MAXIMAL PARABOLIC SUBGROUPS IN CLASSICAL GROUPS ARE OF MODALITY ZERO

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Abstract. The principal aim of this paper is to show that every maximal parabolic subgroup P of a classical reductive algebraic group G operates with a finite number of orbits on its unipotent radical. This is a consequence of the fact that each parabolic subgroup of a group of type A_n whose unipotent radical is of nilpotent class at most two has this finiteness property.

1. Introduction

Let G be a reductive algebraic group defined over an algebraically closed field K of characteristic 0. Let $P = LP_u$ be a parabolic subgroup of G with P_u the unipotent radical of P and L a Levi complement in P. As P_u is a normal subgroup of P, the latter group acts on the former via conjugation and on the Lie algebra of P_u via the adjoint action.

In [6] the general problem was posed to determine each parabolic subgroup P of a reductive algebraic group G that has a finite number of orbits on P_u. The cases for Borel and semisimple rank one parabolic subgroups were classified in [5] and [6] respectively.

In [6, Problem 5.6] the question was raised whether every maximal parabolic subgroup of G acts with a finite number of orbits on its unipotent radical. For classical groups the answer is affirmative:

Theorem 1.1. *Let G be a classical simple algebraic group and P a maximal parabolic subgroup of G. Then P acts on P_u with a finite number of orbits.*

Remark 1.2. The analogous statement of Theorem 1.1 for exceptional groups fails. There are two conjugacy classes of maximal parabolic subgroups P in E_8 such that P acts with an infinite number of orbits on P_u; see [9, Remark 3.3].

Theorem 1.1 is a consequence of

Theorem 1.3. *Let G be of type A_n and P a parabolic subgroup of G. If the length of the descending central series of P_u is at most two, then P acts on P_u with a finite number of orbits.*

Finally, we prove a generalization of Theorems 1.1 and 1.3:

Theorem 1.4. *Let G be a classical simple algebraic group and P a parabolic subgroup of G. If the length of the descending central series of P_u is at most two, then P acts on P_u with a finite number of orbits.*

1991 *Mathematics Subject Classification.* 20G15, 20G20, 22E20.

As in [6] and [9] we take the more general point of view in terms of the *modality* of an algebraic group acting morphically on an algebraic variety. Recall from [12] that for an algebraic group R acting morphically on an algebraic variety X the *modality* of the action, $\mathrm{mod}(R : X)$, is defined as

$$\mathrm{mod}(R : X) = \max_{Z} \min_{z \in Z} \mathrm{codim}\,_Z R^0 \cdot z,$$

where Z runs through all irreducible R^0-invariant subvarieties of X, and R^0 denotes the identity component of R. Informally, $\mathrm{mod}(R : X)$ is the maximal number of parameters on which a family of R-orbits on X may depend.

For a parabolic subgroup P of an algebraic group we call $\mathrm{mod}\, P := \mathrm{mod}(P : P_u)$ the *modality* of P, cf. [6]. Observe that the modality of P is 0 precisely when P admits only a finite number of orbits on P_u.

In Section 2 we first recall some basic results from [6] and [9] and apply them to our situation in Section 3 in showing that Theorem 1.1 follows from Theorem 1.3. In the next section we outline a proof of Theorem 1.3 followed by a section where we prove Theorem 1.4.

For an algebraic group R we often denote its Lie algebra by $\mathrm{Lie}\, R$ or simply by the corresponding lower case gothic letter $\mathfrak{r}$ and its unipotent radical by R_u. The Lie algebra of R_u is denoted by $\mathfrak{r}_u$. Throughout the identity element of an algebraic group is denoted by e.

The descending central series $\{C^i R \mid i \geq 0\}$ of R is defined as usual: $C^0 R := R$ and $C^{i+1} R := [C^i R, R]$ for $i \geq 0$. If $C^i R = \{e\}$ for some $i \geq 0$, then R is nilpotent. In that case the *length* of the descending central series of R, denoted by $\ell(R)$, is the smallest integer m such that $C^m R = \{e\}$, i.e., $\ell(R) = m$; equivalently, R is nilpotent of class m.

If R acts morphically on an algebraic variety X, we denote by $R \cdot x$ the orbit of x in X under the action of R and if Z is a subset of X, then $R \cdot Z$ is the union of all orbits $R \cdot z$ in X, where $z \in Z$.

Let G be a connected reductive algebraic group, T a maximal torus in G and Ψ the set of roots of G with respect to T. As usual, for a root $\alpha \in \Psi$ we denote the corresponding root space in $\mathfrak{g}$ by $\mathfrak{g}_\alpha$. Fix a set of simple roots Π of Ψ and let B be the Borel subgroup of G corresponding to Π. A parabolic subgroup of G containing B is called a standard parabolic subgroup. For a subset J of Π we denote by P_J the standard parabolic subgroup corresponding to J such that $P_\varnothing = B$.

As every parabolic subgroup P of G is conjugate to a standard one, and since $\ell(P_u)$ and $\mathrm{mod}\, P$ are invariant under conjugation by elements from G, we may if we wish assume that the parabolic subgroups under consideration are standard.

Because a reductive algebraic group is the direct product of simple ones over its center, and similarly for parabolic subgroups, we can reduce to the case of a simple algebraic group in our study of parabolic subgroups of modality zero. Henceforth we assume that G is a connected, simply connected simple algebraic group, unless stated otherwise.

Since the characteristic of K is 0, the exponential mapping is a P-equivariant isomorphism between $\mathrm{Lie}\, P_u$ and P_u. This allows us to work on $\mathrm{Lie}\, P_u$ instead of P_u, whenever convenient.

For each root $\alpha \in \Psi$ we fix a homomorphism of algebraic groups $u_\alpha : K \longrightarrow G$ and denote the image, i.e., the corresponding one-parameter unipotent subgroup of G, by $U_\alpha = u_\alpha(K)$. The *support* of a subset Y of B_u, $\operatorname{supp} Y$, is the set of all roots $\alpha \in \Psi$ such that the projection of Y into U_α is non-trivial. If $Y = \{y\}$, we write $\operatorname{supp} y$ instead of $\operatorname{supp}\{y\}$.

As a general reference for algebraic groups we cite Borel's book [1] and for information on root systems we refer the reader to Bourbaki [3]. In particular, the simple roots in a base Π of a root system of a classical simple algebraic group are indexed as in [3, Planches I - IV].

I am grateful to T. Springer and V.L. Popov for suggesting several improvements in the paper.

2. Preliminary Results and Recollections

We maintain the notation from the Introduction. Our first result is a special case of [7, Theorem E].

Proposition 2.1. *Let G be a reductive algebraic group and $P = LP_u$ a parabolic subgroup of G. Then L (and thus P) operates on each quotient of consecutive members of the descending central series of P_u with a finite number of orbits.*

We immediately obtain

Corollary 2.2. *Let G be a reductive algebraic group and $P = LP_u$ a parabolic subgroup of G with P_u Abelian. Then L (and thus P) operates on P_u with a finite number of orbits.*

See [8] for a detailed analysis of the orbit structure in this case.

We recall V. Kashin's classification of Borel subgroups with a finite number of orbits [5, Theorem 1]:

Theorem 2.3. *Let G be a simple algebraic group and B a Borel subgroup of G. The number of orbits of B on its unipotent radical B_u is finite if and only if G is of type A_n for $n \leq 4$ or B_2.*

The main result on minimal parabolic subgroups in [6, Corollary 1.4] is

Theorem 2.4. *Let G be a simple algebraic group and P a minimal parabolic subgroup of G. Then the number of P-orbits on P_u is finite if and only if G is of type A_n for $n \leq 5$, B_n for $n \leq 3$, C_n for $n \leq 3$, D_4, or G_2.*

We need an elementary but useful lemma.

Lemma 2.5. *Let G be a reductive algebraic group and $Q \subseteq P$ parabolic subgroups of G. If Q operates with a finite number of orbits on Q_u, then so does P on P_u.*

Proof. Since $Q \subseteq P$, we have $P_u \subseteq Q_u$ and Q normalizes P_u. Thus Q operates on P_u with a finite number of orbits, and so does P.

We follow [4] by calling a closed connected subgroup H of a reductive algebraic group G *regular* if the normalizer of H in G contains a maximal torus T of G. In that case the root spaces of $\mathfrak{h}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of H with respect to T, $\Psi(H)$, is a subset of Ψ, the root system of G relative to T, and is closed under addition in Ψ. If H is a reductive regular subgroup of G, then $\Psi(H)$ is a subsystem of Ψ. Moreover, a set of simple roots $\Pi(H)$ for $\Psi(H)$ can always be chosen such that the Borel subgroup of H relative to $\Pi(H)$ is precisely $H \cap B$, where B is the Borel subgroup of G relative to Π. In particular, $\Psi^+(H) \subseteq \Psi^+$.

Let G be a reductive algebraic group and H a reductive regular subgroup of G relative to T. Let P be a parabolic subgroup of G containing T and let $Q = H \cap P$ which is a parabolic subgroup of H. The following result is Corollary 2.6 from [6].

Proposition 2.6. *Assume that G, P, H, and $Q = H \cap P$ are as above. If the number of P-orbits on P_u is finite, then so is the number of Q-orbits on Q_u.*

Let Θ be a semisimple automorphism of G and let θ denote the corresponding (semisimple) automorphism of $\mathfrak{g}$, i.e., $(d\Theta)_e = \theta$. Let P be a Θ-stable parabolic subgroup of G. Then P_u is Θ-stable as well. Also, then $\mathfrak{p}$ and $\mathfrak{p}_u$ are θ-stable. Let G^Θ be the Θ-fixed point subgroup of G and likewise $\mathfrak{g}^\theta$ the θ-fixed point subalgebra of $\mathfrak{g}$. It is known that G^Θ is reductive and that $\operatorname{Lie} G^\Theta = \mathfrak{g}^\theta$. Moreover, if G is simply connected, then G^Θ is connected [11]. Let $P^\Theta = P \cap G^\Theta$ be the Θ-fixed point subgroup of P. We have $\operatorname{Lie} P^\Theta = \mathfrak{p}^\theta := \mathfrak{g}^\theta \cap \mathfrak{p}$, and $\operatorname{Lie}(P_u \cap P^\Theta) = \mathfrak{p}_u^\theta := \mathfrak{p}_u \cap \mathfrak{p}^\theta$. Our next result is Corollary 2.8 from [6].

Proposition 2.7. *Let Θ be a semisimple automorphism of G. Assume that P is a Θ-stable parabolic subgroup of G. If P has a finite number of orbits on $\mathfrak{p}_u$, then so does P^Θ on $\mathfrak{p}_u^\theta$.*

Our final lemma provides an elementary way to determine $\ell(P_u)$, cf. [9, Lemma 2.5]. Let $\beta \in \Psi^+$. Write $\beta = \sum_{\alpha\in\Pi} c_\alpha(\beta)\alpha$, with $c_\alpha(\beta) \in \mathbb{Z}_0^+$ for each $\alpha \in \Pi$.

Lemma 2.8. *Let $P = P_J$ be a standard parabolic subgroup of G. Let ρ be the highest root of Ψ. Then*

$$\ell(P_u) = \sum_{\alpha\in\Pi\setminus J} c_\alpha(\rho).$$

Proof. This follows readily from the usual Chevalley commutator relations for G.

Remark 2.9. Several of the results in this section do not require the assumption that $\operatorname{char} K = 0$, such as Proposition 2.1, Corollary 2.2, and Lemma 2.5. Also, Lemma 2.8, is valid for groups G defined over an algebraically closed field of characteristic $p > 0$, provided p does not divide the structure constants involved in the Chevalley commutator formulae for G, e.g. see [2, Prop. 4.7(iii)].

3. Proof of Theorem 1.1

Let $G = SL_n(K)$. It is well known that each of the classical groups $SO_n(K)$ and $Sp_n(K)$ can be realized as a fixed point subgroup of G for a suitable semisimple automorphism Θ of G, i.e. G^Θ equals $SO_n(K)$ or $Sp_n(K)$; cf. [10, §11 p. 169].

Moreover, each parabolic subgroup of G^Θ can be realized as a fixed point subgroup P^Θ for some Θ-invariant parabolic subgroup P of G. In case $G^\Theta = SO_n(K)$

with n even, this only works up to symmetry under the graph automorphism of G^Θ.

Granting Theorem 1.3 we can prove Theorem 1.1 using Proposition 2.7:

Proof of Theorem 1.1. Let $G = SL_n(K)$. Every maximal parabolic subgroup of G has an Abelian unipotent radical (cf. Lemma 2.8) and thus acts with a finite number of orbits on its unipotent radical by Corollary 2.2.

As above let Θ be a semisimple automorphism of G such that $G^\Theta = SO_n(K)$ or $Sp_n(K)$ and let Q be a maximal parabolic subgroup of G^Θ. Then $\ell(Q_u) \leq 2$, by Lemma 2.8 and the fact that the coefficient of a simple root in the highest root is at most 2 for root systems of classical type. If $\ell(Q_u) = 1$, then Q_u is Abelian and thus Q has a finite number of orbits on Q_u by Corollary 2.2. So we may assume that $\ell(Q_u) = 2$. In that case we can obtain Q as a fixed point subgroup P^Θ for some Θ-invariant parabolic subgroup P of G with $\ell(P_u) = 2$. The desired result for G^Θ then follows from Proposition 2.7 and Theorem 1.3.

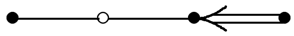
Remark 3.1. We may also obtain the result of Theorem 1.1 for groups of type D_n from that for groups of type B_n. Let G be of type B_n and H the simple regular subgroup of G whose root system consists of the set of long roots in Ψ. Then H is of type D_n. Let Q be a maximal parabolic subgroup of H. There exists a maximal parabolic subgroup P of G such that $P \cap H = Q$ or $P \cap H$ is the image of Q under the graph automorphism of H. The desired result for H then follows from Proposition 2.6 and the result for G.

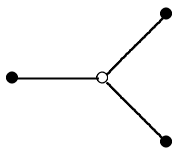
We illustrate the procedure from the proof of Theorem 1.1 with some examples.

Examples 3.2. The colored nodes indicate the simple roots in $\Pi \cap \Psi(P)$ and likewise for $\Psi(G^\Theta)$. In the first example two different choices for the semisimple automorphism lead to different fixed point subgroups.

1.

$G = SL_8(K)$, $P = P_{\Pi \setminus \{\alpha_2, \alpha_6\}}$

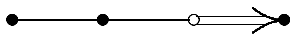




$G^\Theta = Sp_8(K)$, P^Θ $\qquad$ $G^{\Theta'} = SO_8(K)$, $P^{\Theta'}$

2.

$G = SL_9(K)$, $P = P_{\Pi \setminus \{\alpha_3, \alpha_6\}}$



$G^\Theta = SO_9(K)$, P^Θ

4. Proof of Theorem 1.3

We first make an elementary observation. Assume G, P, H, and $Q = H \cap P$ are as in Proposition 2.6. If the number of Q-orbits on Q_u is finite, then P operates (via conjugation) on $P \cdot Q_u := \cup_{x \in Q_u} P \cdot x$ with a finite number of orbits as well.

Proof of Theorem 1.3. Let G be of type A_n and $P = P_J$ a standard parabolic subgroup of G with $\ell(P_u) \leq 2$ ($J \subset \Pi$). If P_u is Abelian, then P acts with a finite number of orbits on P_u by Corollary 2.2. Thus we may assume that $\ell(P_u) = 2$. That is $J = \Pi \setminus \{\alpha_i, \alpha_j\}$ for some $1 \leq i < j \leq n$.

We argue by induction on $\operatorname{rank} G = n$. If $n \leq 5$, then the result follows from Theorem 2.4 and Lemma 2.5.

Let $n > 5$ and assume that the statement of Theorem 1.3 holds in case the rank is less than n. For simplicity we only consider the case when $1 \leq i < j - i$ and $2j - 1 \leq n$. In particular, $j - i \leq n - j + 1$. The purpose of these restrictions on i and j is merely to reduce the number of subcases to be studied. The arguments below equally apply in the other cases *mutatis mutandis*.

We use the following abbreviation for partial sums of simple roots. For $1 \leq k < l \leq n$ let

$$\alpha_{k,l} := \alpha_k + \alpha_{k+1} + \cdots + \alpha_l,$$

and $\alpha_{k,k} := \alpha_k$ for $1 \leq k \leq n$.

For $1 \leq k < l \leq n$ let $H^{k,l}$ be the simple regular subgroup of G with a set of simple roots

$$\Pi(H^{k,l}) := \{\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k,l}, \alpha_{l+1}, \ldots, \alpha_n\}.$$

Whence $H^{k,l}$ is of type A_{n-l+k}. Let $Q^{k,l} := H^{k,l} \cap P$. We write $Q^{k,l}_u$ for the unipotent radical of $Q^{k,l}$. Because $\ell(Q^{k,l}_u) \leq \ell(P_u) = 2$, the number of orbits of $Q^{k,l}$ on its unipotent radical is finite by induction on the rank. By our observation above there is only a finite number of P-orbits on

$$\mathcal{E} := \bigcup_{1 \leq k < l \leq n} P \cdot Q^{k,l}_u \subseteq P_u.$$

Observe that for $1 \leq k < l \leq n$ we have $Q^{k,l}_u \subseteq Q^{k,k+1}_u$. Thus

$$\mathcal{E} = \bigcup_{1 \leq k < n} P \cdot Q^{k,k+1}_u \subseteq P_u.$$

Thus it suffices to show that P operates on $P_u \setminus \mathcal{E}$ with a finite number of orbits.

Let V_1 be the product of all those root subgroups in some fixed order whose roots involve α_i as a summand but not α_j and let V_2 be the product of those root subgroups in some fixed order whose roots involve α_j but not α_i. Observe that $\overline{P}_u := P_u / P'_u$ is a module for L and decomposes as a direct sum of irreducibles $\overline{V}_1 \oplus \overline{V}_2$. By Proposition 2.1 P operates on P'_u and on $\overline{P}_u$ with a finite number of orbits. Observe that the P-orbit structure on P'_u is equivalent to that of $Q^{i,j}$ on its radical.

We define the following sets of roots:

$$\beta_{k+1} := \alpha_{i-k,i+k}, \qquad 0 \le k < i, \tag{1}$$
$$\gamma_{k+1} := \alpha_{j-k,j+k}, \qquad 0 \le k < j-i, \tag{2}$$
$$\delta_{k+1} := \alpha_{i-k,2j-i+k}, \qquad 0 \le k < i, \tag{3}$$

and for a fixed integer s with $1 \le s < i$ let

$$\eta_k := \alpha_{s+k,2j-i+k-1}, \qquad 1 \le k \le i-s. \tag{4}$$

In particular, $\beta_1 = \alpha_i$ and $\gamma_1 = \alpha_j$. Observe that $\{\beta_t \mid 1 \le t \le i\}$ (respectively $\{\gamma_t \mid 1 \le t \le j-i\}$) is a maximal set of mutually orthogonal (long) roots in $\operatorname{supp} V_1$ (respectively $\operatorname{supp} V_2$). As one of the components of L operates trivially on $\overline{V}_1$, the L-orbit structure on $\overline{V}_1$ is equivalent to that of an Abelian unipotent radical case. Likewise for V_2. It follows from [8, Theorem 1.2] that $\{\prod_{k=1}^{t} u_{\beta_{i-k+1}}(1) \mid 0 \le t \le i\}$ is a complete set of L-orbit representatives of $\overline{V}_1$. Likewise $\{\prod_{k=1}^{t} u_{\gamma_{j-i-k+1}}(1) \mid 0 \le t \le j-i\}$ is a complete set of L-orbit representatives of $\overline{V}_2$.

Now we consider the P-orbits in $P_u \setminus P_u'$. Let x be in $P_u \setminus P_u'$.

<u>Case 1</u>: First we consider the P-orbits in $V_1 P_u'$. We claim that $V_1 P_u' \subseteq \mathcal{E}$. Let $x \in V_1 P_u' \setminus P_u'$. In particular, $\operatorname{supp} x \cap \operatorname{supp} V_2 = \varnothing$. By our remarks above, we may assume that x is of the form

$$x = \prod_{k=1}^{s} u_{\beta_{i-k+1}}(1) u' \qquad \text{with } u' \in P_u' \text{ and some fixed } s,\ 1 \le s \le i.$$

If $s < i$, then each of the roots involved in the support of x is of the form $\ldots + \alpha_i + \alpha_{i+1} + \ldots$ by (1) above. Thus $P \cdot x \subseteq P \cdot Q_u^{i,i+1}$, as desired.

If $s = i$, then we argue as follows. By conjugating x with suitable elements from the root subgroups relative to $\alpha_{i+1,j}, \alpha_{i+2,j}, \ldots, \alpha_{2i,j}$ we can remove all roots of the form $\alpha_{k,j}$ for $1 \le k \le i$ from the support of u'. Continuing by conjugating the resulting element with suitable elements from the root subgroups relative to $\alpha_{i+1,j+1}, \alpha_{i+2,j+1}, \ldots, \alpha_{2i,j+1}$ we can remove all roots of the form $\alpha_{k,j+1}$ for $1 \le k \le i$ from the support of u'. Continuing in this fashion we can eliminate all roots in $\operatorname{supp} u'$. That is $z = \prod_{k=1}^{i} u_{\beta_k}(1)$ is a representative in this instance. Observe that $z \in Q_u^{j,j+1}$. In particular, we have $V_1 P_u \subseteq \mathcal{E}$, as claimed.

<u>Case 2</u>: Next we consider all P-orbits in $\overline{V}_2 \setminus P_u'$. Let $x \in \overline{V}_2 \setminus P_u'$. In particular, $\operatorname{supp} x \cap \operatorname{supp} V_1 = \varnothing$.

Analogous to Case 1, we may assume that x is of the form

$$x = \prod_{k=1}^{s} u_{\gamma_{j-i-k+1}}(1) u' \qquad \text{with } u' \in P_u' \text{ and some fixed } s,\ 1 \le s \le j-i.$$

If $s < j-i$, then each of the roots involved in the support of x is of the form $\cdots + \alpha_{j-1} + \alpha_j + \ldots$ by (2) above. Whence, $P \cdot x \subseteq P \cdot Q_u^{j-1,j}$.

If $s = j - i$, then we argue as follows. By conjugating x with suitable elements from root subgroups relative to $\alpha_{i,j-1}$, $\alpha_{i-1,j-1}, \dots, \alpha_{1,j-1}$ we can remove all roots of the form $\alpha_{k,j}$ for $1 \le k \le i$ from the support of u'. Continuing in this manner we can eliminate all roots of the form $\alpha_{k,l}$ for $1 \le k \le i$ and $j \le l \le 2j - i - 1$ from the support of u'. If there are no further roots in $\operatorname{supp} u'$, then $\prod_{k=1}^{j-i} u_{\gamma_k}(1)$ is a representative of the P-orbit through x.

If there are still other roots in $\operatorname{supp} u'$, then by applying reflections s_α from W_L with α orthogonal to $\alpha_{i+1}, \dots, \alpha_{2j-i-1}$, we may assume that $\delta_1 = \alpha_{i,2j-i}$ is in $\operatorname{supp} u'$. Now conjugating the resulting element by suitable elements from root subgroups relative to α_{i-1}, $\alpha_{i-2,i-1}, \dots, \alpha_{1,i-1}$ allows us to remove all roots from $\operatorname{supp} u'$ of the form $\alpha_{k,2j-i}$ for $1 \le k \le i-1$. Then by conjugating the result by suitable elements from root subgroups relative to $\alpha_{2j-i+1}, \alpha_{2j-i+1,2j-i+2}, \dots, \alpha_{2j-i+1,n}$, we can eliminate all roots from $\operatorname{supp} u'$ of the form $\alpha_{i,l}$ for $2j - i < l \le n$. We may continue this procedure until it stops, and once it does, the element

$$z = \prod_{k=1}^{j-i} u_{\gamma_k}(1) \prod_{l=1}^{t} u_{\delta_l}(1) \qquad \text{for some } t,\ 0 \le t \le i,$$

is a representative of the P-orbit through x. In particular, there is only a finite number of orbits in this case. As shown, all parameters can be chosen equal 1 in this instance using the action of T, as $\{\gamma_k, \delta_l \mid 1 \le k \le j - i;\ 0 \le l \le i\}$ is a set of mutually orthogonal (long) roots in Ψ.

Incidentally, we also see that in case $s = j - i$ and $2j < n$, we have $z \in Q_u^{n-1,n}$ for any $0 \le t \le i$. Otherwise, namely when $2j - 1 = n - 1$, or $2j - 1 = n$, there are some P-orbits in P_u which do not pass through any of the $Q_u^{k,l}$. More precisely, if $2j - 1 = n - 1$, then there is one such orbit for $t = i$, while for $2j - 1 = n$, the two orbits of this type corresponding to $t = i - 1$ and $t = i$ do not pass through any of the $Q_u^{k,l}$.

Case 3: Finally let $x \in P_u \setminus \mathcal{E}$ with $\operatorname{supp} x \cap \operatorname{supp} V_1 \neq \varnothing \neq \operatorname{supp} x \cap \operatorname{supp} V_2$. As there are only finitely many L-orbits on P_u/P_u', we may suppose that x is of the form

$$x = \prod_{k=1}^{s} u_{\beta_{i-k+1}}(1) y u' \quad \text{ for some } y \in V_2, u' \in P_u' \text{ and some fixed } s,\ 1 \le s \le i,$$

where $\prod_{k=1}^{s} u_{\beta_{i-k+1}}(1) y$ is one of a given finite list of L-orbit representatives for the L-action on P_u/P_u', and $y \neq 1$.

If $s = i$, then we may argue precisely as in Case 1, for $s = i$ and eliminate u' without changing y. Then $\prod_{k=1}^{i} u_{\beta_k}(1) y$ is a representative for the P-orbit through x.

Now we consider the case $s < i$. If α_{i+1} does not appear as summand for any root in $\operatorname{supp} y$, then $x \in Q_u^{i,i+1}$. So we may assume that α_{i+1} does occur as summand for some root in $\operatorname{supp} y$. After possibly applying elements from W_L, we may suppose that then $\alpha_{i+1,n} \in \operatorname{supp} y$. By conjugating with suitable elements from root subgroups relative to $-\alpha_{k,n}$ for $k > j$ we can eliminate all roots from $\operatorname{supp} y$ of the form $\alpha_{i+1,l}$ for $j \le l < n$. That is we may suppose that $\alpha_{i+1,n}$ is the only root in $\operatorname{supp} y$ which begins in α_{i+1}. Further, if there exists a root $\alpha_{k,n}$ in $\operatorname{supp} y$ with $i+1 < k \leq j$, then by applying a suitable element from the root subgroup relative to $\alpha_{i+1,k-1}$ we can remove $\alpha_{i+1,n}$ from $\operatorname{supp} y$ and again $P \cdot x \subset P \cdot Q_u^{i,i+1}$ follows. Observe that this last process does not alter the V_1-factor of our representative, as $s < i$. Thus we may assume that $\alpha_{i+1,n}$ is the only root in $\operatorname{supp} y$ which begins in α_{i+1} or ends in α_n.

As $x \notin Q_u^{i+1,i+2}$, we may suppose that $\alpha_{i+2,n-1}$ is in the support of y. By the same argument as above we may suppose that $\alpha_{i+2,n-1}$ is the only root in $\operatorname{supp} y$ which begins in α_{i+2} or ends in α_{n-1}. We may continue this argument and may suppose that $\alpha_{i+3,n-2}$, etc. up to $\alpha_{2i-s,n-i+s+1}$ are in $\operatorname{supp} y$ and that these roots are the only ones begining or ending with these particular simple roots respectively.

Further, since $x \notin Q_u^{j-1,j}$, there must be a root in $\operatorname{supp} y$ beginning in α_j, as by assumption $i < j-i$ and whence $2i-1 < j-1$. Thus we may suppose that $\alpha_{j,n-i+s}$ is in $\operatorname{supp} y$. After applying reflections relative to $\alpha_{j-1}, \alpha_{j-2}, \dots, \alpha_{2i-s+1}$, in this order, we may assume that $\alpha_{2i-s+1,n-i+s}$ is in $\operatorname{supp} y$. Applying these reflections alters the β_k's in the support of the V_1-component by adding on a simple root to each β_k. However, because of that changed support, we can argue as before that this is the only root in $\operatorname{supp} y$ which begins in α_{2i-s+1} or ends in α_{n-i+s}. Otherwise we can obtain $x \in Q_u^{2i-s,2i-s+1}$. Using the fact that $x \notin Q_u^{j-1,j}$ repeatedly, we may first continue this process $j-2i$ times. At that stage the roots in the support of the resulting V_1-component are $\tau_k := \alpha_{k,j-k}$ for $1 \leq k \leq s$ and thus far we have $\alpha_{i+1,n}, \alpha_{i+2,n-1}, \dots, \alpha_{j-s,n-j+i+s+1}$ in $\operatorname{supp} y$ and moreover these roots are the only ones beginning or ending with these particular simple roots. We now continue this process by applying pairs of orthogonal reflections relative to $(\alpha_{j-1}, \alpha_1), (\alpha_{j-2}, \alpha_2), \dots, (\alpha_{j-s+1}, \alpha_{s-1})$. These fix the obtained V_1-component by succesively interchanging τ_k with τ_{k+1} for $1 \leq k < s$.

Since $P \cdot x \not\subset P \cdot Q_u^{j-1,j}$, we continue this procedure and once it stops we see that we have $\alpha_{i+1,n}, \alpha_{i+2,n-1}, \dots, \alpha_{j,n-j-i+1}$ in $\operatorname{supp} y$ and these are now all the roots in $\operatorname{supp} y$. Now after applying suitable reflections from W_L and elements from T, we may suppose that the V_2-component of our representative is of the form $y' = \prod_{k=1}^{j-i} u_{\gamma_k}(1)$ without changing the obtained V_1-component.

Finally, we repeatedly apply pairs of orthogonal reflections relative to α_{j-k} and α_{j+k}, for $k = s, s-1, \dots, 1$, in this order. Each of these fixes y', by interchanging consecutive γ_k's, but removes successively simple roots 'from the right' from the τ_k's. We continue until we obtain again the initial β_k's as the support of the V_1-component.

Therefore, we have arrived at a representative of the form

$$z = \prod_{k=1}^{s} u_{\beta_{i-k+1}}(1) \prod_{l=1}^{j-i} u_{\gamma_l}(1) u' \quad \text{for some } u' \in P_u' \text{ and some fixed } s,\ 1 \leq s < i.$$

Next observe that we can remove all roots from the support of u' of the form $\alpha_{k,l}$ with $1 \leq k \leq s$ and $j \leq l \leq n$ as well as for $1 \leq k \leq i$ and $j \leq l \leq 2j-i-1$ without changing the obtained components in V_1 and V_2.

Further, since $s < i$ and $z \notin Q_u^{s,s+1}$, we may suppose that $\eta_1 = \alpha_{s+1,2j-i}$ is in the support of u' (cf. (4) above). As before we can remove all roots from $\operatorname{supp} u'$ of the form $\alpha_{s+1,l}$ with $2j-i < l \leq n$ and $\alpha_{k,2j-i}$ with $s+1 < k \leq i$. Since $z \notin Q_u^{s+1,s+2}$, in case $s < i-1$, we may suppose that η_2 is in $\operatorname{supp} u'$ and may remove all other roots beginning in α_{s+2} or ending in α_{2j-i+1}. We continue in this fashion. Then, since by assumption $P \cdot x \not\subset \mathcal{E}$, we arrive at the following representative

$$z' = \prod_{k=1}^{s} u_{\beta_{i-k+1}}(1) \prod_{l=1}^{j-i} u_{\gamma_l}(1) \prod_{m=1}^{i-s} u_{\eta_m}(1) \quad \text{for some } s \text{ with } 1 \leq s < i.$$

As shown all the parameters can be chosen to equal 1 using the action of the maximal torus T, as the set of roots in the support of z' is linearly independent. In particular, there is only a finite number of orbits in this instance.

The proof of this particular case for $1 \leq i < j-i$ and $2j-1 \leq n$ is now complete.

Remark 4.1. In our proof of Theorem 1.3 above, the assumption that $\operatorname{char} K = 0$ is not necessary, nor has it been used. The argument applies in arbitrary characteristic and thus, so does Theorem 1.3.

5. Proof of Theorem 1.4

First we need to consider several relevant cases for groups of type D_n. Let G be of type B_n and let $J = \Pi \setminus \{\alpha_{n-1}\}$. Then the standard parabolic subgroup $P = P_J$ of G acts on P_u with a finite number of orbits by Theorem 1.1. By Remark 3.1 the regular simple subgroup H of G which is supported by the long roots in Ψ is of type D_n and $\Pi(H) = \{\alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\}$ is a set of simple roots of H. Then $Q = H \cap P$ is the standard (relative to $\Pi(H)$) parabolic subgroup of H corresponding to $\Pi(H) \setminus \{\alpha_{n-1}, \alpha_{n-1} + 2\alpha_n\}$ and by Proposition 2.6 we conclude that the number of Q-orbits on Q_u is finite. Thus we have

Proposition 5.1. *Let G be of type D_n and let $J = \Pi \setminus \{\alpha_{n-1}, \alpha_n\}$. Then $P = P_J$ acts on P_u with a finite number of orbits.*

We may also obtain the result of Proposition 5.1 by means of fixed point subgroups similar to our proof of Theorem 1.1 in Section 3.

Proposition 5.2. *Let G be of type D_n and let $J = \Pi \setminus \{\alpha_1, \alpha_{n-1}\}$ or $\Pi \setminus \{\alpha_1, \alpha_n\}$. Then $P = P_J$ acts on P_u with a finite number of orbits.*

Proof. By symmetry it suffices to only consider the case $J = \Pi \setminus \{\alpha_1, \alpha_{n-1}\}$. We use the same abbreviation as in Section 3 for sums of simple roots. For $1 \leq k < l \leq n$ let

$$\alpha_{k,l} := \alpha_k + \alpha_{k+1} + \cdots + \alpha_l,$$

and $\alpha_{k,k} := \alpha_k$ for $1 \leq k \leq n$. Let $\gamma := \alpha_{1,n-1}$ and $\delta := \gamma + \alpha_n$.

Let V_1 be the product of all those root subgroups in some fixed order whose roots involve α_1 as a summand but not α_{n-1} and let V_2 be the product of those root subgroups in some fixed order whose roots involve α_{n-1} but not α_1. Observe that $\overline{P}_u := P_u/P_u'$ is a module for L and decomposes as a direct sum of irreducibles $\overline{V}_1 \oplus \overline{V}_2$. By Proposition 2.1 P operates on P_u' and on $\overline{P}_u$ with a finite number of orbits. Note that $\overline{V}_1$ is isomorphic to the natural module for L' and $\overline{V}_2$ is isomorphic to the second exterior power of the dual of $\overline{V}_1$.

We define a set of orthogonal roots in $\overline{V}_2$. Let

$$\begin{aligned} \gamma_1 &:= \alpha_{n-1} \qquad \text{and} \\ \gamma_k &:= \alpha_{n-2k+1,n} + \alpha_{n-2k+2,n-2} \qquad \text{for } 1 < k \leq \left[\tfrac{n-1}{2}\right]. \end{aligned}$$

Observe that γ_k is the highest root in the standard subsystem of Ψ of type D_{n-2k+1}.

Note that $\{\gamma_k \mid 1 \leq k \leq \left[\frac{n-1}{2}\right]\}$ is a maximal set of orthogonal long roots in $\operatorname{supp} V_2$. Since the L-orbit structure on $\overline{V}_2$ is equivalent to that of an Abelian unipotent radical case, it follows from [8, Theorem 1.2(a)] that $\{\prod_{k=1}^{s} u_{\gamma_k}(1) \mid 0 \leq s \leq \left[\frac{n-1}{2}\right]\}$ is a complete set of L-orbit representatives of $\overline{V}_2$. Likewise $u_{\alpha_1}(1)$ is a representative for the nontrivial L-orbit on $\overline{V}_1$.

<u>Case 1</u>: Let $x \in V_1 P'_u \setminus P'_u$. Then we may suppose that x is of the form

$$x = u_{\alpha_1}(1)u' \qquad \text{for some } u' \in P'_u.$$

By conjugating with suitable elements from root subgroups relative to $\alpha_{2,n-1}$, $\alpha_{2,n}$, $\alpha_{2,n} + \alpha_{k,n-2}$ for $3 \leq k \leq n-2$ we can eliminate all roots from $\operatorname{supp} u'$ except the highest ρ. Thus $u_{\alpha_1}(1)$ and $u_{\alpha_1}(1)u_\rho(1)$ represent all orbits in this instance. Since α_1 and ρ are orthogonal, the parameters can be chosen to be 1, as shown.

<u>Case 2</u>: Let $x \in V_2 P'_u \setminus P'_u$. Then we may suppose that x is of the form

$$x = \prod_{i=1}^{s} u_{\gamma_i}(1)u' \qquad \text{for some } u' \in P'_u \text{ and some } s,\ 1 \leq s \leq \left[\tfrac{n-1}{2}\right].$$

Since $s \geq 1$, we can remove γ and δ from $\operatorname{supp} u'$. Further, we can eliminate all roots from $\operatorname{supp} u'$ of the form $\delta + \alpha_{k,n-2}$ for $n-2s+1 \leq k \leq n-2$ by conjugating x with suitable elements from the root subgroups relative to $\alpha_{1,n-3}$, $\alpha_{1,n-4}, \ldots,$ $\alpha_{1,n-2s+1}$, $\alpha_{1,n-2s}$.

Suppose there are additional roots in $\operatorname{supp} u'$. Then after applying reflections in W_L orhogonal to $\gamma_1, \ldots, \gamma_s$, we may suppose that $\rho \in \operatorname{supp} u'$. Finally, all roots of the form $\delta + \alpha_{i,n-2}$ for $3 \leq i \leq n-2s$ can be eliminated from the support of u' by conjugation with suitable elements from the root subgroups relative to $-\alpha_2, -\alpha_{2,3}, \ldots, -\alpha_{2,n-2s-1}$. Finally, for each $1 \leq s \leq \left[\frac{n-1}{2}\right]$ we obtain

$$\prod_{i=1}^{s} u_{\gamma_i}(1) \qquad \text{or} \qquad \prod_{i=1}^{s} u_{\gamma_i}(1)u_\rho(1)$$

as an orbit representative, and the latter case does not occur when n is odd and $s = \left[\frac{n-1}{2}\right]$. Thus there is only a finite number of orbits in this case. Again, all parameters can be chosen equal 1, as the roots involved are mutually orthogonal.

<u>Case 3</u>: Let $x \in P_u \setminus P'_u$, but x not in any of the two subsets considered in the previous cases. Thus we may assume that x is of the form

$$x = u_{\alpha_1}(1)yu' \qquad \text{for some } y \in V_2,\ u' \in P'_u,$$

where $u_{\alpha_1}(1)y$ is one element of a given finite set of L-orbit representatives of $\overline{P}_u$ and $y \neq 1$. As in Case 1 above we can remove all roots in $\operatorname{supp} u'$ with the possible exception of ρ. This process does not alter y, as all elements used in this conjugation procedure fix V_2 pointwise. Therefore, we have as a representative of the P-orbit through x

$$u_{\alpha_1}(1)y \quad \text{or} \quad u_{\alpha_1}(1)yu_\rho(a) \quad \text{for some } a \in K \setminus \{0\}.$$

In addition, if in the second case there is a root in $\operatorname{supp} y$ which is not orthogonal to ρ, then we can eliminate ρ from the support of the latter representative. Otherwise, all roots in the support of $u_{\alpha_1}(1)y$ are orthogonal to ρ, and thus, using the action of T we may obtain $u_{\alpha_1}(1)yu_\rho(1)$ as representative in this instance.

Altogether there is only a finite number of orbits of P on P_u. This completes the proof of the proposition.

Remark 5.3. The configurations in Propositions 5.1 and 5.2 are the only ones for classical groups, apart from those in type A_n, when $\ell(P_u) = 2$ for a parabolic subgroup P which is not maximal. This follows from Lemma 2.8.

Proof of Theorem 1.4. Let G be as in Theorem 1.4. For G of type A_n the statement is precisely Theorem 1.3. Otherwise, P is a maximal parabolic subgroup of G, or G is of type D_n and P is as in Proposition 5.1 or 5.2, by Remark 5.3. The desired result follows from Theorem 1.1 and Propositions 5.1 and 5.2.

Remark 5.4. The statement of Theorem 1.4 also holds for simple algebraic groups of type G_2 and F_4, see [6, Remark 5.5]. There is good evidence that it holds for all other exceptional groups as well. For instance, if P is the standard parabolic subgroup of G such that $P_u' = U_\rho$, then $\ell(P_u) = 2$ and it is shown in [6, Proposition 5.1] that the number of P-orbits on P_u is finite.

Remark 5.5. As for the proof of Theorem 3.1, the argument in the proof of Proposition 5.2 shows that the statement holds without any restrictions on the characteristic of the field.

The overall assumption that $\operatorname{char} K = 0$ is due to the fact that some of the crucial ingredients used in our analysis, namely Propositions 2.6 and 2.7, require characteristic zero [6]. However, it is likely that Theorems 1.1 and 1.4 apply in arbitrary characteristic as well.

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