VARIETIES OF DISCRETE REPRESENTATIONS OF HYPERBOLIC 3-LATTICES

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ABSTRACT. We survey global properties of the varieties of conjugacy classes of discrete faithful representations of the fundamental groups of closed (or finite volume) hyperbolic 3-manifolds M in linear groups $O(1,m), m \geq 3$, especially for $m=4$ and $m=5$ and their relation to Teichmüller spaces of geometric structures. Of special interest are questions of connectedness of these spaces, cohomology groups $H^1(\pi_1(M), o(1,m))$ (infinitesimal deformations), smooth paths in the varieties and a link with Platonov's conjecture on arithmeticity and super-rigidity. For the last problem, we find a Coxeter group which is non-arithmetic but "hyperbolically super-rigid". Also, in contrast to Teichmüller spaces of hyperbolic surfaces, we show non-connectedness of varieties $\mathcal{T}(\pi_1(M), m), m = 4, 5$. For $m = 4$, this effect is related to an equivariant and everywhere wild embedding of a closed 3-ball into 3-sphere S^3. For $m = 5$, such phenomenon has been discovered by a recent author's construction of "twisted" 3-lattice representations in $O(1,5)$. It uses so called block-building method and a topological background related to 2-knots in the 4-sphere. This has also a relation to first constructions of non-trivial circle and 2-plane bundles (with geometric structures) over a closed hyperbolic 3-manifold and wild (even locally wild) quasi-symmetric 2-knots in the 4-sphere.

1. Varieties of group representations

Let $\Gamma = \langle g_1, \dots, g_d | s_\alpha, \alpha \in A\rangle$ be a finitely generated torsion-free group, $\mathbf{H}$ be a classical simple algebraic group defined over $\mathbb{R}$ with real points H and $\rho_l : \Gamma \to H$ be a fixed embedding of Γ as a lattice in H. If H is not locally isomorphic to $PSL_2(\mathbb{R})$ then Γ is rigid by the Mostow-Margulis rigidity theorem [**M, MA**]. In addition, we suppose that $\mathbf{H}$ is represented in another algebraic group $\mathbf{G}$, $\rho_a : \mathbf{H} \to \mathbf{G}$, also defined over $\mathbb{R}$ where the image of $\mathbf{H}$ is not contained in a proper parabolic subgroup of $\mathbf{G}$. Then the composition of embeddings $\rho_0 = \rho_a \rho_l$ embeds Γ into G as a discrete subgroup (usually no longer a lattice). The main problem is to study the deformation space $\mathcal{R}_G(\Gamma) = \mathrm{Hom}(\Gamma, G)$ (with the algebraic convergence topology) which has the structure of an affine algebraic variety over $\mathbb{R}$ (obtained by embedding it in a product of d copies of G by sending the representation ρ to the d–tuple $(\rho(g_i))_{i=1,\dots,d}$). Its geometric properties reflect the relationships among the same

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degree representations of Γ. Of most interest is the quotient $\mathcal{S}_G(\Gamma) = R_G(\Gamma)/G$ of this variety by the conjugation action of G. The points of $\mathcal{S}_G(\Gamma)$ correspond to isomorphism classes of simple representations of Γ. A curve in $\mathcal{S}_G(\Gamma)$ through the (isomorphism class $[\rho]$ of) simple representation ρ can be thought of as a deformation of ρ. For a general introduction to representation varieties, see [**LM**].

As applications of representations varieties (see [**GM**]), it is possible to consider a space of locally homogeneous structures on the original locally symmetric space $M = \Gamma\backslash H/K$ where K is a maximal compact subgroup of H. Of most interest are the groups either $SO(1,n)$ or $SU(1,n)$ or $PGL_n(\mathbb{R})$ because the following observation ([**JM**]):

Theorem 1.1. *The representation ρ is locally rigid in G, i.e. its orbit is open in* $\mathrm{Hom}(\Gamma, G)$ *unless H is locally isomorphic to $SO(1,n)$ or $SU(1,n)$.*

It makes sense to consider Γ as a lattice in n-dimensional hyperbolic space (either real or complex). Due to the fact that the case of complex hyperbolic structures is much more rigid than the real one ([**GMI**]), our basic object will be the fundamental group $\Gamma = \pi_1(M)$ of a closed (or finite volume) hyperbolic n-manifold $M = \mathbb{H}^n/\Gamma$, or a hyperbolic n-lattice $\Gamma \subset \mathrm{O}^\circ(1,n) = H, \quad n \geq 3$.

2. Representations of hyperbolic lattices and geometric structures

Now we assume that Γ is a hyperbolic lattice (in a semi-simple Lie group $H = \mathrm{O}^\circ(1,n)$ of real rank 1, $n \geq 3$) and we have its representations in another semi-simple Lie group G (especially, $G = \mathrm{O}^\circ(1,m)$, $m \geq 4$).

Inside of $\mathcal{R}_m(\Gamma) = \mathcal{R}_{\mathrm{O}^\circ(1,m)}(\Gamma)$ we have two sequential subspaces of faithful representations with discrete (correspondingly, discontinuous somewhere at $\partial\mathbb{H}^m = S^{m-1}$) images. $\mathrm{O}^\circ(1,m)$ acts on $\mathcal{R}_m(\Gamma)$ by conjugation leaving both subspaces invariant. We denote their quotient spaces as

$$\mathcal{H}_m(\Gamma), \quad \mathcal{T}_{m-1}(\Gamma) \subset \mathcal{R}_m(\Gamma)/\,\mathrm{O}^\circ(1,m), \tag{2.1}$$

correspondingly.

There is a natural identification of the biggest space $\mathcal{H}_m(\Gamma)$ with the space of n-dimensional hyperbolic structures on Γ. An element of this space is determined by an equivalence class of pairs $\{N, \phi\}$ where N is an m-dimensional hyperbolic manifold and $\phi : \Gamma \to \pi_1(N)$ is an isomorphism (up to isometries of manifolds N and inner automorphisms of Γ).

As an example, if $\Gamma = \pi_1(S_g)$ for a closed surface of genus $g > 1$, then $\mathcal{H}_2(\Gamma)$ is the Teichmüller space of Γ and homeomorphic to $\mathbb{R}^{6g-6}$. Another example of $\Gamma = \pi_1(M)$ for a hyperbolic closed n-manifold M provides the space $\mathcal{H}_n(\Gamma)$ which degenerates to a point due to the Mostow-Margulis rigidity ([**M**], [**MA**]):

Theorem 2.1. *Let M_1, M_2 be closed hyperbolic n-manifolds with isomorphic fundamental groups. Then M_1 and M_2 are isometric.*

Similarly, via holonomy, we have a natural identification of the space $\mathcal{T}_{m-1}(\Gamma)$ with the (Teichmüller) space of uniformizable $(m-1)$-dimensional conformal structures on Γ (cf. [**A1**], [**A3**]). An element of this space is determined by a pair

$\{N, \phi\}$, where N is a conformal (=conformally flat) $(m-1)$-manifold with a non-surjective developing map $d : \tilde{N} \to S^{m-1}$ of its universal covering space $\tilde{N}$ into $d(\tilde{N}) = \Omega \subset S^{m-1}$ and $\phi : \Gamma \to \pi_1(N)$ is a monomorphism corresponding to the short exact sequence

$$0 \to \pi_1(\Omega) \to \pi_1(N) \to \Gamma \to 0\,, \tag{2.2}$$

Here, due to [**KA**] and [**KP**], the developing map d and the natural projection $\pi : \Omega \to \Omega/G \cong N$ (where $\Gamma \cong G = d_*(\pi_1(N)) \subset \mathrm{O}^\circ(1,m)$) are covering maps which factor the universal projection $\tilde{N} \to N$. Two pairs $\{N_0, \phi_0\}$ and $\{N_1, \phi_1\}$ determine the same conformal structure on Γ if there is an orientation preserving conformal homeomorphism $f : N_0 \to N_1$ such that $f_* \circ \phi_0$ and ϕ_1 differ (up to the isotropy subgroup $Z(\rho_0)$ of the inclusion $\rho_0 : \Gamma \to \mathrm{O}^\circ(1,m)$) by an inner automorphism of Γ. (Notice that $f_* : \pi_1(N_0) \to \pi_1(N_1)$ is only well defined up to inner automorphism of $\pi_1(N_0)$.)

3. Deformations of representations

The basic object of this survey is the varieties $\mathcal{T}_{m-1}(\Gamma) \subset \mathcal{H}_m(\Gamma)$. The first surprising result (beyond Mostow-Margulis rigidity) is that there are large classes of groups Γ for which these varieties are positive-dimensional. So the Margulis super-rigidity theorem [**M2**] fails for such lattices, and a question of A.Borel and N. Wallach [**BW**] on triviality of the first cohomology group $H^1(\Gamma, o(1,m))$ (i.e., of the tangent space for the variety $\mathcal{T}_{m-1}(\Gamma)$, consisting of infinitesimal deformations) has a negative answer.

Namely, in [**A2**] the author gave a method for constructing nontrivial deformations under some assumptions on lattices Γ. Referring to more later results ([**A4**], [**A1**], [**K**], [**JM**]), some of these non-rigid lattices $\Gamma \subset \mathrm{O}^\circ(1,n)$ might be described as the fundamental groups of hyperbolic n-manifolds M with at least one embedded totally geodesic hypersurface $N \subset M = \mathbb{H}^n/\Gamma$. For any such hypersurface, the construction assigns a real analytic curve in $\mathcal{T}_{m-1}(\Gamma)$ through the representation ρ_0, consisting of so called bending representations of Γ. We shall outline this construction from an algebraic point of view.

Let $\Gamma_N \subset \Gamma$ be the fundamental group of the hypersurface $N \subset M$. Then Γ_N is a lattice in $\mathrm{O}^\circ(1,n-1) \subset \mathrm{O}^\circ(1,n)$ because N is totally geodesic. It shows that the center $\mathbb{Z}(\Gamma_N, \mathrm{O}^\circ(1,m))$, $\quad m > n$, is not trivial. Now, due to Seifert-Van Kampen theorem, Γ is represented as either a free product with Γ_N as the amalgamated subgroup or a HNN-extension, correspondingly to the property of the complement $M \backslash N$ either to have two connected components M_0 and M_1 or to be connected ($M \backslash N = M_0$). Let us denote corresponding subgroups of Γ (the fundamental groups of M_i) by Γ_i, $\quad i = 0, 1$.

In the first case of a free amalgamated product, $\Gamma = \Gamma_0 *_{\Gamma_N} \Gamma_1$, we have constructed a non-trivial smooth curve through $[\rho_0] \in \mathcal{T}_{m-1}(\Gamma)$ so that its points correspond to conjugacy classes of the following representations:

$$\rho_\alpha \Gamma \;=\; \Gamma_0 *_{\Gamma_N} g_\alpha \Gamma_1 g_\alpha^{-1}, \tag{3.1}$$

where α, $\;-\epsilon < \alpha < \epsilon$, is the angle of rotation $g_\alpha \in \mathbb{Z}(\Gamma_N, \mathrm{O}^\circ(1,m))$.

In the second case of HNN-extension,

$$\Gamma = \Gamma_0 *_{\Gamma_N} = \langle \Gamma_0, \gamma | \quad \gamma \Gamma_N \gamma^{-1} = \Gamma_{N'} \rangle, \quad \Gamma_{N'} \subset \Gamma_0,$$

we have constructed a non-trivial smooth curve through $[\rho_0] \in \mathcal{T}_{m-1}(\Gamma)$ so that its points correspond to conjugacy classes of the following representations:

$$\rho_\alpha \Gamma \ = \langle \Gamma_0, \gamma_\alpha | \quad \gamma_\alpha \Gamma_N \gamma_\alpha^{-1} = \Gamma_{N'} \rangle, \tag{3.2}$$

where the element $\gamma_\alpha = g_\alpha \gamma (g'_\alpha)^{-1} \in \mathrm{O}^\circ(1, m)$ and the small enough parameter α, $-\epsilon < \alpha < \epsilon$, is the angle of rotations $g_\alpha \in \mathbb{Z}(\Gamma_N, \mathrm{O}^\circ(1, m))$ and $g'_\alpha \in \mathbb{Z}(\Gamma_{N'}, \mathrm{O}^\circ(1, m))$.

However, despite such nontriviality of the representation variety and in contrast to the classical Teichmüller theory for surface group representation, we have the next surprising result, so called the weak rigidity theorem of J. Morgan, P. Shalen and W. Thurston ([**MS**], [**MO**], [**TH**]). Due to this theorem, the space $\mathcal{H}_m(\Gamma)$ of conjugacy classes of faithful discrete representations of an uniform lattice $\Gamma \subset \mathrm{O}^\circ(1, 3)$ into $\mathrm{O}^\circ(1, m)$, $m \geq 3$, is compact, and we have (cf. [**A3**]):

Theorem 3.1. *Given uniform lattice $\Gamma \subset \mathrm{O}^\circ(1, 3)$ and $m \geq 3$, the variety $\mathcal{T}_{m-1}(\Gamma)$ has a natural compactification $\overline{\mathcal{T}_{m-1}(\Gamma)}$ whose boundary representations are faithful and have discrete images.*

A special interest for us is a question about the action of boundary representations of Γ (distinct from cusps) on the sphere at infinity of $\mathbb{H}^m$, $m \geq 4$. Is it always discontinuous? (For boundary cusp representations, see [**A1, A2**]).

For study of such kind of questions as well as the question ([**JM**]) about singularities of the representation varieties $\mathcal{T}_{m-1}(\Gamma) \subset \mathcal{H}_m(\Gamma)$ for a hyperbolic 3-lattice Γ, it is important to find new kinds of deformation of hyperbolic lattices which are different from "bendings" defined in (3.1) and (3.2) (see [**A1, A7, AT, T**]). Actually, using of so called stamping deformations defined by the author [**A7**] in connection with cone hyperbolic structures on the manifold $M = \mathbb{H}^3/\Gamma$ together with a new Apanasov-Tetenov's construction [**AT**] of "bendings" along geodesic hypersurfaces with boundary (or along pleated hypersurfaces) in M shows how it might be possible to overcome (in dimension 3) the main obstacle for smoothness of the varieties which has been discovered [**JM**] for some lattices $\Gamma \subset \mathrm{O}^\circ(1, n)$, $\quad n \geq 4$.

Super-rigid groups and Platonov conjecture on arithmeticity

For a given finitely generated group Γ, we may consider another affine algebraic variety over the complex field $\mathbb{C}$, $\mathcal{R}_n(\Gamma) = \mathrm{Hom}(\Gamma, GL_n(\mathbb{C}))$ as well as its open subvariety $\mathcal{R}_n(\Gamma)^s$ of irreducible representations and the quotient $\mathcal{S}_n(\Gamma) = \mathcal{R}_n(\Gamma)^s / PGL_n(\mathbb{C})$ by the conjugation action. The points of $\mathcal{S}_n(\Gamma)$ correspond to isomorphism classes of simple n–dimensional representations of Γ. As before, a curve in $\mathcal{S}_n(\Gamma)$ through the (isomorphism class $[\rho]$ of) simple representation ρ can be thought of as a deformation of ρ.

Groups Γ for which $\mathcal{S}_n(\Gamma)$ is zero–dimensional for all n are of special interest. They are called super-rigid because an analogy with Margulis super-rigidity theorem [**M2**]. This class includes $SL(n, \mathbb{Z})$, the reductive groups (groups for which every representation is semi-simple, due to [**LM**, Cor 2.7] and, due to a result of H.Bass [**B**], the groups of *integral representation type* (that is, finitely generated groups such that all the values of all the characters of all their representations are integral over $\mathbb{C}$). Additionally, Bass [**B**] has shown that super-rigid groups also have the property (F-Ab), namely that every finite index subgroup has finite abelianization. Recently, A.Rapinchuk [**RA**, Thm. 3] has shown that (F-Ab) groups of *bounded generation* are super-rigid. Here a group Γ has bounded generation if there exist elements $\gamma_1, \ldots, \gamma_t \in \Gamma$ with $\Gamma = \langle\gamma_1\rangle \cdots \langle\gamma_t\rangle$ where $\langle\gamma_i\rangle$ is the cyclic subgroup generated by γ_i.

Analyzing such known examples of finite representation-type groups, V.Platonov [**P1, P2**] formulated the following

Platonov Conjecture. *Finitely generated linear super-rigid groups are of arithmetic type.*

Here, by an arithmetic-type group we mean a group commensurable with a direct product of a finite number of S–arithmetic groups (perhaps, for different number sets S). A survey of background on the Platonov conjecture and its connection with the Grothendieck's problem on profinite completions can be found in [**PT**]. It is also known [**TA**, Thm. 1] that linear groups of arithmetic type are of bounded generation.

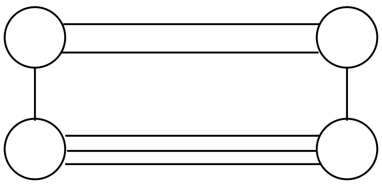
Now let us consider a hyperbolic lattice as the finitely generated linear group Γ above. It is naturally to ask whether non-arithmetic lattices $\Gamma \subset \mathrm{O}^\circ(1,n)$ are super-rigid. Concerning the well known M.Gromov–I.Piatetski-Shapiro interbreeding construction [**GPS**] of non-arithmetic hyperbolic lattices in any dimension $n \geq 3$, we see non-rigidity of such n-lattices in $GL_{n+2}(\mathbb{R})$ because the bending deformations (3.1) and (3.2). Due to an observation by V.Platonov [**P3**], it implies non-rigidity of such lattices in $GL_{n+2}(\mathbb{C})$, too.

Nevertheless, there are non-arithmetic hyperbolic lattices Γ which are "hyperbolically super-rigid" (in the sense of rigidity of their representations in any $\mathrm{O}^\circ(1,m)$, $m \geq n$).

Theorem 4.1. *Let $\Gamma \subset \mathrm{O}^\circ(1,3)$ be the uniform lattice (with the basic field $\mathbb{Q}(\sqrt{2},\sqrt{5})$) generated by four reflections in sides of a hyperbolic simplex $\Delta \subset \mathbb{H}^3$ with dihedral angles $\pi/2, \pi/3, \pi/5, \pi/2, \pi/3$ and $\pi/4$, i.e.*

$$\Gamma = \langle\gamma_1, \ldots, \gamma_4 | \gamma_i^2 = (\gamma_i\gamma_j)^{m_{ij}} = 1, \quad m_{ij} = 2,3,4 \text{ or } 5; \quad i,j = 1,2,3,4\rangle, \quad (4.1)$$

which corresponds to the following Coxeter diagram:



Then Γ is non-arithmetic and hyperbolically super-rigid.

Actually, due to a result of E.Vinberg [**V**], our group Γ in (4.1) is the unique non-arithmetic uniform hyperbolic lattice generated by reflections in sides of a hyperbolic (bounded) n-simplex $\Delta \subset \mathbb{H}^n$. There are few additional hyperbolic (cofinite) lattices which are non-arithmetic: six of them are in $\mathrm{O}^\circ(1,3)$ and one in

$\mathrm{O}^\circ(1,5)$ [**V**]. All of these lattices are hyperbolically super-rigid. Namely, due to the author observation [**A5**], they are rigid in $\mathrm{O}^\circ(1,n+1)$: all their faithful representations in $\mathrm{O}^\circ(1,n+1)$ are conjugated to the inclusion. General rigidity of Γ in $\mathrm{O}^\circ(1,m)$, $m \geq n$, follows ¿from the fact that a subgroup of $\mathrm{O}^\circ(1,m)$ generated by $(n+1)$ reflections always lies (up to conjugation in $\mathrm{O}^\circ(1,m)$) either in $\mathrm{O}^\circ(1,n)$ or in $O(n+1)$ or in Isom $\mathbb{R}^{n+1}$.

These hyperbolically super-rigid but non-arithmetic lattices are likely providing counter examples to the above conjecture because they cannot be S-arithmetic for any finite number set S (for Γ in (4.1), $S=\{\sqrt{2},\sqrt{5}\}$) while they are rigid in the most probable non-rigid cases of the hyperbolic linear groups $\mathrm{O}^\circ(1,m)$) (compare Theorem 1.1).

5. Global properties of representation varieties

Here we shall to show how may geometric structures on manifolds be used to shed light on topology of varieties of group representations. Namely, we are studying a natural question whether or not the varieties $\mathcal{T}_{m-1}(\Gamma) \subset \mathcal{H}_m(\Gamma)$ are connected (like Teichmüller spaces for surface groups, which are cells). Here we consider uniform hyperbolic lattices $\Gamma \subset \mathrm{O}^\circ(1,3)$ (as before, compare Theorem 1.1).

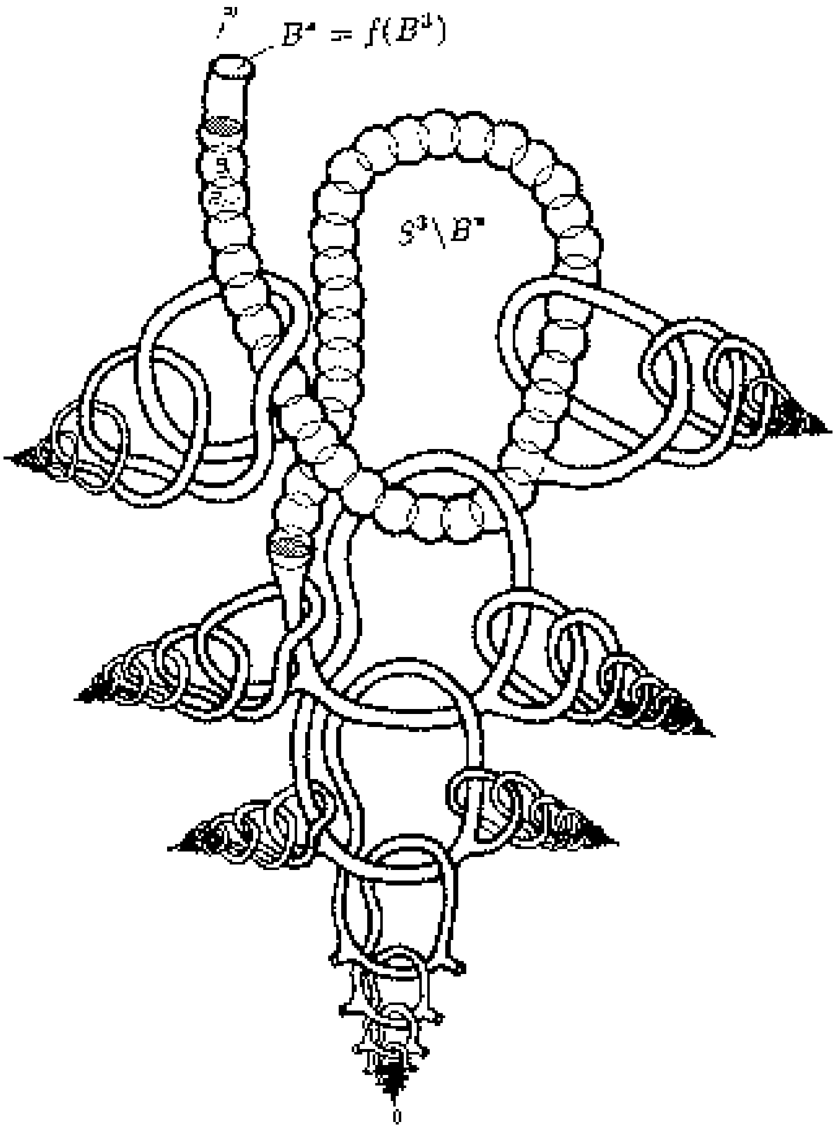
In contrast to Teichmüller spaces of hyperbolic surfaces, Apanasov's construction [**A6**] of strange conformal structures on closed hyperbolic 3-manifolds M, $\pi_1(M)=\Gamma$ (with large enough number of disjoint totally geodesic surfaces), and using of the Sullivan's stability theorem for geometrically finite hyperbolic groups [**S**] show that the space of uniformized conformal structures on M with faithful holonomy representations $d_* : \pi_1(M) \to \mathrm{O}^\circ(1,4)$ is not connected. This implies non-connectedness of our varieties:

Theorem 5.1. *There are (uniform) hyperbolic lattices $\Gamma \subset \mathrm{O}^\circ(1,3)$ whose faithful discrete representation varieties $\mathcal{T}_3(\Gamma) \subset \mathcal{H}_4(\Gamma)$ are non-connected.*

The nature of that non-connectedness of the varieties is a "wild" geometric realization of some representations $\rho^* : \Gamma \to \mathrm{O}^\circ(1,4)$. In fact, both discrete groups Γ and $\rho^*(\Gamma)$ act discontinuously in some 3-dimensional balls $B=B^3(0,1)$ and B^* in the 3-sphere S^3. Moreover, there is a (*qusi-symmetric*, see [**A8**]) homeomorphism $f:\overline{B}\to\overline{B^*}$ between their closures, which is compatible with the group action:

$$\rho^*(\gamma)=f^*(\gamma)=f\gamma f^{-1}, \quad \gamma\in\Gamma. \tag{5.1}$$

However, this embedding f of the closed round ball $\overline{B}$ is wild everywhere on the boundary sphere ∂B so that the complement $S^3\backslash\overline{B^*}$ is not simply connected - see figure below:



Algebraically, this "wild" representation ρ^* is obtained from the inclusion ρ_0 by means of (large enough number of) consecutive bending deformations (3.1) (geometrically, only in the ball B and with some special values of parameters α_i) and a final bending deformation deformation of the type (3.2).

It is still unknown, how many connected components are there in our varieties, and what invariants may distinguish them? One of possible approaches is using such geometric invariants (for conformal structures on corresponding 3-manifolds) as the η- and Chern-Simons invariants which are locally constant in the space of conformally flat structures [**APS**], [**CS**] (compare [**A9**]).

Now, passing to the dimension $m = 5$, we firstly remark that the composition of the constructed representation (5.1) and the inclusion $\mathrm{O}^\circ(1,4) \subset \mathrm{O}^\circ(1,5)$ provides only a representation lying in the same component of $\mathcal{T}_4(\Gamma) \subset \mathcal{H}_5(\Gamma)$ that contains the inclusion representation ρ_0 because the nerv of the knotting of the wild ball B^* in S^3 is one dimensional (it is unknotted in S^4). That is why the problem of connectedness of the varieties $\mathcal{T}_4(\Gamma) \subset \mathcal{H}_5(\Gamma)$ cannot be reduced to Theorem 5.1.

However, the non-connectedness effect takes still place for these varieties $\mathcal{T}_4(\Gamma) \subset \mathcal{H}_5(\Gamma)$. It has been recently discovered by Apanasov's construction [**A10**] of hyperbolic 3-lattice representations in $\mathrm{O}^\circ(1,5)$. This construction also establishes a connection between variety components and some 2-dimensional knots in S^4 [**A11**].

We shall describe the main idea of our construction in general situation of faithful discrete representations of hyperbolic n-dimensional lattices Γ, $n \geq 3$, into $\mathrm{O}^\circ(1,n+2)$. This construction is based on the author's "block-building method" (cf. [**A8**], [**A6**]) and PL-approximation of smooth $(n-1)$-knots in the $(n+1)$-sphere S^{n+1} with a geometric control. It is shortly as follows.

Suppose that a tame $(n-1)$-knot $K \subset S^{n+1}$ belongs to the union of boundaries of a finite set of metric n-balls $\mathcal{C} = \{B_1, \dots, B_k\}$ such that $\partial B_i \cap \partial B_j \neq \emptyset$ iff the balls B_i, B_j are adjacent in the sense $j = i+1 (\mathrm{mod}\, k)$. Additionally suppose that,in each ball B_i, there is a discrete hyperbolic isometry group action $G_i \subset \mathrm{O}^\circ(1,n) \subset \mathrm{O}^\circ(1,n+2)$ such that, upon a geometric control on the isotopy class of K and the family $\mathcal{C}$, the free (amalgamated) product $G \subset \mathrm{O}^\circ(1,n+1)$ of groups $j_i^*(G_i)$ ($j_i : B_i \to S^{n+1}$ are inclusions) is a discrete group isomorphic to our given lattice $\Gamma \subset \mathrm{O}^\circ(1,n)$. Then the limit set $\Lambda(G) \subset S^{n+1}$ of the group G is a certain infinite compounding, K_∞, of the original knot K but still homeomorphic to $(n-1)$sphere because Tukia isomorphism theorem [**TU**]. Due to the ergodicity of G-action on K_∞, this knot should be wild everywhere if K is not trivial. To prove this fact, we have to use additivity of Alexander knot invariants with respect to connected sum of knots [**R**]. Then the Sullivan stability theorem for geometrically finite hyperbolic groups [**S**] completes the proof on non-connectedness of the varieties $\mathcal{T}_{n+1}(\Gamma) \subset \mathcal{H}_{n+2}(\Gamma)$ because the limit set $\Lambda(\rho(\Gamma))$ of each quasi-Fuchsian representation is an unknotted 2-sphere in S^4.

Particularly, due to Andreev-Rivin characterization of hyperbolic 3-polyhedra ([**AN**], [**RI**]), this approach is very successful for non-trivial 2-knots, especially for ribbon knots [**R**]. Namely, we have the following result [**A10**]:

Theorem 5.2. *For a given non-trivial ribbon 2-knot $K \subset S^4$ there exists a discrete faithful representation $\rho : \Gamma \to G = \rho(\Gamma) \subset \mathrm{O}^\circ(1,5)$ of an uniform lattice $\Gamma \subset \mathrm{O}^\circ(1,3)$ with ergodic action of G on the infinitely compounded, everywhere wild 2-knot $K_\infty = \Lambda(G) \subset S^4$. Furthermore, this action is compatible with a quasi-symmetric everywhere wild embedding $f : S^2 \hookrightarrow K_\infty \subset S^4$.*

So we have the following:

Corollary 5.3. *Let $\Gamma \subset \mathrm{O}^\circ(1,3)$ be any uniform lattice ¿from Theorem 5.2. Then the varieties $\mathcal{T}^4(\Gamma) \subset \mathcal{H}^5(\Gamma)$ of discrete faithful representations are non-connected.*

Another kind of discrete faithful representations of a hyperbolic 3-lattice Γ into $\mathrm{O}^\circ(1,5)$ has a close relation to first author's constructions [**A10**] of non-trivial fiber bundles over closed hyperbolic 3-manifolds (or orbifolds) with geometric structures.

Namely, given a closed hyperbolic 3-manifold (orbifold) M with $\pi_1(M) = \Gamma \subset \mathrm{O}^\circ(1,3)$, each conjugacy class $[\rho]$ in the variety $\mathcal{T}^4(\Gamma) \subset \mathcal{H}^5(\Gamma)$ defines a 5-dimensional hyperbolic manifold $E = \mathbb{H}^5/\rho(\Gamma)$ and a closed conformal 4-manifold N at infinity of E, $N = \Omega(\rho(\Gamma))/\rho(\Gamma)$ ($\Omega(\rho(\Gamma))$ is the discontinuity set of $\rho(\Gamma)$ in the sphere at infinity $S^4 = \partial\mathbb{H}^5$), whose fundamental group satisfies to the exact sequence (2.2). Moreover, each such manifold E coming from our general construction of the group

$G = \rho(\Gamma) \subset O^\circ(1,5)$ in Theorem 5.2 (see [**A10**]) has a PL-embedding $M \hookrightarrow E$ (in general, non-locally-flat) which is a homotopy equivalence. E and N themselves are 2-plane and S^1 bundles over M if the exterior of the knot is homeomorphic to $B^3 \times S^1$ (i.e., the knot is trivial [**GO**]). Moreover, it turns out that (besides trivial bundles corresponding to quasi-Fuchsian representation of $\Gamma = \pi_1(M)$) there are non-trivial fiber bundles over M associated with compact convex hyperbolic 3-polyhedra P whose sides have compatible involutions. We construct them (for hyperbolic uniform lattices Γ_0 generated by that involutions in sides of P) by using non-trivial automorphisms $\eta(x,\theta) = (\tau(\theta)(x),\theta)$ of $S^2 \times S^1$ compatible with $\Gamma_0 \supset \Gamma$ (cf. [**KU**], [**A1**, p.419-423]). Here $\tau(\theta)$ is rotation of S^2 about its polar S^0 so that $G = \rho(\Gamma)$ is the free acting subgroup of the group $\rho(\Gamma_0)$ generated by compositions of Γ-generating involutions and rotations τ_i associated with i-th sides of P (see [**A10**]).

Due to [**GO**], our η-Dehn surgery defines unknotted $S^2 \subset S^4$, so the limit set $\Lambda(G)$ is unknotted, hence $\Omega(G) = S^4 \backslash \Lambda(G) \cong B^3 \times S^1$ is homotopy equivalent to S^1, and the exact sequence (2.2) is just the one coming from the fibration $S^1 \to N \to M$.

Nevertheless, fiber bundles over 3-manifolds M are more rigid than their analogues for surfaces (see [**GLT**]). Namely, the group of pseudo-isotopy classes of homeomorphisms of $S^2 \times S^1$ (the boundary of a regular neighborhood of $S^2 \subset S^4$) is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. there are at most two non-equivalent locally flat embeddings of S^2 in S^4 with homeomorphic exteriors; see [**G**], [**GO**]. It implies:

Theorem 5.4. *For a given closed hyperbolic 3-manifold M there are at most two non-equivalent circle (or 2-plane) bundles over M allowing an uniformizable conformal structure developed onto a 2-knot complement (a complete hyperbolic metric, correspondingly). Moreover, there are at most finite number of equivalence classes of conformal 4-manifolds N (hyperbolic 5-manifolds E, $\partial E = N$) homotopy equivalent to M and whose development is onto a 2-knot complement.*

We finally remark that the number of connected components of the varieties $\mathcal{T}^4(\Gamma) \subset \mathcal{H}^5(\Gamma)$ in Corollary 5.3 (as well as the number of manifold equivalence classes in Theorem 5.4) depends of geometry of $M = H^3/\Gamma$, especially, of the number $b(M)$ of disjoint totally geodesic surfaces in M. In our examples of non-connected varieties in [**A10**], this number $b(M)$ is more than 100, and we think it cannot be less than $C_n \cdot e$ where $C_n > 1$ is an universal constant and $e = e(K)$ is the potential energy of the n-knot K, see [**BFHW**] and [**AS**], (actually, for the used trefoil knot $k \subset S^3$, $\quad e(k) \approx 74$.).

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