that $\langle h, y\rangle = 0$ and, hence,

$$\begin{aligned}\mathbf{E}\exp\big(\langle h,\xi\rangle - \langle h, x+y\rangle\big) &= \mathbf{E}\exp\big(\langle h,\xi\rangle - \langle h, x\rangle\big)\\ &= \mathbf{E}\exp\big(\langle h',\xi'\rangle - \langle h',\, x'\rangle\big).\end{aligned} \tag{9}$$

Denote by $\overline{p}(\cdot)$, $\overline{p}'(\cdot)$ the densities of conjugate random vectors $\overline{\xi}(h)$, $\overline{\xi}'(h')$ respectively. By (1.2), the densities $p(\cdot)$, $p'(\cdot)$ can be represented as

$$\begin{aligned}p(x) &= \overline{p}(x)\,\mathbf{E}\exp\big(\langle h,\xi\rangle - \langle h, x\rangle\big),\\ p'(x') &= \overline{p}'(x')\,\mathbf{E}\exp\big(\langle h',\xi'\rangle - \langle h', x'\rangle\big).\end{aligned} \tag{10}$$

From (2), (9), (10) it follows that

$$v(x_d) = p^{(d)}(x_d + y_d \,|\, x') = \frac{p(x+y)}{p'(x')} = \frac{\overline{p}(x+y)}{\overline{p}'(x')}. \tag{11}$$

Besides, since $h_d = 0$, from (9), (10) it is easy to deduce that

$$\mathcal{L}\big(\overline{\xi}'(h')\big) = \mathcal{L}\big(\big(\overline{\xi}(h)\big)'\big) \tag{12}$$

and, consequently,

$$\mathbf{E}\,\big(\overline{\xi}(h)\big)' = \mathbf{E}\,\overline{\xi}'(h') = x', \qquad \operatorname{cov}\overline{\xi}'(h') = \big(D(h)\big)'. \tag{13}$$

Further, from the conditions of the lemma it follows that

$$\big\langle Dh, e_d\big\rangle = 0. \tag{14}$$

Finally, it is evident that

$$\|h\| = \|h'\|, \qquad \big\|\,(D')^{1/2}h'\,\big\| = \big\|\,D^{1/2}h\,\big\|. \tag{15}$$

Using (7), (8), (15) and taking $c_2 < 5c_7/12$, we ensure the fulfilment of the inequalities

$$\|h\|\,\tau \leqslant \frac{\tau}{\sigma}\,\big\|\,D^{1/2}h\,\big\| \leqslant \frac{12\,\tau}{5\,\sigma}\,\big\|\,\big(D'\big)^{-1/2}x'\,\big\| \leqslant \frac{c_7}{d}, \tag{16}$$

for any positive absolute constant c_7, which we shall also choose to be sufficiently small.

By (8), (15), Lemma 1.1 is applicable to the distributions of the conjugate random vectors $\overline{\xi}'(h')$ and $\overline{\xi}(h)$. By this lemma,

$$\mathcal{L}\big(\overline{\xi}'(h')\big) \in \mathcal{A}_{d-1}(2\tau), \qquad \mathcal{L}\big(\overline{\xi}(h)\big) \in \mathcal{A}_d(2\tau). \tag{17}$$

Using the relations (1.17), (8), (14)–(16), we obtain the inequality (3):

$$|y_d| = \Big| \mathbf{E}\,\big\langle \overline{\xi}(h), e_d\big\rangle - \big\langle Dh, e_d\big\rangle \Big| \leqslant \frac{1}{2}\ \|h\|\,\tau\,\sigma\,\big\|D^{1/2}h\,\big\| \\ \leqslant 2.88\,\tau\,\Big\|\big(D'\big)^{-1/2}x'\,\Big\|^2 \leqslant 2.88\,\tau\,\frac{\|\,x'\|^2}{\sigma^2}\,. \tag{18}$$

From (1.20), (13), it follows that

$$\begin{aligned} \big(\det D(h)\big)^{1/2} &= \big(\det D\big)^{1/2}\,\exp\big(c\theta\,d\,\|h\|\,\tau\big), \\ \big(\det\,(D(h))'\big)^{1/2} &= \big(\det D'\big)^{1/2}\,\exp\big(c\theta\,d\,\|h'\|\,\tau\big). \end{aligned} \tag{19}$$

By the conditions of Lemma 6.1,

$$\det D = \sigma^2\,\det D'. \tag{20}$$

Denote by $\varkappa_{lm}$ the elements of the matrix $\big(D(h)\big)^{-1}$, $l,m = 1,\dots,d$. Applying the inversion formula for matrices, we can deduce from (15), (19), (20) that

$$\varkappa_{dd} = \frac{\det\big(D(h)\big)'}{\det D(h)} = \sigma^{-2}\,\exp\big(\theta\,c\,d\,\|h\|\,\tau\big). \tag{21}$$

Let us return to the formula (11). Using (13), (17) and applying the $(d-1)$-dimensional version of Lemma 4.1 (with parameter $\widetilde{\tau} = 2\tau$), we see that

$$\overline{p}\,'(x') = \frac{1}{(2\,\pi)^{(d-1)/2}\big(\det\big(D(h)\big)'\big)^{1/2}}\ \exp\big(\theta\,c\,\tau\sigma^{-1}d^{3/2}\big). \tag{22}$$

The fulfilment of the conditions of Lemma 4.1 is ensured by (1), (16), (17), (1.15), (1.20), (1.21), (2.22), by Remark 1.1 and by choosing c_1, c_2, c_7 to be sufficiently small. For estimating $\overline{p}(x+y)$ we shall also apply Lemma 4.1. Introduce the centered random vector $\xi^*(h) = \overline{\xi}(h) - \mathbf{E}\,\overline{\xi}(h) = \overline{\xi}(h) - a$, where we denote $a = (x_1, x_2, \dots, x_{d-1}, y_d) \in \mathbb{R}^d$ (see (13) and the definition of y_d). It is clear that $\mathbf{E}\,\xi^*(h) = 0$, $\mathcal{L}(\xi^*(h)) \in \mathcal{A}_d(2\tau)$ (see (17)). The distribution density $p^*(z)$ of the random vector $\xi^*(h)$ can be expressed as $p^*(z) = \overline{p}(z+a)$. Putting here

$$z = x + y - a = x^{(d)} = (0, 0, \dots, 0, x_d) \in \mathbb{R}^d, \tag{23}$$

we obtain that

$$\overline{p}(x+y) = p^*\big(x^{(d)}\big). \tag{24}$$

By (16), (21), (23), choosing c_7 to be sufficiently small, we can easily derive the relation

$$\| \big(D(h)\big)^{-1/2} x^{(d)} \|^2 = \big\langle \big(D(h)\big)^{-1} x^{(d)}, x^{(d)} \big\rangle = \varkappa_{dd}\, x_d^2 = \frac{x_d^2}{\sigma^2}\, \exp\big(\theta\, c d\, \|h\|\, \tau\big)$$
$$= \frac{x_d^2}{\sigma^2}\, \big(1 + \theta\, c d\, \|h\|\, \tau\big). \tag{25}$$

By Lemma 4.1 (with parameter $\widetilde{\tau} = 2\tau$), we have:

$$p^*\big(x^{(d)}\big) = \frac{1}{(2\,\pi)^{d/2}\,\big(\det D(h)\big)^{1/2}}\, \exp\Big(-\frac{1}{2}\, \| \big(D(h)\big)^{-1/2} x^{(d)} \|^2$$
$$+ \theta\, c\, \tau\, \sigma^{-1}\, \Big(d^{3/2} + \| \big(D(h)\big)^{-1/2} x^{(d)} \|^3\Big)\Big). \tag{26}$$

The fulfilment of the conditions of this lemma can be ensured with the help of Remark 1.1, (1), (16), (17), (25), (1.15), (1.20), (1.21), (2.22) by choosing c_1, c_3, c_7 to be sufficiently small.

The relations (4) follow now from (11), (16), (21), (22), (24)–(26). The inequalities (5) follow from (4), if we choose sufficiently small c_1, c_2, c_3.

For proving the inequalities (6) (for $d \geqslant 1$) we define the parameter $h^* \in \mathbb{R}^d$ by the relations $h_j^* = 0,\ j = 1, \ldots, d-1,$ and $h_d^* = c_3 \big/ 4\,\tau d$. Below we take into account the possibility to choose $c_3 = 4 c_2$ so small as it will be necessary. Denote by $\overline{p}_*(\cdot)$ the density of the conjugate random vector $\overline{\xi}(h^*)$, By (1.2), the density $p(\cdot)$ can be represented as

$$p(x) = \overline{p}_*(x)\ \mathbf{E}\, \exp\Big(\big\langle h^*, \xi\big\rangle - \big\langle h^*, x\big\rangle\Big). \tag{27}$$

It is evident that

$$\big\langle h^*, x\big\rangle = \frac{c_3\, x_d}{4\,\tau\, d}\,. \tag{28}$$

Taking $c_3 < 1$, we have

$$\|h^*\|\, \tau = \frac{c_3}{4d} < \frac{1}{3}\,. \tag{29}$$

Therefore, we can apply Lemma 1.1. By this lemma, for $x_d \geqslant \dfrac{c_3\, \sigma^2}{d\,\tau}$,

$$\log \mathbf{E}\, e^{\langle h^*, \xi\rangle} = \frac{1}{2}\, \big\langle D h^*, h^*\big\rangle \Big(1 + \frac{1}{3}\theta\ \|h^*\|\, \tau\Big) \leqslant \frac{c_3^2\, \sigma^2}{16\,\tau^2\, d^2} \leqslant \frac{c_3\, x_d}{16\,\tau\, d} \tag{30}$$

and

$$\big(\det D(h^*)\big)^{1/2} = \big(\det D\big)^{1/2}\, \exp\big(c\theta\, d\ \|h^*\|\, \tau\big). \tag{31}$$

Besides,

$$\mathcal{L}\big(\overline{\xi}(h^*)\big) \in \mathcal{A}_d(2\tau). \tag{32}$$

Taking into account (1), (29), (31), (32), (1.15), (1.21), (2.22), and choosing c_1, c_2, c_3 to be sufficiently small, we can apply to $\overline{p}_*(\cdot)$ Lemma 2.1 with parameter $\widetilde{\tau} = 2\tau$. By this lemma and by (29), (31), (1.21),

$$\sup_{w\in\mathbb{R}^d} \bigl|\overline{p}_*(w)\bigr| \leqslant \frac{\bigl(1 + c\,\tau\, d^{3/2}\,\sigma^{-1}\bigr)}{(2\,\pi)^{d/2}\bigl(\det D(h^*)\bigr)^{1/2}} \leqslant \frac{c}{(2\,\pi)^{d/2}\,(\det D)^{1/2}}\,, \tag{33}$$

if we choose c_1 to be sufficiently small.

Let now $d \geqslant 2$. Applying the $(d-1)$-dimensional version of Lemma 4.1, we see that

$$p'(x') = \frac{1}{(2\,\pi)^{(d-1)/2}\bigl(\det D'\bigr)^{1/2}}\ \exp\Bigl(-\frac{1}{2}\,\bigl\|\,\bigl(D'\bigr)^{-1/2}x'\,\bigr\|^2 + \theta\,c\,\tau\sigma^{-1}\,\Bigl(d^{3/2} + \bigl\|\,\bigl(D'\bigr)^{-1/2}x'\,\bigr\|^3\Bigr)\Bigr). \tag{34}$$

The fulfilment of the conditions of Lemma 4.1 is ensured by (1), (29), (31), (32), (1.15), (1.21), (2.22) and by choosing c_1, c_2, c_3 to be sufficiently small. Besides, using an available choice of c_1, c_2 once again, we obtain from (34) that

$$p'(x') \geqslant \frac{2}{(2\,\pi)^{(d-1)/2}\bigl(\det D'\bigr)^{1/2}}\ \exp\Bigl(-\bigl\|\,\bigl(D'\bigr)^{-1/2}x'\,\bigr\|^2\Bigr). \tag{35}$$

By the conditions of lemma, since we choose c_2, c_3 so that $4\,c_2 = c_3$, for $x_d \geqslant \frac{c_3\,\sigma^2}{d\,\tau}$, we have

$$\bigl\|\,\bigl(D'\bigr)^{-1/2}x'\,\bigr\|^2 \leqslant \frac{c_2^2\,\sigma^2}{d^2\,\tau^2} \leqslant \frac{c_3\,x_d}{16\,\tau\,d}\,. \tag{36}$$

Taking c_3 to be sufficiently small and using (3), (29), (36), we obtain

$$\bigl|\bigl\langle h^*, y\bigr\rangle\bigr| = \frac{c_3\,|y_d|}{4\,\tau\,d} \leqslant \frac{3\,c_3\,\bigl\|\,\bigl(D'\bigr)^{-1/2}x'\,\bigr\|^2}{4d} \leqslant \frac{3\,c_3^2\,x_d}{64\,d^2\,\tau} \leqslant \frac{c_3\,x_d}{16\,\tau\,d}\,. \tag{37}$$

From (11), (20), (27), (28), (30), (33), (35)–(37) it follows that, for $x_d \geqslant \frac{c_3\,\sigma^2}{d\,\tau}$,

$$v(x_d) \leqslant \frac{c}{\sigma}\ \exp\Bigl(-\frac{c_3\,|x_d|}{16\,d\,\tau}\Bigr). \tag{38}$$

For $d = 1$ the inequality (38) can be derived from (27), (28), (30), (33), since in this case $v(x_d) = p(x)$. It is clear that the inequality (38) can be analogously obtained in the case $x_d \leqslant -\frac{c_3\,\sigma^2}{d\,\tau}$.

If $|x_d| \leqslant \frac{c_3\,\sigma^2}{d\,\tau}$, we can use the inequalities (5). The inequality (6) follows from (5), (38).

7. Non-uniform estimate for the closeness of conditional densities for $(F_1, F_2) \in \mathcal{R}_d\big(\tau, 0\big)$

The notations introduced at the beginning of Section 6 will be used in this section too.

Lemma 7.1. *Suppose that* $\tau \geqslant 0$, $(F_1, F_2) \in \mathcal{R}_d\big(\tau, 0\big)$. *Let, for* $k = 1, 2$,

$$F_k = \mathcal{L}(\xi_k) \in \mathcal{A}_d^*(\tau, 4, 4) \qquad \text{and, if } d > 1, \quad F_k' = \mathcal{L}(\xi_k') \in \mathcal{A}_{d-1}^*(\tau, 4, 4). \tag{1}$$

Assume that $\mathbf{E}\,\xi_1 = \mathbf{E}\,\xi_2 = 0$ *and the last coordinates* $\xi_{k,d}$ *of the vectors* ξ_k *are not correlated with the previous ones. Let* $\sigma^2 = \big\langle De_d, e_d\big\rangle = \mathbf{E}\,\xi_{k,d}^2 > 0$ *be the minimal eigenvalue of the common covariance operator* $D = D_k = \operatorname{cov} F_k$. *Let* $p_k(v)$, $v \in \mathbb{R}^d$, *be the probability densities of distributions* F_k. *Denote by* $p_k'(v')$, $v' \in \mathbb{R}^{d-1}$, *the densities of the vectors* ξ_k'. *Let*

$$p_k^{(d)}(x_{k,d}\,|\,x_k') = p_k(x_k)/p_k'(x_k'), \qquad x_k = (x_{k,1}, \ldots, x_{k,d}), \tag{2}$$

be the corresponding conditional densities of the d^{th} *coordinates* $\xi_{k,d}$ *of the vectors* $\xi_k \in \mathbb{R}^d$ *for fixed values of* $\xi_k' = x_k'$. *For* $d = 1$ *we denote* $p_k^{(1)}(x_{k,1}\,|\,x_k') = p_k(x_{k,1})$. *Denote by* $K_{k,d}$ *the corresponding (depending on* x_k'*) one-dimensional conditional distributions.*

Then there exist absolute positive constants $c_1, \ldots, c_5$ *such that the following statements are true for* $\tau\, d^{3/2}/\sigma \leqslant c_1$, $\Big|\big(D'\big)^{-1/2} x_k'\Big| \leqslant c_2\, \sigma/d^{3/2}\tau$, $k = 1, 2$.

For $d \geqslant 2$ *there exist parameters* $h_k' = h_k'(x_k') \in \mathbb{R}^{d-1}$ *giving the solutions of the equations* $\mathbf{E}\,\overline{\xi}_k'(h_k') = x_k'$. *Introduce the parameters* $h_k \in \mathbb{R}^d$ *which have the first* $d-1$ *coordinates coinciding with corresponding coordinates of* h_k' *and the last* d^{th} *coordinates* $h_{k,d} = 0$. *If* $d = 1$, *we take* $h_k = 0$. *Let*

$$\Lambda = \big\|\,(D')^{-1/2} x_1' - (D')^{-1/2} x_2'\,\big\|, \qquad \Gamma = \max_{k=1,2} \big\|\,(D')^{-1/2} x_k'\,\big\|. \tag{3}$$

Denote $y_k = (0, 0, \ldots, 0, y_{k,d}) \in \mathbb{R}^d$, *where* $y_{k,d} = \mathbf{E}\,\big\langle \overline{\xi}_k(h_k), e_d\big\rangle$. *Then*

$$\big|y_{1,d} - y_{2,d}\big| \leqslant c_3\,\big(\tau^2\,\sigma^{-1}\,\Gamma^3 + \tau\,\Lambda\,\Gamma\big). \tag{4}$$

Denote by $v_k(w)$ *the densities of the distributions* $H_{k,d} = K_{k,d}\,E_{-y_{k,d}}$. *Then the following representation is valid for* $|w| \leqslant \frac{c_4\,\sigma^2}{d\,\tau}$:

$$v_1(w) = v_2(w)\,\exp\!\left(\theta\, c_5\,\frac{\tau^2}{\sigma^2}\,\left(d^2 + \big(\Gamma^2 + \sigma\,\tau^{-1}\Lambda\big)\Big(d + \frac{w^2}{\sigma^2}\Big) + \frac{w^4}{\sigma^4}\right)\right). \tag{5}$$

Besides, for $8\sigma \leqslant w \leqslant \frac{c_4\,\sigma^2}{d\,\tau}$,

$$\begin{aligned} \frac{d}{dz}\,\log\,v_k(z)\Big|_{z=w} &\leqslant -\,\frac{w}{8\sigma^2}\,, \\ \frac{d}{dz}\,\log\,v_k(z)\Big|_{z=-w} &\geqslant \frac{w}{8\sigma^2}\,, \end{aligned} \qquad k=1,2. \tag{6}$$

Proof. For $d=1$ the statement of the lemma can be easily derived from Lemmas 4.1 and 5.1 (see (4.3) and (5.10) with $\varepsilon=0$). Note that in this case $\Gamma=\Lambda=0$.

Let $d\geqslant 2$. Fix $x'_k\in\mathbb{R}^{d-1}$, $\left|\left(D'\right)^{-1/2}x'_k\right|\leqslant c_2\,\sigma/d^{3/2}\tau$, $k=1,2$, where c_2 is so small as it is necessary for the following below arguments. Choosing $c_2<5/24$, we ensure the fulfilment of the inequality

$$\frac{12\,\tau}{5\,\sigma}\left\|\left(D'\right)^{-1/2}x'_k\right\|\leqslant\frac{12\,\tau\,d^{1/2}}{5\,\sigma}\left|\left(D'\right)^{-1/2}x'_k\right|\leqslant\frac{12\,c_2}{5\,d}\leqslant\frac{1}{2}\,. \tag{7}$$

Choosing c_1,c_2 to be sufficiently small, using Corollary 1.1 and comparing (1), (5.9), we see that the distributions $F'_k=\mathcal{L}(\xi'_k)$, satisfy the $(d-1)$-dimensional version of Lemma 5.1 (with $\varepsilon=0$). By this lemma, there exist parameters $h'_k\in\mathbb{R}^{d-1}$ satisfying the relations

$$\begin{gathered} \mathbf{E}\,\overline{\xi}'_k(h'_k)=x'_k, \qquad \left\|\,h'_k\,\right\|\tau\leqslant{}^1\!/_2, \\ \sigma\left\|\,h'_k\,\right\|\leqslant\left\|\,(D')^{1/2}h'_k\,\right\|\leqslant 2.4\left\|\,(D')^{-1/2}x'_k\,\right\|, \\ \left\|\,h'_1-h'_2\,\right\|\tau\leqslant\tau\,\sigma^{-1}\left\|\,(D')^{1/2}h'_1-(D')^{1/2}h'_2\,\right\|\leqslant c\,\left(\tau\,\sigma^{-1}\Gamma\right)^2+\tau\,\sigma^{-1}\Lambda, \\ \Big(\mathcal{L}\big(\overline{\xi}'_1(h'_1)\big),\,\mathcal{L}\big(\overline{\xi}'_2(h'_2)\big)\Big)\in\mathcal{R}_{d-1}\Big(2\tau,\,c\,\left(\tau\,\sigma^{-1}\Gamma+\sqrt{\tau\,\sigma^{-1}\Lambda}\right)\Big) \end{gathered} \tag{8}$$

(it is evident that the minimal eigenvalue of the operator D' is at least σ^2).

Now we can define the parameters $h_k\in\mathbb{R}^d$ which have the first $d-1$ coordinates coinciding with corresponding coordinates of h'_k and the d^{th} coordinates $h_{k,d}=0$. It is clear that $\big\langle h_k,\,y_k\big\rangle=0$ and, hence,

$$\begin{aligned} \mathbf{E}\,\exp\Big(\big\langle h_k,\,\xi_k\big\rangle-\big\langle h_k,\,x_k+y_k\big\rangle\Big) &= \mathbf{E}\,\exp\Big(\big\langle h_k,\,\xi_k\big\rangle-\big\langle h_k,\,x_k\big\rangle\Big) \\ &= \mathbf{E}\,\exp\Big(\big\langle h'_k,\,\xi'_k\big\rangle-\big\langle h'_k,\,x'_k\big\rangle\Big). \end{aligned} \tag{9}$$

Denote by $\overline{p}_k(\cdot)$, $\overline{p}'_k(\cdot)$ the densities of conjugate random vectors $\overline{\xi}_k(h_k)$, $\overline{\xi}'_k(h'_k)$ respectively. By (1.2), the densities $p_k(\cdot)$, $p'_k(\cdot)$ can be represented as

$$\begin{aligned} p_k(x_k) &= \overline{p}_k(x_k)\;\mathbf{E}\,\exp\Big(\big\langle h_k,\,\xi_k\big\rangle-\big\langle h_k,\,x_k\big\rangle\Big), \\ p'_k(x'_k) &= \overline{p}'_k(x'_k)\;\mathbf{E}\,\exp\Big(\big\langle h'_k,\,\xi'_k\big\rangle-\big\langle h'_k,\,x'_k\big\rangle\Big). \end{aligned} \tag{10}$$

From (2), (9), (10) it follows that

$$v_k(x_{k,d}) = p_k^{(d)}(x_{k,d} + y_{k,d}\,|\,x'_k) = \frac{p_k(x_k + y_k)}{p'_k(x'_k)} = \frac{\overline{p}_k(x_k + y_k)}{\overline{p}'_k(x'_k)}. \tag{11}$$

Besides, since $h_{k,d} = 0$, one can easily deduce from (9), (10) that

$$\mathcal{L}\big(\overline{\xi}'_k(h'_k)\big) = \mathcal{L}\Big(\big(\overline{\xi}_k(h_k)\big)'\Big) \tag{12}$$

and, consequently,

$$\mathbf{E}\,\big(\overline{\xi}_k(h_k)\big)' = \mathbf{E}\,\overline{\xi}'_k(h'_k) = x'_k, \qquad \operatorname{cov}\overline{\xi}'_k(h'_k) = \big(D_k(h_k)\big)'. \tag{13}$$

Here $D_k(h_k) = \operatorname{cov}\overline{\xi}_k(h_k)$. Although $D = D_1 = D_2$, the operators $D_k(h_k)$ may be different, even when $h_1 = h_2$. Further, from the conditions of the lemma it follows that

$$\big\langle Dh_k, e_d\big\rangle = 0. \tag{14}$$

Finally, it is evident that

$$\begin{gathered}\|h_k\| = \|h'_k\|, \qquad \big\|\,(D')^{1/2}h'_k\,\big\| = \big\|\,D^{1/2}h_k\,\big\|, \qquad \|h'_1 - h'_2\| = \|h_1 - h_2\|, \\ \big\|\,(D')^{1/2}h'_1 - (D')^{1/2}h'_2\,\big\| = \big\|\,D^{1/2}h_1 - D^{1/2}h_2\,\big\|.\end{gathered} \tag{15}$$

Using (7), (8), (15) and taking $c_2 < 5c_6/12$, we ensure the fulfilment of the inequalities

$$\|h_k\|\,\tau \leqslant \frac{\tau}{\sigma}\,\big\|\,D^{1/2}h_k\,\big\| \leqslant \frac{12\,\tau}{5\,\sigma}\,\big\|\,\big(D'\big)^{-1/2}x'_k\,\big\| \leqslant \frac{c_6}{d}, \tag{16}$$

for any positive absolute constant c_6, which we shall also choose to be sufficiently small.

Introduce the functions

$$g_k(z) = \varphi(F_k, z), \qquad \overline{g}_{kl}(z) = \varphi\big(\overline{F}_k\big(h_l\big),\, z\big), \qquad k, l = 1, 2. \tag{17}$$

(see (0.4)). From (1.9), (1.16), (3), (8), (14)–(17) (with sufficiently small c_6) it follows that, for $0 \leqslant \alpha \leqslant 1$,

$$\begin{aligned}\Big|\,d_{h_2-h_1}\,d_{e_d}\,g_1(z)\,\Big|_{z=h_1+\alpha(h_2-h_1)}\,\Big| &= \Big\langle\,D_1\big(h_1 + \alpha(h_2 - h_1)\big)\,(h_2 - h_1),\, e_d\Big\rangle \\ &\leqslant \max_{k=1,2}\ \|h_k\|\,\tau\,\big\|\,D^{1/2}e_d\,\big\|\,\big\|\,D^{1/2}h_1 - D^{1/2}h_2\,\big\| \\ &= \max_{k=1,2}\ \|h_k\|\,\tau\,\big\|\,D^{1/2}e_d\,\big\|\,\big\|\,(D')^{1/2}h'_1 - (D')^{1/2}h'_2\,\big\| \\ &\leqslant c\,\big(\tau^2\,\sigma^{-1}\,\Gamma^3 + \tau\,\Lambda\,\Gamma\big).\end{aligned} \tag{18}$$

Using (17), (18), (1.8), we see that

$$\begin{aligned}\Big|\big\langle \mathbf{E}\,\overline{\xi}_1(h_1), e_d\big\rangle - \big\langle \mathbf{E}\,\overline{\xi}_1(h_2), e_d\big\rangle\Big| &= \Big|\, d_{e_d} g_1\big(h_1\big) - d_{e_d} g_1\big(h_2\big)\,\Big| \\ &= \left|\int_0^1 d_{h_2-h_1} d_{e_d} g_1(z)\Big|_{z=h_1+\alpha(h_2-h_1)} d\alpha\right| \\ &\leqslant c\big(\tau^2\sigma^{-1}\Gamma^3 + \tau\Lambda\Gamma\big). \end{aligned} \tag{19}$$

From (1.30), (3), (8), (16) (with sufficiently small c_6) it follows that

$$\Big|\big\langle \mathbf{E}\,\overline{\xi}_1(h_2), e_d\big\rangle - \big\langle \mathbf{E}\,\overline{\xi}_2(h_2), e_d\big\rangle\Big| \leqslant 2\,\big(\|h_2\|\,\tau\big)^2\,\big\|D^{1/2}e_d\,\big\|\,\big\|D^{1/2}h_2\,\big\| \leqslant c\tau^2\sigma^{-1}\Gamma^3. \tag{20}$$

The inequality (4) is an evident consequence of (19), (20).

Applying the $(d-1)$-dimensional version of Lemma 5.1 (with parameters $\widetilde{\tau} = 2\tau$, $\widetilde{\varepsilon} = c\,\big(\tau\sigma^{-1}\Gamma + \sqrt{\tau\sigma^{-1}\Lambda}\big)$, see (8), (13), (5.10)), we obtain that

$$\overline{p}_1'(x_1') = \overline{p}_2'(x_2')\,\exp\Big(\theta\, c\,\frac{\tau^2}{\sigma^2}\,d\,\Big(d + \Gamma^2 + \sigma\tau^{-1}\Lambda\Big)\Big). \tag{21}$$

The fulfilment of the conditions of Lemma 5.1 is ensured by (1), (3), (8), (16), (1.15), (1.20), (1.21), (2.22), by Remark 1.1 and by choosing c_1, c_2, c_6 to be sufficiently small.

Taking into account (15)–(17), (1.27), we obtain that, for all $u, v \in \mathbb{R}^d$,

$$\begin{aligned}\Big|d_u d_v^2\,\overline{g}_{11}(0) - d_u d_v^2\,\overline{g}_{12}(0)\Big| &= \Big|\, d_u d_v^2\, g_1\big(h_1\big) - d_u d_v^2\, g_1\big(h_2\big)\,\Big| \\ &= \left|\int_0^1 d_{h_2-h_1} d_u d_v^2\, g_1(z)\Big|_{z=h_1+\alpha(h_2-h_1)} d\alpha\right| \\ &\leqslant 2\,\|h_1-h_2\|\,\tau\,\|u\|\,\tau\big\langle Dv, v\big\rangle \\ &= 2\,\|h_1'-h_2'\|\,\tau\,\|u\|\,\tau\big\langle Dv, v\big\rangle. \end{aligned} \tag{22}$$

From (8), (22) it follows that

$$\Big|d_u d_v^2\,\overline{g}_{11}(0) - d_u d_v^2\,\overline{g}_{12}(0)\Big| \leqslant c\,\Big(\tau^2\sigma^{-2}\Gamma^2 + \tau\sigma^{-1}\Lambda\Big)\,\|u\|\,\tau\big\langle Dv, v\big\rangle. \tag{23}$$

Analogously, using (8), (17), (0.5), we have

$$\begin{aligned}\Big|d_v^2\,\overline{g}_{11}(0) - d_v^2\,\overline{g}_{12}(0)\Big| &= \Big|\, d_v^2\, g_1\big(h_1\big) - d_v^2\, g_1\big(h_2\big)\,\Big| \\ &= \left|\int_0^1 d_{h_2-h_1} d_v^2\, g_1(z)\Big|_{z=h_1+\alpha(h_2-h_1)} d\alpha\right| \\ &\leqslant \|h_1'-h_2'\|\,\tau\big\langle Dv, v\big\rangle \end{aligned} \tag{24}$$

and, consequently,

$$\left|d_v^2\,\overline{g}_{11}(0)-d_v^2\,\overline{g}_{12}(0)\right|\leqslant c\,\left(\tau^2\,\sigma^{-2}\,\Gamma^2+\tau\,\sigma^{-1}\Lambda\right)\left\langle Dv,v\right\rangle. \tag{25}$$

Choosing c_2 to be sufficiently small, we can ensure the fulfilment of the inequality

$$\varepsilon=\tau\,\sigma^{-1}\Gamma+\sqrt{\tau\,\sigma^{-1}\Lambda}\leqslant 1. \tag{26}$$

Taking into account (16), we can use the items b) of Lemmas 1.1, 1.4. By these lemmas,

$$\left(\overline{F}_1(h_2),\overline{F}_2(h_2)\right)\in\mathcal{R}_d\big(2\,\tau,\,2\,\|h_2\|\,\tau\big),\qquad \overline{F}_k(h_k)\in\mathcal{A}_d(2\tau),\quad k=1,2. \tag{27}$$

It follows from (3), (16), (17), (23), (26), (27), (1.10), (1.15) that

$$\begin{aligned}\left|d_u d_v^2\,\overline{g}_{11}(0)-d_u d_v^2\,\overline{g}_{22}(0)\right|&\\ \leqslant\left|d_u d_v^2\,\overline{g}_{11}(0)-d_u d_v^2\,\overline{g}_{12}(0)\right|&+\left|d_u d_v^2\,\overline{g}_{12}(0)-d_u d_v^2\,\overline{g}_{22}(0)\right|\\ \leqslant c\,\varepsilon\,\|u\|\,\tau\left\langle Dv,v\right\rangle\leqslant c\,\varepsilon\,\|u\|\,\tau\left\langle D_k\big(h_k\big)\,v,\,v\right\rangle,&\qquad k=1,2,\end{aligned} \tag{28}$$

for all $u,v\in\mathbb{R}^d$ (remember that $D=D_1=D_2$). Analogously (see also (25), (1.11)),

$$\begin{aligned}\left|d_v^2\,\overline{g}_{11}(0)-d_v^2\,\overline{g}_{22}(0)\right|&\leqslant\left|d_v^2\,\overline{g}_{11}(0)-d_v^2\,\overline{g}_{12}(0)\right|+\left|d_v^2\,\overline{g}_{12}(0)-d_v^2\,\overline{g}_{22}(0)\right|\\ &\leqslant c\,\varepsilon^2\left\langle Dv,v\right\rangle\leqslant c\,\varepsilon^2\left\langle D_k\big(h_k\big)\,v,\,v\right\rangle,\qquad k=1,2.\end{aligned} \tag{29}$$

It follows from (17), (27)–(29) and from the definition of classes $\mathcal{R}_d\big(\tau,\varepsilon\big)$ that

$$\Big(\mathcal{L}\big(\overline{\xi}_1(h_1)\big),\,\mathcal{L}\big(\overline{\xi}_2(h_2)\big)\Big)=\left(\overline{F}_1(h_1),\overline{F}_2(h_2)\right)\in\mathcal{R}_d\big(2\,\tau,\,c\varepsilon\big). \tag{30}$$

Let now $x_{1,d}=x_{2,d}=w$. For comparing of $v_k(w)$ with the help of (11), (21) we need to compare $\overline{p}_k(x_k+y_k)$, $k=1,2$. For this we shall use again Lemma 5.1. Introduce the centered random vectors

$$\xi_k^*(h_k)=\overline{\xi}_k(h_k)-\,\mathbf{E}\,\overline{\xi}_k(h_k)=\overline{\xi}_k(h_k)-a_k, \tag{31}$$

where $a_k=(x_{k,1},x_{k,2},\ldots,x_{k,d-1},y_{k,d})\in\mathbb{R}^d$ (see (13) and the definition of $y_{k,d}$). It is clear that

$$\mathbf{E}\,\xi_1^*(h_1)=\,\mathbf{E}\,\xi_2^*(h_2)=0. \tag{32}$$

Besides, the density $p_k^*(t)$ of the random vector $\xi_k^*(h_k)$ can be represented as $p_k^*(t)=\overline{p}_k(t+a_k)$. Putting here

$$t=t_w=x_k+y_k-a_k=(0,0,\ldots,0,w)\in\mathbb{R}^d, \tag{33}$$

we obtain that

$$\overline{p}_k(x_k + y_k) = p_k^*\big(t_w\big). \tag{34}$$

From (30), (31) it follows that

$$\Big(\mathcal{L}\big(\xi_1^*(h_1)\big),\, \mathcal{L}\big(\xi_2^*(h_2)\big)\Big) \in \mathcal{R}_d\big(2\tau,\, c\varepsilon\big). \tag{35}$$

By (15), (16) (with sufficiently small c_6), Lemma 1.1 is applicable to the distributions of the conjugate random vectors $\overline{\xi}_k'(h_k')$ and $\overline{\xi}_k(h_k)$, $k = 1, 2$. Using this lemma (formula (1.20)) and taking into account (13), we see that

$$\begin{aligned} \big(\det D_k(h_k)\big)^{1/2} &= \big(\det D\big)^{1/2}\, \exp\big(c\theta d\, \|h_k\|\, \tau\big), \\ \big(\det\, (D_k(h_k))'\big)^{1/2} &= \big(\det D'\big)^{1/2}\, \exp\big(c\theta d\, \|h_k'\|\, \tau\big). \end{aligned} \tag{36}$$

By the conditions of Lemma 7.1,

$$\det D = \sigma^2\, \det D'. \tag{37}$$

Denote by $\varkappa_{lm}^{(k)}$ the elements of the matrix $\big(D_k(h_k)\big)^{-1}$, $l, m = 1, \ldots, d$, $k = 1, 2$. Applying the inversion formula for matrices, we can deduce from (15), (36), (37) that

$$\varkappa_{dd}^{(k)} = \frac{\det\big(D_k(h_k)\big)'}{\det D_k(h_k)} = \sigma^{-2}\, \exp\big(\theta c d\, \|h_k\|\, \tau\big). \tag{38}$$

By (16), (33), (38), choosing c_6 to be sufficiently small, we can easily derive the relation

$$\Big\|\, \big(D_k(h_k)\big)^{-1/2} t_w \,\Big\|^2 = \big\langle \big(D_k(h_k)\big)^{-1} t_w,\, t_w \big\rangle = \varkappa_{dd}^{(k)}\, w^2 \leqslant c\, \frac{w^2}{\sigma^2}\,. \tag{39}$$

By Lemma 5.1 (with parameters $\widetilde{\tau} = 2\tau$, $\widetilde{\varepsilon} = c\varepsilon$), we have:

$$\begin{aligned} p_1^*\big(t_w\big) = p_2^*\big(t_w\big)\, \exp\Bigg(\theta\, c\Big(\varepsilon\Big(\sqrt{d} + \Big\|\, \big(D_1(h_1)\big)^{-1/2} t_w \,\Big\|\Big) \\ + \tau\sigma^{-1}\Big(d + \Big\|\, \big(D_1(h_1)\big)^{-1/2} t_w \,\Big\|^2\Big)\Big)^2\Bigg). \end{aligned} \tag{40}$$

The fulfilment of the conditions of this lemma can be ensured by using (1), (3), (16), (32), (33), (35), (36), (39), (1.15), (1.21), (2.22), by Remark 1.1 and by choosing c_1, c_4, c_6 to be sufficiently small. The relation (5) follows now from (11), (21), (26), (34), (39), (40).

Using now (11) (with $x_{k,d} = w$), (33)–(35), and the inequality (4.3) of Lemma 4.1, we see that

$$\frac{d}{dz}\log v_k(z)\Big|_{z=w} = \frac{d}{dw}\log p_k^*\big(t_w\big) = w^{-1}d_{t_w}\log p_k^*\big(t\big)\big|_{t=t_w}$$
$$\leqslant -\frac{1}{7\,w}\,\Big\|\big(D_k(h_k)\big)^{-1/2}t_w\Big\|^2, \tag{41}$$

for $7 < \Big\|\big(D_k(h_k)\big)^{-1/2}t_w\Big\| \leqslant \frac{c\,\sigma}{d\,\tau}$, $w > 0$. The fulfilment of the conditions of Lemma 4.1 can be ensured by the available choice of the constants c_1, c_4, c_6 and by using Remark 1.1 and the relations (1), (16), (35), (36), (38), (39), (1.15), (1.21), (2.22). The same arguments are needed to derive from (16), (38), (39), (41) the first of the inequalities (6). The second one can be obtained analogously.

8. Estimates for the closeness of quantiles of conditional distributions for $F \in \mathcal{A}_d(\tau)$

Lemma 8.1. *Let the conditions of Lemma* 6.1 *be satisfied. Then there exist absolute positive constants* $c_1, \dots, c_4$ *such that*

$$\Phi_\sigma\big(z - \gamma(z)\big) \leqslant \int\limits_{-\infty}^{z} v(y)\,dy \leqslant \Phi_\sigma\big(z + \gamma(z)\big), \tag{1}$$

for $\frac{\tau\,d^{3/2}}{\sigma} \leqslant c_1$, $\Big|\big(D'\big)^{-1/2}x'\Big| \leqslant \frac{c_2\,\sigma}{d^{3/2}\tau}$, $|z| \leqslant \frac{c_3\,\sigma^2}{d\,\tau}$, *where*

$$\gamma(z) = c_4\,\tau\left(d^{3/2} + d\,\delta\left(1 + \frac{|z|}{\sigma}\right) + \frac{z^2}{\sigma^2}\right), \qquad \delta = \Big\|\big(D'\big)^{-1/2}x'\Big\|. \tag{2}$$

The proof of Lemma 8.1 consists of several steps. For each of them we formulate the corresponding lemma.

Lemma 8.2. *Let the conditions of Lemma* 8.1 *be satisfied. Then there exist absolute positive constants* $c_5, \dots, c_8$ *such that if*

$$\frac{\tau\,d^{3/2}}{\sigma} \leqslant c_5, \qquad \Big|\big(D'\big)^{-1/2}x'\Big| \leqslant \frac{c_6\,\sigma}{d^{3/2}\tau}, \qquad 4\sigma \leqslant z \leqslant b = \frac{c_7\,\sigma^2}{d\,\tau}, \tag{3}$$

then

$$\int\limits_{z+\lambda(2z)}^{b} \varphi_\sigma(u)\,du \leqslant I = I(z,b) = \int_z^b v(y)\,dy \leqslant \int\limits_{z-\lambda(z)}^{\infty} \varphi_\sigma(u)\,du, \tag{4}$$

$$\int\limits_{z+\lambda(2z)}^{b} \varphi_\sigma(u)\,du \leqslant I(-b,-z) = \int_{-b}^{-z} v(y)\,dy \leqslant \int\limits_{z-\lambda(z)}^{\infty} \varphi_\sigma(u)\,du, \tag{5}$$

where

$$\lambda(z) = c_8\,\tau\left(d^{3/2} + d\,\delta + z\left(d\,\delta/\sigma + z/\sigma^2\right)\right) \leqslant z/8. \tag{6}$$

Proof. At first we note that by Lemma 6.1 for $2\sigma \leqslant |y| \leqslant c_9\,\sigma^2/d\,\tau$ we have

$$v(y) = \varphi_\sigma(y)\,\exp\left(\theta\,c_{10}\,\frac{\tau}{\sigma}\,\left(d^{3/2} + \frac{d\,\delta\,y^2}{\sigma^2} + \frac{|y|^3}{\sigma^3}\right)\right), \tag{7}$$

if c_5, c_6 are small enough. Recall that

$$\varphi_\sigma(u) = \frac{1}{\sqrt{2\,\pi}\,\sigma}\,\exp\left(-\frac{u^2}{2\,\sigma^2}\right), \qquad u \in \mathbb{R}^1. \tag{8}$$

We take $c_8 = 16\,c_{10}$. Thus,

$$\lambda(u) = 16\,c_{10}\,\tau\left(d^{3/2} + d\,\delta + u\left(d\,\delta/\sigma + u/\sigma^2\right)\right). \tag{9}$$

Consider the following functions:

$$\begin{aligned} f_-(u) &= u - \lambda(u), \\ f_+(u) &= u + \lambda(u). \end{aligned} \tag{10}$$

We shall study some properties of the function $f_-(u)$ for $4\sigma \leqslant u \leqslant 2b$ and of the function $f_+(u)$ for $2\sigma \leqslant u \leqslant b$. It is clear that choosing c_5, c_6, c_7 to be sufficiently small we can ensure the validity of the relations

$$2b \leqslant c_9\,\sigma^2/d\,\tau, \tag{11}$$

$$0 \leqslant \lambda(u) \leqslant u/8 \qquad \text{for} \quad 2\sigma \leqslant u \leqslant 2b, \tag{12}$$

and, consequently,

$$\begin{aligned} 2\sigma \leqslant u/2 \leqslant f_-(u) \leqslant u \leqslant 2b \leqslant c_9\,\sigma^2/d\,\tau \qquad &\text{for} \quad 4\sigma \leqslant u \leqslant 2b, \\ 2\sigma \leqslant u \leqslant f_+(u) \leqslant 2u \leqslant 2b \leqslant c_9\,\sigma^2/d\,\tau \qquad &\text{for} \quad 2\sigma \leqslant u \leqslant b. \end{aligned} \tag{13}$$

Besides, taking again c_5, c_6, c_7 to be small enough, we can ensure the validity of the relation

$$\lambda'(u) = 16\,c_{10}\,\tau\left(d\delta/\sigma + 2\,u/\sigma^2\right) \leqslant {}^1\!/_2 \qquad \text{for} \quad 2\sigma \leqslant u \leqslant 2b. \tag{14}$$

Since $e^{-2a} \leqslant 1 - a$ for $0 \leqslant a \leqslant {}^1\!/_2$, we have

$$\begin{aligned} f'_-(u) &= 1 - 16\,c_{10}\,\tau\left(d\delta/\sigma + 2\,u/\sigma^2\right) \geqslant \exp\Big(-32\,c_{10}\,\tau\left(d\delta/\sigma + 2\,u/\sigma^2\right)\Big) \\ &\geqslant \exp\Big(-2\,c_{10}\,\tau\,u^2\sigma^{-2}\left(d\delta/\sigma + 2\,u/\sigma^2\right)\Big), \end{aligned} \tag{15}$$

if $4\sigma \leqslant u \leqslant 2b$. Analogously,

$$\begin{aligned} f'_+(u) &= 1 + 16\,c_{10}\,\tau\left(d\delta/\sigma + 2\,u/\sigma^2\right) \leqslant \exp\Big(16\,c_{10}\,\tau\left(d\delta/\sigma + 2\,u/\sigma^2\right)\Big) \\ &\leqslant \exp\Big(4\,c_{10}\,\tau\,u^2\sigma^{-2}\left(d\delta/\sigma + 2\,u/\sigma^2\right)\Big), \end{aligned} \tag{16}$$

if $2\sigma \leqslant u \leqslant b$. The relations (9), (10), (14) imply that, for $2\sigma \leqslant u \leqslant 2b$,

$$0 \leqslant \lambda'(u) \leqslant {}^1\!/_2, \qquad {}^1\!/_2 \leqslant f'_-(u) \leqslant 1, \qquad 1 \leqslant f'_+(u) \leqslant {}^3\!/_2. \tag{17}$$

Therefore, there exist the inverse functions $f_+^{-1}(y)$ (for $f_+(2\sigma) \leqslant y \leqslant f_+(b)$) and $f_-^{-1}(y)$ (for $f_-(4\sigma) \leqslant y \leqslant f_-(2b)$). From (13) it follows that

$$y/2 \leqslant f_+^{-1}(y) \leqslant y, \qquad y \leqslant f_-^{-1}(y) \leqslant 2y. \tag{18}$$

Using (10), (17), (18), one can easily obtain that

$$\begin{aligned} f_-\big(f_-^{-1}(y)\big) &= f_-^{-1}(y) - \lambda\big(f_-^{-1}(y)\big), \\ f_+\big(f_+^{-1}(y)\big) &= f_+^{-1}(y) + \lambda\big(f_+^{-1}(y)\big), \end{aligned} \tag{19}$$

and, consequently,

$$\begin{aligned} f_-^{-1}(y) &= y + \lambda\big(f_-^{-1}(y)\big) \leqslant y + \lambda(2y), \\ f_+^{-1}(y) &= y - \lambda\big(f_+^{-1}(y)\big) \geqslant y - \lambda(y). \end{aligned} \tag{20}$$

Consider the integral I for $4\sigma \leqslant z \leqslant b$. Then, by (13), (18), we have:

$$z \geqslant 4\sigma \geqslant f_+(2\sigma), \qquad f_+^{-1}(z) \geqslant 2\sigma, \qquad f_+^{-1}(b) \leqslant b \tag{21}$$

and

$$f_-^{-1}(z) \geqslant z \geqslant 4\sigma, \qquad b \leqslant f_-^{-1}(b) \leqslant 2b. \tag{22}$$

By (7), (8), (11), the following estimates hold:

$$I \leqslant \int_z^b \frac{1}{\sqrt{2\,\pi}\,\sigma}\ \exp\Big(-\frac{y^2}{2\,\sigma^2}+c_{10}\ \frac{\tau}{\sigma}\ \Big(d^{3/2}+\frac{d\,\delta\,y^2}{\sigma^2}+\frac{y^3}{\sigma^3}\Big)\Big)\,dy \tag{23}$$

and

$$I \geqslant \int_z^b \frac{1}{\sqrt{2\,\pi}\,\sigma}\ \exp\Big(-\frac{y^2}{2\,\sigma^2}-c_{10}\ \frac{\tau}{\sigma}\ \Big(d^{3/2}+\frac{d\,\delta\,y^2}{\sigma^2}+\frac{y^3}{\sigma^3}\Big)\Big)\,dy. \tag{24}$$

In the right-hand side of the inequality (23) we make the change of variable $y = f_+(u)$. It is clear that $dy = f_+'(u)\,du$. Besides, by (9), (10),

$$y^2 = \big(u+\lambda(u)\big)^2 \geqslant u^2+2\,u\,\lambda(u) = u^2+32c_{10}\,u\,\tau\,\Big(d^{3/2}+d\,\delta+u\,\big(d\,\delta/\sigma+u/\sigma^2\big)\Big). \tag{25}$$

From (13) it follows that

$$c_{10}\ \frac{\tau}{\sigma}\ \Big(d^{3/2}+\frac{d\,\delta\,y^2}{\sigma^2}+\frac{y^3}{\sigma^3}\Big) \leqslant 8\,c_{10}\ \frac{\tau}{\sigma}\ \Big(\frac{d^{3/2}\,u}{\sigma}+\frac{d\,\delta\,u^2}{\sigma^2}+\frac{u^3}{\sigma^3}\Big). \tag{26}$$

Using (8), (16), (20), (21), (25), (26), we derive from (23) that

$$I \leqslant \int\limits_{f_+^{-1}(z)}^{f_+^{-1}(b)} \varphi_\sigma(u)\,du \leqslant \int\limits_{z-\lambda(z)}^{\infty} \varphi_\sigma(u)\,du. \tag{27}$$

The upper estimate of $I(z,b)$ in (4) is proved. It is clear that the upper estimate of $I(-z,-b)$ in (5) can be obtained analogously.

In the right-hand side of the inequality (24) we make the change of variable $y = f_-(u)$. It is clear that $dy = f_-'(u)\,du$. Besides, by virtue of (9), (10), (12), we have

$$y^2 = \big(u-\lambda(u)\big)^2 \leqslant u^2-u\,\lambda(u) = u^2-16\,c_{10}\,u\,\tau\,\Big(d^{3/2}+d\,\delta+u\,\big(d\,\delta/\sigma+u/\sigma^2\big)\Big). \tag{28}$$

From (13) it follows that

$$c_{10}\ \frac{\tau}{\sigma}\ \Big(d^{3/2}+\frac{d\,\delta\,y^2}{\sigma^2}+\frac{y^3}{\sigma^3}\Big) \leqslant c_{10}\ \frac{\tau}{\sigma}\ \Big(\frac{d^{3/2}\,u}{\sigma}+\frac{d\,\delta\,u^2}{\sigma^2}+\frac{u^3}{\sigma^3}\Big). \tag{29}$$

Using (8), (15), (20), (22), (28), (29), we derive from (24) that

$$I \geqslant \int\limits_{f_-^{-1}(z)}^{f_-^{-1}(b)} \varphi_\sigma(u)\,du \geqslant \int\limits_{z+\lambda(2z)}^{b} \varphi_\sigma(u)\,du. \tag{30}$$

The lower estimate of $I(z,b)$ in (4) is proved. It is evident that the bound for $I(-z,-b)$ can be obtained in a similar way. The inequality (6) follows from (12).

Lemma 8.3. *Let the conditions of Lemma* 8.1 *be satisfied. Then there exist absolute positive constants* $c_{11},\dots,c_{14}$ *such that, if*

$$\frac{\tau\, d^{3/2}}{\sigma} \leqslant c_{11}, \qquad \left|\left(D'\right)^{-1/2} x'\right| \leqslant \frac{c_{12}\,\sigma}{d^{3/2}\tau}, \qquad 4\sigma \leqslant z \leqslant \frac{c_{13}\,\sigma^2}{d\,\tau}, \tag{31}$$

then

$$\int\limits_{z+\mu(2z)}^{\infty} \varphi_\sigma(u)\,du \leqslant I = I(z,\infty) = \int_z^\infty v(y)\,dy \leqslant \int\limits_{z-\mu(z)}^{\infty} \varphi_\sigma(u)\,du, \tag{32}$$

$$\int\limits_{z+\mu(2z)}^{\infty} \varphi_\sigma(u)\,du \leqslant I(-\infty,-z) = \int\limits_{-\infty}^{-z} v(y)\,dy \leqslant \int\limits_{z-\mu(z)}^{\infty} \varphi_\sigma(u)\,du, \tag{33}$$

where

$$\mu(z) = c_{14}\,\tau\left(d^{3/2} + d\,\delta + z\left(d\,\delta/\sigma + z/\sigma^2\right)\right) \leqslant z/4. \tag{34}$$

Proof. Choosing c_{11}, c_{12} to be small enough, we ensure the possibility to estimate the integral $I(b,\infty)$ with the help of inequality (6.6) of Lemma 6.1. By this inequality, for $y > b$ we have

$$\begin{aligned} v(y) &\leqslant \frac{c_{15}}{\sigma}\ \exp\Big(-\min\Big\{\frac{y^2}{4\sigma^2}, \frac{c_{16}\,y}{d\,\tau}\Big\}\Big) \\ &\leqslant \frac{c_{15}}{\sigma}\ \exp\Big(-\min\Big\{\frac{c_7\,y}{4\,d\,\tau}, \frac{c_{16}\,y}{d\,\tau}\Big\}\Big) = \frac{c_{15}}{\sigma}\ \exp\Big(-\frac{c_{17}\,y}{d\,\tau}\Big). \end{aligned} \tag{35}$$

Thus, we can estimate $I(b,\infty)$ as follows:

$$\begin{aligned} I(b,\infty) = \int_b^\infty v(y)\,dy &\leqslant \int_b^\infty \frac{c_{15}}{\sigma}\ \exp\Big(-\frac{c_{17}\,y}{d\,\tau}\Big)\,dy \\ &= \frac{c_{15}\,d\,\tau}{c_{17}\,\sigma}\ \exp\Big(-\frac{c_{17}\,c_7\,\sigma^2}{d^2\,\tau^2}\Big). \end{aligned} \tag{36}$$

We shall choose c_{11}, c_{12}, c_{13} so that $c_{11} \leqslant c_5,\ c_{12} \leqslant c_6,\ c_{13} = \min\{c_7/2,\ \sqrt{c_7\, c_{17}}\}$. Thus, we can use the inequalities (4)–(6). From (4), (36) it follows that

$$I(z,\infty) = I(z,b) + I(b,\infty)$$
$$\leqslant \frac{c_{15}\, d\, \tau}{c_{17}\, \sigma}\ \exp\Big(-\frac{c_{17}\, c_7\, \sigma^2}{d^2\, \tau^2}\Big) + \int\limits_{z-\mu(z)}^{\infty} \varphi_\sigma(u)\, du - \int\limits_{z-\mu(z)}^{z-\lambda(z)} \varphi_\sigma(u)\, du \tag{37}$$

and

$$I(z,\infty) \geqslant I(z,b) \geqslant \int\limits_{z+\mu(2z)}^{\infty} + \int\limits_{z+\lambda(2z)}^{z+\mu(2z)} - \int\limits_{b}^{\infty} \varphi_\sigma(u)\, du. \tag{38}$$

We take $c_{14} = 2c_8$. Taking into account (6), (31), (34), we have:

$$\begin{aligned} \mu(z) &= 2\,\lambda(z) \leqslant z/4, & z - \mu(z) &\geqslant 3z/4, \\ \mu(2z) &= 2\,\lambda(2z) \leqslant 8\,\lambda(z) \leqslant z, & z + \mu(2z) &\leqslant 2z. \end{aligned} \tag{39}$$

Therefore, taking into account that $c_{13} = \min\{c_7/2,\ \sqrt{c_7\, c_{17}}\}$, choosing c_{11} to be sufficiently small, and using (6), (8), (31), (39), we can ensure the validity of the inequalities

$$\int\limits_{z-\mu(z)}^{z-\lambda(z)} \varphi_\sigma(u)\, du \geqslant \frac{\lambda(z)}{\sqrt{2\,\pi}\,\sigma}\ \exp\Big(-\frac{z^2}{2\,\sigma^2}\Big) \geqslant \frac{c_8\, \tau\, d^{3/2}}{\sqrt{2\,\pi}\,\sigma}\ \exp\Big(-\frac{c_{13}^2\, \sigma^2}{2\, d^2\, \tau^2}\Big)$$
$$\geqslant \frac{c_{15}\, d\, \tau}{c_{17}\, \sigma}\ \exp\Big(-\frac{c_{17}\, c_7\, \sigma^2}{d^2\, \tau^2}\Big), \tag{40}$$

$$\int\limits_{z+\lambda(2z)}^{z+\mu(2z)} \varphi_\sigma(u)\, du \geqslant \frac{\lambda(2z)}{\sqrt{2\,\pi}\,\sigma}\ \exp\Big(-\frac{2\,z^2}{\sigma^2}\Big) \geqslant \frac{c_8\, \tau\, d^{3/2}}{\sqrt{2\,\pi}\,\sigma}\ \exp\Big(-\frac{2\, c_{13}^2\, \sigma^2}{d^2\, \tau^2}\Big)$$
$$\geqslant \exp\Big(-\frac{c_7^2\, \sigma^2}{d^2\, \tau^2}\Big) = \exp\Big(-\frac{b^2}{4\,\sigma^2}\Big) \geqslant \int\limits_{b}^{\infty} \varphi_\sigma(u)\, du. \tag{41}$$

The upper bound in (32) follows now from (37), (40). The lower one can be easily derived from (38), (41). The inequalities (33) can be obtained analogously. The inequality (34) is contained in (39).

Proof of Lemma 8.1. We shall choose c_1, c_2, c_3 so small that $c_1 \leqslant c_{11}$, $c_2 \leqslant c_{12}$, $c_3 \leqslant c_{13}$. Thus, we can use the inequalities (32)–(34).

Let $0 \leqslant z \leqslant 4\sigma$. It is evident that, if c_1, c_2 are small enough, then from (34) it follows that

$$\mu(8\sigma) \leqslant 4\,\mu(4\sigma) \leqslant c_{18}\,\tau\left(d^{3/2} + d\,\delta\right) \leqslant \sigma, \tag{42}$$

and, by virtue of (2) and Lemma 6.1,

$$v(y) = \varphi_\sigma(y)\,\exp\Big(\theta\,c_{19}\,\frac{\tau}{\sigma}\,\left(d^{3/2} + d\,\delta\right)\Big) \qquad \text{for} \quad |y| \leqslant 4\sigma, \tag{43}$$

where

$$c_{19}\,\frac{\tau}{\sigma}\,\left(d^{3/2} + d\,\delta\right) \leqslant {}^1\!/_2. \tag{44}$$

From (8), (43), (44) it follows that we can write

$$v(y) = \varphi_\sigma(y)\Big(1 + \theta\,c_{20}\,\frac{\tau}{\sigma}\,\left(d^{3/2} + d\,\delta\right)\Big) \qquad \text{for} \quad |y| \leqslant 4\sigma. \tag{45}$$

Thus, the integral $I(z, 4\sigma)$ can be estimated as follows:

$$I(z, 4\,\sigma) = \int_z^{4\sigma} v(y)\,dy \leqslant \int_z^{4\sigma} \varphi_\sigma(u)\,du + c_{20}\,\frac{\tau}{\sigma}\,\left(d^{3/2} + d\,\delta\right). \tag{46}$$

Analogously,

$$I(z, 4\,\sigma) \geqslant \int_z^{4\sigma} \varphi_\sigma(u)\,du - c_{20}\,\frac{\tau}{\sigma}\,\left(d^{3/2} + d\,\delta\right). \tag{47}$$

Using Lemma 8.3, we see that

$$I(4\,\sigma, \infty) = \int_{4\,\sigma}^{\infty} v(y)\,dy \leqslant \int\limits_{4\sigma - \mu(4\sigma)}^{\infty} \varphi_\sigma(u)\,du \tag{48}$$

and

$$I(4\,\sigma, \infty) \geqslant \int\limits_{4\sigma + \mu(8\sigma)}^{\infty} \varphi_\sigma(u)\,du. \tag{49}$$

Taking into account (8), (42), we obtain the inequalities

$$\int\limits_{4\,\sigma - \mu(4\,\sigma)}^{4\sigma} \varphi_\sigma(u)\,du \leqslant \frac{e^{-7}\,\mu(4\sigma)}{\sqrt{2\,\pi}\,\sigma} \leqslant \frac{c_{18}\,e^{-7}}{4\,\sqrt{2\,\pi}}\,\frac{\tau}{\sigma}\,\left(d^{3/2} + d\,\delta\right) \tag{50}$$

and

$$\int\limits_{4\sigma}^{4\sigma+\mu(8\sigma)} \varphi_\sigma(u)\,du \leqslant \frac{e^{-8}\,\mu(8\sigma)}{\sqrt{2\,\pi}\,\sigma} \leqslant \frac{c_{18}\,e^{-8}}{\sqrt{2\,\pi}}\,\frac{\tau}{\sigma}\left(d^{3/2}+d\,\delta\right). \tag{51}$$

Now from (46), (48), (50) it follows that

$$I(z,\infty)=\int_z^\infty v(y)\,dy \leqslant \int_z^\infty \varphi_\sigma(u)\,du + c_{21}\,\frac{\tau}{\sigma}\left(d^{3/2}+d\,\delta\right). \tag{52}$$

Analogously, from (47), (49), (51) one can easily derive that

$$I(z,\infty) \geqslant \int_z^\infty \varphi_\sigma(u)\,du - c_{22}\,\frac{\tau}{\sigma}\left(d^{3/2}+d\,\delta\right). \tag{53}$$

Put $c_{23}=\sqrt{2\pi}\,\max\bigl\{c_{21}\,e^8,\ c_{22}\,e^{25/2}\bigr\}$. It is clear that choosing c_1, c_2 to be sufficiently small we can ensure the validity of the inequality

$$\varkappa = c_{23}\,\tau\left(d^{3/2}+d\,\delta\right) \leqslant \sigma. \tag{54}$$

Hence,

$$z-\varkappa \geqslant -\sigma \qquad \text{for} \quad z \geqslant 0. \tag{55}$$

Taking into account (8), (54), (55), we see that, for $0 \leqslant z \leqslant 4\,\sigma$,

$$\int\limits_{z-\varkappa}^{z} \varphi_\sigma(u)\,du \geqslant \frac{e^{-8}\,\varkappa}{\sqrt{2\,\pi}\,\sigma} \geqslant c_{21}\,\frac{\tau}{\sigma}\left(d^{3/2}+d\,\delta\right) \tag{56}$$

and, analogously,

$$\int\limits_{z}^{z+\varkappa} \varphi_\sigma(u)\,du \geqslant \frac{e^{-25/2}\,\varkappa}{\sqrt{2\,\pi}\,\sigma} \geqslant c_{22}\,\frac{\tau}{\sigma}\left(d^{3/2}+d\,\delta\right). \tag{57}$$

Now from (52), (53), (56), (57) it follows that

$$\int\limits_{z+\varkappa}^{\infty} \varphi_\sigma(u)\,du \leqslant \int_z^\infty v(y)\,dy \leqslant \int\limits_{z-\varkappa}^{\infty} \varphi_\sigma(u)\,du \qquad \text{for} \quad 0 \leqslant z \leqslant 4\sigma. \tag{58}$$

The statement of Lemma 8.1 for $z \geqslant 0$ can be easily derived from Lemma 8.3 and from (54), (58). The case $z \leqslant 0$ can be considered in a similar way.

9. Estimates for the closeness of quantiles of conditional distributions for $(F_1, F_2) \in \mathcal{R}_d(\tau, 0)$

For $-\infty \leqslant z_1 \leqslant z_2 \leqslant \infty$ we denote

$$J_k(z_1, z_2) = \int_{z_1}^{z_2} v_k(y)\, dy, \qquad k = 1, 2.$$

Lemma 9.1. *Let the conditions of Lemma* 7.1 *be satisfied. Then there exist absolute positive constants* $c_1, \dots, c_4$ *such that*

$$\int_{-\infty}^{z-\varkappa(z)} v_1(y)\, dy \leqslant \int_{-\infty}^{z} v_2(y)\, dy \leqslant \int_{-\infty}^{z+\varkappa(z)} v_1(y)\, dy, \tag{1}$$

for $\dfrac{\tau\, d^{3/2}}{\sigma} \leqslant c_1$, $\left|\left(D'\right)^{-1/2} x_k'\right| \leqslant \dfrac{c_2\, \sigma}{d^{3/2}\, \tau}$, $k = 1, 2$, $|z| \leqslant \dfrac{c_3\, \sigma^2}{d\, \tau}$, *where*

$$\varkappa(z) = c_4\, \frac{\tau^2}{\sigma} \left(d^2 + \delta \left(d + \frac{|z|}{\sigma}\right) + \frac{|z|^3}{\sigma^3}\right), \qquad \delta = \Gamma^2 + \sigma\, \tau^{-1} \Lambda. \tag{2}$$

The proof of Lemma 9.1 consists of several steps. For each of them we formulate the corresponding lemma.

Lemma 9.2. *Let the conditions of Lemma* 9.1 *be satisfied. Then there exist absolute positive constants* $c_5, \dots, c_8$ *such that, if*

$$\frac{\tau\, d^{3/2}}{\sigma} \leqslant c_5, \qquad \left|\left(D'\right)^{-1/2} x_k'\right| \leqslant \frac{c_6\, \sigma}{d^{3/2}\, \tau}, \quad k = 1, 2, \qquad 10\, \sigma \leqslant z \leqslant b = \frac{c_7\, \sigma^2}{d\, \tau}, \tag{3}$$

then

$$\int_{z+\beta(5z/4)}^{b} v_1(u)\, du \leqslant J = J_2(z, b) = \int_{z}^{b} v_2(y)\, dy \leqslant \int_{z-\beta(z)}^{\infty} v_1(u)\, du, \tag{4}$$

$$\int_{z+\beta(5z/4)}^{b} v_1(u)\, du \leqslant J_2(-b, -z) = \int_{-b}^{-z} v_2(y)\, dy \leqslant \int_{z-\beta(z)}^{\infty} v_1(u)\, du, \tag{5}$$

where

$$\beta(z) = c_8\, \tau^2 \sigma^{-1} \left(d^2 + d\delta + z\left(\delta/\sigma + z^2/\sigma^3\right)\right) \leqslant z/8. \tag{6}$$

Proof. At first we note that, if c_5, c_6 are small enough, then, by Lemma 7.1 and according to (2), for $|y| \leqslant c_9\, \sigma^2/d\tau$ we have

$$v_2(y) = v_1(y)\, \exp\left(\theta\, c_{10}\, \frac{\tau^2}{\sigma^2} \left(d^2 + d\delta + \frac{\delta\, y^2}{\sigma^2} + \frac{y^4}{\sigma^4}\right)\right) \tag{7}$$

and, for $8\sigma \leqslant y \leqslant \frac{c_9\,\sigma^2}{d\,\tau}$,

$$\begin{aligned} \frac{d}{dx}\log\, v_k(x)\Big|_{x=y} &\leqslant -\frac{y}{8\,\sigma^2}\,, \\ \frac{d}{dx}\log\, v_k(x)\Big|_{x=-y} &\geqslant \frac{y}{8\,\sigma^2}\,, \end{aligned} \qquad k=1,2. \tag{8}$$

We take $c_8 = 256\,c_{10}$. Thus,

$$\beta(u) = 256\,c_{10}\,\tau^2\sigma^{-1}\left(d^2 + d\delta + u\left(\delta/\sigma + u^2/\sigma^3\right)\right). \tag{9}$$

Consider the following functions:

$$\begin{aligned} g_-(u) &= u - \beta(u), \\ g_+(u) &= u + \beta(u). \end{aligned} \tag{10}$$

We shall study some properties of the function $g_-(u)$ for $10\sigma \leqslant u \leqslant 2\,b$ and of the function $g_+(u)$ for $8\sigma \leqslant u \leqslant b$. It is clear that, choosing c_5, c_6, c_7 to be sufficiently small and taking into account (2), (9), (10), (7.3), we can ensure the validity of the relations

$$2b \leqslant c_9\,\sigma^2/d\,\tau, \tag{11}$$

$$0 \leqslant \beta(u) \leqslant u/8 \qquad \text{for} \quad 8\sigma \leqslant u \leqslant 2\,b, \tag{12}$$

and, consequently,

$$\begin{aligned} &8\sigma \leqslant 4\,u/5 \leqslant g_-(u) \leqslant u \leqslant 2b \leqslant c_9\,\sigma^2/d\,\tau \qquad \text{for} \quad 10\sigma \leqslant u \leqslant 2b, \\ &8\sigma \leqslant u \leqslant g_+(u) \leqslant 5\,u/4 \leqslant 2b \leqslant c_9\,\sigma^2/d\,\tau \qquad \text{for} \quad 8\sigma \leqslant u \leqslant b. \end{aligned} \tag{13}$$

Besides, taking again c_5, c_6, c_7 to be small enough, we can ensure the validity of the relation

$$\beta'(u) = 256\,c_{10}\,\tau^2\sigma^{-1}\left(\delta/\sigma + 3\,u^2/\sigma^3\right) \leqslant {}^1\!/_2 \qquad \text{for} \quad 8\sigma \leqslant u \leqslant 2\,b. \tag{14}$$

Since $e^{-2a} \leqslant 1 - a$ for $0 \leqslant a \leqslant {}^1\!/_2$, we have

$$\begin{aligned} g'_-(u) &= 1 - 256\,c_{10}\,\tau^2\sigma^{-1}\left(\delta/\sigma + 3\,u^2/\sigma^3\right) \geqslant \exp\Big(-512\,c_{10}\,\tau^2\sigma^{-1}\left(\delta/\sigma + 3\,u^2/\sigma^3\right)\Big) \\ &\geqslant \exp\Big(-6\,c_{10}\,\tau^2 u^2\sigma^{-3}\left(\delta/\sigma + 3\,u^2/\sigma^3\right)\Big), \end{aligned} \tag{15}$$

if $10\sigma \leqslant u \leqslant 2b$. Analogously,

$$\begin{aligned} g'_+(u) &= 1 + 256\,c_{10}\,\tau^2\sigma^{-1}\left(\delta/\sigma + 3\,u^2/\sigma^3\right) \leqslant \exp\Big(256\,c_{10}\,\tau^2\sigma^{-1}\left(\delta/\sigma + 3\,u^2/\sigma^3\right)\Big) \\ &\leqslant \exp\Big(4\,c_{10}\,\tau^2\,u^2\sigma^{-3}\left(\delta/\sigma + 3\,u^2/\sigma^3\right)\Big), \end{aligned} \tag{16}$$

if $8\sigma \leqslant u \leqslant b$. The relations (9), (10), (14) imply that, for $8\sigma \leqslant u \leqslant 2b$,

$$0 \leqslant \beta'(u) \leqslant {}^1\!/_2, \qquad 1 \leqslant g'_+(u) \leqslant {}^3\!/_2, \qquad {}^1\!/_2 \leqslant g'_-(u) \leqslant 1. \tag{17}$$

Therefore, there exist the inverse functions $g_+^{-1}(y)$ (for $g_+(8\sigma) \leqslant y \leqslant g_+(b)$) and $g_-^{-1}(y)$ (for $g_-(10\sigma) \leqslant y \leqslant g_-(2b)$). From (13) it follows that

$$4y/5 \leqslant g_+^{-1}(y) \leqslant y, \qquad y \leqslant g_-^{-1}(y) \leqslant 5y/4. \tag{18}$$

Using (10), (17), (18), one can easily obtain that

$$\begin{aligned} g_-\big(g_-^{-1}(y)\big) &= g_-^{-1}(y) - \beta\big(g_-^{-1}(y)\big), \\ g_+\big(g_+^{-1}(y)\big) &= g_+^{-1}(y) + \beta\big(g_+^{-1}(y)\big), \end{aligned} \tag{19}$$

and, consequently,

$$\begin{aligned} g_-^{-1}(y) &= y + \beta\big(g_-^{-1}(y)\big) \leqslant y + \beta\big(5y/4\big), \\ g_+^{-1}(y) &= y - \beta\big(g_+^{-1}(y)\big) \geqslant y - \beta(y). \end{aligned} \tag{20}$$

Consider the integral J for $10\sigma \leqslant z \leqslant b$. Then, by (13), (18), we have:

$$z \geqslant 10\sigma \geqslant g_+(8\sigma), \qquad g_+^{-1}(z) \geqslant 8\sigma, \qquad g_+^{-1}(b) \leqslant b \tag{21}$$

and

$$g_-^{-1}(z) \geqslant z \geqslant 10\sigma, \qquad b \leqslant g_-^{-1}(b) \leqslant 5b/4. \tag{22}$$

By (7), (11), the following estimates hold:

$$J \leqslant \int_z^b v_1(y)\, \exp\Big(c_{10}\, \frac{\tau^2}{\sigma^2}\, \Big(d^2 + d\delta + \frac{\delta\, y^2}{\sigma^2} + \frac{y^4}{\sigma^4} \Big) \Big)\, dy \tag{23}$$

and

$$J \geqslant \int_z^b v_1(y)\, \exp\Big(-c_{10}\, \frac{\tau^2}{\sigma^2}\, \Big(d^2 + d\delta + \frac{\delta\, y^2}{\sigma^2} + \frac{y^4}{\sigma^4} \Big) \Big)\, dy. \tag{24}$$

In the right-hand side of the inequality (23) we make the change of variable $y = g_+(u)$. It is clear that $dy = g'_+(u)\, du$. Besides, using (8)–(10), (13), we see that

$$\begin{aligned} \log v_1(y) &= \log v_1\big(u + \beta(u)\big) \\ &= \log v_1(u) + \beta(u)\, \frac{d}{dx}\, \log v_1(x)\Big|_{x = u + |\theta|\, \beta(u)} \\ &\leqslant \log v_1(u) - \frac{u\beta(u)}{8\sigma^2} \\ &= \log v_1(u) - 32 c_{10}\, \frac{\tau^2 u}{\sigma^3}\, \Big(d^2 + d\delta + \frac{u\delta}{\sigma} + \frac{u^3}{\sigma^3} \Big). \end{aligned} \tag{25}$$

From (13) it follows that

$$c_{10}\,\frac{\tau^2}{\sigma^2}\,\Big(d^2+d\,\delta+\frac{\delta\,y^2}{\sigma^2}+\frac{y^4}{\sigma^4}\Big)\leqslant 3\,c_{10}\,\frac{\tau^2}{\sigma^2}\,\Big(\frac{d^2\,u}{\sigma}+\frac{d\,\delta\,u}{\sigma}+\frac{\delta\,u^2}{\sigma^2}+\frac{u^4}{\sigma^4}\Big). \tag{26}$$

Using (16), (20), (21), (25), (26), we derive from (23) that

$$J\leqslant\int\limits_{g_+^{-1}(z)}^{g_+^{-1}(b)}v_1(u)\,du\leqslant\int\limits_{z-\beta(z)}^{\infty}v_1(u)\,du. \tag{27}$$

The upper estimate of $J_2(z,b)$ in (4) is proved. It is clear that the upper estimate of $J_2(-z,-b)$ in (5) can be obtained analogously.

In the right-hand side of the inequality (24) we make the change of variable $y=g_-(u)$. It is clear that $dy=g_-'(u)\,du$. Besides, using (8)–(10), (13), we see that

$$\begin{aligned}\log v_1(y)&=\log v_1\big(u-\beta(u)\big)\\&=\log v_1(u)-\beta(u)\,\frac{d}{dx}\,\log v_1(x)\Big|_{x=u-|\theta|\,\beta(u)}\\&\geqslant\log v_1(u)+\frac{\big(u-\beta(u)\big)\,\beta(u)}{8\,\sigma^2}\geqslant\log v_1(u)+\frac{u\,\beta(u)}{10\,\sigma^2}\\&\geqslant\log v_1(u)+25\,c_{10}\,\frac{\tau^2\,u}{\sigma^3}\,\Big(d^2+d\,\delta+\frac{u\,\delta}{\sigma}+\frac{u^3}{\sigma^3}\Big).\end{aligned} \tag{28}$$

From (13) it follows that

$$c_{10}\,\frac{\tau^2}{\sigma^2}\,\Big(d^2+d\,\delta+\frac{\delta\,y^2}{\sigma^2}+\frac{y^4}{\sigma^4}\Big)\leqslant c_{10}\,\frac{\tau^2}{\sigma^2}\,\Big(\frac{d^2\,u}{\sigma}+\frac{d\,\delta\,u}{\sigma}+\frac{\delta\,u^2}{\sigma^2}+\frac{u^4}{\sigma^4}\Big). \tag{29}$$

Using (15), (20), (22), (28), (29), we derive from (24) that

$$J\geqslant\int\limits_{g_-^{-1}(z)}^{g_-^{-1}(b)}v_1(u)\,du\geqslant\int\limits_{z+\beta(5z/4)}^{b}v_1(u)\,du. \tag{30}$$

The lower estimate of $J_2(z,b)$ in (4) is proved. It is evident that the bound for $J_2(-z,-b)$ can be obtained in a similar way. The inequality (6) follows from (12).

Lemma 9.3. *Let the conditions of Lemma* 9.1 *be satisfied. Then there exist absolute positive constants* $c_{11}, \dots, c_{14}$ *such that, if*

$$\frac{\tau\, d^{3/2}}{\sigma} \leqslant c_{11}, \qquad \left|\left(D'\right)^{-1/2} x_k'\right| \leqslant \frac{c_{12}\,\sigma}{d^{3/2}\tau}, \quad k = 1, 2, \qquad 10\sigma \leqslant z \leqslant \frac{c_{13}\,\sigma^2}{d\,\tau}, \tag{31}$$

then

$$\int\limits_{z+\nu(5z/4)}^{\infty} v_1(u)\,du \leqslant J = J_2(z,\infty) = \int_z^{\infty} v_2(y)\,dy \leqslant \int\limits_{z-\nu(z)}^{\infty} v_1(u)\,du, \tag{32}$$

$$\int\limits_{z+\nu(5z/4)}^{\infty} v_1(u)\,du \leqslant J_2(-\infty,-z) = \int\limits_{-\infty}^{-z} v_2(y)\,dy \leqslant \int\limits_{z-\nu(z)}^{\infty} v_1(u)\,du, \tag{33}$$

where

$$\nu(z) = c_{14}\,\tau^2\sigma^{-1}\left(d^2 + d\,\delta + z\left(\delta/\sigma + z^2/\sigma^3\right)\right) \leqslant z/4. \tag{34}$$

Proof. Choosing c_{11}, c_{12} to be small enough, we ensure the possibility to estimate the integrals $J_k(b,\infty)$, $k = 1, 2$, with the help of inequality (6.6) of Lemma 6.1. By virtue of (6.6), we have (for $y > b$, $k = 1, 2$)

$$\begin{aligned} v_k(y) &\leqslant \frac{c_{15}}{\sigma}\,\exp\left(-\min\left\{\frac{y^2}{4\,\sigma^2}, \frac{c_{16}\,y}{d\,\tau}\right\}\right) \\ &\leqslant \frac{c_{15}}{\sigma}\,\exp\left(-\min\left\{\frac{c_7\,y}{4\,d\,\tau}, \frac{c_{16}\,y}{d\,\tau}\right\}\right) = \frac{c_{15}}{\sigma}\,\exp\left(-\frac{c_{17}\,y}{d\,\tau}\right). \end{aligned} \tag{35}$$

Thus, we can estimate $J_k(b,\infty)$ as follows:

$$\begin{aligned} J_k(b,\infty) = \int_b^{\infty} v_k(y)\,dy &\leqslant \int_b^{\infty} \frac{c_{15}}{\sigma}\,\exp\left(-\frac{c_{17}\,y}{d\,\tau}\right) dy \\ &= \frac{c_{15}\,d\,\tau}{c_{17}\,\sigma}\,\exp\left(-\frac{c_{17}\,c_7\,\sigma^2}{d^2\,\tau^2}\right). \end{aligned} \tag{36}$$

We shall choose c_{11}, c_{12}, c_{13} so that

$$c_{11} \leqslant \min\{c_5, c_{18}\}, \quad c_{12} \leqslant \min\{c_6, c_{19}\}, \quad c_{13} = \min\left\{c_7, \frac{\sqrt{c_7\,c_{17}}}{2}, \frac{2\,c_{20}}{3}\right\}, \tag{37}$$

where c_{18}, c_{19}, c_{20} are the constants c_1, c_2, c_3 from the formulation of Lemma 6.1. Thus, we can use the inequalities (6.5) for $0 \leqslant |x_d| \leqslant 3\,c_{13}\,\sigma^2/2d\tau$ (with the change

of $v(\cdot)$ by $v_k(\cdot)$, $k = 1, 2)$ and the inequalities (4)–(6). From (4), (36) it follows that

$$J_2(z,\infty) = J_2(z,b) + J_2(b,\infty)$$
$$\leqslant \frac{c_{15}\,d\,\tau}{c_{17}\,\sigma}\,\exp\Big(-\frac{c_{17}\,c_7\,\sigma^2}{d^2\,\tau^2}\Big) + \int\limits_{z-\nu(z)}^{\infty} v_1(u)\,du - \int\limits_{z-\nu(z)}^{z-\beta(z)} v_1(u)\,du \tag{38}$$

and

$$J_2(z,\infty) \geqslant J_2(z,b) \geqslant \int\limits_{z+\nu(5z/4)}^{\infty} + \int\limits_{z+\beta(5z/4)}^{z+\nu(5z/4)} - \int\limits_{b}^{\infty} v_1(u)\,du. \tag{39}$$

We take $c_{14} = 2c_8$. Taking into account the relations (3), (6), (31), (34), (37), we have:

$$\begin{aligned} \nu(z) &= 2\,\beta(z) \leqslant z/4, & z - \nu(z) &\geqslant 3\,z/4, \\ \nu(5z/4) &= 2\,\beta(5z/4) \leqslant 4\,\beta(z) \leqslant z/2, & z + \nu(5z/4) &\leqslant 3\,z/2. \end{aligned} \tag{40}$$

Therefore, using (6), (31), (36), (37), (40), (6.5) and choosing c_{11} to be sufficiently small, we can ensure the validity of the inequalities

$$\int\limits_{z-\nu(z)}^{z-\beta(z)} v_1(u)\,du \geqslant \frac{\beta(z)}{2\sqrt{2\,\pi}\,\sigma}\,\exp\Big(-\frac{z^2}{\sigma^2}\Big) \geqslant \frac{c_8\,\tau^2\,d^2}{2\sqrt{2\,\pi}\,\sigma^2}\,\exp\Big(-\frac{c_{13}^2\,\sigma^2}{d^2\,\tau^2}\Big)$$
$$\geqslant \frac{c_{15}\,d\,\tau}{c_{17}\,\sigma}\,\exp\Big(-\frac{c_{17}\,c_7\,\sigma^2}{d^2\,\tau^2}\Big), \tag{41}$$

$$\int\limits_{z+\beta(5z/4)}^{z+\nu(5z/4)} v_1(u)\,du \geqslant \frac{\beta(5\,z/4)}{2\sqrt{2\,\pi}\,\sigma}\,\exp\Big(-\frac{9z^2}{4\,\sigma^2}\Big) \geqslant \frac{c_8\,\tau^2\,d^2}{2\sqrt{2\,\pi}\,\sigma^2}\,\exp\Big(-\frac{9c_{13}^2\,\sigma^2}{4\,d^2\,\tau^2}\Big)$$
$$\geqslant \frac{c_{15}\,d\,\tau}{c_{17}\,\sigma}\,\exp\Big(-\frac{c_{17}\,c_7\,\sigma^2}{d^2\,\tau^2}\Big) \geqslant \int\limits_{b}^{\infty} v_1(u)\,du. \tag{42}$$

The upper bound in (32) follows now from (38), (41). The lower one can be easily derived from (39), (42). The inequalities (33) can be obtained analogously. The inequality (34) is contained in (40).

Proof of Lemma 9.1. We shall choose c_1, c_2, c_3 so small that $c_1 \leqslant c_{11}$, $c_2 \leqslant c_{12}$, $c_3 \leqslant c_{13}$. Thus, we can use the inequalities (32)–(34).

Let $0 \leqslant z \leqslant 10\sigma$. It is evident that, if c_1, c_2 are small enough, then from (2), (34), (7.3) it follows that

$$\nu(25\sigma/2) \leqslant 2\nu(10\sigma) \leqslant c_{21}\,\tau^2\sigma^{-1}\left(d^2 + d\delta\right) \leqslant \sigma, \tag{43}$$

and, by virtue of (2) and Lemma 7.1,

$$v_2(y) = v_1(y)\,\exp\Big(\theta\, c_{22}\,\frac{\tau^2}{\sigma^2}\,\left(d^2 + d\delta\right)\Big) \qquad \text{for} \quad |y| \leqslant 10\sigma, \tag{44}$$

where

$$c_{22}\,\frac{\tau^2}{\sigma^2}\,\left(d^2 + d\delta\right) \leqslant {}^1\!/_2. \tag{45}$$

From (44), (45) it follows that we can write

$$v_2(y) = v_1(y)\,\Big(1 + \theta\, c_{23}\,\frac{\tau^2}{\sigma^2}\,\left(d^2 + d\delta\right)\Big) \qquad \text{for} \quad |y| \leqslant 10\sigma. \tag{46}$$

Thus, the integral $J_2(z, 10\sigma)$ can be estimated as follows:

$$J_2(z, 10\sigma) = \int_z^{10\sigma} v_2(y)\,dy \leqslant \int_z^{10\sigma} v_1(y)\,dy + c_{23}\,\frac{\tau^2}{\sigma^2}\,\left(d^2 + d\delta\right). \tag{47}$$

Analogously,

$$J_2(z, 10\sigma) \geqslant \int_z^{10\sigma} v_1(y)\,dy - c_{23}\,\frac{\tau^2}{\sigma^2}\,\left(d^2 + d\delta\right). \tag{48}$$

Using Lemma 9.3, we see that

$$J_2(10\sigma, \infty) = \int_{10\sigma}^{\infty} v_2(y)\,dy \;\leqslant \int\limits_{10\sigma - \nu(10\sigma)}^{\infty} v_1(y)\,dy \tag{49}$$

and

$$J_2(10\sigma, \infty) \;\geqslant \int\limits_{10\sigma + \nu(25\sigma/2)}^{\infty} v_1(y)\,dy. \tag{50}$$

Taking into account (43), choosing c_1, c_2, c_3 to be small enough and applying the inequality (6.5) of Lemma 6.1, we obtain the inequalities

$$\begin{aligned}\int\limits_{10\sigma - \nu(10\sigma)}^{10\sigma} v_1(y)\,dy &\leqslant \int\limits_{10\sigma - \nu(10\sigma)}^{10\sigma} \frac{2}{\sqrt{2\,\pi}\,\sigma}\,\exp\Big(-\frac{y^2}{4\sigma^2}\Big)\,dy \\ &\leqslant \frac{2\,e^{-22}\,\nu(10\sigma)}{\sqrt{2\,\pi}\,\sigma} \leqslant \frac{c_{21}\,e^{-22}}{\sqrt{2\,\pi}}\,\frac{\tau^2}{\sigma^2}\,\left(d^2 + d\delta\right)\end{aligned} \tag{51}$$

and

$$\int\limits_{10\sigma}^{10\sigma+\nu(25\sigma/2)} v_1(y)\,dy \leqslant \int\limits_{10\sigma}^{10\sigma+\nu(25\sigma/2)} \frac{2}{\sqrt{2\,\pi}\,\sigma}\,\exp\Big(-\frac{y^2}{4\,\sigma^2}\Big)\,dy$$
$$\leqslant \frac{2\,e^{-25}\,\nu(25\,\sigma/2)}{\sqrt{2\,\pi}\,\sigma} \leqslant \frac{2\,c_{21}\,e^{-25}}{\sqrt{2\,\pi}}\,\frac{\tau^2}{\sigma^2}\,\Big(d^2+d\,\delta\Big). \tag{52}$$

Now from (47), (49), (51) it follows that

$$J_2(z,\infty) = \int_z^\infty v_2(y)\,dy \leqslant \int_z^\infty v_1(y)\,dy + c_{24}\,\frac{\tau^2}{\sigma^2}\,\Big(d^2+d\,\delta\Big). \tag{53}$$

Analogously, from (48), (50), (52) one can easily derive that

$$J_2(z,\infty) \geqslant \int_z^\infty v_1(y)\,dy - c_{25}\,\frac{\tau^2}{\sigma^2}\,\Big(d^2+d\,\delta\Big). \tag{54}$$

Put $c_{26} = 2\,\sqrt{2\pi}\,\max\big\{c_{24}\,e^{100},\ c_{25}\,e^{121}\big\}$. It is clear that choosing c_1, c_2 to be sufficiently small (see (2), (7.3)) we can ensure the validity of the inequality

$$r = c_{26}\,\tau^2\sigma^{-1}\,\Big(d^2+d\,\delta\Big) \leqslant \sigma. \tag{55}$$

Hence,

$$z - r \geqslant -\sigma \qquad \text{for} \quad z \geqslant 0. \tag{56}$$

Taking into account (55), (56), choosing c_1, c_2, c_3 to be small enough and applying the inequality (6.5) of Lemma 6.1, we see that, for $0 \leqslant z \leqslant 10\,\sigma$,

$$\int\limits_{z-r}^{z} v_1(y)\,dy \geqslant \int\limits_{z-r}^{z} \frac{1}{2\,\sqrt{2\,\pi}\,\sigma}\,\exp\Big(-\frac{y^2}{\sigma^2}\Big)\,dy \geqslant \frac{e^{-100}\,r}{2\,\sqrt{2\,\pi}\,\sigma} \geqslant c_{24}\,\frac{\tau^2}{\sigma^2}\,\Big(d^2+d\,\delta\Big) \tag{57}$$

and, analogously,

$$\int\limits_{z}^{z+r} v_1(y)\,dy \geqslant \int\limits_{z}^{z+r} \frac{1}{2\,\sqrt{2\,\pi}\,\sigma}\,\exp\Big(-\frac{y^2}{\sigma^2}\Big)\,dy \geqslant \frac{e^{-121}\,r}{2\,\sqrt{2\,\pi}\,\sigma} \geqslant c_{25}\,\frac{\tau^2}{\sigma^2}\,\Big(d^2+d\,\delta\Big). \tag{58}$$

Now from (53), (54), (57), (58) it follows that

$$\int\limits_{z+r}^{\infty} v_1(y)\,dy \leqslant \int_z^\infty v_2(y)\,dy \leqslant \int\limits_{z-r}^{\infty} v_1(y)\,dy \tag{59}$$

for $0 \leqslant z \leqslant 10\,\sigma$. The statement of Lemma 9.1 for $z \geqslant 0$ can be easily derived from Lemma 9.3 and from (55), (59). The case $z \leqslant 0$ can be considered in a similar way.

10. Dyadic scheme

Below we describe an extension of the multivariate version of the KMT [10] dyadic scheme due to Einmahl [9].

Let N be a positive integer and $\{\xi_1, \dots, \xi_{2^N}\}$, $\{Y_1, \dots, Y_{2^N}\}$ are two collections of independent random vectors. We shall always suppose that the random vectors $\xi_1, \dots, \xi_{2^N}$; $Y_1, \dots, Y_{2^N}$ have some known distributions from $\mathcal{A}_d(c\tau)$. For simplicity we shall assume that these distributions are absolutely continuous. Denote

$$\widetilde{S}_k = \sum_{l=1}^{k} \xi_l, \qquad 1 \leqslant k \leqslant 2^N; \tag{1}$$

$$U^*_{m,k} = \widetilde{S}_{(k+1)\cdot 2^m} - \widetilde{S}_{k\cdot 2^m}, \qquad 0 \leqslant k < 2^{N-m}, \quad 0 \leqslant m \leqslant N. \tag{2}$$

In particular, $U^*_{0,k} = \xi_{k+1}$, $U^*_{N,0} = \widetilde{S}_{2^N} = \xi_1 + \dots + \xi_{2^N}$. In the sequel we shall use the term *block of summands* for a collection of summands with indices of the form $k \cdot 2^m + 1, \dots, (k+1) \cdot 2^m$, where $0 \leqslant k < 2^{N-m}$, $0 \leqslant m \leqslant N$. Thus, $U^*_{m,k}$ is the sum over a block containing 2^m summands. Put also

$$\widetilde{U}^*_{n,k} = U^*_{n-1,2k} - U^*_{n-1,2k+1}, \qquad 0 \leqslant k < 2^{N-n}, \quad 1 \leqslant n \leqslant N. \tag{3}$$

Note that

$$U^*_{n-1,2k} + U^*_{n-1,2k+1} = U^*_{n,k}, \qquad 0 \leqslant k < 2^{N-n}, \quad 1 \leqslant n \leqslant N. \tag{4}$$

Below we describe the procedure of constructing the random vectors $\{U_{n,k}\}$ with $\mathcal{L}\big(\{U_{n,k}\}\big) = \mathcal{L}\big(\{U^*_{n,k}\}\big)$, provided that the vectors $Y_1, \dots, Y_{2^N}$ are already constructed. For this purpose we shall use the so-called Rosenblatt quantile transformation (see [9, 19]).

Introduce now the coordinate mappings $\pi_i(x_1, \dots, x_d) = x_i$, $\widehat{\pi}_j = (\pi_1, \dots, \pi_j)$, $i, j = 1, \dots, d$. Denote by $F^{(1)}_{N,0}(x_1) = \mathbf{P}\big(\pi_1\big(U^*_{N,0}\big) < x_1\big)$, $x_1 \in \mathbb{R}^1$, the distribution function of the first coordinate of the vector $U^*_{N,0}$. Introduce the conditional distributions, denoting by $F^{(j)}_{N,0}\big(\,\cdot\,\big|x_1, \dots, x_{j-1}\big)$, $2 \leqslant j \leqslant d$, the regular conditional distribution function (r.c.d.f.) of $\pi_j\big(U^*_{N,0}\big)$, given $\widehat{\pi}_{j-1}\big(U^*_{N,0}\big) = (x_1, \dots, x_{j-1})$. Let $\widetilde{F}^{(1)}_{n,k}\big(\,\cdot\,\big|y_1, \dots, y_d\big)$ be the r.c.d.f. of $\pi_1\big(\widetilde{U}^*_{n,k}\big)$, given $U^*_{n,k} = (y_1, \dots, y_d)$, and $\widetilde{F}^{(j)}_{n,k}\big(\,\cdot\,\big|y_1, \dots, y_d;\, x_1, \dots, x_{j-1}\big)$ the r.c.d.f. of $\pi_j\big(\widetilde{U}^*_{n,k}\big)$, given $U^*_{n,k} = (y_1, \dots, y_d)$ and $\widehat{\pi}_{j-1}\big(\widetilde{U}^*_{n,k}\big) = (x_1, \dots, x_{j-1})$, $0 \leqslant k < 2^{N-n}$, $1 \leqslant n \leqslant N$, $2 \leqslant j \leqslant d$. Put

$$T_k = \sum_{l=1}^{k} Y_l, \qquad 1 \leqslant k \leqslant 2^N; \tag{5}$$

$$V_{m,k} = \big(V_{m,k}^{(1)}, \dots, V_{m,k}^{(d)}\big) = T_{(k+1)\cdot 2^m} - T_{k\cdot 2^m}, \qquad 0 \leqslant k < 2^{N-m}, \quad 0 \leqslant m \leqslant N;$$
$$\widetilde{V}_{n,k} = \big(\widetilde{V}_{n,k}^{(1)}, \dots, \widetilde{V}_{n,k}^{(d)}\big) = V_{n-1,2k} - V_{n-1,2k+1}, \qquad 0 \leqslant k < 2^{N-n}, \quad 1 \leqslant n \leqslant N. \tag{6}$$

For the r.c.d.f. of these vectors we shall use the analogous notations. Denote by $G_{N,0}^{(1)}(x_1) = \mathbf{P}\big(\pi_1\big(V_{N,0}\big) < x_1\big)$, $x_1 \in \mathbb{R}^1$, the distribution function of the first coordinate of the vector $V_{N,0}$. Let $G_{N,0}^{(j)}\big(\,\cdot\,\big|x_1, \dots, x_{j-1}\big)$, $2 \leqslant j \leqslant d$, be the r.c.d.f. of $\pi_j\big(V_{N,0}\big)$, given $\widehat{\pi}_{j-1}\big(V_{N,0}\big) = (x_1, \dots, x_{j-1})$. Let $\widetilde{G}_{n,k}^{(1)}\big(\,\cdot\,\big|y_1, \dots, y_d\big)$ be the r.c.d.f. of $\pi_1\big(\widetilde{V}_{n,k}\big)$, given $V_{n,k} = (y_1, \dots, y_d)$, and $\widetilde{G}_{n,k}^{(j)}\big(\,\cdot\,\big|y_1, \dots, y_d;\ x_1, \dots, x_{j-1}\big)$ the r.c.d.f. of $\pi_j\big(\widetilde{V}_{n,k}\big)$, given $V_{n,k} = (y_1, \dots, y_d)$ and $\widehat{\pi}_{j-1}\big(\widetilde{V}_{n,k}\big) = (x_1, \dots, x_{j-1})$, $0 \leqslant k < 2^{N-n}$, $1 \leqslant n \leqslant N$, $2 \leqslant j \leqslant d$.

Denote now the new collection of random vectors X_k as follows. At first we define

$$U_{N,0}^{(1)} = \big(F_{N,0}^{(1)}\big)^{-1}\big(G_{N,0}^{(1)}\big(V_{N,0}^{(1)}\big)\big) \qquad \text{and,} \quad \text{for} \quad 2 \leqslant j \leqslant d,$$
$$U_{N,0}^{(j)} = \big(F_{N,0}^{(j)}\big)^{-1}\big(G_{N,0}^{(j)}\big(V_{N,0}^{(j)}\,\big|\,V_{N,0}^{(1)}, \dots, V_{N,0}^{(j-1)}\big)\,\big|\,U_{N,0}^{(1)}, \dots, U_{N,0}^{(j-1)}\big)$$

(here $\big(F_{N,0}^{(1)}\big)^{-1}(t) = \sup\big\{x : F_{N,0}^{(1)}(x) \leqslant t\big\}$, $0 < t < 1$, and so on). This formula can be rewritten in a more natural "symmetric" form (see [21, p. 30–31]):

$$F_{N,0}^{(1)}\big(U_{N,0}^{(1)}\big) = G_{N,0}^{(1)}\big(V_{N,0}^{(1)}\big) \qquad \text{and,} \quad \text{for} \quad 2 \leqslant j \leqslant d,$$
$$F_{N,0}^{(j)}\big(U_{N,0}^{(j)}\,\big|\,U_{N,0}^{(1)}, \dots, U_{N,0}^{(j-1)}\big) = G_{N,0}^{(j)}\big(V_{N,0}^{(j)}\,\big|\,V_{N,0}^{(1)}, \dots, V_{N,0}^{(j-1)}\big). \tag{7}$$

Suppose that the random vectors $U_{n,k} = \big(U_{n,k}^{(1)}, \dots, U_{n,k}^{(d)}\big)$, $0 \leqslant k < 2^{N-n}$, $1 \leqslant n \leqslant N$, corresponding to blocks containing each 2^n summands, are already constructed. Now we want to pass to the sums containing each 2^{n-1} summands. For this for $0 \leqslant k < 2^{N-n}$ we define

$$\widetilde{U}_{n,k}^{(1)} = \big(\widetilde{F}_{n,k}^{(1)}\big)^{-1}\big(\widetilde{G}_{n,k}^{(1)}\big(\widetilde{V}_{n,k}^{(1)}\,\big|\,V_{n,k}\big)\,\big|\,U_{n,k}\big), \tag{8}$$

and, for $2 \leqslant j \leqslant d$,

$$\widetilde{U}_{n,k}^{(j)} = \big(\widetilde{F}_{n,k}^{(j)}\big)^{-1}\big(\widetilde{G}_{n,k}^{(j)}\big(\widetilde{V}_{n,k}^{(j)}\,\big|\,V_{n,k};\ \widetilde{V}_{n,k}^{(1)}, \dots, \widetilde{V}_{n,k}^{(j-1)}\big)\,\big|\,U_{n,k};\ \widetilde{U}_{n,k}^{(1)}, \dots, \widetilde{U}_{n,k}^{(j-1)}\big). \tag{9}$$

It is clear that we can rewrite the relations (8), (9) in the symmetric form, similar to (7). Then we put $\widetilde{U}_{n,k} = \big(\widetilde{U}_{n,k}^{(1)}, \dots, \widetilde{U}_{n,k}^{(d)}\big)$,

$$U_{n-1,2k} = \frac{1}{2}\big(U_{n,k} + \widetilde{U}_{n,k}\big),$$
$$U_{n-1,2k+1} = \frac{1}{2}\big(U_{n,k} - \widetilde{U}_{n,k}\big). \tag{10}$$

Thus, we have constructed the random vectors $U_{n-1,k}$, $0 \leqslant k < 2^{N-n-1}$. After N steps we shall obtain the random vectors $U_{0,k}$, $0 \leqslant k < 2^N$. Now we set

$$X_k = U_{0,k-1}, \qquad S_k = \sum_{l=1}^{k} X_l, \qquad 1 \leqslant k \leqslant 2^N. \tag{11}$$

It can be checked that the mutual distribution of the constructed vectors $U_{n,k}$ coincides with the mutual distribution of the vectors $U^*_{n,k}$. In particular, X_k, $k = 1, \ldots, 2^N$, are independent with $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$. Besides, according to (3), (4), we have

$$\begin{aligned} \widetilde{U}_{n,k} &= U_{n-1,2k} - U_{n-1,2k+1}, \\ U_{n,k} &= U_{n-1,2k} + U_{n-1,2k+1}, \end{aligned} \qquad 0 \leqslant k < 2^{N-n}, \quad 1 \leqslant n \leqslant N, \tag{12}$$

(it is clear, that (12) follows from (10)).

The scheme described above is somewhat more general than that from [9]. In [9] it was supposed that the vectors Y_k have independent coordinates. However, the fact that our scheme gives the vectors with needed mutual distributions can be justified with the help of [9, Theorem 6], [10, p. 117], [21, pp. 30–33]. For this we note that we can begin with independent Gaussian vectors $\{ Z_1, \ldots, Z_{2^N} \}$ with $\mathcal{L}(Z_k) = \mathcal{N}(0, I)$ and construct the vectors $\{ X_1, \ldots, X_{2^N} \}$ and $\{ Y_1, \ldots, Y_{2^N} \}$ as the functions of vectors $\{ Z_1, \ldots, Z_{2^N} \}$ using our scheme. It is clear that the vectors Z_k have independent coordinates and the vectors $\{ X_1, \ldots, X_{2^N} \}$ and $\{ Y_1, \ldots, Y_{2^N} \}$ are connected by the relations (5)–(11). This is especially clear for the relations (7). Now we can exclude the vectors $\{ Z_1, \ldots, Z_{2^N} \}$ from the scheme.

Below we give several auxiliary lemmas. The following Lemma 10.1 can be considered as an analog of Lemma 2 from Einmahl's paper [9, p. 53].

Lemma 10.1. *Suppose that* $\tau > 0$ *and* $\mathcal{L}(X_i) \in \mathcal{A}_d(\tau)$, $\mathbf{E}\, X_i = 0$, $\operatorname{cov} X_i = I$ *for* $1 \leqslant i \leqslant 2^N$. *Let* $1 \leqslant m \leqslant 2^N$, $0 \leqslant k \leqslant 2^N - m$ *and* $0 \leqslant x < \dfrac{8}{3\tau}$. *Then*

$$\mathbf{P}\Big\{ \max_{1 \leqslant l \leqslant m} \big| S_{k+l} - S_k \big| \geqslant m\, x \Big\} \leqslant (4d+1) \exp\Big(- \frac{3\, m\, x^2}{32} \Big). \tag{13}$$

Proof. We shall denote by $S_k^{(i)}$ the i^{th} coordinates of the vectors S_k. Then, using a version the Lévy inequality (see [13, Theorem 12, Ch. III]) and supposing without

loss of generality that $x \geqslant 4/\sqrt{m}$, we obtain

$$\begin{aligned}\mathbf{P}\Big\{\max_{1\leqslant l\leqslant m}\big|S_{k+l}-S_k\big|\geqslant m\,x\Big\} &\leqslant \sum_{i=1}^{d}\mathbf{P}\Big\{\max_{1\leqslant l\leqslant m}\big|S_{k+l}^{(i)}-S_k^{(i)}\big|\geqslant m\,x\Big\}\\ &\leqslant 2\sum_{i=1}^{d}\mathbf{P}\Big\{\big|S_{k+m}^{(i)}-S_k^{(i)}\big|\geqslant m\,x-2\sqrt{m}\Big\}\\ &\leqslant 2\sum_{i=1}^{d}\mathbf{P}\Big\{\big|S_{k+m}^{(i)}-S_k^{(i)}\big|\geqslant m\,x/2\Big\}. \qquad (14)\end{aligned}$$

From the completeness of the classes $\mathcal{A}_d(\tau)$ with respect to convolution and from Lemma 1.6 it follows that $\mathcal{L}\big(S_{k+m}^{(i)}-S_k^{(i)}\big)\in\mathcal{A}_1(\tau)$. Therefore, using Lemma 1.1, inequality (1.18), we see that, for $0\leqslant t\tau<1$,

$$\mathbf{E}\exp\Big(t\big(S_{k+m}^{(i)}-S_k^{(i)}\big)\Big)\leqslant\exp\Big(\frac{1}{2}\,m\,t^2\Big(1+\frac{1}{3}\,t\tau\Big)\Big)\leqslant\exp\Big(\frac{2}{3}\,m\,t^2\Big)$$

and, applying the exponential Chebyshev inequality,

$$\mathbf{P}\Big\{S_{k+m}^{(i)}-S_k^{(i)}\geqslant m\,x/2\Big\}\leqslant\exp\Big(\frac{2}{3}\,m\,t^2-\frac{1}{2}\,t\,m\,x\Big).$$

We take here $t=3\,x/8<\tau^{-1}$. Then we have

$$\mathbf{P}\Big\{S_{k+m}^{(i)}-S_k^{(i)}\geqslant m\,x/2\Big\}\leqslant\exp\Big(-\frac{3}{32}\,m\,x^2\Big). \qquad (15)$$

One can analogously prove that

$$\mathbf{P}\Big\{S_{k+m}^{(i)}-S_k^{(i)}\leqslant -m\,x/2\Big\}\leqslant\exp\Big(-\frac{3}{32}\,m\,x^2\Big). \qquad (16)$$

Inequality (13) follows now from (14)–(16).

The following Lemma 10.2 can be considered as an analog of Lemma 3 from Einmahl's paper [9, p. 54].

Lemma 10.2. *Suppose that* $\tau>0$ *and* $\mathcal{L}(X_j)\in\mathcal{A}_d(\tau)$, $\mathbf{E}\,X_i=0$, $\operatorname{cov}X_i=I$ *for* $1\leqslant j\leqslant 2^N$. *Let for* $0\leqslant k<2^{N-n}$, $1\leqslant n\leqslant N$,

$$\varkappa_{n,k}=\begin{cases}2^{-n}\big|U_{n,k}\big|^2, & \text{if}\quad \big|U_{n,k}\big|\leqslant\dfrac{8\cdot 2^n}{3\tau},\\ 0 & \text{otherwise.}\end{cases} \qquad (17)$$

Then for $0\leqslant t\leqslant\dfrac{1}{32}$ *we have*

$$\mathbf{E}\exp(t\,\varkappa_{n,k})\leqslant 2\,d+2. \qquad (18)$$

Proof. Integrating by parts, we obtain

$$\mathbf{E}\,\exp(t\,\varkappa_{n,k}) = 1 + \int\limits_0^{64\cdot 2^n/9\tau^2} t\,\exp(tx)\;\mathbf{P}\left\{\,\varkappa_{n,k} \geqslant x\,\right\} dx$$

$$\leqslant 1 + \int\limits_0^{64\cdot 2^n/9\tau^2} t\,\exp(tx)\;\mathbf{P}\left\{\,\left|\,U_{n,k}\,\right| \geqslant 2^{n/2}\sqrt{x}\,\right\} dx. \tag{19}$$

Since the sum $U_{n,k}$ contains 2^n summands, using Lemma 10.1, we obtain that

$$\mathbf{P}\left\{\,\left|\,U_{n,k}\,\right| \geqslant 2^{n/2}\sqrt{x}\,\right\} = \mathbf{P}\left\{\,\left|\,U_{n,k}\,\right| \geqslant 2^n\left(\sqrt{x}\cdot 2^{-n/2}\right)\right\}$$

$$\leqslant (4d+1)\,\exp\left(-\,\frac{3\cdot 2^n\left(\sqrt{x}\cdot 2^{-n/2}\right)^2}{32}\right)$$

$$= (4d+1)\,\exp\left(-\,\frac{3\,x}{32}\right), \tag{20}$$

if $0 \leqslant \sqrt{x}\cdot 2^{-n/2} < \dfrac{8}{3\tau}$ (that can be rewritten in the form $0 \leqslant x < 64\cdot 2^n/9\,\tau^2$). Then for $0 \leqslant t \leqslant \dfrac{1}{32}$ we derive from (19), (20) that

$$\mathbf{E}\,\exp(t\,\varkappa_{n,k}) \leqslant 1 + \frac{4\,d+1}{32}\int_0^{\infty}\exp\left(\frac{x}{32} - \frac{3\,x}{32}\right) dx \leqslant 2\,d+2.$$

Lemma is proved.

Lemma 10.3 [9, Lemma 5, p. 55]**.** *Denote* $S_m^* = \dfrac{m}{2^N}\,S_{2^N}$, *for* $1 \leqslant m \leqslant 2^N$. *If* $m = (2\,k+1)\cdot 2^r$, *where* k, r *are non-negative integers, then the following representation is valid*:

$$S_m = S_m^* + \sum_{n=r+1}^{N} \gamma_n\,\widetilde{U}_{n,l\,(n,m)}, \tag{21}$$

where $\gamma_n = \gamma_n(m) \in [\,0,1\,]$ *and the integers* $l(n,m)$ *are defined by the relations*

$$l(n,m)\cdot 2^n < m \leqslant \big(l(n,m)+1\big)\cdot 2^n. \tag{22}$$

Proof. At first we note that the statement of the lemma is evident for $m = 2^N$. We shall use the induction method. For $N = 1$ and $N = 2$ the statement of the lemma can be directly verified. Suppose that it holds for the case of 2^N summands. We shall prove that this statement remains true in the case of 2^{N+1} summands. We consider separately two possible situations: $m \leqslant 2^N$ and $2^N < m < 2^{N+1}$.

Let $m \leqslant 2^N$. Then, by the induction hypothesis,

$$S_m = \frac{m}{2^N} S_{2^N} + \sum_{n=r+1}^{N} \gamma_n \widetilde{U}_{n,l(n,m)}, \qquad \gamma_n \in [0,1]. \tag{23}$$

Besides,

$$S_{2^N} = \frac{1}{2} S_{2^{N+1}} + \frac{1}{2} \widetilde{U}_{N+1,0}. \tag{24}$$

From (23), (24) it follows that

$$S_m = \frac{m}{2^{N+1}} S_{2^{N+1}} + \frac{m}{2^{N+1}} \widetilde{U}_{N+1,0} + \sum_{n=r+1}^{N} \gamma_n \widetilde{U}_{n,l(n,m)}. \tag{25}$$

Denoting $\gamma_{N+1} = \frac{m}{2^{N+1}}$ and taking into account that $l(N+1,m) = 0$, we can rewrite (25) in the form

$$S_m = \frac{m}{2^{N+1}} S_{2^{N+1}} + \sum_{n=r+1}^{N+1} \gamma_n \widetilde{U}_{n,l(n,m)}. \tag{26}$$

Thus, we proved the statement of the lemma in the case of 2^{N+1} summands, $m \leqslant 2^N$.

Let now $2^N < m < 2^{N+1}$. Denote $\zeta_l = X_{2^N+l}$, $P_l = \sum\limits_{i=1}^{l} \zeta_l$, $l = 1, \ldots, 2^N$. Then, applying the induction hypothesis to the sums of random vectors ζ_l, we obtain that

$$P_{m-2^N} = S_m - S_{2^N} = \frac{m-2^N}{2^N} \left(S_{2^{N+1}} - S_{2^N} \right) + \sum_{n=r+1}^{N} \gamma_n \widetilde{U}_{n,l(n,m-2^N)+2^{N-n}}, \tag{27}$$

where $\gamma_n \in [0,1]$. We have used that if $m = (2k+1) \cdot 2^r$ for $2^N < m < 2^{N+1}$, then $k \geqslant 1$, $r < N$ and $m - 2^N = (2k - 2^{N-r} + 1) \cdot 2^r$. Note also that

$$l(n, m-2^N) + 2^{N-n} = l(n,m). \tag{28}$$

From (24), (27), (28) it follows that

$$\begin{aligned} S_m &= \frac{1}{2} S_{2^{N+1}} + \frac{1}{2} \widetilde{U}_{N+1,0} + \frac{m-2^N}{2^N} \left(\frac{1}{2} S_{2^{N+1}} - \frac{1}{2} \widetilde{U}_{N+1,0} \right) + \sum_{n=r+1}^{N} \gamma_n \widetilde{U}_{n,l(n,m)} \\ &= \frac{m}{2^{N+1}} S_{2^{N+1}} + \frac{2^{N+1}-m}{2^{N+1}} \widetilde{U}_{N+1,0} + \sum_{n=r+1}^{N} \gamma_n \widetilde{U}_{n,l(n,m)} \\ &= \frac{m}{2^{N+1}} S_{2^{N+1}} + \sum_{n=r+1}^{N+1} \gamma_n \widetilde{U}_{n,l(n,m)}, \end{aligned} \tag{27}$$

where $\gamma_{N+1} = \dfrac{2^{N+1} - m}{2^{N+1}}$ and $l(N+1, m) = 0$. Thus, we established the statement of the lemma in the case of 2^{N+1} summands when $2^N < m < 2^{N+1}$ and the proof is completed.

Remark 10.1. U. Einmahl informed the author that the shortest proof of Lemma 10.3 can be obtained with the help of geometrical approach due to Massart [12, p. 275].

Corollary 10.1. *Under the conditions of Lemma* 10.3

$$\left| S_m - T_m \right| \leqslant \left| U_{N,0} - V_{N,0} \right| + \sum_{n=r+1}^{N} \left| \widetilde{U}_{n,l(n,m)} - \widetilde{V}_{n,l(n,m)} \right|, \tag{28}$$

for $m = 1, \ldots, 2^N$.

This statement evidently follows from Lemma 10.3 and from the relations (1)–(6).

11. Estimation of the rate of approximation in constructing the vectors with needed moments up to the third order

Below we shall need the following useful fact.

Lemma 11.1. *Assume that* $0 < \alpha < 1$ *and a random variable* μ *has values* 0 *with probability* $1 - \alpha^3$ *and* α^{-1} *with probability* α^3. *Let* $X, K \in \mathbb{R}^d$ *be random vectors with* $\mathbf{E}\, X = \mathbf{E}\, K = 0$, $\operatorname{cov} X = D$, $\operatorname{cov} K = (1 - \alpha) D$, $\mathbf{E}\, \|X\|^3 < \infty$, $\mathbf{E}\, \|K\|^3 < \infty$. *Suppose also that* X, K, μ *are mutually independent and the distribution of* K *is symmetric. Then the random vector* $W = \mu X + K$ *has mean zero, covariance operator* D *and, for any* $u, v, w \in \mathbb{R}^d$, *we have*

$$\mathbf{E} \langle W, u \rangle \langle W, v \rangle \langle W, w \rangle = \mathbf{E} \langle X, u \rangle \langle X, v \rangle \langle X, w \rangle \tag{1}$$

(*in other words,* W *and* X *have the same moments of the first three orders*).

The proof of Lemma 11.1 is elementary and, therefore, is omitted. One should note that Lemma 11.1 is a simplified (and modified) version of (due to Yurinskii [28]) Lemma 4 from [23, p. 85][3].

Consider a one-dimensional symmetric probability distribution $H = \mathcal{L}(\eta)$ concentrated on the interval $[-1, 1]$ and having an everywhere infinitely differentiable density $w(x)$. Suppose that $\mathbf{E}\, \eta^2 = 1/2$. It is evident that $\mathbf{E}\, \eta = 0$ and such a distribution can be easily constructed. Below we shall assume that the distrubution H with properties indicated above is fixed. Consider a two-dimensional random vector $\eta^* = (\eta_1^*, \eta_2^*)$ having independent coordinates η_j^*, with $\mathcal{L}(\eta_j^*) = H$, $j = 1, 2$. Denote $\zeta = (\eta_1^* + \eta_2^*,\ \eta_1^* - \eta_2^*)$, $H^* = \mathcal{L}(\zeta)$. Introduce the corresponding conjugate distributions $H_h = \overline{H}(h)$, $h \in \mathbb{R}^1$; $H_h^* = \overline{H}^*(h)$, $h \in \mathbb{R}^2$.

[3]The author is grateful to B. A. Zalesskii who attracted the author's attention to the book [23].

Lemma 11.2. *There exist absolute positive constants $c_1, \dots, c_{10}$ such that the following relations are true:*

$$\sup_{|h|\leqslant c_1} \sup_{|t|\geqslant c_2} \left|\widehat{H}_h(t)\right| \leqslant c_3 < 1, \tag{2}$$

$$\sup_{|h|\leqslant c_1} \left|\widehat{H}_h(t)\right| \leqslant \exp\Big(-\frac{1}{6}t^2\Big) \qquad \textit{for} \quad |t| \leqslant c_2, \tag{3}$$

$$\sup_{|h|\leqslant 1} \int\limits_{\mathbb{R}^1} \left|\widehat{H}_h(t)\right| dt \leqslant c_4, \qquad \sup_{|h|\leqslant 1} \int\limits_{\mathbb{R}^1} t^2 \left|\widehat{H}_h(t)\right| dt \leqslant c_5, \tag{4}$$

$$\sup_{\|h\|\leqslant c_6} \sup_{|t|\geqslant c_7} \left|\widehat{H}^*_h(t)\right| \leqslant c_8 < 1, \tag{5}$$

$$\sup_{\|h\|\leqslant c_6} \left|\widehat{H}^*_h(t)\right| \leqslant \exp\Big(-\frac{1}{3}\,\|t\|^2\Big) \qquad \textit{for} \quad |t| \leqslant c_7, \tag{6}$$

$$\sup_{\|h\|\leqslant 1} \int\limits_{\mathbb{R}^2} \left|\widehat{H}^*_h(t)\right| dt \leqslant c_9, \qquad \sup_{\|h\|\leqslant 1} \int\limits_{\mathbb{R}^2} \|t\|^2 \left|\widehat{H}^*_h(t)\right| dt \leqslant c_{10}. \tag{7}$$

Proof. We shall prove the relations (5)–(7) only. The inequalities (2)–(4) can be proved analogously. Denote by $w^*(x)$, $x \in \mathbb{R}^2$, the density of the distribution H^*. Putting $w^*(h,x) = e^{\langle h,x\rangle} w^*(x)$, $h, x \in \mathbb{R}^2$, and integrating by parts, we see that (for $t = (t_1, t_2) \in \mathbb{R}^2$, $t_j \neq 0$, $j = 1, 2$)

$$\begin{aligned}\widehat{H}^*_h(t) &= \Big(\mathbf{E}\, e^{\langle h,\zeta\rangle}\Big)^{-1} \int\limits_{|x|\leqslant\sqrt{2}} e^{i\langle t,x\rangle} w^*(h,x)\, dx \\ &= \Big(\mathbf{E}\, e^{\langle h,\zeta\rangle}\Big)^{-1} \int\limits_{|x|\leqslant\sqrt{2}} t_1^{-4} t_2^{-4}\, e^{i\langle t,x\rangle}\, \frac{\partial^8 w^*(h,x)}{\partial x_1^4\, \partial x_2^4}\, dx. \end{aligned} \tag{8}$$

Besides,

$$\sup_{x\in\mathbb{R}^2} \sup_{\|h\|\leqslant 1} \left| \frac{\partial^8 w^*(h,x)}{\partial x_1^4\, \partial x_2^4} \right| dx \leqslant c_{11} \tag{9}$$

(constants here and below are absolute, since the distribution H^* is fixed). From (8), (9) it follows that

$$\sup_{\|h\|\leqslant 1} \left|\widehat{H}^*_h(t)\right| \leqslant 2c_{11}\, t_1^{-4}\, t_2^{-4} \tag{10}$$

(we take into account that, by the Jensen inequality, $\mathbf{E}\, e^{\langle h,\zeta\rangle} \geqslant e^{\mathbf{E}\,\langle h,\zeta\rangle} = 1$). It is evident that with the help of analogous integrations by parts (different for different values of t) one can prove that

$$\sup_{\|h\|\leqslant 1} \left|\widehat{H}_h^*(t)\right| \leqslant \frac{c_{12}}{(1+t_1^4)\,(1+t_2^4)}\,. \tag{11}$$

Hence,

$$\sup_{\|h\|\leqslant 1}\ \sup_{|t|\geqslant c_{13}} \left|\widehat{H}_h^*(t)\right| \leqslant {}^1\!/_2. \tag{12}$$

(one can take $c_{13} = (2\,c_{12})^{1/4}$). Inequalities (7) follow from (11).

By Lemma 1.5, $H^* \in \mathcal{A}_2(c)$. It is easy to see that $\operatorname{cov} H^* = I : \mathbb{R}^2 \to \mathbb{R}^2$. Using Lemma 1.1 (item b) and the relations (1.15), (1.19)), we see that the relation (6) is valid, if $c_6 \leqslant 1$ and $c_7 \leqslant c_{13}$ are sufficiently small.

Since the distributions H_h^* are absolutely continuous, the relation $\left|\widehat{H}_h^*(t)\right| = 1$ can be valid for $t = 0$ only. Besides, the function $\left|\widehat{H}_h^*(t)\right|$ considered as function of two variables h and t is continuous for all $h, t \in \mathbb{R}^2$. Therefore,

$$\sup_{\|h\|\leqslant c_6}\ \sup_{c_7\leqslant|t|\leqslant c_{13}} \left|\widehat{H}_h^*(t)\right| \leqslant c < 1. \tag{13}$$

The inequalities (12), (13) imply (5). Thus, Lemma 11.2 is proved.

Assume that the conditions of Lemma 11.1 are satisfied with $D = I$, $\alpha = {}^1\!/_2$, and the vector K has independent coordinates K_j with $\mathcal{L}(K_j) = H$, $j = 1, \dots, d$. In this case we shall use below the notation $\Psi\big(\mathcal{L}(X)\big) = \mathcal{L}(W)$. It is clear that the distribution $\Psi\big(\mathcal{L}(X)\big)$ is always absolutely continuous with infinitely differentiable density. Suppose that $\mathcal{L}(X) \in \mathcal{A}_d(\tau)$, $\tau \geqslant 1$. By Lemma 1.5, item b), this implies that $\mathcal{L}(X) \in \mathcal{B}_d(c\tau)$. Using (1.35), we see that, for all $u, v \in \mathbb{R}^d$ and for all integer $m \geqslant 3$,

$$\begin{aligned}\left|\mathbf{E}\left\langle \mu X, v\right\rangle^2\left\langle \mu X, u\right\rangle^{m-2}\right| &= \mathbf{E}\,\mu^m\left|\mathbf{E}\left\langle X, v\right\rangle^2\left\langle X, u\right\rangle^{m-2}\right| \leqslant \frac{1}{2}\,m!\,\big(c\tau\,\|u\|\big)^{m-2}\mathbf{E}\left\langle X, v\right\rangle^2 \\ &\leqslant \frac{1}{2}\,m!\,\big(c\tau\,\|u\|\big)^{m-2}\,\mathbf{E}\left\langle \mu X, v\right\rangle^2.\end{aligned}$$

Therefore, by (1.35), $\mathcal{L}(\mu X) \in \mathcal{B}_d(c\tau)$ (remember that $\mathbf{E}\, X = 0$). By Lemma 1.5, item a), this implies that $\mathcal{L}(\mu X) \in \mathcal{A}_d(c\tau)$. Since $\mathbf{P}\left\{|K_j| \leqslant 1\right\} = 1$, $j = 1, \dots, d$, from Lemmas 1.5, 1.7 and Remark 1.3 it follows that

$$\mathcal{L}(K) \in \mathcal{A}_d(c_{14}), \qquad \mathcal{L}(W) = \Psi\big(\mathcal{L}(X)\big) \in \mathcal{A}_d(c_{14}\,\tau), \qquad c_{14} \geqslant 1 \tag{14}$$

(we used the completeness of classes $\mathcal{A}_d(\tau)$ with respect to convolution and their monotonicity with respect to τ). It is evident that $\mathbf{E}\, K = 0$, $\operatorname{cov} K = I/2$.

Let now $\tau \geqslant 1$ and $\widetilde{\xi}_1, \dots, \widetilde{\xi}_{2^N}$ be independent random vectors with known absolutely continuous distributions such that $\mathcal{L}(\widetilde{\xi}_k) \in \mathcal{A}_d(\tau)$, $\mathbf{E}\,\widetilde{\xi}_k = 0$, $\operatorname{cov}\widetilde{\xi}_k = I$, $k = 1, \dots, 2^N$. Introduce the independent random vectors $\xi_1, \dots, \xi_{2^N}$ with distributions satisfying one of the following conditions:

A1. $\mathcal{L}(\xi_k) = \Psi\big(\mathcal{L}(\widetilde{\xi}_k)\big)$, $k = 1, \dots, 2^N$;

A2. $N \geqslant 1$ and $\mathcal{L}(\xi_{2m-1}) = \Phi\big(\mathcal{L}(\widetilde{\xi}_{2m-1})\big)$, $\mathcal{L}(\xi_{2m}) = \mathcal{L}(\widetilde{\xi}_{2m})$, $m = 1, \dots, 2^{N-1}$.

According to Lemma 11.1, ξ_k and $\widetilde{\xi}_k$ have in the case A1 the same moments of the first three orders. Taking into account (14), we have

$$\mathcal{L}(\xi_k) \in \mathcal{A}_d(c_{14}\,\tau), \qquad k = 1, \dots, 2^N. \tag{15}$$

Let $Y_1, \dots, Y_{2^N}$ be independent Gaussian random vectors with $\mathcal{L}(Y_k) = \Phi\big(\mathcal{L}(\widetilde{\xi}_k)\big)$. Assume that the random vectors X_k, $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$; $V_{n,k}, \widetilde{V}_{n,k}$; $U_{n,k}, \widetilde{U}_{n,k}$ are constructed as functions of the vectors $Y_1, \dots, Y_{2^N}$ as in Section 10.

Theorem 11.1. *Let the conditions described above be fulfilled. Then there exist absolute positive constants c_{15}, c_{16} such that*

$$\mathbf{E}\,\exp\Big(\frac{c_{15}\,\Delta(X,Y)}{d^{3/2}\,\tau}\Big) \leqslant \exp\big(c_{16}\,N\,\log^+ d\big).$$

The rest of this section is devoted to the proof of Theorem 11.1. Therefore, below we assume that its conditions are fulfilled. The following Lemma 11.3 can be considered as an analog of Lemma 4 from Einmahl's paper [9, p. 55].

Lemma 11.3. *There exist absolute positive constants $c_{17}, \dots, c_{22}$ such that*

a) *If $\tau\, d^{3/2}/2^{N/2} \leqslant c_{17}$, then*

$$\big|U_{N,0} - V_{N,0}\big| \leqslant c_{18}\,d^{3/2}\,\tau\,\big(1 + 2^{-N}\,\big|U_{N,0}\big|^2\big) \tag{16}$$

provided that $\big|U_{N,0}\big| \leqslant \frac{c_{19}\cdot 2^N}{d^{3/2}\,\tau}$;

b) *If $1 \leqslant n \leqslant N$, $0 \leqslant k < 2^{N-n}$, $\tau\, d^{3/2}/2^{n/2} \leqslant c_{20}$, then*

$$\big|\widetilde{U}_{n,k} - \widetilde{V}_{n,k}\big| \leqslant c_{21}\,d^{3/2}\,\tau\,\Big(1 + 2^{-n}\,\max\Big\{\big|U_{n,k}\big|^2,\ \big|\widetilde{U}_{n,k}\big|^2\Big\}\Big) \tag{17}$$

provided that $\max\Big\{\big|U_{n,k}\big|,\ \big|\widetilde{U}_{n,k}\big|\Big\} \leqslant \frac{c_{22}\cdot 2^n}{d^{3/2}\,\tau}$.

Proof. At first we prove item a). From the properties of classes $\mathcal{A}_d(\tau)$ and from (15) it follows that $\mathcal{L}(U_{N,0}) \in \mathcal{A}_d(c_{14}\,\tau)$, $\mathcal{L}(V_{N,0}) = \Phi\big(\mathcal{L}(U_{N,0})\big)$. Below we prove that if c_{17} is small enough, then these distributions satisfy the conditions of Lemmas 6.1 and 8.1 with $D = 2^N I$, $\sigma^2 = 2^N$ and with the change of τ by $c_{23}\,\tau$,

where $c_{23} = \max\{c_{14}, c_1^{-1}, 1\}$. For this we shall verify that for $h \in \mathbb{R}^d$, $\|h\|\, c_{23}\,\tau < 1$, and for all $v \in \mathbb{R}^d$ the following inequalities hold:

$$(2\pi)^{-d} \int\limits_T \bigl|\widehat{F}_h(t)\bigr|\, dt \leqslant \frac{c_{23}^2\,\tau^2\,d^2}{2^N\,(2\pi)^{d/2}\cdot 2^{Nd/2}}\,, \tag{18}$$

$$(2\pi)^{-d} \int\limits_T \bigl|\langle t, v\rangle\, \widehat{F}_h(t)\bigr|\, dt \leqslant \frac{\|v\|}{4\cdot 2^{N/2}\,(2\pi)^{d/2}\cdot 2^{Nd/2}}\,, \tag{19}$$

where $F_h = \overline{F}(h)$, $F = \mathcal{L}(U_{N,0})$,

$$T = \Bigl\{ t \in \mathbb{R}^d : 4\,\|t\|\, c_{23}\,\tau\, d \geqslant 1 \Bigr\}. \tag{20}$$

Besides, we need the validity of analogous relations in the $(d-1)$-dimensional situation for the distributions F'.

Let the distributions $\mathcal{L}(\xi_k)$ satisfy the condition A1. Then from the definition of distributions $\Psi(\cdot)$ and from (1.4) it follows that

$$\bigl|\widehat{F}_h(t)\bigr| \leqslant \prod_{j=1}^{d} \bigl|\widehat{H}_{h_j}(t_j)\bigr|^{2^N}. \tag{21}$$

Since $c_{23} \geqslant c_1^{-1}$, $\tau \geqslant 1$, we can use for estimating $\bigl|\widehat{H}_{h_j}(t_j)\bigr|$ the inequalities (2)–(4). In particular, from (2), (3), (20) one can easily derive that

$$\sup_{t\in T} \prod_{j=1}^{d} \bigl|\widehat{H}_{h_j}(t_j)\bigr| \leqslant \max\Bigl\{ \exp\bigl(-(96\,c_{23}^2\,\tau^2\,d^2)^{-1}\bigr),\ c_3 \Bigr\} \leqslant \exp\Bigl(-\frac{c_{24}}{\tau^2\,d^2}\Bigr). \tag{22}$$

From (21), (22) it follows that

$$\bigl|\widehat{F}_h(t)\bigr| \leqslant \exp\Bigl(-\frac{2^N\,c_{24}}{2\,\tau^2\,d^2}\Bigr) \prod_{j=1}^{d} \bigl|\widehat{H}_{h_j}(t_j)\bigr|^{2^{N-1}}, \qquad t \in T. \tag{23}$$

Choosing c_{17} to be sufficiently small and taking into account that $\tau \geqslant 1$, we ensure the validity of the inequality

$$2^{N-1} \geqslant 2. \tag{24}$$

It is clear that the set T can be represented as the join of 2^d disjoint sets T_m, $m = 1, \ldots, 2^d$, of the form

$$T_m = T \cap \bigcap_{j\in\Xi_m} \Bigl\{ t \in \mathbb{R}^d : |t_j| \leqslant c_2 \Bigr\} \cap \bigcap_{j\notin\Xi_m} \Bigl\{ t \in \mathbb{R}^d : |t_j| > c_2 \Bigr\}. \tag{25}$$

Here Ξ_m, $m = 1, \ldots, 2^d$, are all possible different subsets of the set $\{1, \ldots, d\}$.

Fix a set Ξ_m. For $j \in \Xi_m$, using (3), we obtain

$$\int\limits_{|t_j|\leqslant c_2} \bigl|\widehat{H}_{h_j}(t_j)\bigr|^{2^{N-1}} dt_j \leqslant \int\limits_{|t|\leqslant c_2} \exp\Bigl(-\frac{2^N t^2}{12}\Bigr)\, dt \leqslant \frac{c}{2^{N/2}}\,. \tag{26}$$

If $j \notin \Xi_m$, then, using (2), (4), (24), we have

$$\int\limits_{|t_j|>c_2} \bigl|\widehat{H}_{h_j}(t_j)\bigr|^{2^{N-1}} dt_j \leqslant \int\limits_{|t|>c_2} c_3^{2^{N-2}} \bigl|\widehat{H}_{h_j}(t)\bigr|\, dt \leqslant c\exp\bigl(-c\cdot 2^N\bigr) \leqslant \frac{c}{2^{N/2}}\,. \tag{27}$$

From (23), (25)–(27) it follows that, for $m = 1, \ldots, 2^d$,

$$(2\pi)^{-d} \int\limits_{T_m} \bigl|\widehat{F}_h(t)\bigr|\, dt \leqslant \frac{c^d}{2^{Nd/2}\,(2\pi)^{d/2}}\ \exp\Bigl(-\frac{c\cdot 2^N}{\tau^2 d^2}\Bigr). \tag{28}$$

Using now (25), (28), we see that

$$\begin{aligned}(2\pi)^{-d} \int\limits_{T} \bigl|\widehat{F}_h(t)\bigr|\, dt &\leqslant (2\pi)^{-d} \sum_{m=1}^{2^d} \int\limits_{T_m} \bigl|\widehat{F}_h(t)\bigr|\, dt \\ &\leqslant \frac{1}{2^{Nd/2}\,(2\pi)^{d/2}}\ \exp\Bigl(c_{25}\, d - \frac{c_{26}\cdot 2^N}{\tau^2 d^2}\Bigr).\end{aligned} \tag{29}$$

It is evident that, choosing c_{17} to be small enough, we can ensure the validity of the inequality

$$\exp\Bigl(c_{25}\, d - \frac{c_{26}\cdot 2^N}{\tau^2 d^2}\Bigr) \leqslant \frac{c_{23}^2\,\tau^2 d^2}{2^N} \leqslant \frac{1}{4}\,. \tag{30}$$

The inequalities (29), (30) imply (18).

For the proof of (19) we apply the Hölder inequality and obtain that

$$\int\limits_{T} \bigl|\bigl\langle t, v\bigr\rangle\, \widehat{F}_h(t)\bigr|\, dt \leqslant \|v\| \left(\int\limits_{T} \bigl|\widehat{F}_h(t)\bigr|\, dt\right)^{1/2} \left(\int\limits_{T} \|t\|^2\, \bigl|\widehat{F}_h(t)\bigr|\, dt\right)^{1/2}. \tag{31}$$

For estimating the first integral in the right-hand side of (31) we note that, according to (29), (30),

$$(2\pi)^{-d} \int\limits_{T} \bigl|\widehat{F}_h(t)\bigr|\, dt \leqslant \frac{1}{4\,(2\pi)^{d/2}\cdot 2^{Nd/2}}\,. \tag{32}$$

Let us estimate the second integral. For this, using (3), we obtain

$$\int\limits_{|t_j|\leqslant c_2} t_j^2 \left|\widehat{H}_{h_j}(t_j)\right|^{2^{N-1}} dt_j \leqslant \int\limits_{|t|\leqslant c_2} t^2 \exp\Big(-\frac{2^N t^2}{12}\Big)\, dt \leqslant \frac{c}{2^{3N/2}}. \tag{33}$$

Besides, using (2), (4), (24), we have

$$\int\limits_{|t_j|>c_2} t_j^2 \left|\widehat{H}_{h_j}(t_j)\right|^{2^{N-1}} dt_j \leqslant \int\limits_{|t|>c_2} c_3^{2^{N-2}} t^2 \left|\widehat{H}_{h_j}(t)\right| dt$$
$$\leqslant c \exp\big(-c\cdot 2^N\big) \leqslant \frac{c}{2^{3N/2}}. \tag{34}$$

From (23), (25)–(27), (33), (34) it follows that, for $m = 1, \ldots, 2^d$,

$$(2\pi)^{-d} \int\limits_{T_m} \|t\|^2 \left|\widehat{F}_h(t)\right| dt \leqslant \frac{c^d}{2^N (2\pi)^{d/2}\cdot 2^{Nd/2}} \exp\Big(-\frac{c\cdot 2^N}{\tau^2 d^2}\Big). \tag{35}$$

Using now (25), (35), we see that

$$(2\pi)^{-d} \int\limits_{T} \|t\|^2 \left|\widehat{F}_h(t)\right| dt \leqslant (2\pi)^{-d} \sum_{m=1}^{2^d} \int\limits_{T_m} \|t\|^2 \left|\widehat{F}_h(t)\right| dt$$
$$\leqslant \frac{1}{2^N (2\pi)^{d/2}\cdot 2^{Nd/2}} \exp\Big(c_{27} d - \frac{c_{28}\cdot 2^N}{\tau^2 d^2}\Big). \tag{36}$$

It is evident that, choosing c_{17} to be small enough, we can ensure the validity of the inequality

$$\exp\Big(c_{27} d - \frac{c_{28}\cdot 2^N}{\tau^2 d^2}\Big) \leqslant \frac{1}{4}. \tag{37}$$

The inequalities (31), (32), (36), (37) imply (19).

The analogs of the relations (18), (19) for the distributions F' follows, in fact, from (18), (19) applied in the $(d-1)$-dimensional case. Thus, the validity of the relations (18), (19) is established in the case A1.

Let now the distributions $\mathcal{L}(\xi_k)$ satisfy the condition A2. Then, using (1.4), (1.7), we obtain

$$\left|\widehat{F}_h(t)\right| \leqslant \exp\Big(-\frac{\|t\|^2\cdot 2^{N-1}}{2}\Big). \tag{38}$$

The relations (2.19), (20), (38) and the corresponding analog of (30) imply

$$(2\pi)^{-d} \int\limits_{T} \left|\widehat{F}_h(t)\right| dt \leqslant (2\pi)^{-d} \exp\Big(-\frac{2^{N-2}}{32\, c_{23}^2\, \tau^2 d^2}\Big) \int\limits_{T} \exp\Big(-\frac{\|t\|^2\cdot 2^{N-2}}{2}\Big)\, dt$$
$$\leqslant \frac{c^d}{(2\pi)^{d/2}\cdot 2^{Nd/2}} \exp\Big(-\frac{c\cdot 2^N}{\tau^2 d^2}\Big)$$
$$\leqslant \frac{c_{23}^2\, \tau^2 d^2}{2^N (2\pi)^{d/2}\cdot 2^{Nd/2}} \leqslant \frac{1}{4\,(2\pi)^{d/2}\cdot 2^{Nd/2}}, \tag{39}$$

if c_{17} is small enough.

Analogously, using (2.22), (20), (38), we obtain

$$(2\pi)^{-d}\int\limits_T \langle t,v\rangle^2 \bigl|\widehat{F}_h(t)\bigr|\,dt \leqslant (2\pi)^{-d}\exp\Bigl(-\frac{2^{N-2}}{32\,c_{23}^2\,\tau^2\,d^2}\Bigr)\int\limits_T \langle t,v\rangle^2\exp\Bigl(-\frac{\|t\|^2\cdot 2^{N-2}}{2}\Bigr)\,dt$$

$$\leqslant \frac{c^d\,\|v\|^2}{2^N\,(2\,\pi)^{d/2}\cdot 2^{Nd/2}}\ \exp\Bigl(-\frac{c\cdot 2^N}{\tau^2\,d^2}\Bigr)$$

$$\leqslant \frac{\|v\|^2}{4\cdot 2^N\,(2\,\pi)^{d/2}\cdot 2^{Nd/2}}\,. \tag{40}$$

The inequalities (39), (40) and the Hölder inequality imply (18), (19) in the case A2. In a similar way one can prove the analogs of the inequalities (18), (19) for the distributions F' in the $(d-1)$-dimensional case.

Using Lemma 1.6, we can analogously prove that the conditions of Lemmas 6.1 and 8.1 with $D=2^N I$, $\sigma^2=2^N$ and with the change of τ by $c_{23}\,\tau$ are satisfied in the j-dimensional situation for the distributions $\mathcal{L}(U_{N,0}^j)$, $\mathcal{L}(V_{N,0}^j)$, $j=1,\ldots,d$, where the vectors

$$U_{N,0}^j=\bigl(U_{N,0}^{(1)},\ldots,U_{N,0}^{(j)}\bigr),\qquad V_{N,0}^j=\bigl(V_{N,0}^{(1)},\ldots,V_{N,0}^{(j)}\bigr) \tag{41}$$

consist of the first j coordinates of the vectors $U_{N,0}$, $V_{N,0}$ respectively. Note that all distribution functions $G_{N,0}^{(j)}$ coincide with $\Phi_\sigma(\,\cdot\,)$, $j=1,\ldots,d$.

Taking into account (10.7) and applying Lemma 8.1 and the inequality (6.3) of Lemma 6.1 in the j-dimensional case, $j=1,\ldots,d$, we obtain that

$$\bigl|U_{N,0}^{(1)}-V_{N,0}^{(1)}\bigr|\leqslant c_{29}\,\tau\Bigl(1+\frac{\bigl|U_{N,0}^{(1)}\bigr|^2}{2^N}\Bigr), \tag{42}$$

if $\dfrac{\tau}{2^{N/2}}\leqslant c_{30}$, $\bigl|U_{N,0}^{(1)}\bigr|\leqslant\dfrac{c_{31}\cdot 2^N}{\tau}$;

$$\bigl|U_{N,0}^{(j)}-V_{N,0}^{(j)}\bigr|\leqslant c_{29}\,\tau\left(j^{3/2}+j^{3/2}\,\frac{\bigl|U_{N,0}^{j-1}\bigr|}{2^{N/2}}\,\Bigl(1+\frac{\bigl|U_{N,0}^{(j)}\bigr|}{2^{N/2}}\Bigr)+\frac{\bigl|U_{N,0}^{(j)}\bigr|^2}{2^N}+j\,\frac{\bigl|U_{N,0}^{j-1}\bigr|^2}{2^N}\right), \tag{43}$$

if

$$\frac{\tau\,j^{3/2}}{2^{N/2}}\leqslant c_{30},\quad \bigl|U_{N,0}^{(j)}\bigr|\leqslant\frac{c_{31}\cdot 2^N}{j\,\tau},\quad \frac{\bigl|U_{N,0}^{j-1}\bigr|}{2^{N/2}}\leqslant\frac{c_{32}\cdot 2^{N/2}}{j^{3/2}\tau},\qquad 2\leqslant j\leqslant d. \tag{44}$$

Checking the inequality (43), one should take into account that in Lemmas 6.1 and 8.1 the densities $v(\,\cdot\,)$ are the densities of shifted conditional distributions. The possibility to write the inequality (43) and the conditions (44) without shift is ensured by an available choice of sufficiently small c_{31} and c_{32} with the use of inequality (6.3).

Inequality (16) with $c_{17}=c_{30}$, $c_{18}=4c_{29}$, $c_{19}=\min\bigl\{c_{31},\,c_{32}\bigr\}$ follows from (42), (43) and from the definitions of $U_{N,0}$, $U_{N,0}^j$.

Now we shall prove item b). Let $L^*_{n,k} = \big(U_{n-1,2k},\, U_{n-1,2k+1}\big) \in \mathbb{R}^{2d}$ be the vector with first d coordinates coinciding with those of $U_{n-1,2k}$ and with last d coordinates coinciding with those of $U_{n-1,2k+1}$. Let $L_{n,k} = \big(U_{n,k},\, \widetilde{U}_{n,k}\big) \in \mathbb{R}^{2d}$ be the vector constructed analogously from the coordinates of the vectors $U_{n,k}$, $\widetilde{U}_{n,k}$. It is evident that the vectors $U_{n-1,2k}$, $U_{n-1,2k+1}$ are independent. Besides, from the relations (10.2), (15) it follows that these vectors are sums of independent summands with distributions from $\mathcal{A}_d(c_{14}\,\tau)$. From the properties of classes $\mathcal{A}_d(\tau)$ and from Lemma 1.7 it follows that $\mathcal{L}(L^*_{n,k}) \in \mathcal{A}_{2d}(c_{14}\,\tau)$. The relations (10.12) show that $L_{n,k} = A\,L^*_{n,k}$, where $A : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is a linear operator with $\|A\| = \sqrt{2}$. Therefore, using Lemma 1.6, we see that $\mathcal{L}(L_{n,k}) \in \mathcal{A}_{2d}(c_{14}\sqrt{2}\,\tau)$.

Introduce the corresponding Gaussian random vectors $R_{n,k} = \big(V_{n,k},\, \widetilde{V}_{n,k}\big) \in \mathbb{R}^{2d}$. It can be easily verified that (see (10.1)–(10.6)) $\mathcal{L}(R_{n,k}) = \Phi\big(\mathcal{L}(L_{n,k})\big)$. Define also the vectors

$$L^j_{n,k} = \big(L^{(1)}_{n,k}, \dots, L^{(j)}_{n,k}\big), \qquad R^j_{n,k} = \big(R^{(1)}_{n,k}, \dots, R^{(j)}_{n,k}\big), \qquad j = d, \dots, 2d, \tag{45}$$

consisting of the first j coordinates of the vectors $L_{n,k}$, $R_{n,k}$ respectively.

One can show that the distributions $\mathcal{L}(L^j_{n,k})$, $\mathcal{L}(R^j_{n,k})$, $j = d+1, \dots, 2d$, satisfy in j-dimensional situation the conditions of Lemmas 6.1, 8.1 with $D = 2^n I$, $\sigma^2 = 2^n$ and with the change of d by j and τ by $c_{33}\,\tau$, where $c_{33} = \max\big\{c_{14}\sqrt{2},\, c_1^{-1},\, c_6^{-1}, 1\big\}$. For this one should fix j and verify that for $h^* \in \mathbb{R}^j$, $\|h^*\|\, c_{33}\,\tau < 1$, and, for all $v^* \in \mathbb{R}^j$, the following inequalities hold:

$$(2\,\pi)^{-j} \int\limits_T \big|\widehat{F}_{h^*}(t^*)\big|\, dt^* \leqslant \frac{c_{33}^2\,\tau^2\,j^2}{2^n\,(2\,\pi)^{j/2}\cdot 2^{nj/2}}\,, \tag{46}$$

$$(2\,\pi)^{-j} \int\limits_T \big|\big\langle t^*, v^*\big\rangle\, \widehat{F}_{h^*}(t^*)\big|\, dt^* \leqslant \frac{\|\,v^*\|}{4\cdot 2^{n/2}\,(2\,\pi)^{j/2}\cdot 2^{nj/2}}\,, \tag{47}$$

where $F_{h^*} = \overline{F}(h^*)$, $F = \mathcal{L}(L^j_{n,k})$, $h^* = \big(h^*_1, \dots, h^*_d\big) \in \mathbb{R}^j$,

$$\begin{gathered} T = \Big\{t^* = \big(t^*_1, \dots, t^*_d\big) \in \mathbb{R}^j : 4\,\|t^*\|\, c_{33}\,\tau\, j \geqslant 1\Big\}; \\ h^*_m = (h_m, h_{d+m}) \in \mathbb{R}^2, \quad t^*_m = (t_m, t_{d+m}) \in \mathbb{R}^2, \qquad m = 1, \dots, j-d; \\ h^*_m = h_m \in \mathbb{R}^1, \quad t^*_m = t_m \in \mathbb{R}^1, \qquad m = j-d+1, \dots, d. \end{gathered} \tag{48}$$

Besides, we need the validity of analogous relations in the $(j-1)$-dimensional situation for the distributions F'. Thus, one should verify the fulfilment of (46), (47) for $j = d, \dots, 2d$.

Let the conditions A1 be satisfied. Then, using the definition of distributions $\Psi(\cdot)$ and of random vectors $L_{n,k}^{j}$ and the relations (1.4), (10.1), (10.2), (10.12), we obtain that

$$\bigl|\widehat{F}_{h^*}(t^*)\bigr| \leqslant \prod_{m=1}^{j-d} \bigl|\widehat{H}^*_{h^*_m}(t^*_m)\bigr|^{2^n} \prod_{m=j-d+1}^{d} \bigl|\widehat{H}_{h_m}(t_m)\bigr|^{2^{n+1}}. \tag{49}$$

From the definition of c_{33} it follows that we can use for estimating $\bigl|\widehat{H}_{h_m}(t_m)\bigr|$ the inequalities (2)–(4) and for estimating $\bigl|\widehat{H}^*_{h^*_m}(t^*_m)\bigr|$ the inequalities (5)–(7). The rest of the proof of inequalities (46), (47) in the case A1 does not differ from the proof of (18), (19) and therefore is omitted. The only difference is in the use of (5)–(7) instead of (2)–(4) and in the change of constants and notations (see also the proof of (46), (47) in somewhat more complicated situation in Section 12). The proof of inequalities (46), (47) in the case A2 is also similar to the corresponding proof of (18), (19).

Using (10.8), (10.9), taking into account that all distribution functions $\widetilde{G}_{n,k}^{(j-d)}(\cdot\,|\,\cdot)$ coincide with $\Phi_\sigma(\cdot)$ and applying Lemma 8.1 and the inequality (6.3) of Lemma 6.1 in the j-dimensional case, $j = d+1, \ldots, 2d$, we obtain that

$$\bigl|L^{(j)}_{n,k} - R^{(j)}_{n,k}\bigr| \leqslant c_{34}\,\tau \left(j^{3/2} + j^{3/2}\,\frac{\bigl|L^{j-1}_{n,k}\bigr|}{2^{n/2}} \left(1 + \frac{\bigl|L^{(j)}_{n,k}\bigr|}{2^{n/2}}\right) + \frac{\bigl|L^{(j)}_{n,k}\bigr|^2}{2^n} + j\,\frac{\bigl|L^{j-1}_{n,k}\bigr|^2}{2^n} \right), \tag{50}$$

if

$$\frac{\tau\, j^{3/2}}{2^{n/2}} \leqslant c_{35}, \quad \bigl|L^{(j)}_{n,k}\bigr| \leqslant \frac{c_{36}\cdot 2^n}{j\,\tau}, \quad \frac{\bigl|L^{j-1}_{n,k}\bigr|}{2^{n/2}} \leqslant \frac{c_{37}\cdot 2^{n/2}}{j^{3/2}\tau}. \tag{51}$$

Checking the inequality (50) should be carried out analogously to the proof of (43). The possibility to write the inequality (50) and the conditions (51) without shift is also ensured by the available choice of sufficiently small constants (c_{36} and c_{37}) with the use of inequality (6.3).

Inequality (17) follows from (45), (50) and from the definition of $L_{n,k}$, $R_{n,k}$ with $c_{20} = 2^{-3/2}c_{35}$, $c_{21} = 2^{3/2}\cdot 4c_{34}$, $c_{22} = 2^{-3/2}\min\bigl\{c_{36}, c_{37}\bigr\}$. The proof of Lemma 11.3 is completed.

Suppose that

$$c_{38} = \min\bigl\{4c_{19},\, 4c_{22},\, 32/3c_{14}\bigr\}, \quad c_{39} = \frac{c_{38}}{\min\{c_{17}^2,\, c_{20}^2,\, 2^{-1}\}}, \quad x \geqslant c_{39}\,d^{3/2}\tau, \tag{52}$$

and choose the integer M from the conditions

$$x < \frac{c_{38}\cdot 2^M}{d^{3/2}\,\tau} \leqslant 2\,x. \tag{53}$$

It is clear that $M \geqslant 1$ since (52) implies the inequality $x \geqslant \frac{2\,c_{38}}{d^{3/2}\,\tau}$. Denote

$$\Delta = \Delta(X,Y) = \max_{1 \leqslant k \leqslant 2^N} \bigl| S_k - T_k \bigr|. \tag{54}$$

We consider separately two possible cases: $M \geqslant N$ and $M < N$. Let, at first, $M \geqslant N$. Denote

$$\begin{aligned} \Delta_1 &= \max_{1 \leqslant k \leqslant 2^N} \bigl| S_k \bigr|, \\ \Delta_2 &= \max_{1 \leqslant k \leqslant 2^N} \bigl| T_k \bigr|. \end{aligned} \tag{55}$$

It is easy to see that

$$\Delta \leqslant \Delta_1 + \Delta_2. \tag{56}$$

From (56) it follows that

$$\mathbf{P}\Bigl\{ \Delta \geqslant x \Bigr\} \leqslant \mathbf{P}\Bigl\{ \Delta_1 \geqslant x/2 \Bigr\} + \mathbf{P}\Bigl\{ \Delta_2 \geqslant x/2 \Bigr\}. \tag{57}$$

Applying Lemma 10.1 with parameters $\tau^* = c_{14}\,\tau$, $x^* = \frac{x}{4 \cdot 2^M} \leqslant \frac{8}{3\,c_{14}\,\tau}$, $N^* = M$, $m^* = 2^M$ and using (53), we obtain in the case $M \geqslant N$ that

$$\begin{aligned} \mathbf{P}\Bigl\{ \Delta_1 \geqslant x/2 \Bigr\} \leqslant \mathbf{P}\Bigl\{ \max_{1 \leqslant k \leqslant 2^N} \bigl| S_k \bigr| \geqslant m^*\,x^* \Bigr\} &\leqslant \mathbf{P}\Bigl\{ \max_{1 \leqslant k \leqslant 2^M} \bigl| S_k \bigr| \geqslant m^*\,x^* \Bigr\} \\ &\leqslant (4\,d+1)\,\exp\Bigl(-\,\frac{3\,x^2}{512 \cdot 2^M} \Bigr) \\ &\leqslant (4\,d+1)\exp\Bigl(-\,\frac{3\,c_{38}\,x}{1024\,d^{3/2}\,\tau} \Bigr) \end{aligned} \tag{58}$$

(we can continue the sequence of independent random vectors X_k, $1 \leqslant k \leqslant 2^N$, for example, we can assume that $\bigl\{ X_l,\ l > 2^N \bigr\}$ is a sequence of independent copies of X_{2^N}). Since Gaussian distributions from $\mathfrak{G}_d$ belong to all classes $\mathcal{A}_d(\tau)$, we automatically obtain that

$$\mathbf{P}\Bigl\{ \Delta_2 \geqslant x/2 \Bigr\} \leqslant (4\,d+1)\,\exp\Bigl(-\,\frac{3\,c_{38}\,x}{1024\,d^{3/2}\,\tau} \Bigr). \tag{59}$$

From (57)–(59) it follows in the case $M \geqslant N$ that

$$\mathbf{P}\Bigl\{ \Delta \geqslant x \Bigr\} \leqslant 10\,d\,\exp\Bigl(-\,\frac{3\,c_{38}\,x}{1024\,d^{3/2}\,\tau} \Bigr). \tag{60}$$

Let now $M < N$. It is easy to see that from (52), (53) it follows that $M \geqslant 1$. Denote

$$\Delta_3 = \max_{0 \leqslant k < 2^{N-M}} \max_{1 \leqslant l \leqslant 2^M} \bigl| S_{k \cdot 2^M + l} - S_{k \cdot 2^M} \bigr|, \tag{61}$$

$$\Delta_4 = \max_{0 \leqslant k < 2^{N-M}} \max_{1 \leqslant l \leqslant 2^M} \bigl| T_{k \cdot 2^M + l} - T_{k \cdot 2^M} \bigr|, \tag{62}$$

$$\Delta_5 = \max_{1 \leqslant k < 2^{N-M}} \bigl| S_{k \cdot 2^M} - T_{k \cdot 2^M} \bigr|. \tag{63}$$

It is evident that

$$\Delta \leqslant \Delta_3 + \Delta_4 + \Delta_5. \tag{64}$$

Introduce the event

$$A = \Big\{\omega : \big|U_{M,k}\big| < \tfrac{c_{38}\cdot 2^M}{4\,d^{3/2}\,\tau},\ 0 \leqslant k < 2^{N-M}\Big\} \tag{65}$$

(we shall always assume that all considered random vectors are measurable mappings of elementary event $\omega \in \Omega$). By virtue of (53), (61), for the opposite event $\overline{A}$ we have

$$\overline{A} \subset \big\{\omega : \Delta_3 \geqslant x/4\big\}. \tag{66}$$

From (64), (66) it follows that

$$\mathbf{P}\big\{\Delta \geqslant x\big\} \leqslant \mathbf{P}\big\{\Delta_3 \geqslant x/4\big\} + \mathbf{P}\big\{\Delta_4 \geqslant x/4\big\} + \mathbf{P}\big\{\Delta_5 \geqslant x/2,\ A\big\}. \tag{67}$$

Applying Lemma 10.1 with parameters $\tau^* = c_{14}\,\tau$, $x^* = \frac{x}{4\cdot 2^M} \leqslant \frac{8}{3\,c_{14}\,\tau}$, $N^* = N$, $m^* = 2^M$ and using (53), we obtain in the case $M < N$ that

$$\begin{aligned}\mathbf{P}\Big\{\max_{1\leqslant l\leqslant 2^M}\big|S_{k\cdot 2^M+l} - S_{k\cdot 2^M}\big| \geqslant x/4\Big\} &= \mathbf{P}\Big\{\max_{1\leqslant l\leqslant 2^M}\big|S_{k\cdot 2^M+l} - S_{k\cdot 2^M}\big| \geqslant m^*\,x^*\Big\}\\ &\leqslant (4d+1)\,\exp\Big(-\frac{3\,x^2}{512\cdot 2^M}\Big)\\ &\leqslant (4d+1)\,\exp\Big(-\frac{3\,c_{38}\,x}{1024\,d^{3/2}\,\tau}\Big).\end{aligned} \tag{68}$$

Since $\mathfrak{G}_d \subset \mathcal{A}_d(\tau)$ for all $\tau \geqslant 0$, we obtain immediately that

$$\mathbf{P}\Big\{\max_{1\leqslant l\leqslant 2^M}\big|T_{k\cdot 2^M+l} - T_{k\cdot 2^M}\big| \geqslant x/4\Big\} \leqslant (4d+1)\,\exp\Big(-\frac{3\,c_{38}\,x}{1024\,d^{3/2}\,\tau}\Big). \tag{69}$$

From (61), (62), (68), (69) it follows that

$$\mathbf{P}\big\{\Delta_3 \geqslant x/4\big\} + \mathbf{P}\big\{\Delta_4 \geqslant x/4\big\} \leqslant 2^N\cdot 10d\,\exp\Big(-\frac{3\,c_{38}\,x}{1024\,d^{3/2}\,\tau}\Big). \tag{70}$$

According to Corollary 10.3, for $1 \leqslant k \leqslant 2^{N-M}$ we have

$$\big|S_{k\cdot 2^M} - T_{k\cdot 2^M}\big| \leqslant \big|U_{N,0} - V_{N,0}\big| + \sum_{n=M+1}^{N}\big|\widetilde{U}_{n,l_k(n)} - \widetilde{V}_{n,l_k(n)}\big|, \tag{71}$$

where $l_k(n)$ are integers, defined by $l_k(n)\cdot 2^n < k\cdot 2^M \leqslant \big(l_k(n)+1\big)\cdot 2^n$.

By virtue of (52), (53), we have $c_{39}\, d^{3/2}\tau \leqslant x \leqslant \frac{c_{38}\cdot 2^M}{d^{3/2}\tau}$ and, consequently,

$$\tau \leqslant \frac{\min\{c_{17}, c_{20}\}\cdot 2^{M/2}}{d^{3/2}}, \qquad \frac{\tau\, d^{3/2}}{2^{M/2}} \leqslant \min\{c_{17}, c_{20}\}. \tag{72}$$

According to (52), (65), for $\omega \in A$ we have

$$\bigl|U_{M,k}\bigr| < \frac{c_{38}\cdot 2^M}{4\, d^{3/2}\tau} \leqslant \frac{\min\{c_{19},\, c_{22}\}\cdot 2^M}{d^{3/2}\tau}, \qquad 0 \leqslant k < 2^{N-M}, \tag{73}$$

and, by (10.2), (10.10)–(10.12), $U_{M,k}$ are sums over blocks consisting of 2^M summands. Besides, $U_{n,l}$ (resp. $\widetilde{U}_{n,l}$), $M+1 \leqslant n \leqslant N$, $0 \leqslant l < 2^{N-n}$, are sums (resp. differences) of two sums over blocks containing each 2^{n-1} summands. These sums can be represented as sums of 2^{n-M-1} sums over blocks containing each 2^M summands and satisfying (73). Therefore, for $\omega \in A$, $0 \leqslant l < 2^{N-n}$ we have

$$\max\Bigl\{\bigl|U_{n,l}\bigr|,\ \bigl|\widetilde{U}_{n,l}\bigr|\Bigr\} \leqslant 2\cdot 2^{n-M-1}\,\frac{\min\{c_{19},\, c_{22}\}\cdot 2^M}{d^{3/2}\tau} = \frac{\min\{c_{19},\, c_{22}\}\cdot 2^n}{d^{3/2}\tau}. \tag{74}$$

Using (72), (74), we see that if $\omega \in A$, the conditions of Lemma 11.3 are satisfied for $U_{n,l}$, $\widetilde{U}_{n,l}$, $M+1 \leqslant n \leqslant N$, $0 \leqslant l < 2^{N-n}$. By (71) and by Lemma 11.3, for $1 \leqslant k \leqslant 2^{N-M}$, $\omega \in A$ we have

$$\begin{aligned}
\bigl|S_{k\cdot 2^M} - T_{k\cdot 2^M}\bigr| &\leqslant c_{18}\, d^{3/2}\tau\Bigl(1 + 2^{-N}\bigl|U_{N,0}\bigr|^2\Bigr) \\
&\quad + \sum_{n=M+1}^{N} c_{21}\, d^{3/2}\tau\Bigl(1 + 2^{-n}\max\Bigl\{\bigl|U_{n,l_k(n)}\bigr|^2,\ \bigl|\widetilde{U}_{n,l_k(n)}\bigr|^2\Bigr\}\Bigr) \\
&\leqslant c d^{3/2}\tau\left(N + 2^{-N}\bigl|U_{N,0}\bigr|^2 + \sum_{n=M}^{N-1} 2^{-n}\Bigl(\bigl|U_{n,l_k(n)}\bigr|^2 + \bigl|U_{n,l'_k(n)}\bigr|^2\Bigr)\right),
\end{aligned} \tag{75}$$

where

$$l'_k(n-1) = \begin{cases} 2\,l_k(n), & \text{if} \quad l_k(n-1) = 2\,l_k(n)+1, \\ 2\,l_k(n)+1, & \text{if} \quad l_k(n-1) = 2\,l_k(n), \end{cases} \qquad M < n \leqslant N, \tag{76}$$

(it is not hard to verify that $l_k(n-1)$ can be equal either to $2\,l_k(n)$ or to $2\,l_k(n)+1$). In other words, $U_{n,l_k(n)}$, $M \leqslant n \leqslant N$, is the sum over the block of 2^n summands which contains $X_{k\cdot 2^M}$. The sum $U_{n,l'_k(n)}$ does not contain $X_{k\cdot 2^M}$ and

$$U_{n+1,l_k(n+1)} = U_{n,l_k(n)} + U_{n,l'_k(n)}, \qquad M \leqslant n < N. \tag{77}$$

The equality (77) implies

$$U_{n,l_k(n)} = U_{M,l_k(M)} + \sum_{s=0}^{n-M-1} U_{M+s,l'_k(M+s)}, \qquad M \leqslant n \leqslant N. \tag{78}$$

It is important that all summands in the right-hand side of (78) are the sums of disjoint blocks of independents summands. Therefore, they are themselves independent. This argument will be used below in similar situations.

Put $q = 1/\sqrt{2}$. Then from (78) one can easily derive, using the Cauchy–Bunyakovskii–Schwarz inequality that, for $M \leqslant n \leqslant N$,

$$\left|U_{n,l_k(n)}\right|^2 \leqslant c\left(q^{-(n-M)}\left|U_{M,l_k(M)}\right|^2 + \sum_{s=0}^{n-M-1} q^{-(n-M-1)+s}\left|U_{M+s,l'_k(M+s)}\right|^2\right), \tag{79}$$

with $c = \sum\limits_{j=0}^{\infty} q^j = \dfrac{\sqrt{2}}{\sqrt{2}-1}$. It is easy to see that

$$\sum_{n=M}^{N} 2^{-n}\, q^{-(n-M)}\left|U_{M,l_k(M)}\right|^2 \leqslant c\cdot 2^{-M}\left|U_{M,l_k(M)}\right|^2. \tag{80}$$

Besides,

$$\begin{aligned}\sum_{n=M+1}^{N}\sum_{s=0}^{n-M-1} 2^{-n} q^{-(n-M-1)+s}\left|U_{M+s,l'_k(M+s)}\right|^2 &= \sum_{s=0}^{N-M-1}\sum_{n=M+1+s}^{N} 2^{-n} q^{-(n-M-1)+s}\left|U_{M+s,l'_k(M+s)}\right|^2 \\ &\leqslant c\sum_{s=0}^{N-M-1} 2^{-(M+1+s)}\left|U_{M+s,l'_k(M+s)}\right|^2.\end{aligned} \tag{81}$$

It is clear that the inequalities (79)–(81) imply

$$\begin{aligned}&2^{-N}\left|U_{N,0}\right|^2 + \sum_{n=M}^{N-1} 2^{-n}\left(\left|U_{n,l_k(n)}\right|^2 + \left|U_{n,l'_k(n)}\right|^2\right) \\ &\quad\leqslant c\left(\frac{\left|U_{M,l_k(M)}\right|^2}{2^M} + \sum_{s=0}^{N-M-1}\frac{\left|U_{M+s,l'_k(M+s)}\right|^2}{2^{M+1+s}}\right) + \sum_{n=M}^{N-1}\frac{\left|U_{n,l'_k(n)}\right|^2}{2^n} \\ &\quad\leqslant c\left(\frac{\left|U_{M,l_k(M)}\right|^2}{2^M} + \sum_{n=M}^{N-1}\frac{\left|U_{n,l'_k(n)}\right|^2}{2^n}\right).\end{aligned} \tag{82}$$

From (75), (82) it follows that for $1 \leqslant k \leqslant 2^{N-M}$, $\omega \in A$ we have

$$\left| S_{k\cdot 2^M} - T_{k\cdot 2^M} \right| \leqslant c_{40}\, d^{3/2}\, \tau \left(N + \frac{\left| U_{M,l_k(M)} \right|^2}{2^M} + \sum_{n=M}^{N-1} \frac{\left| U_{n,l'_k(n)} \right|^2}{2^n} \right). \tag{83}$$

Besides, using (52), (65), we see that, for $M \leqslant n \leqslant N$, $0 \leqslant l < 2^{N-n}$, $\omega \in A$, $\tau^* = c_{14}\,\tau$,

$$\left| U_{n,l} \right| \leqslant 2^{n-M} \max_{0 \leqslant m < 2^{N-M}} \left| U_{M,m} \right| \leqslant 2^{n-M}\, \frac{c_{38}\cdot 2^M}{4\, d^{3/2}\, \tau} \leqslant \frac{8\cdot 2^n}{3 c_{14}\, \tau} = \frac{8\cdot 2^n}{3\,\tau^*}. \tag{84}$$

This implies (see (10.17)) that

$$\varkappa_{n,l} = 2^{-n} \left| U_{n,l} \right|^2, \qquad \omega \in A, \tag{85}$$

where $\varkappa_{n,l}$ are the random variables from Lemma 10.2, which will be applied with parameter $\tau^* = c_{14}\,\tau$. Thus, we can rewrite (83) in the form

$$\left| S_{k\cdot 2^M} - T_{k\cdot 2^M} \right| \leqslant c_{40}\, d^{3/2}\, \tau \left(N + \varkappa_{M,l_k(M)} + \sum_{n=M}^{N-1} \varkappa_{n,l'_k(n)} \right), \tag{86}$$

if $1 \leqslant k \leqslant 2^{N-M}$, $\omega \in A$. Therefore, choosing $t^* = (32\, c_{40}\, d^{3/2}\, \tau)^{-1}$ and $t = t^* \cdot c_{40}\, d^{3/2}\, \tau = 1/32$, taking into account that the random variables $\varkappa_{M,l_k(M)}$, $\varkappa_{M,l'_k(M)}$, $\ldots$, $\varkappa_{N-1,l'_k(N-1)}$ are independent and applying the inequality (10.18) of Lemma 10.2, we obtain

$$\begin{aligned}
&\mathbf{P}\left\{ \left\{ \omega : \left| S_{k\cdot 2^M} - T_{k\cdot 2^M} \right| \geqslant x/2 \right\} \cap A \right\} \\
&\qquad \leqslant \mathbf{P}\left\{ c_{40}\, d^{3/2}\, \tau \left(N + \varkappa_{M,l_k(M)} + \sum_{n=M}^{N-1} \varkappa_{n,l'_k(n)} \right) \geqslant x/2 \right\} \\
&\qquad \leqslant \mathbf{P}\left\{ t \left(\varkappa_{M,l_k(M)} + \sum_{n=M}^{N-1} \varkappa_{n,l'_k(n)} \right) \geqslant t^* x/2 - tN \right\} \\
&\qquad \leqslant \mathbf{E} \exp\left(t \left(\varkappa_{M,l_k(M)} + \sum_{n=M}^{N-1} \varkappa_{n,l'_k(n)} \right) \right) \Big/ \exp\left(t^* x/2 - tN \right) \\
&\qquad = \mathbf{E} \exp\left(t\, \varkappa_{M,l_k(M)} \right) \prod_{n=M}^{N-1} \mathbf{E} \exp\left(t\, \varkappa_{n,l'_k(n)} \right) \Big/ \exp\left(t^* x/2 - tN \right) \\
&\qquad \leqslant (4d)^N \exp\left(\frac{N}{32} - \frac{x}{64\, c_{40}\, d^{3/2}\, \tau} \right).
\end{aligned} \tag{87}$$

From (63), (87) it follows that

$$\mathbf{P}\left\{ \Delta_5 \geqslant x/2,\ A \right\} \leqslant 2^N \cdot (4d)^N \exp\left(\frac{N}{32} - \frac{x}{64\, c_{40}\, d^{3/2}\, \tau} \right). \tag{88}$$

Using (52), (67), (70), (88), we obtain that

$$\mathbf{P}\Big\{\Delta \geqslant x\Big\} \leqslant (30d)^N \exp\Big(-\frac{x}{c_{41}\, d^{3/2}\,\tau}\Big), \qquad x \geqslant 0, \tag{89}$$

where we can take $c_{41} = \max\big\{64c_{40},\ 342c_{38}^{-1},\ c_{39},\ 2\big\}$ (it is clear that, since $c_{41} \geqslant c_{39}$, the inequality (89) is automatically valid for $0 \leqslant x \leqslant c_{39}\, d^{3/2}\,\tau$). Let the quantities $\varepsilon,\ x_0 > 0$ be defined by the relations

$$\varepsilon = \frac{1}{2\,c_{41}\, d^{3/2}\,\tau} \leqslant \frac{1}{4\,\tau} \leqslant \frac{1}{4}, \qquad e^{\varepsilon x_0} = (30d)^N. \tag{90}$$

Integrating by parts and using (89), (90), we obtain

$$\mathbf{E}\, e^{\varepsilon\Delta} = \int_0^\infty \varepsilon e^{\varepsilon x}\ \mathbf{P}\Big\{\Delta \geqslant x\Big\}\, dx + 1, \tag{91}$$

$$\int_0^{x_0} \varepsilon e^{\varepsilon x}\ \mathbf{P}\Big\{\Delta \geqslant x\Big\}\, dx \leqslant \int_0^{x_0} \varepsilon e^{\varepsilon x}\, dx = e^{\varepsilon x_0} - 1 = (30d)^N - 1, \tag{92}$$

$$\int_{x_0}^\infty \varepsilon e^{\varepsilon x}\ \mathbf{P}\Big\{\Delta \geqslant x\Big\}\, dx \leqslant \int_{x_0}^\infty \varepsilon e^{-\varepsilon(x-x_0)}\, dx = 1, \tag{93}$$

and, hence,

$$\mathbf{E}\, e^{\varepsilon\Delta} \leqslant (30d)^N + 1 \leqslant (31d)^N. \tag{94}$$

Together with (54), (90) this completes the proof of Theorem 11.1.

12. Estimates in the dyadic scheme under the condition of coincidence of moments of the first three orders

Suppose that p, N are natural numbers, $p \leqslant N$, $\tau \geqslant 1$, and $\widetilde{\xi}_1, \dots, \widetilde{\xi}_{2^N}$ are random vectors with known distributions $\mathcal{L}(\widetilde{\xi}_k) \in \mathcal{A}_d(\tau)$, $\mathbf{E}\,\widetilde{\xi}_k = 0$, $\operatorname{cov}\widetilde{\xi}_k = I$, $k = 1, \dots, 2^N$. Let $\widetilde{\eta}_k$ be random vectors which have the same moments of the first three orders as the vectors $\widetilde{\xi}_k$, $\mathcal{L}(\widetilde{\eta}_k) \in \mathcal{A}_d(\tau)$, $k = 1, \dots, 2^N$. Assume that the distributions $\mathcal{L}(\widetilde{\xi}_k)$ are absolutely continuous. Let $Y_1, \dots, Y_{2^N}$ be independent random vectors with distributions $\mathcal{L}(Y_k) = \Psi\big(\mathcal{L}(\widetilde{\eta}_k)\big)$, $k = 1, \dots, 2^N$, where the distributions $\Psi(\cdot)$ are defined in Section 11. Assume that the random vectors $\xi_1, \dots, \xi_{2^N}$ have distributions $\mathcal{L}(\xi_l) = \mathcal{L}(Y_l)$ for $l = (m-1)\cdot 2^p + 1$, $m = 1, \dots, 2^{N-p}$, and $\mathcal{L}(\xi_l) = \mathcal{L}(\widetilde{\xi}_l)$ otherwise. Let the random vectors X_k, $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$; $V_{n,k}, \widetilde{V}_{n,k}$, $U_{n,k}, \widetilde{U}_{n,k}$ be constructed as functions of the vectors $Y_1, \dots, Y_{2^N}$ as in Section 10.

Remark 12.1. From Lemma 11.1 and from the definition of the distributions $\Psi(\cdot)$ it follows that the vectors ξ_k, X_k, Y_k have the same moments of the first three orders for each $k = 1, \dots, 2^N$.

Let $c_1, \dots, c_{41}$ be absolute positive constants from Section 11. By virtue of (11.14) we have

$$\mathcal{L}(X_k),\ \mathcal{L}(Y_k) \in \mathcal{A}_d(c_{14}\tau), \qquad k = 1, \dots, 2^N. \tag{1}$$

Lemma 12.1. *There exist absolute positive constants* $c_{42}, \dots, c_{47}$ *such that*

a) *If* $1 \leqslant n \leqslant N$, $0 \leqslant k < 2^{N-n}$, $\tau^2 d^3 p / 2^{n-p} \leqslant c_{42}$, *then*

$$\left|\widetilde{U}_{n,k} - \widetilde{V}_{n,k}\right| \leqslant \frac{c_{43}\, d^2\, \tau^2}{2^{n/2}} \left(1 + 2^{-3n/2} \max\left\{ \left|U_{n,k}\right|^3, \left|\widetilde{U}_{n,k}\right|^3 \right\}\right) + \frac{1}{8} \left|U_{n,k} - V_{n,k}\right| \quad (2)$$

provided that $\max\left\{ \left|U_{n,k}\right|, \left|V_{n,k}\right|, \left|\widetilde{U}_{n,k}\right| \right\} \leqslant \frac{c_{44} \cdot 2^n}{d^{3/2}\, \tau}$;

b) *If* $\tau^2 d^3 p / 2^{N-p} \leqslant c_{45}$, *then*

$$\left|U_{N,0} - V_{N,0}\right| \leqslant c_{46}\, d^2 \tau^2 \cdot 2^{-N/2} \left(1 + 2^{-3N/2} \left|U_{N,0}\right|^3\right) \quad (3)$$

provided that $\left|U_{N,0}\right| \leqslant \frac{c_{47} \cdot 2^N}{d^{3/2}\, \tau}$.

Proof. At first we prove item a). Suppose that the vectors $L^*_{n,k}, L_{n,k}$ are constructed from coordinates of vectors $U_{n-1,2k}$, $U_{n-1,2k+1}$; $U_{n,k}$, $\widetilde{U}_{n,k}$ as in the proof of Lemma 11.3. Let the vectors $R^*_{n,k}, R_{n,k}$ be analogously constructed from coordinates of vectors $V_{n-1,2k}$, $V_{n-1,2k+1}$; $V_{n,k}$, $\widetilde{V}_{n,k}$. It is evident that the vectors $U_{n-1,2k}$, $U_{n-1,2k+1}$ are as before independent. The same may be said about the vectors $V_{n-1,2k}$, $V_{n-1,2k+1}$. Using (10.2), (1), the properties of classes $\mathcal{A}_d(\tau)$, the definition of classes $\mathcal{R}_d\big(\tau, \varepsilon\big)$, Lemma 1.8 and Remarks 1.2, 12.1 we can easily verify that $\Big(\mathcal{L}(L^*_{n,k}),\, \mathcal{L}(R^*_{n,k})\Big) \in \mathcal{R}_{2d}(c_{14}\, \tau,\, 0)$. The relations (10.6), (10.12) show that $L_{n,k} = A\, L^*_{n,k}$, $R_{n,k} = A\, R^*_{n,k}$, where $A : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is a linear operator with $\|A\| = \sqrt{2}$. Therefore, using Corollary 1.1, we see that

$$\Big(\mathcal{L}(L_{n,k}),\, \mathcal{L}(R_{n,k})\Big) \in \mathcal{R}_{2d}\big(c_{14}\sqrt{2}\, \tau,\, 0\big). \quad (4)$$

Now we define the vectors $L^j_{n,k}, R^j_{n,k}$, $j = d, \dots, 2d$, consisting of the first j coordinates of the vectors $L_{n,k}$, $R_{n,k}$ respectively (see (11.45)). Below we show that the distributions $\mathcal{L}(L^j_{n,k})$, $\mathcal{L}(R^j_{n,k})$, $j = d+1, \dots, 2d$, satisfy in j-dimensional situation the conditions of Lemmas 7.1, 9.1 with $D = 2^n I$, $\sigma^2 = 2^n$ and with the change of d by j and τ by $c_{33}\tau$, where $c_{33} = \max\big\{c_{14}\sqrt{2},\, c_1^{-1},\, c_6^{-1}, 1\big\}$ was already defined above. For this one should fix j and verify that, if $\tau^2 d^3 p / 2^{n-p} \leqslant c_{48}$ with sufficiently small c_{48}, then for $h^* \in \mathbb{R}^j$, $\|h^*\|\, c_{33}\, \tau < 1$, and for all $v^* \in \mathbb{R}^j$ the inequalities (11.46), (11.47) hold, where $F_{h^*} = \overline{F}(h^*)$, $F = \mathcal{L}(L^j_{n,k})$ or $F = \mathcal{L}(R^j_{n,k})$, $h^* = \big(h^*_1, \dots, h^*_d\big)$ and the relations (11.48) are satisfied. Besides, we need the validity of analogous relations in the $(j-1)$-dimensional situation for the distributions F'. Finally, one should take into account the relation (4). It is clear that it suffices to check the conditions (11.46), (11.47) for $F = \mathcal{L}(L^j_{n,k})$, $j = d, \dots, 2d$, if c_{48} is small enough. For $F = \mathcal{L}(R^j_{n,k})$ these conditions can be verified analogously.

Choosing c_{48} to be sufficiently small and taking into account that $\tau \geqslant 1$, we ensure the validity of the inequality

$$2^{n-p-1} \geqslant 2. \tag{5}$$

From the definition of distributions $\Psi(\cdot)$ and random vectors $L_{n,k}^{j}$ and from (5), (1.4), (10.1), (10.2), (10.12) it is easy to deduce that (in the conditions of Section 12)

$$\bigl|\widehat{F}_{h^*}(t^*)\bigr| \leqslant \prod_{m=1}^{j-d} \bigl|\widehat{H}^*_{h^*_m}(t^*_m)\bigr|^{2^{n-p}} \prod_{m=j-d+1}^{d} \bigl|\widehat{H}_{h_m}(t_m)\bigr|^{2^{n-p+1}}. \tag{6}$$

From the definition of c_{33} it follows that we can use for estimating $\bigl|\widehat{H}_{h_m}(t_m)\bigr|$ the inequalities (11.2)–(11.4) and for estimating $\bigl|\widehat{H}^*_{h^*_m}(t^*_m)\bigr|$ the inequalities (11.5)–(11.7). In particular, from (11.2), (11.3), (11.5), (11.6), (11.48) one can easily derive that

$$\begin{aligned}\sup_{t\in T} \prod_{m=1}^{j-d} \bigl|\widehat{H}^*_{h^*_m}(t^*_m)\bigr| \prod_{m=j-d+1}^{d} \bigl|\widehat{H}_{h_m}(t_m)\bigr|^2 &\leqslant \max\Bigl\{ \exp\bigl(-(48\,c_{33}^2\,\tau^2\,d^2)^{-1}\bigr),\ c_3^2,\ c_8\Bigr\} \\ &\leqslant \exp\Bigl(-\frac{c_{49}}{\tau^2\,d^2}\Bigr).\end{aligned} \tag{7}$$

From (6), (7) it follows that, for $t^* \in T$,

$$\bigl|\widehat{F}_{h^*}(t^*)\bigr| \leqslant \exp\Bigl(-\frac{2^{n-p-1}\,c_{49}}{\tau^2\,d^2}\Bigr) \prod_{m=1}^{j-d} \bigl|\widehat{H}^*_{h^*_m}(t^*_m)\bigr|^{2^{n-p-1}} \prod_{m=j-d+1}^{d} \bigl|\widehat{H}_{h_m}(t_m)\bigr|^{2^{n-p}}. \tag{8}$$

It is clear that the set T (defined by (11.48)) can be represented as the join of 2^d disjoint sets T_s, $s = 1, \dots, 2^d$, of the form

$$\begin{gathered}T_s = T \cap \bigcap_{m\in\Xi_s} \bigl\{ t^* \in \mathbb{R}^j : |t^*_m| \leqslant b_m \bigr\} \cap \bigcap_{m\notin\Xi_s} \bigl\{ t^* \in \mathbb{R}^j : |t^*_m| > b_m \bigr\}, \\ b_m = \begin{cases} c_7, & \text{if} \quad m = 1, \dots, j-d; \\ c_2, & \text{if} \quad m = j-d+1, \dots, d. \end{cases}\end{gathered} \tag{9}$$

Here Ξ_s, $s = 1, \dots, 2^d$, are all possible different subsets of the set $\bigl\{1, \dots, d\bigr\}$.

Fix a set Ξ_s. For $m \in \Xi_s$, using (11.3), (11.6), we obtain

$$\begin{aligned}\int\limits_{|t^*_m|\leqslant c_7} \bigl|\widehat{H}^*_{h^*_m}(t^*_m)\bigr|^{2^{n-p-1}} dt^*_m &\leqslant \int\limits_{|t|\leqslant c_7,\, t\in\mathbb{R}^2} \exp\Bigl(-\frac{2^{n-p}\,\|t\|^2}{6}\Bigr)\,dt \leqslant \frac{c}{2^{n-p}}, \\ \int\limits_{|t_m|\leqslant c_2} \bigl|\widehat{H}_{h_m}(t_m)\bigr|^{2^{n-p}} dt_m &\leqslant \int\limits_{|t|\leqslant c_2,\, t\in\mathbb{R}^1} \exp\Bigl(-\frac{2^{n-p}\,t^2}{6}\Bigr)\,dt \leqslant \frac{c}{2^{(n-p)/2}}.\end{aligned} \tag{10}$$

If $m \notin \Xi_s$, then, using (11.2), (11.4), (11.5), (11.7), (5), we have

$$\int\limits_{|t_m^*|>c_7} \bigl|\widehat{H}^*_{h^*_m}(t^*_m)\bigr|^{2^{n-p-1}} dt^*_m \leqslant \int\limits_{|t|>c_7} c_8^{2^{n-p-2}} \bigl|\widehat{H}^*_{h^*_m}(t)\bigr|\,dt \leqslant c\,\exp\bigl(-c\cdot 2^{n-p}\bigr) \leqslant \frac{c}{2^{n-p}}\,,$$

$$\int\limits_{|t_m|>c_2} \bigl|\widehat{H}_{h_m}(t_m)\bigr|^{2^{n-p}} dt_m \leqslant \int\limits_{|t|>c_2} c_3^{2^{n-p-1}} \bigl|\widehat{H}_{h_m}(t)\bigr|\,dt \leqslant c\,\exp\bigl(-c\cdot 2^{n-p}\bigr) \leqslant \frac{c}{2^{(n-p)/2}}\,. \tag{11}$$

From (8)–(11) it follows that, for $s = 1, \dots, 2^d$,

$$(2\pi)^{-j} \int\limits_{T_s} \bigl|\widehat{F}_{h^*}(t^*)\bigr|\,dt^* \leqslant \frac{c^d}{2^{(n-p)j/2}\,(2\pi)^{j/2}}\ \exp\Bigl(-\frac{c\cdot 2^{n-p}}{\tau^2\,d^2}\Bigr). \tag{12}$$

Using now (9), (12), we see that

$$\begin{aligned}(2\pi)^{-j} \int\limits_{T} \bigl|\widehat{F}_{h^*}(t^*)\bigr|\,dt^* &\leqslant (2\pi)^{-j} \sum_{s=1}^{2^d} \int\limits_{T_s} \bigl|\widehat{F}_{h^*}(t^*)\bigr|\,dt^* \\ &\leqslant \frac{1}{2^{nj/2}\,(2\pi)^{j/2}}\ \exp\Bigl(c_{50}\,dp - \frac{c_{51}\cdot 2^{n-p}}{\tau^2\,d^2}\Bigr). \end{aligned} \tag{13}$$

Choosing c_{48} to be small enough, we can ensure the validity of the inequality

$$\exp\Bigl(c_{50}\,dp - \frac{c_{51}\cdot 2^{n-p}}{\tau^2\,d^2}\Bigr) \leqslant \frac{c_{33}^2\,\tau^2\,d^2}{2^{n-p}} \leqslant \frac{c_{33}^2\,\tau^2\,j^2}{2^{n-p}} \leqslant \frac{4\,c_{33}^2\,\tau^2\,d^2}{2^{n-p}} \leqslant \frac{1}{4}\,. \tag{14}$$

The inequalities (13), (14) imply (11.46).

For the proof of (11.47) we apply the Hölder inequality and obtain that

$$\int\limits_{T} \bigl|\bigl\langle t^*, v^*\bigr\rangle\,\widehat{F}_{h^*}(t^*)\bigr|\,dt^* \leqslant \|v^*\| \left(\int\limits_{T} \bigl|\widehat{F}_{h^*}(t^*)\bigr|\,dt^* \int\limits_{T} \|t^*\|^2\,\bigl|\widehat{F}_{h^*}(t^*)\bigr|\,dt^*\right)^{1/2}. \tag{15}$$

For estimating the first integral in the right-hand side of (15) we note that, according to (13), (14),

$$(2\pi)^{-j} \int\limits_{T} \bigl|\widehat{F}_{h^*}(t^*)\bigr|\,dt^* \leqslant \frac{1}{4\,(2\pi)^{j/2}\cdot 2^{nj/2}}\,. \tag{16}$$

Let us estimate the second integral. By (11.3), (11.6), we have

$$\int\limits_{|t_m^*|\leqslant c_7} \|t^*_m\|^2\,\bigl|\widehat{H}^*_{h^*_m}(t^*_m)\bigr|^{2^{n-p-1}} dt^*_m \leqslant \int\limits_{|t|\leqslant c_7,\,t\in\mathbb{R}^2} \|t\|^2\,\exp\Bigl(-\frac{2^{n-p}\,\|t\|^2}{6}\Bigr)\,dt \leqslant \frac{c}{2^{2(n-p)}}\,,$$

$$\int\limits_{|t_m|\leqslant c_2} t_m^2\,\bigl|\widehat{H}_{h_m}(t_m)\bigr|^{2^{n-p}} dt_m \leqslant \int\limits_{|t|\leqslant c_2,\,t\in\mathbb{R}^1} t^2\,\exp\Bigl(-\frac{2^{n-p}\,t^2}{6}\Bigr)\,dt \leqslant \frac{c}{2^{3(n-p)/2}}\,. \tag{17}$$

Besides, using (11.2), (11.4), (11.5), (11.7), (5), we obtain

$$\begin{aligned}\int\limits_{|t_m^*|>c_7} \|t_m^*\|^2 \left|\widehat{H}^*_{h^*_m}(t_m^*)\right|^{2^{n-p-1}} dt_m^* &\leqslant \int\limits_{|t|>c_7} c_8^{2^{n-p-2}} \|t\|^2 \left|\widehat{H}^*_{h^*_m}(t)\right| dt \\ &\leqslant c\,\exp\left(-c\cdot 2^{n-p}\right) \leqslant \frac{c}{2^{2(n-p)}}, \\ \int\limits_{|t_m|>c_2} t_m^2 \left|\widehat{H}_{h_m}(t_m)\right|^{2^{n-p}} dt_m &\leqslant \int\limits_{|t|>c_2} c_3^{2^{n-p-1}} t^2 \left|\widehat{H}_{h_m}(t)\right| dt \\ &\leqslant c\,\exp\left(-c\cdot 2^{n-p}\right) \leqslant \frac{c}{2^{3(n-p)/2}}.\end{aligned} \tag{18}$$

From (8)–(11), (17), (18) it follows that, for $s = 1, \ldots, 2^d$,

$$(2\pi)^{-j} \int\limits_{T_s} \|t^*\|^2 \left|\widehat{F}_{h^*}(t^*)\right| dt^* \leqslant \frac{c^d}{2^{n-p}\,(2\pi)^{j/2}\cdot 2^{(n-p)j/2}} \exp\Big(-\frac{c\cdot 2^{n-p}}{\tau^2 d^2}\Big). \tag{19}$$

Using now (9), (19), we see that

$$\begin{aligned}(2\pi)^{-j} \int\limits_{T} \|t^*\|^2 \left|\widehat{F}_{h^*}(t^*)\right| dt^* &\leqslant (2\pi)^{-j} \sum_{s=1}^{2^d} \int\limits_{T_s} \|t\|^2 \left|\widehat{F}_h(t)\right| dt \\ &\leqslant \frac{1}{2^n\,(2\pi)^{j/2}\cdot 2^{nj/2}} \exp\Big(c_{52}\,dp - \frac{c_{53}\cdot 2^{n-p}}{\tau^2 d^2}\Big).\end{aligned} \tag{20}$$

Choosing c_{48} to be small enough, we can ensure the validity of the inequality

$$\exp\Big(c_{52}\,dp - \frac{c_{53}\cdot 2^{n-p}}{\tau^2 d^2}\Big) \leqslant \frac{1}{4}. \tag{21}$$

The inequalities (15), (16), (20), (21) imply (11.47).

Using (7.3), (9.2), (10.8), (10.9) and applying Lemma 9.1 and the inequality (7.4) of Lemma 7.1 (for the parameter $\widetilde{\tau} = c_{33}\tau$) in the j-dimensional case, $j = d+1, \ldots, 2d$, we obtain that

$$\begin{aligned}\left|L_{n,k}^{(j)} - R_{n,k}^{(j)}\right| \leqslant \frac{c\tau^2}{2^{n/2}} \Bigg(j^2 + \left(\Gamma_{j-1}^2 + 2^{n/2}\tau^{-1}\Lambda_{j-1}\right)\left(j + \frac{\left|L_{n,k}^{(j)}\right|}{2^{n/2}}\right) + \frac{\left|L_{n,k}^{(j)}\right|^3}{2^{3n/2}} \\ + \Gamma_{j-1}^3 + 2^{n/2}\tau^{-1}\Lambda_{j-1}\Gamma_{j-1}\Bigg),\end{aligned} \tag{22}$$

where

$$\Lambda_{j-1} = 2^{-n/2}\,j^{1/2}\left|L_{n,k}^{j-1} - R_{n,k}^{j-1}\right|, \quad \Gamma_{j-1} = 2^{-n/2}\,j^{1/2}\max\left\{\left|L_{n,k}^{j-1}\right|, \left|R_{n,k}^{j-1}\right|\right\}, \tag{23}$$

if

$$\frac{\tau^2 j^3 p}{2^{n-p}} \leqslant c_{55}, \qquad \bigl|L_{n,k}^{(j)}\bigr| \leqslant \frac{c_{56}\cdot 2^n}{j\tau}, \qquad \frac{\max\{\bigl|L_{n,k}^{j-1}\bigr|, \bigl|R_{n,k}^{j-1}\bigr|\}}{2^{n/2}} \leqslant \frac{c_{57}\cdot 2^{n/2}}{j^{3/2}\tau} \tag{24}$$

and c_{55}, c_{56}, c_{57} are small enough. Verifying the inequality (22), one should take into account that in Lemmas 7.1 and 9.1 the densities $v_m(\cdot)$, $m = 1, 2$, are the densities of shifted conditional distributions. The possibility to write the inequality (22) and the conditions (24) without shift is ensured by choosing c_{56}, c_{57} to be sufficiently small and by using the inequality (6.3) to $y_{m,d}$, $m = 1, 2$.

Using again the possibility to choose c_{55}, c_{56}, c_{57} to be small enough, we can easily derive from (22)–(24) that if the conditions (24) are satisfied, then

$$\bigl|L_{n,k}^{(j)} - R_{n,k}^{(j)}\bigr| \leqslant \frac{c_{54}\,\tau^2 d^2}{2^{n/2}}\left(1 + \frac{\bigl|L_{n,k}^{j}\bigr|^3}{2^{3n/2}}\right) + \frac{1}{16}\bigl|L_{n,k}^{j-1} - R_{n,k}^{j-1}\bigr|. \tag{25}$$

For the proof of (25) one should estimate each summand in the right-hand side of (22) separately. We use, in particular, that $\Gamma_{j-1} \leqslant \Lambda_{j-1} + 2^{-n/2} j^{1/2} \bigl|L_{n,k}^{j-1}\bigr|$.

Using the induction with respect to j, we shall obtain that, for $j = d+1, \ldots, 2d$,

$$\bigl|L_{n,k}^{(j)} - R_{n,k}^{(j)}\bigr| \leqslant \frac{c_{43}\,\tau^2 d^2}{2\cdot 2^{n/2}}\left(1 + \frac{\bigl|L_{n,k}^{j}\bigr|^3}{2^{3n/2}}\right) + \frac{1}{16}\bigl|L_{n,k}^{j-1} - R_{n,k}^{j-1}\bigr|, \tag{26}$$

$$\bigl|R_{n,k}^{(j)}\bigr| \leqslant \frac{c_{57}\cdot 2^n}{4d^{3/2}\tau}, \tag{27}$$

if

$$\frac{\tau^2 d^3 p}{2^{n-p}} \leqslant c_{42}, \qquad \max\Bigl\{\bigl|U_{n,k}\bigr|, \bigl|V_{n,k}\bigr|, \bigl|\widetilde{U}_{n,k}\bigr|\Bigr\} \leqslant \frac{c_{44}\cdot 2^n}{d^{3/2}\tau}, \tag{28}$$

where

$$c_{43} = 2c_{54}, \quad c_{44} = \min\left\{\frac{c_{56}}{2}, \frac{c_{57}}{16}, \frac{1}{\sqrt{c_{43}}}\right\}, \quad c_{42} = \min\left\{\frac{c_{55}}{8}, \left(\frac{c_{44}}{c_{43}}\right)^{2/3}\right\}. \tag{29}$$

It is clear that (29) implies

$$B = c_{44} + \frac{c_{43}}{2}(c_{42}^{3/2} + c_{44}^3) + \frac{c_{44}}{8} + \frac{c_{57}}{64} \leqslant \frac{c_{57}}{4}. \tag{30}$$

Inequality (26) for $j = d+1$ follows directly from (25), (29) and from the definition of vectors $L_{n,k}$, $R_{n,k}$, since the conditions (24) are in this case the consequences of (28), (29). Besides, using (26), (28)–(30), we see that

$$\bigl|R_{n,k}^{(d+1)}\bigr| \leqslant \bigl|L_{n,k}^{(d+1)}\bigr| + \bigl|L_{n,k}^{(d+1)} - R_{n,k}^{(d+1)}\bigr| \leqslant \frac{B\cdot 2^n}{d^{3/2}\tau} \leqslant \frac{c_{57}\cdot 2^n}{4d^{3/2}\tau}. \tag{31}$$

This completes the proof of (26), (27) for $j = d + 1$.

Suppose now that the inequalities (26), (27) are proved for $j = d+1, \dots, s-1$, $d+2 \leqslant s \leqslant 2d$. We shall prove that these inequalities are valid for $j = s$. Inequality (26) for $j = s$ follows from (25), (29) since the validity of (24) can be easily derived in this case from (27) (with $j = d+1, \dots, s-1$), (28), (29). Besides, using (26) (with $j = s$), (27) (with $j = d+1, \dots, s-1$), (28)–(30), we obtain the relation (31) with the change of the superscript $(d+1)$ by (s). This completes the proof of (26), (27) for $j = d+1, \dots, 2d$.

Inequality (2) follows from (26), (11.45) and from the definition of $L_{n,k}$, $R_{n,k}$.

Now we shall prove item b). Consider the vectors $U^j_{N,0}, V^j_{N,0}$, $j = 1, \dots, d$, consisting of the first j coordinates of the vectors $U_{N,0}$, $V_{N,0}$ respectively (see (11.41)). From the properties of classes $\mathcal{R}_d\big(\tau, \varepsilon\big)$, the relations (10.2), (1), Corollary 1.1 and Remarks 1.2, 12.1 it follows that

$$\big(\mathcal{L}(U^j_{N,0}),\, \mathcal{L}(V^j_{N,0})\big) \in \mathcal{R}_j\big(c_{14}\,\tau,\, 0\big). \tag{32}$$

One can verify that, if $\tau^2 j^3 p / 2^{N-p} \leqslant c_{58}$ with sufficiently small c_{58}, then the pairs of distributions $\big(\mathcal{L}(U^j_{N,0}),\, \mathcal{L}(V^j_{N,0})\big)$ satisfy in j-dimensional situation the conditions of Lemmas 7.1 and 9.1 with $D = 2^N I$, $\sigma^2 = 2^N$ and with the change of τ by $c_{23}\,\tau$, where $c_{23} = \max\big\{c_{14},\, c_1^{-1}, 1\big\}$ was defined in Section 11. For this one should verify that, for $h \in \mathbb{R}^j$, $\|h\|\, c_{23}\,\tau < 1$ and for all $v \in \mathbb{R}^j$, the following inequalities hold, if c_{58} is small enough:

$$(2\,\pi)^{-j} \int\limits_T \big|\widehat{F}_h(t)\big|\, dt \leqslant \frac{c_{23}^2\,\tau^2\, j^2}{2^N\,(2\,\pi)^{j/2}\cdot 2^{Nj/2}}\,, \tag{33}$$

$$(2\,\pi)^{-j} \int\limits_T \big|\big\langle t, v\big\rangle\, \widehat{F}_h(t)\big|\, dt \leqslant \frac{\|v\|}{4\cdot 2^{N/2}\,(2\,\pi)^{j/2}\cdot 2^{Nj/2}}\,, \tag{34}$$

where $F_h = \overline{F}(h)$, $F = \mathcal{L}(U^j_{N,0})$ or $F = \mathcal{L}(V^j_{N,0})$,

$$T = \Big\{t \in \mathbb{R}^j \,:\, 4\,\|t\|\, c_{23}\,\tau\, j \geqslant 1\Big\}. \tag{35}$$

The inequalities (33), (34) automatically imply the validity of analogous relations in the $(j-1)$-dimensional situation, $j = 2, \dots, d$, for the distributions F'. It is clear that it suffices to check the conditions (33), (34) for $F = \mathcal{L}(U^j_{N,0})$, $j = 1, \dots, d$, if c_{58} is small enough. For $F = \mathcal{L}(V^j_{N,0})$ these conditions can be verified analogously.

From the definition of distributions $\Psi(\,\cdot\,)$ and random vectors $U^j_{N,0}$ and from (1.4), (10.1), (10.2) one can easily derive that

$$\big|\widehat{F}_h(t)\big| \leqslant \prod_{m=1}^{j} \big|\widehat{H}_{h_m}(t_m)\big|^{2^{N-p}}. \tag{36}$$

From the definition of c_{23} it follows that we can use for estimating $\bigl|\widehat{H}_{h_m}(t_m)\bigr|$ the inequalities (11.2)–(11.4). The rest of the proof of inequalities (33), (34) does not practically differ from the proof of (11.18), (11.19) and is simpler than the proof of (11.46), (11.47) in Section 12. Therefore it is omitted. The only difference is in the change of constants and notations.

Using (7.3), (9.2), (10.7) and applying Lemma 9.1 and the inequality (7.4) of Lemma 7.1 in the j-dimensional case, $j = 1, \dots, d$, we obtain that

$$\bigl|U_{N,0}^{(1)} - V_{N,0}^{(1)}\bigr| \leqslant \frac{c_{59}\,\tau^2}{2^{N/2}} \left(1 + \frac{\bigl|U_{N,0}^{(1)}\bigr|^3}{2^{3N/2}}\right), \tag{37}$$

if

$$\frac{\tau^2 p}{2^{N-p}} \leqslant c_{60}, \qquad \bigl|U_{N,0}^{(1)}\bigr| \leqslant \frac{c_{61}\cdot 2^N}{\tau}, \tag{38}$$

and, for $2 \leqslant j \leqslant d$,

$$\bigl|U_{N,0}^{(j)} - V_{N,0}^{(j)}\bigr| \leqslant \frac{c_{59}\,\tau^2 d^2}{2^{N/2}} \left(1 + \frac{\bigl|U_{N,0}^{j}\bigr|^3}{2^{3N/2}}\right) + \frac{1}{2}\bigl|U_{N,0}^{j-1} - V_{N,0}^{j-1}\bigr|, \tag{39}$$

if

$$\frac{\tau^2 j^3 p}{2^{N-p}} \leqslant c_{60}, \quad \bigl|U_{N,0}^{(j)}\bigr| \leqslant \frac{c_{61}\cdot 2^N}{j\tau}, \quad \frac{\max\bigl\{\bigl|U_{N,0}^{j-1}\bigr|, \bigl|V_{N,0}^{j-1}\bigr|\bigr\}}{2^{N/2}} \leqslant \frac{c_{62}\cdot 2^{N/2}}{j^{3/2}\tau} \tag{40}$$

and c_{60}, c_{61}, c_{62} are small enough. Checking inequalities (37), (39) should be carried out analogously to the proof of (25).

Using the induction with respect to j, we shall prove that, for $j = 1, \dots, d$,

$$\bigl|V_{N,0}^{(j-1)}\bigr| \leqslant \frac{c_{62}\cdot 2^N}{d^{3/2}\tau}, \tag{41}$$

$$\bigl|U_{N,0}^{(j)} - V_{N,0}^{(j)}\bigr| \leqslant \frac{c_{46}\,\tau^2 d^2}{2\cdot 2^{N/2}} \left(1 + \frac{\bigl|U_{N,0}^{j}\bigr|^3}{2^{3N/2}}\right) + \frac{1}{2}\bigl|U_{N,0}^{j-1} - V_{N,0}^{j-1}\bigr|, \tag{42}$$

if

$$\frac{\tau^2 d^3 p}{2^{N-p}} \leqslant c_{45}, \qquad \bigl|U_{N,0}\bigr| \leqslant \frac{c_{47}\cdot 2^N}{d^{3/2}\,\tau}, \tag{43}$$

where

$$\begin{gathered} c_{46} = 2c_{59}, \quad c_{47} = \min\left\{c_{61},\ \frac{c_{62}}{8},\ \frac{1}{\sqrt{c_{46}}}\right\}, \quad c_{45} = \min\left\{c_{60}, \left(\frac{c_{47}}{c_{46}}\right)^{2/3}\right\}, \\ \bigl|V_{N,0}^{(0)}\bigr| = \bigl|U_{N,0}^{0} - V_{N,0}^{0}\bigr| = 0. \end{gathered} \tag{44}$$

It is clear that (44) implies

$$B_0 = c_{47} + \frac{c_{46}}{2}(c_{45}^{3/2} + c_{47}^3) + \frac{c_{47}}{2} + \frac{c_{62}}{2} \leqslant c_{62}. \tag{45}$$

Inequality (42) for $j = 1$ follows directly from (37), (44), since the conditions (38) are in this case the consequences of (43), (44). Inequality (41) for $j = 1$ is valid automatically (see (44)).

Suppose now that the inequalities (41), (42) are proved for $j = 1, \dots, s-1$, $2 \leqslant s \leqslant d$. We shall prove that these inequalities are valid for $j = s$. Using (41) (with $j = 1, \dots, s-1$), (42) (with $j = s-1$), (43)–(45), we obtain

$$\left| V_{N,0}^{(s-1)} \right| \leqslant \left| U_{N,0}^{(s-1)} \right| + \left| U_{N,0}^{(s-1)} - V_{N,0}^{(s-1)} \right| \leqslant \frac{B_0 \cdot 2^N}{d^{3/2}\tau} \leqslant \frac{c_{62} \cdot 2^N}{d^{3/2}\tau}. \tag{46}$$

Besides, it is clear that the inequality (42) for $j = s$ can be easily derived from (39), (44) since the validity of (40) follows in this case from (41) (with $j = 1, \dots, s-1$), (43), (44), (46). This completes the proof of (41), (42) for $j = 1, \dots, d$.

Inequality (3) follows now from (42)–(44) and from the definition of $U_{N,0}$, $U_{N,0}^j$, $U_{N,0}^{(j)}$, $V_{N,0}$, $V_{N,0}^j$, $V_{N,0}^{(j)}$.

13. Proof of Theorem 2

In this section it is convenient to assume that the constants $c_1, \dots, c_{62}$ are the same as in Sections 11, 12. Let u, v, w be arbitrary numbers satisfying the relations

$$\frac{5}{8} \leqslant v < \frac{2}{3}, \qquad u = 1 - v, \qquad w = 2 - 3v. \tag{1}$$

It is clear that

$$\frac{1}{3} < u \leqslant \frac{3}{8}, \qquad 0 < w \leqslant \frac{1}{8}, \qquad 3u - w = 1. \tag{2}$$

Below the symbols $a_1, \dots, a_9$ will be used for denoting positive quantities depending only on v. The letter a can denote different quantities depending only on v when we do not need to fix their numerical values. It is clear that, if v is fixed, $a, a_1, \dots, a_9$ are absolute constants. Denote

$$\begin{gathered} c_{63} = \max\{c_{43}, c_{46}\}, \qquad c_{64} = \min\{c_{44}, c_{47}, 1\}, \qquad c_{65} = \min\{c_{42}, c_{45}\}, \\ a_1 = \min\left\{ \frac{c_{64}}{5}, \ \frac{1}{(126\, c_{63})^{1/2}}, \ \frac{1 - 2^{-w}}{2} \right\}, \qquad a_2 = \min\{c_{65}, a_1^2, 1\}. \end{gathered} \tag{3}$$

Let $\tau \geqslant 1$, $\lambda \geqslant 1$. Define p as the minimal integer satisfying the inequalities

$$p \geqslant 4, \qquad \frac{a_1 \cdot 2^{vp}}{d^{3/2}\tau} \geqslant 4\lambda. \tag{4}$$

Denote by M the maximal integer such that

$$\frac{\tau^2 d^3 p}{2^{M-p}} \geqslant a_2 \ . \tag{5}$$

Let N be a positive integer such that

$$\frac{\tau^2 d^3 p}{2^{N-p}} < a_2 \ , \qquad \frac{N-M}{p} \quad \text{is integer.} \tag{6}$$

Obviously, the relations (3)–(6) imply

$$N > M \geqslant p, \qquad N \geqslant 2p. \tag{7}$$

Let $\widetilde{\xi}_1, \dots, \widetilde{\xi}_{2^N}$ be random vectors with distributions $\mathcal{L}(\widetilde{\xi}_k) \in \mathcal{A}_d(\tau)$, $\mathbf{E}\,\widetilde{\xi}_k = 0$, $\operatorname{cov}\widetilde{\xi}_k = I$, $k = 1, \dots, 2^N$. Assume that the distributions $\mathcal{L}(\widetilde{\xi}_k)$ are absolutely continuous. Let $\widetilde{\eta}_k$ be random vectors which have the same moments of the first three orders as the vectors $\widetilde{\xi}_k$, $\mathcal{L}(\widetilde{\eta}_k) \in \mathcal{A}_d(\tau)$ and $\mathbf{P}\left\{ \left|\widetilde{\eta}_k\right| \leqslant \lambda \right\} = 1$, $k = 1, \dots, 2^N$. Suppose that the independent random vectors $\widetilde{Y}_k$, $k = 1, \dots, 2^N$, with distributions $\mathcal{L}(\widetilde{Y}_k) = \Psi\big(\mathcal{L}(\widetilde{\eta}_k)\big)$ are already constructed (the distributions $\Psi(\cdot)$ are defined in Section 11). According to Lemma 11.1, $\widetilde{Y}_k$ and $\widetilde{\xi}_k$ have the same moments of the first three orders. Our aim is to construct independent random vectors $\widetilde{X}_k$, $k = 1, \dots, 2^N$, with $\mathcal{L}(\widetilde{X}_k) = \mathcal{L}(\widetilde{\xi}_k)$ so that the sequential sums of $\widetilde{X}_k$ and $\widetilde{Y}_k$ would be close with large probability. For this we define the random vectors

$$\left\{ Y_k^{[N-lp]},\ Z_{(k-1)\cdot 2^{lp}+1}^{[N-lp]},\quad 1 \leqslant k \leqslant 2^{N-lp},\quad 0 \leqslant l \leqslant (N-M-p)/p \right\} \tag{8}$$

with the help of a recurrent procedure described below. Note that in one-dimensional case a very similar procedure was used by Sakhanenko [21].

At first, we set $Y_k^{[N]} = \widetilde{Y}_k$, $k = 1, \dots, 2^N$. After this, using the procedure described in Section 12, we define the independent random vectors $Z_k^{[N]}$, $k = 1, \dots, 2^N$, with distributions $\mathcal{L}\big(Z_r^{[N]}\big) = \Psi\big(\mathcal{L}(\widetilde{\eta}_r)\big)$ for $r = (m-1)\cdot 2^p + 1$, $m = 1, \dots, 2^{N-p}$. For $r = 1, \dots, 2^N$, $r \neq (m-1)\cdot 2^p + 1$, $m = 1, \dots, 2^{N-p}$, we take $\mathcal{L}\big(Z_r^{[N]}\big) = \mathcal{L}(\widetilde{\xi}_r)$ and define $\widetilde{X}_r = Z_r^{[N]}$. Thus, the quantity of vectors $\widetilde{X}_k$, which are not constructed, is decreased in 2^p times. If $N = M + p$, the first step of applying this procedure is simultaneously the last one, since the random vectors (8) with $l = (N-M-p)/p = 0$ are already constructed.

Suppose now that $N > M + p$. Then we repeat this procedure, working only with vectors which have the distributions $\Psi\big(\mathcal{L}(\widetilde{\eta}_r)\big)$ and diminishing their number after each step in 2^p times.

Namely, if the vectors $\left\{ Y_k^{[N-lp+p]},\ Z_{(k-1)\cdot 2^{lp-p}+1}^{[N-lp+p]},\ 1 \leqslant k \leqslant 2^{N-lp+p} \right\}$ are already defined for an integer l and $1 \leqslant l \leqslant (N-M-p)/p$, we set $Y_k^{[N-lp]} = Z_{(k-1)\cdot 2^{lp}+1}^{[N-lp+p]}$, $k = 1, \dots, 2^{N-lp}$, and, using again the procedure described in Sections 10, 12, construct the independent random vectors $Z_{(k-1)\cdot 2^{lp}+1}^{[N-lp]}$, $k = 1, \dots, 2^{N-lp}$, with distributions described below. If $r = (m-1)\cdot 2^{lp+p}+1$, $m = 1, \dots, 2^{N-lp-p}$ we choose $\mathcal{L}\big(Z_r^{[N-lp]}\big) = \Psi\big(\mathcal{L}(\widetilde{\eta}_r)\big)$. In the case if $r = (s-1)\cdot 2^{lp}+1$, $s = 1, \dots, 2^{N-lp}$, and $r \neq (m-1)\cdot 2^{lp+p}+1$, $m = 1, \dots, 2^{N-lp-p}$, we take $\mathcal{L}\big(Z_r^{[N-lp]}\big) = \mathcal{L}(\widetilde{\xi}_r)$ and define $\widetilde{X}_r = Z_r^{[N-lp]}$.

After the last step (with $l = (N-M-p)/p$) all independent random vectors $\widetilde{X}_k$, $k = 1, \dots, 2^N$, are constructed except $\widetilde{X}_r$ having indices of the form $r = (m-1)\cdot 2^{N-M}+1$, $m = 1, \dots, 2^M$. Instead of this we have constructed the vectors with distributions $\mathcal{L}\big(Z_r^{[M+p]}\big) = \Psi\big(\mathcal{L}(\widetilde{\eta}_r)\big)$ for $r = (m-1)\cdot 2^{N-M}+1$, $m = 1, \dots, 2^M$. Together with $\widetilde{X}_r$ these vectors form a collection of mutually independent vectors. This can be easily verified by induction, if we take into account that, in each step, the new collection of independent vectors is constructed as a function of the vectors which are independent of all already constructed vectors $\widetilde{X}_r$.

Now we can add the missing elements to the sequence of already constructed $\widetilde{X}_r$. For this we define the random vectors $Y_k^* = Y_k^{[M]} = Z_{(k-1)\cdot 2^{N-M}+1}^{[M+p]}$, $k = 1, \dots, 2^M$, and, using again the procedure described in Section 10, construct the independent random vectors X_k^*, $k = 1, \dots, 2^M$, with distributions $\mathcal{L}(X_k^*) = \mathcal{L}(\widetilde{\xi}_{(k-1)\cdot 2^{N-M}+1})$. Then we set $\widetilde{X}_{(k-1)\cdot 2^{N-M}+1} = X_k^*$. It is clear that the vectors X_k^* are mutually independent and they do not depend on $\widetilde{X}_r$ constructed earlier, since, by construction, X_k^* are functions of Y_k^*. This completes the procedure of constructing $\widetilde{X}_r$, $r = 1, \dots, 2^N$.

By analogy with [21], we shall prove that the procedure described above gives good approximation, using induction with respect to N. It is clear that the procedure of constructing the independent random vectors $X_{*,k} = \widetilde{X}_{(k-1)\cdot 2^p+1}$, $k = 1, \dots, 2^{N-p}$, coincides with the procedure of constructing of $\widetilde{X}_k$, $k = 1, \dots, 2^N$. The only difference is in the change of N by $N-p$. Thus, below we shall assume that we have the good approximation in the case when we construct 2^{N-p} summands and prove the analogous statement in the case of 2^N summands.

We shall use the notations which was already used in Sections 10 and 12 with $Y_k = Y_k^{[N]}$, $X_k = Z_k^{[N]}$, $k = 1, \dots, 2^N$. Denote $S = S_j$, $T = T_j$ (see (10.5), (10.11)). Sometimes we shall omit for simplicity the index j, $j = 1, \dots, 2^N$, keeping in mind that it is fixed. By virtue of (11.14), by construction and by the definition of distributions $\Psi(\cdot)$ we have

$$\mathbf{P}\left\{ |Y_k| \leqslant 4\lambda \right\} = 1, \qquad k = 1, \dots, 2^N. \tag{9}$$

It is clear that from (10.5), (9) it follows that

$$\mathbf{P}\left\{\,|T| \leqslant 4\lambda \cdot 2^N\,\right\} = 1. \tag{10}$$

For fixed $j,\ j = 1, \ldots, 2^N,$ and $s,\ s = 0, \ldots, N,$ define the integer $k_s = k_s(j)$ by the relation $(k_s - 1) \cdot 2^s < j \leqslant k_s \cdot 2^s$. Now we can represent the sums S and T in the form

$$S = S^{(s)} - S^{\{s\}}, \qquad T = T^{(s)} - T^{\{s\}}, \tag{11}$$

where

$$\begin{aligned} S^{(s)} &= S_{k_s \cdot 2^s}, \qquad & S^{\{s\}} &= S^{(s)} - S, \\ T^{(s)} &= T_{k_s \cdot 2^s}, \qquad & T^{\{s\}} &= T^{(s)} - T. \end{aligned} \tag{12}$$

According to Corollary 10.1 and Lemma 10.3,

$$\left|\,S^{(s)} - T^{(s)}\,\right| \leqslant \left|U_{N,0} - V_{N,0}\right| + \sum_{n=s+1}^{N} \left|\,\widetilde{U}^{(n)} - \widetilde{V}^{(n)}\,\right|, \tag{13}$$

where $\widetilde{U}^{(n)} = \widetilde{U}_{n,l_{n,s}},\ \widetilde{V}^{(n)} = \widetilde{V}_{n,l_{n,s}},\ l_{n,s} \cdot 2^n < k_s \cdot 2^s \leqslant \left(l_{n,s}+1\right) \cdot 2^n$ (see (10.22)). As before, $l_{n,s}$ is the integer such that the sum of a block of 2^n summands $U_{n,l_{n,s}}$ contains the summand $X_{k_s \cdot 2^s}$ and, consequently, the summand X_j. In particular, this implies that the $l_{n,s}$ for $s < n \leqslant N$ do not depend on s. Denote also $U^{(n)} = U_{n,l_{n,s}}$, $V^{(n)} = V_{n,l_{n,s}}$. It is evident that $U^{(N)} = U_{N,0},\ V^{(N)} = V_{N,0}$. Therefore, from (11), (13) it follows that

$$\left|\,S - T\,\right| \leqslant \left|U^{(N)} - V^{(N)}\right| + \sum_{n=s+1}^{N} \left|\,\widetilde{U}^{(n)} - \widetilde{V}^{(n)}\,\right| + \left|\,S^{\{s\}}\,\right| + \left|\,T^{\{s\}}\,\right|. \tag{14}$$

Introduce the random integer $\mu = \mu_j$ by $\mu = N+1$, if at least one of the following inequalities is valid:

$$\frac{\tau^2\, d^3\, p}{2^{N-p}} > a_2\,, \qquad \left|\,S^{\{N\}}\,\right| > \frac{2\,a_1 \cdot 2^N}{d^{3/2}\,\tau}\,, \qquad \left|U^{(N)}\right| > \frac{2\,a_1 \cdot 2^N}{d^{3/2}\,\tau} \tag{15}$$

and by

$$\begin{aligned} \mu = \min\Big\{\, m :\ & p \leqslant m \leqslant N,\ \ \frac{\tau^2\, d^3\, p}{2^{m-p}} \leqslant a_2\,; \\ & \left|\,S^{\{n\}}\,\right| \leqslant \frac{2\,a_1 \cdot 2^n}{d^{3/2}\,\tau}\,,\ \ \left|U^{(n)}\right| \leqslant \frac{2\,a_1 \cdot 2^{um+vn}}{d^{3/2}\,\tau}\,,\ n = m, \ldots, N \,\Big\} \end{aligned} \tag{16}$$

otherwise. Here the numbers u, v are defined by (1). Note that the first of inequalities (15) is not satisfied in our conditions (see (6)). The same can be said about inequalities (31), (35).

Denote

$$Y_{*,k} = Y_k^{[N-p]} = Z^{[N]}_{(k-1)\cdot 2^p+1}, \qquad k = 1, \dots, 2^{N-p}. \tag{17}$$

By construction and by virtue of (11.14), (9), we have

$$\mathbf{P}\left\{ |Y_{*,k}| \leqslant 4\lambda \right\} = 1, \qquad k = 1, \dots, 2^{N-p}. \tag{18}$$

Let $i = i(j)$ be the largest integer such that $(i-1)\cdot 2^p + 1 \leqslant j$. We shall denote by $V_*^{(n)}$, $n = 0, \dots, N-p$, the sum of a block of 2^n summands $Y_{*,k}$ which contains the summand $Y_{*,i}$. Let

$$T_* = T_{*,i} = Y_{*,1} + \cdots + Y_{*,i}\,. \tag{19}$$

Remember that, by construction,

$$\bigcup_{k=1}^{2^{N-p}} \left\{ \left\{ \widetilde{X}_r = Z_r^{[N]},\ r = (k-1)\cdot 2^p + 2, \dots, k\cdot 2^p \right\} \cup \left\{ Y_{*,k} = Z^{[N]}_{(k-1)\cdot 2^p+1} \right\} \right\} \tag{20}$$

is a collection of random vectors, which are mutually independent. Set

$$U_0^{(n)} = U^{(n)} - V_*^{(n-p)}, \qquad n = p, \dots, N. \tag{21}$$

It can be easily verified that the sum $U_0^{(n)}$ consists of the summands of the sum $U^{(n)}$ with indices which can not be represented in the form $r = (k-1)\cdot 2^p + 1$ with integer k. In other words, the sum $U_0^{(n)}$ is the sum of contained in $U^{(n)}$ summands $Z_r^{[N]}$ which are equal to $\widetilde{X}_r$. For the proof of this fact one should take into account that, by construction, the numbers j and $(i-1)\cdot 2^p + 1$ are contained in the same block of indices $\left\{ m :\ m = (i-1)\cdot 2^p + 1, \dots, i\cdot 2^p \right\}$. Remember that, according to our construction, after the first step, when the independent random vectors (20) are constructed, we define the rest of the random vectors $\widetilde{X}_r$ as the (d-dimensional) functions of the random vectors $Y_{*,k}$, $k = 1, \dots, 2^{N-p}$. Denote

$$\begin{aligned} X_{*,k} &= \widetilde{X}_{(k-1)\cdot 2^p+1}, \qquad k = 1, \dots, 2^{N-p}; \\ S_* = S_{*,i} &= X_{*,1} + \cdots + X_{*,i}\,, \qquad S_0 = S - T_*\,. \end{aligned} \tag{22}$$

By analogy with (11), (12), for $s = p, \dots, N$ we can represent the sums S_*, S_0 and T_* in the form

$$S_0 = S_0^{(s)} - S_0^{\{s\}}, \qquad T_* = T_*^{(s-p)} - T_*^{\{s-p\}}, \tag{23}$$

where

$$\begin{aligned} T_*^{(s-p)} &= T_{*,k_s\cdot 2^{s-p}}, & T_*^{\{s-p\}} &= T_*^{(s-p)} - T_*, \\ S_0^{(s)} &= S^{(s)} - T_*^{(s-p)}, & S_0^{\{s\}} &= S_0^{(s)} - S_0 = S^{\{s\}} - T_*^{\{s-p\}}, \end{aligned} \tag{24}$$

and the numbers k_s are defined above, before the formulas (11), (12).

From the definition of numbers i, j, k_s, $s = 0, \dots, N$, it follows that

$$\begin{aligned} 0 \leqslant k_s \cdot 2^s - j < 2^s, &\qquad s = 0, \dots, N; \\ 0 \leqslant k_s \cdot 2^{s-p} - i < 2^{s-p}, &\qquad s = p, \dots, N. \end{aligned} \tag{25}$$

It is easy to see that from (10.5), (9), (11), (12), (18), (19), (24), (25) one can derive that, with probability one,

$$\begin{aligned} \left| T^{\{s\}} \right| \leqslant 4\lambda \cdot 2^s, &\qquad s = 0, \dots, N; \\ \left| T_*^{\{s-p\}} \right| \leqslant 4\lambda \cdot 2^{s-p}, &\qquad s = p, \dots, N. \end{aligned} \tag{26}$$

Analogously, using (18) and the definition of $V_*^{(n)}$, we see that

$$\left| V_*^{(n)} \right| \leqslant 4\lambda \cdot 2^n, \qquad n = 0, \dots, N - p. \tag{27}$$

Define also

$$\begin{aligned} \widetilde{S} = \sum_{k=1}^{j} \widetilde{X}_k, \qquad \delta_j(N) &= \left| \sum_{k=1}^{j} \widetilde{Y}_k - \sum_{k=1}^{j} \widetilde{X}_k \right| = \left| T - \widetilde{S} \right|, \\ \delta_i(N-p) &= \left| \sum_{k=1}^{i} Y_{*,k} - \sum_{k=1}^{i} X_{*,k} \right| = \left| T_* - S_* \right|, \\ \psi_j(N) &= \left| \sum_{k=1}^{j} \widetilde{Y}_k - \sum_{k=1}^{j} Z_k^{[N]} \right| = \left| T - S \right| \end{aligned} \tag{28}$$

(see (10.5), (10.11), (19), (22)). It is clear that from (17), (19), (20), (22), (28) and from the definition of i and S it follows that

$$S = \sum_{k=1}^{j} Z_k^{[N]} = S_0 + T_*, \qquad \widetilde{S} = S_0 + S_*, \qquad S - \widetilde{S} = T_* - S_*. \tag{29}$$

It is easy to see that the relations (28), (29) imply

$$\delta_j(N) \leqslant \psi_j(N) + \delta_i(N-p). \tag{30}$$

Introduce the random integer $\beta_j(N)$ by $\beta_j(N) = N + 1$, if at least one of the following inequalities is valid:

$$\frac{\tau^2 d^3 p}{2^{N-p}} > a_2, \qquad \left| V^{(N)} \right| > \frac{4 a_1 \cdot 2^N}{d^{3/2} \tau} \tag{31}$$

and by

$$\beta_j(N) = \min\Big\{ m :\ p \leqslant m \leqslant N, \quad \frac{\tau^2 d^3 p}{2^{m-p}} \leqslant a_2, \\ \big|V^{(n)}\big| \leqslant \frac{4a_1 \cdot 2^{um+vn}}{d^{3/2}\tau}, \quad n = m, \dots, N \Big\} \tag{32}$$

otherwise.

Analogously, taking into account (7), we introduce the random integer $\beta_i(N-p)$ by the equality $\beta_i(N-p) = N-p+1$, if at least one of the following inequalities is valid:

$$\frac{\tau^2 d^3 p}{2^{N-2p}} > a_2, \qquad \big|V_*^{(N-p)}\big| > \frac{4a_1 \cdot 2^{N-p}}{d^{3/2}\tau} \tag{33}$$

and by

$$\beta_i(N-p) = \min\Big\{ m :\ p \leqslant m \leqslant N-p, \quad \frac{\tau^2 d^3 p}{2^{m-p}} \leqslant a_2, \\ \big|V_*^{(n)}\big| \leqslant \frac{4a_1 \cdot 2^{um+vn}}{d^{3/2}\tau}, \quad n = m, \dots, N-p \Big\} \tag{34}$$

otherwise.

Finally, introduce the random integer $\nu = \nu_j$ by $\nu = N+1$, if at least one of the following inequalities is valid:

$$\frac{\tau^2 d^3 p}{2^{N-p}} > a_2, \qquad \big|U_0^{(N)}\big| > \frac{a_1 \cdot 2^N}{d^{3/2}\tau}, \qquad \big|S_0^{\{N\}}\big| > \frac{a_1 \cdot 2^N}{d^{3/2}\tau} \tag{35}$$

and by

$$\nu = \min\Big\{ m :\ p \leqslant m \leqslant N, \quad \frac{\tau^2 d^3 p}{2^{m-p}} \leqslant a_2; \\ \big|S_0^{\{n\}}\big| \leqslant \frac{a_1 \cdot 2^n}{d^{3/2}\tau}, \quad \big|U_0^{(n)}\big| \leqslant \frac{a_1 \cdot 2^{um+vn}}{d^{3/2}\tau}, \ n = m, \dots, N \Big\} \tag{36}$$

otherwise.

Put now

$$\rho_j(N) = 5\lambda \cdot 2^{\beta_j(N)}, \qquad \rho_i(N-p) = 5\lambda \cdot 2^{\beta_i(N-p)}, \tag{37}$$

$$\gamma_j(N) = \begin{cases} 10\lambda \cdot 2^{\nu}, & \text{if} \quad \nu \leqslant N, \\ U + 22\lambda \cdot 2^N, & \text{if} \quad \nu = N+1, \end{cases} \tag{38}$$

where

$$U = \big|U_0^{(N)}\big| + \big|S_0^{\{N\}}\big|. \tag{39}$$

Lemma 13.1. *The following inequalities are valid*:

$$\delta_j(N) + \rho_j(N) \leqslant \delta_i(N-p) + \rho_i(N-p) + \gamma_j(N) \tag{40}$$

and, for each $\varepsilon > 0$,

$$\mathbf{E}\exp\Big(\varepsilon\big(\delta_j(N)+\rho_j(N)\big)\Big) \leqslant \mathbf{E}\exp\Big(\varepsilon\big(\delta_i(N-p)+\rho_i(N-p)\big)\Big)\,\mathbf{E}\exp\big(\varepsilon\,\gamma_j(N)\big). \tag{41}$$

Proof. Assume that $\nu \leqslant N$ and, taking into account (7), introduce the integer valued random variables

$$\beta = \beta_i(N-p) - 1 \leqslant N-p, \qquad \varkappa = \max\Big\{\beta,\ \nu\Big\} \leqslant N. \tag{42}$$

The relations (1), (4), (26) imply

$$\big|T_*^{\{s-p\}}\big| \leqslant \tfrac{a_1\cdot 2^s}{d^{3/2}\,\tau}, \qquad s = p,\dots,N. \tag{43}$$

Using (24), (36), (42), (43), we obtain

$$\big|S^{\{n\}}\big| \leqslant \big|S_0^{\{n\}}\big| + \big|T_*^{\{n-p\}}\big| \leqslant \tfrac{2\,a_1\cdot 2^n}{d^{3/2}\,\tau}, \qquad n = \varkappa,\dots,N, \tag{44}$$

(here and below we use that (36), (42) imply the inequality $\varkappa \geqslant p$). By virtue of (1), (4), (27), (42), for $n = \beta,\dots,\beta+p$, $n \geqslant p$ we have

$$\big|V_*^{(n-p)}\big| \leqslant 4\,\lambda\cdot 2^{n-p} \leqslant \tfrac{a_1\cdot 2^{u\beta+vn}}{d^{3/2}\,\tau} \leqslant \tfrac{a_1\cdot 2^{u\varkappa+vn}}{d^{3/2}\,\tau}. \tag{45}$$

If $n = \beta+p+1,\dots,N$, then, applying (1), (2), (4), (34), (42), we obtain

$$\big|V_*^{(n-p)}\big| \leqslant \tfrac{4\,a_1\cdot 2^{u(\beta+1)+v(n-p)}}{d^{3/2}\,\tau} \leqslant \tfrac{a_1\cdot 2^{u\varkappa+vn}}{d^{3/2}\,\tau} \tag{46}$$

(we use that the inequalities $p \geqslant 4$, (1), (2) imply $2^{vp} \geqslant 2^{5p/8} > 4\cdot 2^{3/8} \geqslant 4\cdot 2^u$). From (21), (36), (42), (45), (46) it follows that

$$\begin{aligned}\big|U^{(n)}\big| &\leqslant \big|U_0^{(n)}\big| + \big|V_*^{(n-p)}\big| \leqslant \tfrac{a_1\cdot 2^{u\varkappa+vn}}{d^{3/2}\,\tau} + \tfrac{a_1\cdot 2^{u\varkappa+vn}}{d^{3/2}\,\tau}\\ &= \tfrac{2\,a_1\cdot 2^{u\varkappa+vn}}{d^{3/2}\,\tau}, \qquad n = \varkappa,\dots,N.\end{aligned} \tag{47}$$

Taking into account (16), (36), (42), (44), (47), we see that

$$\mu \leqslant \varkappa \leqslant N. \tag{48}$$

It is clear that

$$U^{(m)} = \frac{1}{2}U^{(m+1)} + \frac{b_{m+1}}{2}\widetilde{U}^{(m+1)}, \qquad m = 0, \dots, N-1, \tag{49}$$

where the factors $b_{m+1} = b_{m+1}(j)$ can be equal to 1 or to -1 (see (10.10)). From (49) one can easily derive by induction that the following representation is valid:

$$\begin{gathered} U^{(m)} = \frac{U^{(N)}}{2^{N-m}} + \sum_{k=m+1}^{N} \frac{b_k\,\widetilde{U}^{(k)}}{2^{k-m}}, \qquad m = 0, \dots, N-1, \\ |b_k| = 1, \qquad k = 1, \dots, N. \end{gathered} \tag{50}$$

It is evident that we can write the analogous equality with change of U by V.

Suppose now that an integer n satisfies the inequalities

$$\mu \leqslant n \leqslant N. \tag{51}$$

Then, taking into account (1)–(4), (16), (51), we see that

$$\tau^2 d^3 \leqslant a_2 \cdot 2^{\mu} \leqslant a_1^2 \cdot 2^{2u\mu+(2v-1)n} \leqslant a_1^2 \cdot 2^{2u\mu+(2v-1)N}. \tag{52}$$

Using (1), (3), (16), (51), we obtain the inequalities

$$\left|U^{(n)}\right| \leqslant \frac{2\,a_1 \cdot 2^{u\mu+vn}}{d^{3/2}\,\tau} \leqslant \frac{c_{64} \cdot 2^n}{d^{3/2}\,\tau}. \tag{53}$$

Taking here $n = N$ and using (1)–(3), (16), (51)–(53) and Lemma 12.1 (item b)), we obtain

$$\begin{aligned} \left|U^{(N)} - V^{(N)}\right| &\leqslant c_{46}\,d^2\,\tau^2 \cdot 2^{-N/2}\left(1 + 2^{-3N/2}\left|U^{(N)}\right|^3\right) \\ &\leqslant c_{63}\,d^2\,\tau^2\left(2^{-N/2} + \frac{8a_1^3 \cdot 2^{3u\mu-wN}}{d^{9/2}\,\tau^3}\right) \\ &\leqslant \frac{9\,a_1^3\,c_{63} \cdot 2^{3u\mu-wN}}{d^{5/2}\,\tau} \leqslant \frac{a_1 \cdot 2^{3u\mu-wN}}{14\,d^{5/2}\,\tau}. \end{aligned} \tag{54}$$

From (1), (53), (54) it follows that

$$\left|V^{(N)}\right| \leqslant \left|U^{(N)}\right| + \left|V^{(N)} - U^{(N)}\right| \leqslant \frac{4\,a_1 \cdot 2^{u\mu+vN}}{d^{3/2}\,\tau}. \tag{55}$$

Analogously, if we suppose that

$$\mu \leqslant n < N, \tag{56}$$

then from (1)–(3), (16), (49), (56) one can easily derive that

$$\left|\widetilde{U}^{(n+1)}\right| = \left|U^{(n+1)} - 2U^{(n)}\right| \leqslant \frac{2\,(1+2^{3/8})\,a_1 \cdot 2^{u\mu+v(n+1)}}{d^{3/2}\,\tau} \leqslant \frac{c_{64} \cdot 2^{n+1}}{d^{3/2}\,\tau}. \tag{57}$$

Using the induction with respect to n, we shall prove that, if $\mu < N$, then, for $n = N-1, N-2, \ldots, \mu$,

$$\left|\widetilde{U}^{(n+1)} - \widetilde{V}^{(n+1)}\right| \leqslant \frac{2\,a_1 \cdot 2^{3u\mu-w(n+1)}}{d^{5/2}\,\tau}, \tag{58}$$

$$\left|U^{(n)} - V^{(n)}\right| \leqslant \frac{2\,a_1 \cdot 2^{3u\mu-wn}}{d^{5/2}\,\tau}, \tag{59}$$

$$\left|V^{(n)}\right| \leqslant \frac{4\,a_1 \cdot 2^{u\mu+vn}}{d^{3/2}\,\tau}. \tag{60}$$

For $n = N$ the relations (59), (60) are already proved (see (54), (55)).

Assume that the inequalities (59), (60) are valid for $n = s+1, \ldots, N$ and the inequality (58) holds for $n = s+1, \ldots, N-1$, $\mu \leqslant s \leqslant N-1$. It suffices to prove the relations (58)–(60) for $n = s$.

Using (1)–(3), (16), (52), (53), (56), (57), (59), (60) and Lemma 12.1 (item a)), we obtain that in this case

$$\begin{aligned}\left|\widetilde{U}^{(s+1)} - \widetilde{V}^{(s+1)}\right| &\leqslant \frac{c_{43}\,d^2\,\tau^2}{2^{(s+1)/2}}\left(1 + \frac{\max\left\{\left|U^{(s+1)}\right|^3, \left|\widetilde{U}^{(s+1)}\right|^3\right\}}{2^{3(s+1)/2}}\right) + \frac{\left|U^{(s+1)} - V^{(s+1)}\right|}{8} \\ &\leqslant c_{63}\,d^2\,\tau^2\left(2^{-(s+1)/2} + \frac{125\,a_1^3 \cdot 2^{3u\mu-w(s+1)}}{d^{9/2}\,\tau^3}\right) + \frac{a_1 \cdot 2^{3u\mu-w(s+1)}}{4d^{5/2}\,\tau^3} \\ &\leqslant \frac{2\,a_1 \cdot 2^{3u\mu-w(s+1)}}{d^{5/2}\,\tau}.\end{aligned}$$

The fulfilment of the conditions of Lemma 12.1 can be easily deduced from (1), (16), (53), (57), (60) (with $n = s+1$) and from the definitions of a_1, c_{64} (see (3)).

From (1), (2), (50), (53), (54), (58) (with $n = s, \ldots, N-1$) it follows that

$$\left|U^{(s)} - V^{(s)}\right| \leqslant \frac{\left|U^{(N)} - V^{(N)}\right|}{2^{N-s}} + \sum_{k=s+1}^{N} \frac{\left|\widetilde{U}^{(k)} - \widetilde{V}^{(k)}\right|}{2^{k-s}} \leqslant \frac{2\,a_1 \cdot 2^{3u\mu-ws}}{d^{5/2}\,\tau}$$

and, consequently,

$$\left|V^{(s)}\right| \leqslant \left|U^{(s)}\right| + \left|U^{(s)} - V^{(s)}\right| \leqslant \frac{2\,a_1 \cdot 2^{u\mu+vs}}{d^{3/2}\,\tau} + \frac{2\,a_1 \cdot 2^{3u\mu-ws}}{d^{5/2}\,\tau} \leqslant \frac{4\,a_1 \cdot 2^{u\mu+vs}}{d^{3/2}\,\tau}.$$

This completes the proof of (58)–(60) for $n = s$.

It is evident that from (16), (32), (55), (60) it follows that

$$\beta_j(N) \leqslant \mu. \tag{61}$$

It is clear that the relations (42), (48), (61) imply

$$2^{\beta_j(N)} + 2^{\mu} \leqslant 2 \cdot 2^{\mu} \leqslant 2 \cdot 2^{\varkappa} \leqslant 2^{\beta_i(N-p)} + 2 \cdot 2^{\nu}. \tag{62}$$

Using (2), (3), (14), (16), (26), (28), (54), (58) and taking into account that $\lambda \geqslant 1$, $\tau \geqslant 1$, we see that

$$\begin{aligned}\psi_j(N) &\leqslant \tfrac{a_1 \cdot 2^{3u\mu - wN}}{14\, d^{5/2}\, \tau} + \sum_{n=\mu+1}^{N} \tfrac{2\, a_1 \cdot 2^{3u\mu - wn}}{d^{5/2}\, \tau} + \tfrac{2\, a_1 \cdot 2^{\mu}}{d^{3/2}\, \tau} + 4\,\lambda \cdot 2^{\mu} \\ &\leqslant \bigl(2\,(1 - 2^{-w})^{-1} a_1 + 4\,\lambda\bigr) \cdot 2^{\mu} \leqslant 5\,\lambda \cdot 2^{\mu}. \end{aligned} \tag{63}$$

The relations (30), (37), (62), (63) imply

$$\delta_j(N) + \rho_j(N) \leqslant \delta_i(N-p) + 5\,\lambda\left(2^{\mu} + 2^{\beta_j(N)}\right) \leqslant \delta_i(N-p) + \rho_i(N-p) + 10\,\lambda \cdot 2^{\nu}. \tag{64}$$

From (38), (64) it is easy to derive the inequality (40) in the case $\nu \leqslant N$.

Let now $\nu = N + 1$. From (12), (21) it follows that

$$S^{(N)} = U^{(N)} = U_0^{(N)} + V_*^{(N-p)}. \tag{65}$$

Using (11), (28), we see that

$$\psi_j(N) \leqslant \bigl|S^{(N)}\bigr| + \bigl|S^{\{N\}}\bigr| + \bigl|T\bigr|. \tag{66}$$

By virtue of (24) we have

$$\bigl|S^{\{N\}}\bigr| \leqslant \bigl|S_0^{\{N\}}\bigr| + \bigl|T_*^{\{N-p\}}\bigr|. \tag{67}$$

It is not hard to see that the relations (10), (26), (27), (39), (65)–(67) imply

$$\psi_j(N) \leqslant U + 12\,\lambda \cdot 2^{N}. \tag{68}$$

From (37) and from definition of $\beta_j(N)$ it follows that $\beta_j(N) \leqslant N + 1$ and

$$\rho_j(N) \leqslant 10\,\lambda \cdot 2^{N}. \tag{69}$$

The relations (30), (38), (68), (69) imply

$$\delta_j(N) + \rho_j(N) \leqslant \delta_i(N-p) + U + 22\,\lambda \cdot 2^{N} = \delta_i(N-p) + \gamma_j(N). \tag{70}$$

Thus, the inequality (40) is proved in the case $\nu = N + 1$.

By construction (see (19)–(24), (28), (33)–(38)), the random variable $\gamma_j(N)$ is a function of

$$\bigcup_{k=1}^{2^{N-p}} \left\{ \widetilde{X}_r = Z_r^{[N]},\ r = (k-1)\cdot 2^p + 2, \dots, k\cdot 2^p \right\} \tag{71}$$

and the random variables $\delta_i(N-p),\ \rho_i(N-p)$ are functions of

$$\left\{ Y_{*,k} = Z^{[N]}_{(k-1)\cdot 2^p+1},\ k = 1, \dots, 2^{N-p} \right\}. \tag{72}$$

Since the sets (71), (72) are disjoint sets consisting of mutually independent vectors (see (20)), the random variables $\gamma_j(N)$ and $\delta_i(N-p) + \rho_i(N-p)$ are independent. Therefore, the inequality (41) follows from (40) and the proof of Lemma 13.1 is completed.

In this section we shall denote by ε the quantity

$$\varepsilon = \frac{a_1^2}{512\,\lambda\, d^3\,\tau^2} \leqslant \frac{a_1}{16\, d^{3/2}\,\tau} \tag{73}$$

(see (3)).

Lemma 13.2. *There exists a positive quantity a_3 depending only on v and such that the following inequality is valid*:

$$\mathbf{E}\,\exp\bigl(\varepsilon\,\gamma_j(N)\bigr) \leqslant \exp\Bigl(a_3\, p\,(\lambda d^{3/2}\tau)^{1/v}\Bigr). \tag{74}$$

Proof. Denote by

$$\zeta = \mathbf{1}_{\{\nu = N+1\}} \tag{75}$$

the indicator function of the event $\left\{\nu = N+1\right\}$ (see (35)). Using (38) and the Hölder inequality, we obtain

$$\mathbf{E}\,\zeta\,\exp\bigl(\varepsilon\,\gamma_j(N)\bigr) \leqslant \bigl(\mathbf{E}\,\zeta^2\bigr)^{1/2}\,\exp\bigl(22\,\varepsilon\,\lambda\cdot 2^N\bigr)\bigl(\mathbf{E}\,e^{2\varepsilon U}\bigr)^{1/2}. \tag{76}$$

From the definitions of $U_0^{(n)}$, $S_0^{\{n\}}$ it follows that $U_0^{(N)}$, $S_0^{\{N\}}$ are sums of not more than 2^N independent random vectors with distributions $\mathcal{L}(\widetilde{X}_k) \in \mathcal{A}_d(\tau),\ \operatorname{cov}\widetilde{X}_k = I$. (see (21), (24)). Therefore, the relations (3), (6), (35), (75) and Lemmas 1.6, 1.9 imply

$$\begin{aligned}
\mathbf{E}\,\zeta^2 &\leqslant \mathbf{P}\left\{\bigl|U_0^{(N)}\bigr| > \frac{a_1\cdot 2^N}{d^{3/2}\,\tau}\right\} + \mathbf{P}\left\{\bigl|S_0^{\{N\}}\bigr| > \frac{a_1\cdot 2^N}{d^{3/2}\,\tau}\right\} \\
&\leqslant 4\,d\,\exp\Bigl(-\min\left\{\frac{a_1^2\cdot 2^N}{4\,d^3\,\tau^2},\ \frac{a_1\cdot 2^N}{4\,d^{3/2}\,\tau^2}\right\}\Bigr) \\
&= 4\,d\,\exp\Bigl(-\frac{a_1^2\cdot 2^N}{4\,d^3\,\tau^2}\Bigr).
\end{aligned} \tag{77}$$

It is clear that (see (39))

$$\mathbf{E}\, e^{2\varepsilon U} \leqslant \Big(\mathbf{E}\, \exp\Big(4\,\varepsilon \,\big|\, U_0^{(N)}\,\big|\Big)\; \mathbf{E}\, \exp\Big(4\,\varepsilon \,\big|\, S_0^{\{N\}}\,\big|\Big)\Big)^{1/2}. \tag{78}$$

By virtue of (3), (73) we have

$$4\,\varepsilon\,\tau < 1. \tag{79}$$

Using Lemma 1.6, taking into account (79) and applying Lemma 1.10 with $r = d$, $\sigma^2 \leqslant 2^N$, $h = 4\,\varepsilon$, we see that

$$\begin{aligned} &\mathbf{E}\, \exp\Big(4\,\varepsilon \,\big|\, U_0^{(N)}\,\big|\Big) \leqslant 2\,d\, \exp\big(16\,\varepsilon^2 \cdot 2^N\big), \\ &\mathbf{E}\, \exp\Big(4\,\varepsilon \,\big|\, S_0^{\{N\}}\,\big|\Big) \leqslant 2\,d\, \exp\big(16\,\varepsilon^2 \cdot 2^N\big). \end{aligned} \tag{80}$$

From (3), (4), (6), (73), (76)–(78), (80) it follows that

$$\begin{aligned} \mathbf{E}\,\zeta\, \exp\big(\varepsilon\,\gamma_j(N)\big) &\leqslant 2\sqrt{2}\,d\, \exp\Big(-\tfrac{a_1^2 \cdot 2^N}{8\,d^3\,\tau^2} + 22\,\varepsilon\,\lambda \cdot 2^N + 8\,\varepsilon^2 \cdot 2^N\Big) \\ &\leqslant 2\sqrt{2}\,d\, \exp\Big(-\tfrac{a_1^2 \cdot 2^N}{32\,d^3\,\tau^2}\Big) \leqslant 2\sqrt{2}\,d\, \exp\Big(-\tfrac{p \cdot 2^p}{32}\Big) \\ &\leqslant 2\sqrt{2}\,d\, \exp\Big(-\tfrac{d^{3/2}\,\tau\,\lambda}{8\,a_1}\Big) \leqslant c. \end{aligned} \tag{81}$$

Denote now by m_0 the minimal integer such that

$$\frac{\tau^2\, d^3\, p}{2^{m_0 - p}} < a_2\ . \tag{82}$$

It is evident that (3), (5)–(7), (82) imply $p \leqslant m_0 = M + 1 \leqslant N$. Using the definition of the random variable ν, we see that

$$\mathbf{P}\,\{\,\nu = m+1\,\} \leqslant \sum_{n=m}^{N} \left(\mathbf{P}\left\{\big|\,U_0^{(n)}\,\big| \geqslant \tfrac{a_1 \cdot 2^{um+vn}}{d^{3/2}\,\tau}\right\} + \mathbf{P}\left\{\big|\,S_0^{\{n\}}\,\big| \geqslant \tfrac{a_1 \cdot 2^n}{d^{3/2}\,\tau}\right\}\right), \qquad m = m_0, \ldots, N-1. \tag{83}$$

From the definitions of vectors $U_0^{(n)}, S_0^{\{n\}}$ it follows that they are sums of not more than 2^n independent random vectors with distributions $\mathcal{L}(\widetilde{X}_k) \in \mathcal{A}_d(\tau)$, $\operatorname{cov} \widetilde{X}_k = I$ (see (21), (24)). Therefore, for $m \geqslant m_0$, the relations (1), (3), (82) and Lemmas 1.6, 1.9 imply

$$\begin{aligned} \mathbf{P}\left\{\big|\,U_0^{(n)}\,\big| \geqslant \tfrac{a_1 \cdot 2^{um+vn}}{d^{3/2}\,\tau}\right\} &\leqslant 2\,d\, \exp\Big(-\min\Big\{\tfrac{a_1^2 \cdot 2^{2um+2vn}}{4\,d^3\,\tau^2 \cdot 2^n},\ \tfrac{a_1 \cdot 2^{um+vn}}{4\,d^{3/2}\,\tau^2}\Big\}\Big) \\ &= 2\,d\, \exp\Big(-\tfrac{a_1^2 \cdot 2^{2um+(2v-1)n}}{4\,d^3\,\tau^2}\Big), \quad n = m, \ldots, N, \end{aligned} \tag{84}$$

and, analogously,

$$\mathbf{P}\left\{\left|\,S_0^{\{n\}}\,\right| \geqslant \frac{a_1\cdot 2^n}{d^{3/2}\,\tau}\right\} \leqslant 2d\,\exp\Big(-\frac{a_1^2\cdot 2^n}{4\,d^3\,\tau^2}\Big), \qquad n=m,\dots,N. \tag{85}$$

Taking into account (38), (75), we see that

$$\mathbf{E}\,(1-\zeta)\,\exp\big(\varepsilon\,\gamma_j(N)\big) \leqslant \exp\big(10\,\lambda\varepsilon\cdot 2^{m_0}\big) + \sum_{m=m_0+1}^{N} e^{10\,\lambda\varepsilon\cdot 2^m}\ \mathbf{P}\left\{\nu=m\right\}. \tag{86}$$

From (1), (3), (73), (82) and from the definition of p (see (4)) it follows that

$$10\,\lambda\varepsilon\cdot 2^{m_0} \leqslant \frac{10\,a_1^2}{512\,d^3\,\tau^2}\cdot\frac{d^3\,\tau^2\,p\cdot 2^{p+1}}{a_2} \leqslant a\,p\cdot 2^p \leqslant a\,p\,(\lambda\,d^{3/2}\,\tau)^{1/v}. \tag{87}$$

Using (1)–(4), (73), (82)–(85), we obtain

$$\begin{aligned}
\sum_{m=m_0+1}^{N} e^{10\,\lambda\varepsilon\cdot 2^m}\,\mathbf{P}\left\{\nu=m\right\} &\leqslant \sum_{m=m_0+1}^{N}\ \sum_{n=m-1}^{N} 4\,d\,\exp\Big(\frac{10\,a_1^2\cdot 2^m}{512\,d^3\,\tau^2} - \frac{a_1^2\cdot 2^{2um+(2v-1)n}}{8\,d^3\,\tau^2}\Big)\\
&\leqslant 4\,d\sum_{m=m_0+1}^{N}\ \sum_{n=m-1}^{N}\exp\Big(-\frac{a_1^2\cdot 2^{2um+(2v-1)n}}{16\,d^3\,\tau^2}\Big)\\
&\leqslant 4\,d\sum_{m=m_0+1}^{\infty} c\,\log^+(a_1^{-1}\,d\tau)\exp\Big(-\frac{a_1^2\cdot 2^m}{32\,d^3\,\tau^2}\Big)\\
&\leqslant cd\,\big(\log^+(a_1^{-1}\,d\tau)\big)^2\,\exp\Big(-\frac{a_1^2\cdot 2^{m_0}}{16\,d^3\,\tau^2}\Big)\\
&\leqslant cd\,\big(\log^+(a_1^{-1}\,d\tau)\big)^2\,\exp\Big(-\frac{p\cdot 2^p}{16}\Big)\\
&\leqslant cd\,\big(\log^+(a_1^{-1}\,d\tau)\big)^2\,\exp\Big(-\frac{d^{3/2}\,\tau\,\lambda}{4\,a_1}\Big) \leqslant c.
\end{aligned} \tag{88}$$

The inequality (74) follows from (81), (86)–(88).

Lemma 13.3. *Assume now* (*instead of* (6)) *that* $N=M$. *Then there exists a positive quantity* a_4 *depending only on* v *and such that the following inequality is valid*:

$$\mathbf{E}\,\exp\Big(\varepsilon\,\big(\rho_j(M)+\delta_j(M)\big)\Big) \leqslant \exp\big(a_4\,p\,(\lambda\,d^{3/2}\,\tau)^{1/v}\big), \qquad j=1,\dots,2^M. \tag{89}$$

Proof. Similarly to (69), one can verify that $\rho_j(M) \leqslant 10\,\lambda \cdot 2^M$. Therefore, using (28), we obtain

$$\mathbf{E}\,\exp\Big(\varepsilon\,\big(\rho_j(M)+\delta_j(M)\big)\Big) \leqslant \exp\big(10\,\varepsilon\,\lambda\cdot 2^M\big)\,\mathbf{E}\,e^{\varepsilon W}, \tag{90}$$

where $W = |\widetilde{S}|+|T|$. From the definitions of $\widetilde{S}$, $T = T_j$ (see (10.5), (28)) it follows that they are sums of not more than 2^M independent random vectors $\widetilde{X}_k$ and $Y_k^{[M]}$ respectively. By analogy with (10), we have $\mathbf{P}\left\{\,|T| \leqslant 4\,\lambda\cdot 2^M\,\right\} = 1$. Hence,

$$\mathbf{E}\,e^{\varepsilon W} \leqslant\ \mathbf{E}\,e^{\varepsilon|\widetilde{S}|}\ \exp\big(4\,\varepsilon\,\lambda\cdot 2^M\big). \tag{91}$$

Remember that $\operatorname{cov}\widetilde{X}_k = I$, $\mathcal{L}(\widetilde{X}_k) \in \mathcal{A}_d(\tau)$. Using Lemma 1.6, taking into account (79) and applying Lemma 1.10 with $r = d$, $\sigma^2 \leqslant 2^M$, $h = \varepsilon$, we see that

$$\mathbf{E}\,e^{\varepsilon|\widetilde{S}|} \leqslant 2\,d\,\exp\big(\varepsilon^2\cdot 2^M\big). \tag{92}$$

From (1), (3), (5), (73), (90)–(92) and from the definition of p (see (4)) it follows that

$$\begin{aligned}\mathbf{E}\,\exp\Big(\varepsilon\,\big(\rho_j(M)+\delta_j(M)\big)\Big) &\leqslant 2\,d\,\exp\Big(14\,\varepsilon\,\lambda\cdot 2^M+\varepsilon^2\cdot 2^M\Big)\\ &\leqslant 2\,d\,\exp\Big(\tfrac{15\,a_1^2\cdot 2^M}{512\,d^3\,\tau^2}\Big) \leqslant 2\,d\,\exp\big(a\,p\cdot 2^p\big)\\ &\leqslant 2\,d\,\exp\big(a\,p\,(\lambda d^{3/2}\tau)^{1/v}\big).\end{aligned} \tag{93}$$

It is evident that (1), (4), (93) implies (89). The proof of Lemma 13.3 is completed.

Denote

$$\begin{aligned} b(N) &= \max_{1\leqslant m\leqslant 2^N}\ \mathbf{E}\,\exp\Big(\varepsilon\big(\delta_m(N)+\rho_m(N)\big)\Big),\\ b(N-p) &= \max_{1\leqslant m\leqslant 2^{N-p}}\ \mathbf{E}\,\exp\Big(\varepsilon\big(\delta_m(N-p)+\rho_m(N-p)\big)\Big)\\ r(N) &= \max_{1\leqslant m\leqslant 2^N}\ \mathbf{E}\,\exp\big(\varepsilon\,\gamma_m(N)\big).\end{aligned} \tag{94}$$

One can easily verify that the definitions of $b(N)$, $b(N-p)$ are good coordinated: if we begin with 2^{N_0} vectors, where $N_0 = N-p$, then $b(N_0) = b(N-p)$ (see (17), (28)). Since j, $j = 1,\dots,2^N$, is arbitrary, we can easily derive from (41), (94) and Lemma 13.2 that

$$b(N) \leqslant b(N-p)\,r(N) \leqslant b(N-p)\,\exp\Big(a_3\,p\,(\lambda d^{3/2}\tau)^{1/v}\Big). \tag{95}$$

Continuing this procedure and taking into account (5)–(7), we obtain

$$b(N) \leqslant b(M) \exp\left(a_3 (N-M) (\lambda d^{3/2} \tau)^{1/v}\right). \tag{96}$$

From (7), (89), (94), (96) it follows that

$$b(N) \leqslant \exp\left(a_5 N (\lambda d^{3/2} \tau)^{1/v}\right) \tag{97}$$

with $a_5 = \max\{a_3, a_4\}$. Using (28), (94), (97), we obtain

$$\begin{aligned} \mathbf{E} \exp\left(\varepsilon \Delta(\widetilde{X}, \widetilde{Y})\right) &\leqslant \sum_{m=1}^{2^N} \mathbf{E} \exp\left(\varepsilon \delta_m(N)\right) \leqslant \sum_{m=1}^{2^N} \mathbf{E} \exp\left(\varepsilon\big(\delta_m(N) + \rho_m(N)\big)\right) \\ &\leqslant 2^N \exp\left(a_5 N (\lambda d^{3/2} \tau)^{1/v}\right) \leqslant \exp\left(a N (\lambda d^{3/2} \tau)^{1/v}\right). \end{aligned} \tag{98}$$

Assume now, instead of (6), that $N \leqslant M$ and let the vectors $\widetilde{X}_k$, $k = 1, \dots, 2^M$, be constructed with the help of the vectors $\widetilde{Y}_k$, $k = 1, \dots, 2^M$, in the same way as the vectors $X_k^* = \widetilde{X}_{(k-1)\cdot 2^{N-M}+1}$, $k = 1, \dots, 2^M$, was constructed with the help of the vectors $Y_k^* = Z^{[M+p]}_{(k-1)\cdot 2^{N-M}+1}$, $k = 1, \dots, 2^M$. For $k = 2^N + 1, \dots, 2^M$ we can take $\mathcal{L}(\widetilde{X}_k) = \mathcal{L}(\widetilde{X}_1)$, $\mathcal{L}(\widetilde{Y}_k) = \mathcal{L}(\widetilde{Y}_1)$. Now, if we denote

$$\Delta_N(\widetilde{X}, \widetilde{Y}) = \max_{1 \leqslant j \leqslant 2^N} \left| \sum_{k=1}^{j} \widetilde{Y}_k - \sum_{k=1}^{j} \widetilde{X}_k \right|, \tag{99}$$

then, using (28), (89), (94), (99), we obtain

$$\begin{aligned} \mathbf{E} \exp\left(\varepsilon \Delta_N(\widetilde{X}, \widetilde{Y})\right) &\leqslant \sum_{m=1}^{2^N} \mathbf{E} \exp\left(\varepsilon \delta_m(M)\right) \leqslant 2^N \exp\left(a_4 p (\lambda d^{3/2} \tau)^{1/v}\right) \\ &\leqslant \exp\left(a (N+p) (\lambda d^{3/2} \tau)^{1/v}\right). \end{aligned} \tag{100}$$

Proof of Theorem 2. Without loss of generality we suppose that $0 < \alpha \leqslant 1/5$ and $n = 2^N$. Assume that the distributions $\mathcal{L}(\widetilde{\xi}_k)$, $k = 1, \dots, n$, are absolutely continuous. Suppose that independent Gaussian vectors $Y_1, \dots, Y_{2^N}$ are already constructed. According to Theorem 11.1 (case A1), we can construct the independent random vectors $\widetilde{Y}_1, \dots, \widetilde{Y}_{2^N}$ so that $\mathcal{L}(\widetilde{Y}_k) = \Psi(\widetilde{\eta}_k)$, $k = 1, \dots, n$, and

$$\mathbf{E} \exp\Big(\tfrac{c_{15}\, \Delta(\widetilde{Y}, Y)}{d^{3/2}\, \tau}\Big) \leqslant \exp\left(c_{16} N \log^+ d\right). \tag{101}$$

Then we construct the vectors $\widetilde{X}_1, \dots, \widetilde{X}_{2^N}$ as functions of $\widetilde{Y}_1, \dots, \widetilde{Y}_{2^N}$ as it was described above in this section.

Taking $v = \dfrac{2}{3+\alpha}$, we see that v satisfies (1) and

$$v^{-1} = \frac{3+\alpha}{2}. \tag{102}$$

The relations (1), (3), (102) show that $a, a_1, \dots, a_5$ can be considered as quantities depending only on α.

Let M be the maximal integer satisfying the inequality (5) with a_2, p defined by (3), (4). Define N as the minimal integer such that $n \leqslant 2^N$, if $n \leqslant 2^M$, and as the minimal integer satisfying the relations $n \leqslant 2^N$ and (6), if $n > 2^M$. In both cases we have

$$N \leqslant \log n + p. \tag{103}$$

It is evident that from (1), (4), (102) it follows that

$$p \leqslant a_6 \log^+(\lambda d \tau) \leqslant a_7 (\lambda d^{3/2} \tau)^{\alpha/2}, \tag{104}$$

where a_6, a_7 depend only on α. The statement of Theorem 2 in the case of absolutely continuous distributions $\mathcal{L}(\widetilde{\xi}_k)$ follows now from (73), (98)–(104).

Below we show that the technical assumption of absolute continuity of $\mathcal{L}(\widetilde{\xi}_k)$ can be easily removed with the help of smoothing.

Let the conditions of Theorem 2 be fulfilled in the general case. Assume that there exist random vectors $\widetilde{K}_1, \dots, \widetilde{K}_n$ such that the vectors $\widetilde{\xi}_1, \dots, \widetilde{\xi}_n$, $\widetilde{K}_1, \dots, \widetilde{K}_n$ are mutually independent and $\mathcal{L}(\widetilde{K}_k) = \mathcal{L}(K)$, $k = 1, \dots, n$, where K is the vector defined before the formula (11.14). Let $\delta < 1$ be a positive number which will be chosen later to be so small as it will be necessary. Set

$$\xi_k^* = \sqrt{1-\delta}\, \widetilde{\xi}_k + \sqrt{2\delta}\, \widetilde{K}_k, \qquad k = 1, \dots, n. \tag{105}$$

One can verify that the vectors ξ_k^* satisfy all the conditions of Theorem 2 with the change of $\widetilde{\xi}_k$ by ξ_k^* and of λ by 2λ. For the proof of this assertion one should use the definition of K, (11.14), the properties of $\mathcal{A}_d(\tau)$ and the choice of so small δ that $2\lambda \geqslant \lambda_\delta = \sqrt{1-\delta}\,\lambda + \sqrt{2\delta}$, $\tau \geqslant \tau_\delta = \max\{\sqrt{1-\delta}\,\tau, \sqrt{2\delta}\, c_{14}\}$. Besides, the distributions $\mathcal{L}(\xi_k^*)$ are absolutely continuous. This follows from (105) and from the absolute continuity of $\mathcal{L}(\widetilde{K}_k)$.

We have already proved that one can construct on a probability space a sequence of independent random vectors $X_1^*, \dots, X_n^*$ and a corresponding sequence of independent Gaussian random vectors $Y_1, \dots, Y_n$ so that $\mathcal{L}(X_j^*) = \mathcal{L}(\xi_j^*)$, $j = 1, \dots, n$, and

$$\mathbf{E} \exp\Big(\frac{a_8\, \Delta(X^*, Y)}{\lambda d^3 \tau^2} \Big) \leqslant \exp\Big(a_9 (\lambda d^{3/2} \tau)^{3/2+\alpha} \log^+ n \Big), \tag{106}$$

where a_8, a_9 depend only on α.

Using the Berkes–Philipp lemma [3, Lemma A1, p. 53], we can suppose without loss of generality that

$$X_k^* = \sqrt{1-\delta}\,\widetilde{X}_k + \sqrt{2\,\delta}\,K_k^*, \qquad k = 1, \ldots, n, \tag{107}$$

where the mutual distribution of $\left\{\widetilde{X}_1, \ldots, \widetilde{X}_n,\ K_1^*, \ldots, K_n^*\right\}$ coincides with the mutual distribution of $\left\{\widetilde{\xi}_1, \ldots, \widetilde{\xi}_n,\ \widetilde{K}_1, \ldots, \widetilde{K}_n\right\}$. In particular, $\widetilde{X}_1, \ldots, \widetilde{X}_n$ are independent random vectors with $\mathcal{L}(\widetilde{X}_j) = \mathcal{L}(\widetilde{\xi}_j),\ j = 1, \ldots, n$. One can easily verify that (107) implies

$$\widetilde{X}_k - Y_k = \frac{X_k^* - Y_k}{\sqrt{1-\delta}} + \frac{\left(1-\sqrt{1-\delta}\right)Y_k}{\sqrt{1-\delta}} - \frac{\sqrt{2\,\delta}\,K_k^*}{\sqrt{1-\delta}}, \qquad k = 1, \ldots, n. \tag{108}$$

Denote

$$\Delta_Y = \max_{1 \leqslant m \leqslant n} \left|Y_1 + \cdots + Y_m\right|, \qquad \Delta_K = \max_{1 \leqslant m \leqslant n} \left|K_1^* + \cdots + K_m^*\right|. \tag{109}$$

From (108), (109) it follows that

$$\sqrt{1-\delta}\,\Delta(\widetilde{X}, Y) \leqslant \Delta(X^*, Y) + \left(1-\sqrt{1-\delta}\right)\Delta_Y + \sqrt{2\,\delta}\,\Delta_K \tag{110}$$

and, hence,

$$\begin{aligned}\mathbf{E}\,\exp\left(\frac{a_8\,\sqrt{1-\delta}\,\Delta(\widetilde{X}, Y)}{3\,\lambda\,d^3\,\tau^2}\right) &\leqslant \mathbf{E}\,\exp\left(\frac{a_8\,\Delta(X^*, Y)}{3\,\lambda\,d^3\,\tau^2} + \frac{a_8\left(1-\sqrt{1-\delta}\right)\Delta_Y}{3\,\lambda\,d^3\,\tau^2} + \frac{a_8\,\sqrt{2\,\delta}\,\Delta_K}{3\,\lambda\,d^3\,\tau^2}\right) \\ &\leqslant \left(\mathbf{E}\,\exp\left(\frac{a_8\,\Delta(X^*, Y)}{\lambda\,d^3\,\tau^2}\right)\mathbf{E}\,\exp\left(\frac{a_8\left(1-\sqrt{1-\delta}\right)\Delta_Y}{\lambda\,d^3\,\tau^2}\right)\mathbf{E}\,\exp\left(\frac{a_8\,\sqrt{2\,\delta}\,\Delta_K}{\lambda\,d^3\,\tau^2}\right)\right)^{1/3}. \end{aligned} \tag{111}$$

It can be easily verified that $\mathbf{E}\,\exp\left(\frac{a_8\left(1-\sqrt{1-\delta}\right)\Delta_Y}{\lambda\,d^3\,\tau^2}\right)$ and $\mathbf{E}\,\exp\left(\frac{a_8\,\sqrt{2\,\delta}\,\Delta_K}{\lambda\,d^3\,\tau^2}\right)$ tend to unit as $\delta \to 0$. Therefore, the statement of Theorem 2 can be easily derived from (106), (111) if we take δ to be sufficiently small.

14. Proof of Theorems 1, 3–5

Proof of Theorem 3. It is evident that $0 \leqslant M \leqslant N$ (see (0.11)) and the random vectors $\widetilde{\xi}_1, \ldots, \widetilde{\xi}_{2^{N-M}}$ satisfy the conditions of Theorem 2 with the change of n by 2^{N-M} and τ by τ_3 (one should use Lemma 1.6, the completeness of $\mathcal{A}_d(\tau)$ with respect to convolution and the monotonicity of $\mathcal{A}_d(\tau)$ with respect to τ). Therefore, one can construct on a probability space a sequence of independent random

vectors $\widetilde{X}_1, \dots, \widetilde{X}_{2^{N-M}}$ and a sequence of independent Gaussian random vectors $\widetilde{Y}_1, \dots, \widetilde{Y}_{2^{N-M}}$ so that $\mathcal{L}(\widetilde{X}_m) = \mathcal{L}(\widetilde{\xi}_m)$, $\mathcal{L}(\widetilde{Y}_m) = \mathcal{N}(0, I)$, $m = 1, \dots, 2^{N-M}$, and

$$\mathbf{E} \exp\Big(\frac{c_3(\alpha)\, \Delta(\widetilde{X}, \widetilde{Y})}{\lambda d^3 \tau_3^2} \Big) \leqslant \exp\Big(c_4(\alpha)\, (\lambda d^{3/2} \tau_3)^{3/2+\alpha} \log^+(2^{N-M}) \Big), \tag{1}$$

where $c_3(\alpha)$, $c_4(\alpha)$ are quantities depending only on α from Theorem 2. It is clear that we can take

$$c_3(\alpha) \leqslant {}^1\!/_2. \tag{2}$$

Using the Berkes–Philipp lemma [3, Lemma A1, p. 53] and (0.12), we can suppose without loss of generality that

$$\begin{aligned} \widetilde{X}_m &= 2^{-M/2} \big(X_{(m-1)\cdot 2^M + 1} + \dots + X_{m \cdot 2^M} \big), \\ \widetilde{Y}_m &= 2^{-M/2} \big(Y_{(m-1)\cdot 2^M + 1} + \dots + Y_{m \cdot 2^M} \big), \end{aligned} \qquad m = 1, \dots, 2^{N-M}, \tag{3}$$

where $\big\{ X_1, \dots, X_{2^N} \big\}$ and $\big\{ Y_1, \dots, Y_{2^N} \big\}$ are collections of mutually independent random vectors with distributions $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$, $\mathcal{L}(Y_k) = \mathcal{N}(0, I)$, $k = 1, \dots, 2^N$. Denote

$$\Delta_1 = \max_{0 \leqslant k < 2^{N-M}} \max_{1 \leqslant r \leqslant 2^M} \Big| \big(X_{k \cdot 2^M + 1} - Y_{k \cdot 2^M + 1} \big) + \dots + \big(X_{k \cdot 2^M + r} - Y_{k \cdot 2^M + r} \big) \Big|, \tag{4}$$

$$\Delta_2 = \max_{1 \leqslant k \leqslant 2^{N-M}} \Big| \big(X_1 - Y_1 \big) + \dots + \big(X_{k \cdot 2^M} - Y_{k \cdot 2^M} \big) \Big|. \tag{5}$$

It is evident that from (4), (5), (0.1) it follows that

$$\Delta(X, Y) \leqslant \Delta_1 + \Delta_2. \tag{6}$$

Besides, the relations (3), (5), (0.3) imply

$$\Delta_2 = 2^{M/2}\, \Delta(\widetilde{X}, \widetilde{Y}). \tag{7}$$

Using (6), (7) and the Hölder inequality we obtain

$$\begin{aligned} \mathbf{E} \exp\Big(\frac{c_3(\alpha)\, \Delta(X, Y)}{2 \lambda d^3 \tau_3^2 \cdot 2^{M/2}} \Big) &\leqslant \mathbf{E} \exp\Big(\frac{c_3(\alpha)\, \Delta_1}{2 \lambda d^3 \tau_3^2 \cdot 2^{M/2}} + \frac{c_3(\alpha)\, \Delta_2}{2 \lambda d^3 \tau_3^2 \cdot 2^{M/2}} \Big) \\ &\leqslant \Big(\mathbf{E} \exp\Big(\frac{c_3(\alpha)\, \Delta_1}{\lambda d^3 \tau_3^2 \cdot 2^{M/2}} \Big)\, \mathbf{E} \exp\Big(\frac{c_3(\alpha)\, \Delta(\widetilde{X}, \widetilde{Y})}{\lambda d^3 \tau_3^2} \Big) \Big)^{1/2}. \end{aligned} \tag{8}$$

From (2) and from the conditions of Theorem 3 it follows that

$$\frac{c_3(\alpha) \cdot \tau_1}{\lambda d^3 \tau_3^2 \cdot 2^{M/2}} \leqslant \frac{c_3(\alpha) \cdot \tau_1}{\lambda d^3 \tau_2^2 \cdot 2^{M/2}} \leqslant \frac{c_3(\alpha)}{\lambda d^3 \tau_2} \leqslant {}^1\!/_2 < 1. \tag{9}$$

Using (4), (9), the completeness of $\mathcal{A}_d(\tau)$ with respect to convolution and Lemma 1.6, we see that for the estimation of the right-hand side of (8) we can use Lemma 1.10 with $\zeta = \Delta_1$, $\tau = \tau_1$, $h = \frac{c_3(\alpha)}{\lambda d^3 \tau_3^2 \cdot 2^{M/2}}$, $r = 2^N d$, $\sigma^2 = 2^{M+1}$. By Lemma 1.10 and by (2), we have

$$\mathbf{E} \exp\Big(\frac{c_3(\alpha)\, \Delta_1}{\lambda d^3 \tau_3^2 \cdot 2^{M/2}} \Big) \leqslant 2^{N+1} d \exp\Big(\frac{c_3^2(\alpha) \cdot 2^{M+1}}{\lambda^2 d^6 \tau_3^4 \cdot 2^M} \Big) \leqslant 2^{N+1} d\, e^{1/2}. \tag{10}$$

Now the statement of Theorem 3 follows from (0.11), (1), (8), (10).

For the proof of Theorem 1 we need the following Lemma 14.1.

Lemma 14.1. *Let* $F = \mathcal{L}(\xi) \in \mathcal{A}_d(1)$, $\mathbf{E}\,\xi = 0$ *and* $\operatorname{cov} \xi = I$. *Then there exists a random vector* η *and absolute positive constants* $c_{66}, c_{67} \geqslant 1$ *such that*

$$\begin{gathered} \mathbf{E}\,\eta = \mathbf{E}\,\xi = 0, \qquad \operatorname{cov} \eta = \operatorname{cov} \xi = I, \\ \mathbf{P}\big\{ |\eta| \leqslant c_{66}\,(1 + \log d) \big\} = 1, \qquad \mathcal{L}(\eta) \in \mathcal{A}_d(c_{67}), \end{gathered} \tag{11}$$

and, for all $u, v, w \in \mathbb{R}^d$,

$$\mathbf{E} \langle \eta, u \rangle \langle \eta, v \rangle \langle \eta, w \rangle = \mathbf{E} \langle \xi, u \rangle \langle \xi, v \rangle \langle \xi, w \rangle. \tag{12}$$

The following Lemma 14.2 (due to Yurinskii [28]) will be used in the proof of Lemma 14.1.

Lemma 14.2 (see [23, Lemma 3, p. 84][4])**.** *Let* $0 < \alpha < 1$ *and let the random variable* μ *take values* $\frac{1}{2\alpha}\big(1 \pm \big(1 + 4\alpha^3\big)^{1/2}\big)$ *with probabilities* $\frac{1}{2} \mp \frac{1}{2\big(1 + 4\alpha^3\big)^{1/2}}$. *Then* $\mathbf{E}\,\mu = 0$, $\mathbf{E}\,\mu^2 = \alpha$, $\mathbf{E}\,\mu^3 = 1$.

Proof of Lemma 14.1. Denote

$$R = 4 c_{68}\,(1 + \log d), \tag{13}$$

where an absolute positive constant c_{68} is so large as it will be necessary for the arguments below. Define the "cubic layers" A_j by

$$A_j = \big\{ x \in \mathbb{R}^d : R(j-1) \leqslant |x| < Rj \big\}, \qquad j = 1, 2, \ldots, \tag{14}$$

and represent the distribution F as the mixture

$$F = \sum_{j=1}^{\infty} q_j\, G_j, \tag{15}$$

where

$$q_j = F\big\{ A_j \big\}, \qquad G_j \in \mathfrak{F}_d, \qquad G_j\big\{ A_j \big\} = 1, \qquad j = 1, 2, \ldots. \tag{16}$$

[4]The author is grateful to B. A. Zalesskii who attracted the author's attention to the book [23].

Below we shall denote by ξ_j the random vectors with distributions

$$\mathcal{L}(\xi_j) = G_j, \qquad j = 1, 2, \ldots. \tag{17}$$

From Lemmas 1.6, 1.9 it follows that, for $x \geqslant 1$,

$$\mathbf{P}\left\{ |\xi| \geqslant x \right\} \leqslant 2d \exp\Big(-\frac{x}{4}\Big). \tag{18}$$

Using (13), (14), (16), (18), we obtain

$$\begin{aligned} q_j \leqslant \mathbf{P}\left\{ |\xi| \geqslant R(j-1) \right\} \leqslant 2d \exp\Big(-\frac{R(j-1)}{4}\Big) &= \frac{2d}{(e^{c_{68}}\, d^{c_{68}})^{j-1}} \\ &< \frac{1}{d^3 j^3 R^2 (4d)^j}, \qquad j = 2, 3, \ldots, \end{aligned} \tag{19}$$

if c_{68} is sufficiently large.

Below we define a transform of d-dimensional distributions which will be denoted for $H = \mathcal{L}(\zeta) \in \mathfrak{F}_d$, $r > 0$ by $H^{(r)}$. Suppose that μ is a random variable which is independent of ζ and has the distribution described in Lemma 14.1 with $\alpha = 1 - \frac{1}{8d}$. According to Lemma 14.1, we have

$$\mathbf{E}\,\mu = 0, \qquad \mathbf{E}\,\mu^2 = 1 - \frac{1}{8d}, \qquad \mathbf{E}\,\mu^3 = 1, \qquad \mathbf{P}\left\{ |\mu| \leqslant 2 \right\} = 1. \tag{20}$$

Define

$$s = q_1 + \sum_{j=2}^{\infty} d^3 j^3 q_j. \tag{21}$$

The quantity s can be estimated with the help of (13), (15), (16), (19), (21) as follows:

$$1 = \sum_{j=1}^{\infty} q_j \leqslant s \leqslant 1 + \sum_{j=2}^{\infty} d^3 j^3 q_j < \sum_{j=2}^{\infty} \frac{1}{(4d)^j} < 1 + \frac{1}{8d}. \tag{22}$$

Below we shall use the notation $H^{(r)}$ for

$$H^{(r)} = \mathcal{L}\big(s^{1/3} \mu\, r^{-1} \zeta\big). \tag{23}$$

Consider the probability measure

$$F_1 = s^{-1} \left(q_1 G_1^{(1)} + \sum_{j=1}^{\infty} d^3 j^3 q_j\, G_j^{(dj)} \right). \tag{24}$$

Introduce the random vector η_1 with $\mathcal{L}(\eta_1) = F_1$. The relations (15), (17), (20), (23), (24) imply

$$\mathbf{E}\,\eta_1 = \int_{\mathbb{R}^d} x\,F_1\{dx\} = s^{-1}\biggl(\,q_1\,s^{1/3}\,\mathbf{E}\,\mu\,\mathbf{E}\,\xi_1 + \sum_{j=1}^{\infty} d^2 j^{\,2} q_j\,s^{1/3}\,\mathbf{E}\,\mu\,\mathbf{E}\,\xi_j\biggr) = 0 \quad (25)$$

and, for all $u, v, w \in \mathbb{R}^d$,

$$\begin{aligned}\mathbf{E}\,\bigl\langle\eta_1, u\bigr\rangle\bigl\langle\eta_1, v\bigr\rangle\bigl\langle\eta_1, w\bigr\rangle &= \int_{\mathbb{R}^d}\bigl\langle x, u\bigr\rangle\bigl\langle x, v\bigr\rangle\bigl\langle x, w\bigr\rangle\,F_1\{dx\}\\ &= s^{-1}\sum_{j=1}^{\infty} d^3 j^3\,q_j\,\bigl(s^{1/3} d^{-1} j^{-1}\bigr)^3\,\mathbf{E}\,\mu^3\,\mathbf{E}\,\bigl\langle\xi_j, u\bigr\rangle\bigl\langle\xi_j, v\bigr\rangle\bigl\langle\xi_j, w\bigr\rangle\\ &= \sum_{j=1}^{\infty} q_j\;\mathbf{E}\,\bigl\langle\xi_j, u\bigr\rangle\bigl\langle\xi_j, v\bigr\rangle\bigl\langle\xi_j, w\bigr\rangle\\ &= \int_{\mathbb{R}^d}\bigl\langle x, u\bigr\rangle\bigl\langle x, v\bigr\rangle\bigl\langle x, w\bigr\rangle\,F\{dx\}. \qquad (26)\end{aligned}$$

Besides, from (13), (14), (16), (20), (22)–(24) it follows that

$$\mathbf{P}\,\bigl\{\,|\eta_1| > 3\,R\,\bigr\} = F_1\Bigl\{\,\bigl\{\,x : |x| > 3\,R\,\bigr\}\,\Bigr\} = 0. \qquad (27)$$

Below we shall denote by t an arbitrary vector $t \in \mathbb{R}^d$ with $\|t\| = 1$. Using (15)–(17) and taking into account that $\operatorname{cov} F = \operatorname{cov}\xi = I$, we see that

$$q_1\,\mathbf{E}\,\bigl\langle\xi_1, t\bigr\rangle^2 = \int_{\mathbb{R}^d}\bigl\langle x, t\bigr\rangle^2\,q_1\,G_1\{dx\} \leqslant \int_{\mathbb{R}^d}\bigl\langle x, t\bigr\rangle^2\,F\{dx\} = \mathbf{E}\,\bigl\langle\xi, t\bigr\rangle^2 = 1. \qquad (28)$$

The relations (15), (17), (20), (22)–(24), (28) imply

$$\begin{aligned}\mathbf{E}\,\bigl\langle\eta_1, t\bigr\rangle^2 &= s^{-1}\biggl(\,q_1\,s^{2/3}\,\mathbf{E}\,\mu^2\,\mathbf{E}\,\bigl\langle\xi_1, t\bigr\rangle^2 + \sum_{j=2}^{\infty} d^3 j^3 q_j\,\bigl(s^{1/3} d^{-1} j^{-1}\bigr)^2\,\mathbf{E}\,\mu^2\,\mathbf{E}\,\bigl\langle\xi_j, t\bigr\rangle^2\biggr)\\ &= s^{-1/3}\,\mathbf{E}\,\mu^2\biggl(\,q_1\,\mathbf{E}\,\bigl\langle\xi_1, t\bigr\rangle^2 + \sum_{j=2}^{\infty} d\,j\,q_j\,\mathbf{E}\,\bigl\langle\xi_j, t\bigr\rangle^2\biggr)\\ &\geqslant s^{-1}\,\mathbf{E}\,\mu^2\sum_{j=1}^{\infty} q_j\;\mathbf{E}\,\bigl\langle\xi_j, t\bigr\rangle^2 = s^{-1}\,\mathbf{E}\,\mu^2\;\mathbf{E}\,\bigl\langle\xi, t\bigr\rangle^2 \geqslant 1 - \frac{1}{4d}\,. \qquad (29)\end{aligned}$$

According to (14), (16), (17), (19), we have

$$\sum_{j=2}^{\infty} d\,j\,q_j\;\mathbf{E}\,\bigl\langle\xi_j, t\bigr\rangle^2 \leqslant \sum_{j=2}^{\infty} d\,j\,q_j\;\mathbf{E}\,\|\xi_j\|^2 \leqslant d^2 R^2 \sum_{j=2}^{\infty} j^3\,q_j < \sum_{j=2}^{\infty}\frac{1}{(4d)^j} < \frac{1}{8d}\,. \quad (30)$$

From (20), (22), (25), (28)–(30) it follows that

$$\mathbf{E}\,\langle \eta_1, t\rangle^2 \leqslant 1 - \frac{1}{64\,d^2} < 1 \tag{31}$$

and, hence, the maximal eigenvalue of the covariance operator $D_1 = \operatorname{cov}\eta_1$ does not exceed 1. Let $D_2 = I - D_1$. From (25), (29), (31) one can easily derive that

$$0 < \|D_2^{1/2} t\|^2 = \langle D_2\, t, t\rangle = \langle I\, t, t\rangle - \langle D_1\, t, t\rangle = 1 - \mathbf{E}\,\langle \eta_1, t\rangle^2 \leqslant \frac{1}{4\,d}\,. \tag{32}$$

Therefore, D_2 can be considered as covariance operator. Let η_2 be the random vector which is independent of η_1 and

$$\mathcal{L}(\eta_2) = \mathcal{L}\big(\sqrt{2}\,D_2^{1/2}\,K\big), \tag{33}$$

where K is the vector defined before the formula (11.14). From (32) and from the definition of K it follows that

$$\mathbf{E}\,\eta_2 = 0, \qquad \operatorname{cov}\eta_2 = D_2, \qquad \mathbf{P}\,\big\{\,\|\eta_2\| \leqslant 1\,\big\} = 1, \tag{34}$$

and, for all $u, v, w \in \mathbb{R}^d$,

$$\mathbf{E}\,\langle \eta_2, u\rangle\langle \eta_2, v\rangle\langle \eta_2, w\rangle = 0 \tag{35}$$

(we use that the vector K has symmetric distribution).

Now we define the vector η by $\eta = \eta_1 + \eta_2$. The first of the relations (11) follows now from (25), (34). The second one is a simple consequence of (32), (34). The third can be easily derived from (13), (27), (34). Finally, it is evident that the relation (12) follows from (25), (26), (34), (35).

It remains to prove the fourth of the relations (11). For this we shall prove that there exists an absolute positive constant c_{69} such that

$$\mathcal{L}(\eta_1) \in \mathcal{B}_d(c_{69}) \tag{36}$$

(see (1.35)). Let $m \geqslant 3$ be an even integer. Using (15), (17) and Lemma 1.5, item b), we see that, for all $u, v \in \mathbb{R}^d$,

$$q_1\,\mathbf{E}\,\langle \xi_1, v\rangle^2\langle \xi_1, u\rangle^{m-2} \leqslant \mathbf{E}\,\langle \xi, v\rangle^2\langle \xi, u\rangle^{m-2} \leqslant \frac{1}{2}\,m!\,\big(c\,\|u\|\big)^{m-2}\,\mathbf{E}\,\langle \xi, v\rangle^2. \tag{37}$$

Using (13)–(17), (20), (22)–(24), (28), (29), (37), we obtain that, for all $u, v \in \mathbb{R}^d$,

$$\begin{aligned}
\Big|\mathbf{E}\,\langle \eta_1, v\rangle^2\langle \eta_1, u\rangle^{m-2}\Big| &= s^{-1}\Bigg(\,q_1\,s^{m/3}\,\mathbf{E}\,\mu^m\,\mathbf{E}\,\langle \xi_1, v\rangle^2\langle \xi_1, u\rangle^{m-2} \\
&\qquad + \sum_{j=2}^{\infty} d^3 j^3 q_j\,\big(s^{1/3} d^{-1} j^{-1}\big)^m\,\mathbf{E}\,\mu^m\,\mathbf{E}\,\langle \xi_j, v\rangle^2\langle \xi_j, u\rangle^{m-2}\Bigg) \\
&\leqslant m!\,\big(c\,\|u\|\big)^{m-2}\mathbf{E}\,\langle \xi, v\rangle^2 + \big(c d^{-1/2} R\big)^m\,\|v\|^2\,\|u\|^{m-2}\sum_{j=2}^{\infty} d^3 j^3 q_j \\
&\leqslant \frac{1}{2}\,m!\,\big(c\,\|u\|\big)^{m-2}\,\mathbf{E}\,\langle \eta_1, v\rangle^2.
\end{aligned} \tag{38}$$

The analogous inequality for odd m can be easily derived from the case of even m with the help of the Hölder inequality. Together with (25), (1.35) this implies the relation (36). From (34), (36) and from Lemma 1.5 it follows that $\mathcal{L}(\eta_1) \in \mathcal{A}_d(c_{70})$, $\mathcal{L}(\eta_2) \in \mathcal{A}_d(c_{71})$. For the proof of the fourth of the relations (11) it remains to apply the completeness of classes $\mathcal{A}_d(\tau)$ with respect to convolution. This completes the proof of Lemma 14.2.

Proof of Theorem 1. Without loss of generality we can assume that $n = 2^N$, where N is some integer. Let the integer M be defined by the relation

$$\tau^2 \leqslant 2^M < 2\tau^2. \tag{39}$$

Let $M \leqslant N$. Denote

$$\widetilde{\xi}_m = 2^{-M/2}\big(\xi_{(m-1)\cdot 2^M+1} + \cdots + \xi_{m\cdot 2^M}\big), \qquad m = 1, \ldots, 2^{N-M}. \tag{40}$$

From the conditions of Theorem 1 and from the properties of classes $\mathcal{A}_d(\tau)$ it follows that $\mathcal{L}(\widetilde{\xi}_m) \in \mathcal{A}_d(1)$, $\mathbf{E}\,\widetilde{\xi}_m = 0$ and $\operatorname{cov}\widetilde{\xi}_m = I$. According to Lemma 14.1, there exist random vectors $\widetilde{\eta}_m$ satisfying the relation

$$\mathbf{P}\left\{|\widetilde{\eta}_m| \leqslant \rho\right\} = 1, \qquad \rho = c_{66}\,(1 + \log d), \qquad \mathcal{L}(\widetilde{\eta}_m) \in \mathcal{A}_d(c_{67}), \quad c_{67} \geqslant 1, \tag{41}$$

and having the same moments of the first three orders as $\widetilde{\xi}_m$. Thus, we see that the conditions of Theorem 3 are fulfilled with $\tau_1 = \tau$, $\tau_2 = 1$, $\tau_3 = c_{67}$, $\lambda = \rho$. Therefore, for any $\beta > 0$ one can construct on a probability space a sequence of independent random vectors $X_1, \ldots, X_n$ and a corresponding sequence of independent Gaussian random vectors $Y_1, \ldots, Y_n$ so that $\mathcal{L}(X_k) = \mathcal{L}(\xi_k)$, $k = 1, \ldots, n$, and

$$\mathbf{E}\,\exp\Big(\frac{c_5(\beta)\,\Delta(X,Y)}{c_{67}^2 d^3\,\tau\,\rho}\Big) \leqslant \exp\Big(c_6(\beta)\,\big(c_{67} d^{3/2}\rho\big)^{3/2+\beta}\log^+ n\Big). \tag{42}$$

The relations (41), (42) imply the statement of Theorem 1 in the case $M \leqslant N$.

Let now $M > N$. Then from (39) it follows that

$$2^N < 2\tau^2. \tag{43}$$

For any method of construction of random vectors $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ with needed distributions (for example, one can take mutually independent vectors), we can apply Lemma 1.10 with $\zeta = \Delta(X,Y)$, $h = \frac{1}{2\tau}$, $r = 2^N d$, $\sigma^2 = 2^{N+1}$. By Lemmas 1.6, 1.10 and by (43),

$$\mathbf{E}\,\exp\Big(\frac{\Delta(X,Y)}{2\tau}\Big) \leqslant 2^{N+1} d\,\exp\Big(\frac{2^{N+1}}{4\tau^2}\Big) \leqslant 2\,n\,d\,e. \tag{44}$$

The inequality (44) implies the statement of Theorem 1 in the case $M > N$.

Proof of Theorem 4. It repeats the proof of Theorem 1. The only difference is in the construction of the vectors $\widetilde{\eta}_m$. Let $n = 2^N$. Assume that the integer $M \leqslant N$ and the vectors $\widetilde{\xi}_m$, $m = 1, \dots, 2^{N-M}$, are defined by (39), (40). From the conditions of Theorem 4 it follows that, for all $u, v, w \in \mathbb{R}^d$,

$$\mathbf{E}\,\langle\widetilde{\xi}_m, u\rangle\langle\widetilde{\xi}_m, v\rangle\langle\widetilde{\xi}_m, w\rangle = 0, \qquad m = 1, \dots, 2^{N-M}. \tag{45}$$

Introduce the random vectors $\widetilde{\eta}_m$, $m = 1, \dots, 2^{N-M}$, with $\mathcal{L}(\widetilde{\eta}_m) = \mathcal{L}\big(\sqrt{2}\,K\big)$, where K is the vector with symmetric distribution defined before the formula (11.14). The random vectors $\widetilde{\eta}_m$ have the same moments of the first three orders as $\widetilde{\xi}_m$, $\mathbf{P}\big\{\,|\widetilde{\eta}_m| \leqslant \sqrt{2}\big\} = 1$, $\mathcal{L}(\widetilde{\eta}_m) \in \mathcal{A}_d\big(\sqrt{2}\,c_{14}\big)$. The last inclusion follows from (11.14) and from the properties of classes $\mathcal{A}_d(\tau)$. Thus, we see that the conditions of Theorem 3 are fulfilled with $\tau_1 = \tau$, $\tau_2 = 1$, $\tau_3 = \sqrt{2}\,c_{14}$, $\lambda = \sqrt{2}$. The application of Theorem 3 completes the proof of Theorem 4 in the case $M \leqslant N$. The case $M > N$ can be considered as in the proof of Theorem 1.

Proof of Theorem 5. Suppose that the distributions $\mathcal{L}(\widetilde{\xi}_k) \in \mathcal{A}_d(\tau)$, $k = 1, \dots, 2^N$, are absolutely continuous. Without loss of generality we can assume that $n = 2^N$, where $N \geqslant 1$ is some integer.

Suppose that the independent random vectors $\widetilde{Y}_k$, $k = 1, \dots, 2^N$, with distributions $\mathcal{L}(\widetilde{Y}_k) = \Phi\big(\mathcal{L}(\widetilde{\xi}_k)\big) = \mathcal{N}(0, I)$ are already constructed. Our aim is to construct independent random vectors $\widetilde{X}_k$, $k = 1, \dots, 2^N$, with $\mathcal{L}(\widetilde{X}_k) = \mathcal{L}(\widetilde{\xi}_k)$ so that the sequential sums of $\widetilde{X}_k$ and $\widetilde{Y}_k$ would be close with large probability. For this we define the random vectors $\big\{Y_k^{[N-l]}, Z_k^{[N-l]},\ 1 \leqslant k \leqslant 2^{N-l},\ 0 \leqslant l \leqslant N-1\big\}$ with the help of a recurrent procedure which is similar to the procedure described in Section 13. The only difference is in the change of $\Psi\big(\mathcal{L}(\widetilde{\eta}_k)\big)$ by $\Phi\big(\mathcal{L}(\widetilde{\xi}_k)\big)$ and in other values of parameters: $p = 1$ and $M = 0$ (we do not assume the fulfilment of the relations (13.4)–(13.7)). This means that after each step of the procedure the number of vectors $\widetilde{X}_k$ which are not constructed becomes smaller in 2 times. Besides, in each step, each second of the vectors being constructed has standard normal distribution $\mathcal{N}(0, I)$.

After the last step all independent random vectors $\widetilde{X}_k$, $k = 1, \dots, 2^N$, are constructed except $\widetilde{X}_1$. Instead of this we have constructed the vector $Z_1^{[1]}$ with distributions $\mathcal{L}\big(Z_1^{[1]}\big) = \Phi\big(\mathcal{L}(\widetilde{\xi}_1)\big)$. Together with $\widetilde{X}_k$, $k = 2, \dots, 2^N$, this vector form a collection of mutually independent vectors. Obviously, we can construct the vector $\widetilde{X}_1$ (independent of $\widetilde{X}_k$, $k = 2, \dots, 2^N$, and with $\mathcal{L}(\widetilde{X}_1) = \mathcal{L}(\widetilde{\xi}_1)$) with the help of the quantile transformation of $Z_1^{[1]}$.

This construction is close to the version of Sakhanenko's construction [21] which was used by Einmahl in [9].

Similarly to Section 13, we denote

$$\begin{aligned} Y_k &= Y_k^{[N]} = \widetilde{Y}_k, \qquad & X_k &= Z_k^{[N]}, \qquad & k &= 1, \dots, 2^N, \\ Y_{*,k} &= Y_k^{[N-1]} = Z_{2k-1}^{[N]}, \qquad & X_{*,k} &= \widetilde{X}_{2k-1}, \qquad & k &= 1, \dots, 2^{N-1}. \end{aligned} \tag{46}$$

and fix the index j, $j = 1, \dots, 2^N$. Let i be the largest integer such that $2i-1 \leqslant j$. Let the quantities $\delta_j(N)$, $\delta_i(N-1)$, $\psi_j(N)$ be defined by (13.28) (with $p = 1$). In the same way as in Section 13 one can obtain the following analog of the inequality (13.30):

$$\delta_j(N) \leqslant \psi_j(N) + \delta_i(N-1). \tag{47}$$

By analogy with [9], we shall prove that the procedure described above gives good approximation, using induction with respect to N. It is clear that the procedure of constructing the independent random vectors $X_{*,k}$, $k = 1, \dots, 2^{N-1}$, coincides with the procedure of constructing of $\widetilde{X}_k$, $k = 1, \dots, 2^N$. The only difference is in the change of N by $N-1$. Define

$$\begin{aligned} \Gamma_N &= \max_{1 \leqslant m \leqslant 2^N} \delta_m(N) = \Delta(\widetilde{Y}, \widetilde{X}), \\ \Gamma_{N-1} &= \max_{1 \leqslant m \leqslant 2^{N-1}} \delta_m(N-1) = \Delta(Y_*, X_*), \\ \Gamma_0 &= \delta_1(0) = \bigl| Z_1^{[1]} - \widetilde{X}_1 \bigr|. \end{aligned} \tag{48}$$

From (13.28), (47), (48) it follows that

$$\Gamma_N \leqslant \Delta\bigl(\widetilde{Y}, Z^{[N]}\bigr) + \Gamma_{N-1}. \tag{49}$$

Now we note that the vectors $\widetilde{Y}_k$, $Z_k^{[N]}$ satisfy the conditions which are fulfilled for the vectors Y_k, X_k, $k = 1, \dots, 2^N$, in Section 11 (under condition A2). Therefore, according to (11.54), (11.90), (11.94),

$$\mathbf{E} \exp\Bigl(\varepsilon\, \Delta\bigl(\widetilde{Y}, Z^{[N]}\bigr)\Bigr) \leqslant (31\,d)^N \qquad \text{for} \quad \varepsilon = \frac{1}{2\, c_{41}\, d^{3/2}\, \tau}\,. \tag{50}$$

Using induction with respect to N, we shall prove that

$$\mathbf{E} \exp\Bigl(\frac{\varepsilon\, \Gamma_N}{N+1}\Bigr) \leqslant (31\,d)^{N+1}, \qquad N = 0, 1, \dots. \tag{51}$$

In the case $N = 0$, taking into account (11.90), we can apply the Hölder inequality and Lemma 1.10 with $\zeta = \Gamma_0$, $r = d$, $\sigma^2 = 2$, $h = \varepsilon$ and obtain

$$\mathbf{E} \exp\bigl(\varepsilon\, \Gamma_0\bigr) \leqslant 2\, d\, e^{2\varepsilon^2} \leqslant 3d. \tag{52}$$

Suppose that the inequality (51) is valid with the change of N by $N-1$. Then, using the Hölder inequality, (49), (50) and the induction hypothesis, we have

$$\begin{aligned}\mathbf{E}\,\exp\Big(\frac{\varepsilon\,\Gamma_N}{N+1}\Big) &\leqslant \mathbf{E}\,\exp\Big(\frac{\varepsilon\,\Delta\big(\widetilde{Y},Z^{[N]}\big)+\varepsilon\,\Gamma_{N-1}}{N+1}\Big)\\ &\leqslant \Big(\mathbf{E}\,\exp\big(\varepsilon\,\Delta\big(\widetilde{Y},Z^{[N]}\big)\big)\Big)^{1/(N+1)}\Big(\mathbf{E}\,\exp\Big(\frac{\varepsilon\,\Gamma_{N-1}}{N}\Big)\Big)^{N/(N+1)}\\ &\leqslant (31\,d)^N < (31\,d)^{N+1}. \qquad (53)\end{aligned}$$

This completes the proof of Theorem 5 for the case when $\widetilde{\xi}_k$ have absolutely continuous distributions. The general case can be analyzed in the same way as in the proof of Theorem 2.

Acknowledgment. The main part of this research was done in Bielefeld University. I would like to thank Professor Friedrich Götze for his hospitality during my stay in Bielefeld. I am grateful to V. Bentkus, F. Götze, A. I. Sakhanenko and B. A. Zalesskii for useful discussions.

References

1. P. Bártfai, *Die Bestimmung der zu einem wiederkehrenden Prozess gehörenden Verteilungfunktion aus den mit Fehlern behafteten Daten einer einzigen Realisation*, Studia Sci. Math. Hungar. **1** (1966), 161–168.
2. E. Berger, *Fast sichere Approximation von Partialsummen unabhängiger und stationärer ergodischer Folgen von Zufallsvectoren*, Dissertation (1982), Universität Göttingen.
3. I. Berkes and W. Philipp, *Approximation theorems for independent and weakly dependent random vectors*, Ann. Probab. (1979), 29–54.
4. A. A. Borovkov, *On the rate of convergence in the invariance principle*, Theor. Probab. Appl. **18** (1973), no. 2, 207–225.
5. M. Csőrgő, and P. Révész, *Strong approximations in probability and statistics*, Academic Press, New York, 1981.
6. S. Csőrgő, and P. Hall, *The Komlós–Major–Tusnády approximations and their applications*, Austral. J. Statist. **26** (1984), no. 2, 189–218.
7. U. Einmahl, *A useful estimate in the multidimensional invariance principle*, Probab. Theor. Rel. Fields **76** (1987), no. 1, 81–101.
8. U. Einmahl, *Strong invariance principles for partial sums of independent random vectors*, Ann. Probab. **15** (1987), 1419–1440.
9. U. Einmahl, *Extensions of results of Komlós, Major and Tusnády to the multivariate case*, J. Multivar. Anal. **28** (1989), 20–68.
10. J. Komlós, P. Major, and G. Tusnády, *An approximation of partial sums of independent RV'-s and the sample DF.* I, Z. Wahrscheinlichkeitstheor. verw. Geb. **32** (1975), 111–131.
11. J. Komlós, P. Major, and G. Tusnády, *An approximation of partial sums of independent RV'-s and the sample DF.* II, Z. Wahrscheinlichkeitstheor. verw. Geb. **34** (1976), 34–58.
12. P. Massart, *Strong approximation for multivariate empirical and related processes, via KMT construction*, Ann. Probab. **17** (1989), no. 1, 266–291.
13. V. V. Petrov, *Sums of independent random variables*, Springer, Berlin–Heidelberg–New York, 1975.

14. W. Philipp, *Almost sure invariance principles for sums of B-valued random variables*, Lect. Notes in Math. **709** (1979), 171–193.
15. Yu. V. Prokhorov, *Convergence of random processes and limit theorem of probability theory*, Theor. Probab. Appl. **1** (1956), no. 2, 157–214.
16. Qi-Man Shao, *Strong approximation theorems for independent random variables and their applications*, J. Multivar. Anal. **52** (1995), no. 1, 107–130.
17. E. Rio, *Strong approximation for set-indexed partial-sum processes via KMT construction.* I., Ann. Probab. **21** (1993), no. 2, 759–790.
18. E. Rio, *Strong approximation for set-indexed partial-sum processes via KMT construction.* II., Ann. Probab. **21** (1993), no. 3, 1706–1727.
19. M. Rosenblatt, *Remarks on a multivariate transformation*, Ann. Math. Statist. **23** (1952), 470–472.
20. R. Rudzkis, *Probabilities of large deviations of random vectors*, Lithuanian Math. J. **23** (1983), no. 1, 113–120.
21. A. I. Sakhanenko, *Rate of convergence in the invariance principles for variables with exponential moments that are not identically distributed*, In: Trudy Inst. Mat. SO AN SSSR, vol. 3, Nauka, Novosibirsk, 1984, pp. 4–49. (Russian)
22. L. Saulis, *Large deviations for random vectors for certain classes of sets.* I, Lithuanian Math. J. **23** (1983), no. 3, 308–317.
23. V. V. Sazonov, *Normal approximation – some recent advances*, Lect. Notes in Math. **879** (1981).
24. A. V. Skorokhod, *Studies in the theory of random processes*, Univ. Kiev, Kiev, 1961 (Russian); Engl. transl., Addison–Wesley, Reading, Mass., 1965.
25. V. A. Statulevičius, *On large deviations*, Z. Wahrscheinlichkeitstheor. verw. Geb. **6** (1966), no. 2, 133–144.
26. V. Strassen, *An invariance principle for the law of iterated logarithm*, Z. Wahrscheinlichkeitstheor. verw. Geb. **3** (1964), 211–226.
27. V. V. Yurinskii, *On approximation of convolutions by normal laws*, Theor. Probab. Appl. **22** (1977), no. 4, 653–667.
28. V. V. Yurinskii, *On the error of the Gaussian approximation to the probability of a ball*, preprint.
29. A. Yu. Zaitsev, *On the Gaussian approximation of convolutions under satisfaction of multidimensional analogs of the Bernstein inequality conditions*, LOMI Preprint P-9-84, Leningrad (1984), 1–54. (Russian)
30. A. Yu. Zaitsev, *Estimates of the Lévy–Prokhorov distance in the multivariate central limit theorem for random variables with finite exponential moments*, Theor. Probab. Appl. **31** (1986), no. 2, 203–220.
31. A. Yu. Zaitsev, *On the Gaussian approximation of convolutions under multidimensional analogues of S. N. Bernstein inequality conditions*, Probab. Theor. Rel. Fields **74** (1987), no. 4, 535–566.
32. A. Yu. Zaitsev, *On the connection between two classes of probability distributions*, In: Rings and modulus. Limit theorems of probability theory. Vol. 2, Leningrad University Press, Leningrad, 1988, pp. 153–158. (Russian)
33. A. Yu. Zaitsev, *An improvement of U. Einmahl estimate in the multidimensional invariance principle*, In: Proceedings of the Kolmogorov semester on probability theory and mathematical statistics, Gordon and Breach, 1995, pp. 109–116 (to appear).

Andrei Zaitsev

St. Petersburg Branch of Steklov Mathematical Institute,

Fontanka 27, St. Petersburg 191011, Russia

E-mail address: zaitsev@pdmi.ras.ru