SOME UNIVERSAL RESULTS ON THE BEHAVIOR OF INCREMENTS OF PARTIAL SUMS

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Abstract. We establish very general one-sided results on the lim sup behavior of increments of suitably normalized partial sums of i.i.d. random variables. Our main results apply to arbitrary nondegenerate positive random variables which need not have any finite moments. As a corollary we can show that such results also hold for not necessarily positive random variables whose negative parts have finite moment-generating functions.

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1. Introduction. Let $X, X_1, X_2, \ldots$ be independent identically distributed (i.i.d.) nondegenerate random variables with distribution function F. As usual, set $S_0 = 0, S_n = X_1 + \ldots + X_n, n \geq 1$, and let $Lt := \log(max(t, e)), LLt := L(Lt), t \geq 0$. Then it follows from the classical Hartman-Wintner LIL that one has under the assumptions $EX = 0, EX^2 = 1$, with probability one,

$$\limsup_{n \to \infty} |S_n|/(2nLLn)^{1/2} = 1. \tag{1.1}$$

It is also known that the above moment assumptions are necessary for (1.1) to hold. Much more general LIL results, however, are attainable, if one considers one-sided versions of (1.1) and uses different centering and norming sequences (see, for instance, Klass (1976), Mason (1994), and Pruitt (1981)).

Mason (1994) has recently shown that if X is an arbitrary nondegenerate positive random variable, then one can find centering constants $\{\mu_n\}$ and norming constants $a_n \nearrow \infty$ such that with probability one,

$$0 \leq \limsup_{n \to \infty} \{n\mu_n - S_n\}/a_n \leq 2^{1/2}. \tag{1.2}$$

Moreover, it follows from a result of Einmahl and Mason (1994) that the lim sup in (1.2) is positive, whenever X is in the Feller class. Recall that a random variable is in the Feller class, if one can find centering constants $\{\delta_n\}$ and norming constants $\{c_n\}$ such that

(1.3) $\qquad (S_n - \delta_n)/c_n$ is tight with nondegenerate subsequential limits.

Note in particular that any random variable in the domain of attraction to a stable law of index $\alpha \in (0, 2]$ belongs to the Feller class.

The purpose of the present paper is to investigate whether one can obtain similar general results on the behavior of increments of partial sums of i.i.d. random variables. Our starting point is a result of Csörgő and Révész (1979), the exact formulation of which requires some additional notation. Let $0 < \kappa_n \leq n$ be a non-decreasing sequence of real numbers satisfying

(1.4) $\qquad \kappa_n/n$ is non-increasing, and

(1.5) $\qquad \kappa_n/\log n \to \infty$ as $n \to \infty$.

Set $k_n := [\kappa_n],\ n \geq 1$, where $[x]$ denotes the integer part of $-\infty < x < \infty$, and let for $1 \leq k \leq n$,

$$M_n(k) := \max_{0 \leq i \leq n-k} \max_{0 \leq j \leq k} |S_{i+j} - S_i|.$$

THEOREM A. *Assuming that X is a random variable with finite moment generating function, mean zero and variance one and $\{k_n\}$ is a sequence as above, we have with probability one*

$$\limsup_{n\to\infty} \alpha_n M_n(k_n) = 1, \text{ and} \tag{1.6}$$

$$\limsup_{n\to\infty} \alpha_n |S_n - S_{n-k_n}| = 1, \tag{1.7}$$

where $\alpha_n := (2k_n\gamma_n)^{-1/2}$ and $\gamma_n := \log(n \log n/\kappa_n)$, $n \geq 1$.

The assumption that X has a finite moment generating function is quite strong, and it is very natural to ask whether one can obtain similar results under less restrictive assumptions. This is in general not possible if one wants to have a result of this type which is valid for any sequence κ_n satisfying (1.4) and (1.5). If one focuses on particular sequences κ_n, however, one can prove (1.6) and (1.7) under weaker assumptions which are specific to the choice of κ_n. It turns out that the larger one chooses κ_n, the weaker are the corresponding assumptions. In particular if $\kappa_n = n$, (1.6) and (1.7) hold if and only if $EX = 0$ and $EX^2 = 1$. (For more information in this direction refer to Section 3.2 of Csörgő and Révész (1981), and Hanson and Russo (1983).)

We shall show that there is a *universal* one-sided version of Theorem A which is valid for *any* nondegenerate positive random variable X and *any* sequence κ_n satisfying (1.4), (1.5) and an additional very mild assumption. Similarly as in the one-sided LIL result (1.2) we have to introduce centering and norming constants, which will be defined in terms of suitable truncated mean and variance functions.

To that end, consider the quantile function $Q(u) = \inf\{x : F(x) \geq u\}, 0 < u < 1$, and set for $0 < s < 1$,

$$\mu(s) := \int_0^{1-s} Q(u)du, \tag{1.8}$$

$$\nu(s) := \mu(s) + sQ(1-s), \tag{1.9}$$

$$\tau^2(s) := \int_0^{1-s} Q^2(u)du - (\mu(s))^2, \tag{1.10}$$

and

$$\sigma^2(s) := \int_0^{1-s} Q^2(u)du + sQ^2(1-s) - \nu^2(s). \tag{1.11}$$

Let k_n and κ_n be as above, and define for each $n \geq 1$,

$$b_n := \gamma_n/(\kappa_n + \gamma_n), \text{ and } \beta_n := (2k_n\gamma_n)^{-1/2}\sigma(b_n)^{-1},$$

where γ_n is as in Theorem A. From assumption (1.13) below it follows that $\{b_n\}$ and $\{\beta_n\}$ are non-increasing sequences.

Finally set for each $n \geq 1, 0 < k \leq n$ and $0 < b < 1$

$$M_n(b,k) := \max_{0\leq i\leq n-k} \max_{0\leq j\leq k} \{j\mu(b) - S_{i+j} + S_i\}. \tag{1.12}$$

Then our first result can be formulated as follows,

THEOREM 1. *Let $X \geq 0$ be a nondegenerate random variable, and let $k_n = [\kappa_n]$, where $0 < \kappa_n \leq n$ is a nondecreasing sequence satisfying* (1.4), (1.5) *and*

$$\gamma_n/\kappa_n \downarrow 0 \text{ as } n \to \infty. \tag{1.13}$$

Then we have with probability one,

$$\limsup_{n\to\infty} \beta_n M_n(b_n, k_n) \leq 1, \text{ and} \tag{1.14}$$

$$\limsup_{n\to\infty} \beta_n\{k_n\mu(b_n) - (S_n - S_{n-k_n})\} \geq 0. \tag{1.15}$$

Note in particular that if we apply Theorem 1 with $\kappa_n = n$, we obtain a version of the universal LIL (1.2), and we can infer from Lemma 4 of Mason (1994) that the constants 1 and 0 in (1.14) and (1.15), respectively, are sharp.

One might next ask whether the lim sup in (1.15) (and consequently that in (1.14)) is positive whenever X is in the Feller class. Our next Theorem 2 shows that this is indeed the case, and it also shows that if the sequence κ_n is small enough, then one can supplement statement (1.14) by a lim inf result.

THEOREM 2. *Assume that $X \geq 0$ is a nondegenerate random variable in the Feller class. Then we have for any sequence k_n as in Theorem 1 with probability one,*

$$\limsup_{n\to\infty} \beta_n\{k_n\mu(b_n) - (S_n - S_{n-k_n})\} \geq C_1, \tag{1.16}$$

where C_1 is a positive constant depending on the distribution of X.

Furthermore, if κ_n satisfies in addition to the above assumptions,

$$\log(n/\kappa_n)/LLn \to \infty \text{ as } n \to \infty, \tag{1.17}$$

we have with probability one,

$$\liminf_{n\to\infty} \beta_n M_n(b_n, k_n) \geq C_2, \tag{1.18}$$

where C_2 is a positive constant depending on the distribution of X.

Using Theorems 1 and 2 in combination with Theorem A, one can easily show that these results are also valid for not necessarily non-negative random variables if one imposes suitable conditions on the negative part X^- of X. In particular one can show,

COROLLARY. *The conclusions of Theorems 1 and 2 remain true when the assumption that X is non-negative is replaced by $E\exp(tX^-) < \infty$ in a neighborhood of 0.*

We note that it is possible to find weaker sufficient conditions if one is only interested in specific sequences k_n. For instance, one can show if $k_n = n$, then it is enough to assume that X^- has a finite second moment.

As we already indicated earlier, such results have been obtained in the setting of Theorem A. Refer to Theorem 2 of Lai (1972), Sect. 3.2 of Csörgő and Révész (1981), or Hanson and Russo (1983). Given the work of these authors, it is straightforward to obtain similar refinements of our results, and we will not provide any details here.

Our proof of Theorems 1 and 2 is based on the so-called quantile transformation method. We first use a martingale argument to obtain suitable (one-sided) maximal inequalities (see Lemma 2.1 and Lemma 2.2). Combining these inequalities with some facts about the truncated mean/variance functions, we obtain statement (1.14) in Sect. 2. The proof of statement (1.15), which will be carried out in Sect. 3, is based on a refinement of the method employed by Mason (1994). Among other things, we will need an extended version of a result of Kiefer (1972) (see Lemma 3.2 below). We then prove Theorem 2 in Sect. 4. The main difficulty is to find a good lower bound for probabilities of the type $P\{k_n\mu_n(b_n) - S_{k_n} \geq C(\beta_n)^{-1}\}$, where $C > 0$. This will be accomplished by Lemma 4.1. Finally, the proof of the Corollary will be given in Sect. 5.

2. Proof of Theorem 1 (Part 1).

2.1. *Some Auxiliary Results*

The following two exponential inequalities will be crucial for the proof.

LEMMA 2.1. *Let W_j, $1 \leq j \leq n$ be independent non-negative random variables satisfying $E\exp(tW_j) < \infty$, $0 < t < \infty$, $EW_j = \mu_j$ and $EW_j^2 \leq \alpha^2$, $1 \leq j \leq n$. Then we have* *for* $x \geq 0$ *and* $p > 1$,

$$\text{(a)} \qquad P\left\{ \max_{1\leq k\leq n} \sum_{j=1}^{k} (\mu_j - W_j) \geq x \right\} \leq \exp(-x^2/2n\alpha^2),$$

$$\text{(b)} \qquad P\left\{ \max_{1\leq k\leq \ell\leq n} \sum_{j=k}^{\ell} (\mu_j - W_j) \geq x \right\} \leq \frac{p}{p-1} \exp(-x^2/2np\alpha^2).$$

Proof. Noting that for any $t > 0$, $\left\{\exp\left(t\sum_{j=1}^{k}(\mu_j - W_j)\right),\ 1 \leq k \leq n\right\}$ is a submartingale, we can infer via Doob's inequality,

$$\text{(2.1)} \qquad P\left\{ \max_{1\leq k\leq n} \sum_{j=1}^{k} (\mu_j - W_j) \geq x \right\} \leq E\exp\left(t\sum_{j=1}^{n}(\mu_j - W_j)\right) \cdot \exp(-tx)$$

Next observe that for some $0 < \bar{t} < t$,

$$E\exp(-tW_j) = 1 - t\mu_j + \frac{1}{2}t^2 EW_j^2 - \frac{t^3}{6}EW_j^3 \exp(-\bar{t}W_j)$$

$$\leq \exp(-t\mu_j + t^2\alpha^2/2),\ 1 \leq j \leq n.$$

Using the independence of the W_j's, we readily obtain that the probability in (2.1) is

$$\leq \exp\left(\frac{1}{2}nt^2\alpha^2 - tx\right),$$

which after choosing $t = x/n\alpha^2$ implies (a).

We now turn to the proof of (b). Let for $1 \leq m \leq n$, and $t > 0$,

$$Y_m(t) = \max_{1\leq j\leq m} \exp\left(t\sum_{i=j}^{m}\{\mu_i - W_i\}\right),$$

and set $\mathcal{F}_m := \sigma(W_1, \ldots, W_m),\ 1 \leq m \leq n$. Then it is easy to see that

$$\text{(2.2)} \qquad \{(Y_m(t), \mathcal{F}_m) : 1 \leq m \leq n\} \text{ is a submartingale.}$$

Using again Doob's inequality, we find that

$$P\left\{ \max_{1\leq k\leq \ell\leq n} \sum_{j=k}^{\ell} (\mu_j - W_j) \geq x \right\}$$

$$= P\left\{ \max_{1\leq m\leq n} Y_m(t) \geq \exp(tx)\right\}$$

$$\leq EY_n(t)\exp(-tx)$$

Next observe that

$$Y_n(t) = \max_{1 \le k \le n} Z_{k,n}(t),$$

where $Z_{k,n}(t) = \exp\left(t \sum_{j=1}^{k} (\mu_{n+1-j} - W_{n+1-j})\right)$, $1 \le k \le n$ is another submartingale. Using Lyapounov's inequality in combination with Doob's L_p-inequality (see, for instance, inequality (35) on page 247 of Chow and Teicher (1988)), we can conclude that

$$EY_n(t) \le \frac{p}{p-1} (EZ_{n,n}(pt))^{1/p}$$

which by the proof of (a) is

$$\le \frac{p}{p-1} \exp\left(\frac{1}{2} n t^2 p \alpha^2\right)$$

Combining this with the above bound for the probability in (b) and setting $t = x/np\alpha^2$, we obtain our assertion. □

Arguing as in the proof of Lemma 2.1.a, one can use the proof of Bernstein's inequality (see, for instance, Dudley (1984), page 14), to obtain the following maximal version of it.

LEMMA 2.2. *Let Z_j, $1 \le j \le n$ be independent mean zero random variables satisfying for some $M > 0$,*

$$|Z_j| \le M, \ 1 \le j \le n. \tag{2.3}$$

Then we have for $x \ge 0$,

$$P\left\{\max_{1 \le k \le n} \sum_{j=1}^{k} Z_j \ge x\right\} \le \exp\left(-x^2 / \left(2 \sum_{j=1}^{n} \sigma_j^2 + 2Mx/3\right)\right),$$

where as usual $\sigma_j^2 := EZ_j^2$, $1 \le j \le n$.

For the sake of easier reference later on we now collect some more or less known facts about the functions $\mu(s)$, $\sigma^2(s)$, and $\tau^2(s)$, $0 < s < 1$ which have been introduced in Section 1.

FACT 2.1. *Let $X \ge 0$ be a nondegenerate random variable. We have,*

(a) $\limsup_{s \downarrow 0} s^{1/2} Q(1-s)/\sigma(s) \le 1$,

(b) $\limsup_{s \downarrow 0} \tau^2(s)/\sigma^2(s) \le 1$.

For a proof refer to relations (2.1) and (2.2) of Mason (1994).

FACT 2.2. *If $X \geq 0$ is a random variable with $EX^2 = \infty$, we have,*

$$\limsup_{s \downarrow 0} \ \mu(s)/\sigma(s) = 0.$$

Fact 2.2 can be proven by the same argument as in Lemma 2.1 of Csörgő, Häusler and Mason (1988b). The next fact is obvious so that we can omit the proof.

FACT 2.3. *If X is a random variable with $EX^2 < \infty$, we have*

(a) $\lim_{s \downarrow 0} s^{1/2} Q(1-s)/\sigma(s) = 0,$

(b) $\lim_{s \downarrow 0} \mu(s)/\sigma(s) = EX/\left(\mathrm{Var}(X)\right)^{1/2}.$

2.2 *Conclusion of the proof.* Let $U_1, U_2, \ldots$ be a sequence of i.i.d. uniform (0,1) random variables. Since $(X_1, X_2, \ldots) \stackrel{\mathcal{D}}{=} (Q(U_1), Q(U_2), \ldots)$, we can and do assume that

$$X_i = Q(U_i), \ i = 1, 2, \ldots \tag{2.4}$$

Let for any $\lambda > 1, \ m_r := [\lambda^r], \ r = 1, 2, \ldots$.

LEMMA 2.3. *For any $\varepsilon > 0$ there exists a $\lambda(\varepsilon) > 1$ such that for $1 < \lambda < \lambda(\varepsilon)$ and large enough r,*

$$\max_{m_r \leq n \leq m_{r+1}} \beta_n k_n \left(\mu(b_n) - \mu(b_{m_r})\right) \leq \varepsilon. \tag{2.5}$$

Proof. Notice that for any $m_r \leq n \leq m_{r+1}$,

$$0 \leq \left(2b_n \sigma^2(b_n)\right)^{-1/2} \left(\mu(b_n) - \mu(b_{m_r})\right) \leq \left(2b_{m_{r+1}}\right)^{-1/2} \int_{1-b_{m_r}}^{1-b_n} \frac{Q(u)}{\sigma(1-u)} du$$

which by Fact 2.1 (a) is for large enough r bounded above by

$$\left(b_{m_{r+1}}\right)^{-1/2} \int_{1-b_{m_r}}^{1-b_{m_{r+1}}} (1-u)^{-1/2} du = 2\left\{\left(b_{m_r}/b_{m_{r+1}}\right)^{1/2} - 1\right\}. \tag{2.6}$$

Now conditions (1.4) and (1.5) imply that as $r \to \infty$,

$$b_{m_r}/b_{m_{r+1}} \sim (\gamma_{m_r}/\kappa_{m_r})/(\gamma_{m_{r+1}}/\kappa_{m_{r+1}}) \leq m_{r+1}/m_r,$$

which in turn is asymptotically equivalent to λ, and we see that if r is large, the quantities in (2.6) are less than $3(\lambda^{1/2} - 1)$ (say). This establishes (2.5). $\square$

Choosing $\lambda(\varepsilon) > 1$ as in Lemma 2.3, we see that for all $1 < \lambda < \lambda(\varepsilon)$ and sufficiently large r,

$$\max_{m_r < n \le m_{r+1}} \beta_n M_n(b_n, k_n) \le \max_{m_r < n \le m_{r+1}} \beta_n M_n(b_{m_r}, k_n) + \varepsilon \tag{2.7}$$

which in turn is less than or equal to

$$\max_{0 \le i \le m_{r+1} - k_{m_r}} \max_{0 \le j \le k_{m_{r+1}}} \beta_{m_r} \{j\mu(b_{m_r}) - (S_{j+i} - S_i)\} + \varepsilon \tag{2.8}$$

$$=: \beta_{m_r} M_r(\lambda) + \varepsilon.$$

Therefore, it is enough to prove,

$$\limsup_{\lambda \downarrow 1} \limsup_{r \to \infty} \beta_{m_r} M_r(\lambda) \le 1 \quad \text{a.s.} \tag{2.9}$$

We now have to introduce some additional notation. Let for any $0 < b < 1$ and $i \ge 1$,

$$W_i(b) := Q(U_i) 1\{U_i < 1 - b\} + Q(1 - b) 1\{U_i \ge 1 - b\},$$

and set

$$T_j = T_j(r) := \sum_{i=1}^{j} \{\nu(b_{m_r}) - W_i(b_{m_r})\}, \ j \ge 1.$$

It is easy to see that we have for any $0 \le i \le m_{r+1} - k_{m_r}$ and $0 \le j \le k_{m_{r+1}}$,

$$j\mu(b_{m_r}) - (S_{j+i} - S_j) \le T_{i+j}(r) - T_i(r). \tag{2.10}$$

Therefore, the proof of (2.9) is further reducible to showing

$$\limsup_{\lambda \downarrow 1} \limsup_{r \to \infty} \ \beta_{m_r} \overline{M}_r(\lambda) \le 1, \tag{2.11}$$

where

$$\overline{M}_r(\lambda) := \max_{0 \le i \le m(r) - k(r-1)} \max_{0 \le j \le k(r)} \{T_{i+j}(r) - T_i(r)\}, \text{ and}$$

$$m(r) := m_{r+1}, \ k(r) := k_{m(r)}, \ r \ge 1.$$

Further, let for any $0 < \delta < 1$ and $r \ge 1$, $k(\delta,r) := [\delta k(r)]$, $\ell(\delta,r) := [m(r)/k(\delta,r)] + 1$. Next choose for any $0 \le i \le m(r) - k(r-1)$ the unique integer $1 \le \ell \le \ell(\delta,r)$ for which

$$(\ell - 1) k(\delta, r) \le i < \ell k(\delta, r). \tag{2.12}$$

If $0 \le j \le k(r)$, we have two possibilities:

CASE 1. $\quad i+j \ge \ell k(\delta, r)$

CASE 2. $\quad i+j < \ell k(\delta, r)$

In the first case we can conclude that

$$T_{i+j} - T_i = T_{i+j} - T_{\ell k(\delta,r)} + T_{\ell k(\delta,r)} - T_i$$

$$\le \max_{1 \le \ell \le \ell(\delta,r)} \max_{0 \le j \le k(r)} (T_{\ell k(\delta,r)+j} - T_{\ell k(\delta,r)}) + \max_{1 \le \ell \le \ell(\delta,r)} \max_{(\ell-1)k(\delta,r) \le i \le j \le \ell k(\delta,r)} (T_j - T_i)$$

$$=: \overline{M}_{r,1}(\lambda) + \overline{M}_{r,2}(\lambda).$$

Noticing that we have in the second case,

$$T_{i+j} - T_i \le \overline{M}_{r,2}(\lambda),$$

we readily obtain in both cases,

$$\overline{M}_r(\lambda) \le \overline{M}_{r,1}(\lambda) + \overline{M}_{r,2}(\lambda). \tag{2.13}$$

We now claim that

$$\limsup_{r \to \infty} \beta_{m(r-1)} \overline{M}_{r,1}(\lambda) \le \lambda \text{ a.s.} \tag{2.14}$$

To prove (2.14) we first assume that $EX^2 = \infty$. In this case it follows from Fact 2.2. that

$$\lim_{r \to \infty} \sigma^2(b_{m(r-1)})/EW_1^2(b_{m(r-1)}) = 1. \tag{2.15}$$

Using Lemma 2.1.a it is easy to see that for large enough r,

$$P\{\overline{M}_{r,1}(\lambda) \ge \lambda \beta_{m(r-1)}^{-1}\} \tag{2.16}$$

$$\le 2\delta^{-1} \left(m(r)/k(r) \right) \exp\left(-\sqrt{\lambda} \gamma_{m(r-1)} \right),$$

where we use the fact that $\limsup_{r \to \infty} k(r)/k(r-1) \le \lambda$. By the definition of the sequence $\{\gamma_n\}$ the last term is bounded above by

$$2\delta^{-1} \left(m(r)/m(r-1) \right) \left(\log m(r-1) \right)^{-\sqrt{\lambda}}.$$

Since this is a summable sequence, (2.14) follows by a straightforward application of the Borel-Cantelli lemma.

If $EX^2 < \infty$, we can use Lemma 2.2. in combination with Fact 2.3.a, and we obtain (2.14) by an obvious modification of the above proof.

We next show that with probability one,

$$\limsup_{r\to\infty} \beta_{m(r-1)}\overline{M}_{r,2}(\lambda) \leq K\lambda\delta^{1/2}, \tag{2.17}$$

where K is a positive constant depending on the distribution of X.

Since $\delta > 0$ can be made arbitrarily small, we obtain (2.11) by combining (2.13), (2.14) and (2.17). The proof of (2.17) is based on Lemma 2.1.b. We set $K = 2^{1/2}$ if $EX^2 = \infty$, and $K = 2^{1/2}\left(1 + (EX)^2/\mathrm{Var}(X)\right)^{1/2}$ if $EX^2 < \infty$. Applying the aforementioned lemma with $p = 2$, we readily obtain that for large r,

$$\begin{aligned} P\{\overline{M}_{r,2}(\lambda) \geq K\lambda\delta^{1/2}(\beta_{m(r-1)})^{-1}\} \\ \leq 4\delta^{-1}\,(m(r)/k(r))\exp\left(-\sqrt{\lambda}\gamma_{m(r-1)}\right), \end{aligned} \tag{2.18}$$

where we use (2.15) if $EX^2 = \infty$ and Fact 2.3 if $EX^2 < \infty$.

Now we can use the same Borel-Cantelli argument as before, and we get (2.17), thereby finishing the first part of the proof of Theorem 1. □

3. Proof of Theorem 1 (Part 2). We first introduce a subsequence $\{n_j : j \geq 1\}$ as follows.

Let $n_1 = 1$, and set $n_{j+1} := \min\{n : n - k_n \geq n_j\},\ j \geq 1$ if $\rho := \lim_{n\to\infty} k_n/n < 1$. (Note that in this case $n - k_n \to \infty$ as $n \to \infty$.)

If $\rho = 1$, we set $n_j := j^j,\ j \geq 1$.

LEMMA 3.1. *We have in both cases,*

(a) $\sum_{j=1}^{\infty} k_{n_j}/(n_j \log n_j) = \infty$, *and*
(b) $\sum_{j=1}^{\infty}(k_{n_j}/n_j \log n_j)^{1+\delta} < \infty,\ \delta > 0$.

Proof. We only prove Lemma 3.1. for the case $\rho < 1$. From the definition of the sequence $\{n_j\}$ it follows that in this case,

$$k_{n_j} \sim n_j - n_{j-1} \text{ as } j \to \infty, \tag{3.1}$$

whence we have for large enough j,

$$(3.2) \qquad (k_{n_j}/n_j \log n_j)^{1+\delta} \leq 2(n_j - n_{j-1})/n_j(\log n_j)^{1+\delta},$$

and (b) is obvious.

To prove (a), note that the assumption $\rho < 1$ implies

$$(3.3) \qquad \limsup_{j\to\infty} n_j/n_{j-1} < \infty,$$

and (a) follows from (3.1) in combination with the fact that

$$\int_2^\infty (x \log x)^{-1} dx = \infty.$$

□

Next consider the events $A_n := \{U_i \leq 1 - b_n,\ n - k_n < i \leq n\},\ n \geq 1$. The following lemma is related to a fact contained in the proof of Theorem 2 of Kiefer (1972) (see relation (2.11), Mason (1994).) For some other result in this direction refer also to Theorem 1 of Rothmann and Russo (1993).

LEMMA 3.2. $P(A_{n_j}$ *infinitely often*$) = 1$

Proof. We first show,

$$(3.4) \qquad \sum_{j=1}^{\infty} P(A_{n_j}) = \infty.$$

To verify (3.4), observe that

$$(3.5) \qquad P(A_{n_j}) = \exp\left(k_{n_j} \log(1 - b_{n_j})\right),\ j \geq 1.$$

Using the inequality

$$\log(1 - b_{n_j}) \geq -b_{n_j} - b_{n_j}^2$$

which holds for large j since $b_n \to 0$ as $n \to \infty$, we get after some calculation,

$$(3.6) \qquad P(A_{n_j}) \geq k_{n_j}/(n_j \log n_j),$$

which in combination with Lemma 3.1a implies (3.4).

Noticing that the events $A_{n_j},\ j \geq 1$ are independent if $\rho < 1$, we obtain the assertion of Lemma 3.2. in this case immediately from (3.4) via the "usual" Borel-Cantelli lemma.

To prove Lemma 3.2 for the case $\rho = 1$, we use a more general version of the Borel-Cantelli lemma (see, e.g. Theorem 6.4, Billingsley (1986)). According to this result it is enough to check that

$$\liminf_{m\to\infty} \sum_{i,j\leq m} P(A_{n_i} \cap A_{n_j}) \Big/ \left(\sum_{i=1}^{m} P(A_{n_i})\right)^2 \leq 1. \tag{3.7}$$

In order to establish (3.7), it is obviously sufficient to prove that for a suitable sequence $\delta_j \to 0$,

$$P(A_{n_j} \cap A_{n_{j+\ell}}) \leq (1+\delta_j) P(A_{n_j}) P(A_{n_{j+\ell}}),\ \ell = 1, 2, \ldots \tag{3.8}$$

If $n_{j+\ell} - k_{n_{j+\ell}} \geq n_j$, the events A_{n_j} and $A_{n_{j+\ell}}$ are independent, and (3.8) is trivial.

If $n_{j+\ell} - k_{n_{j+\ell}} < n_j$, we have,

$$P(A_{n_j} \cap A_{n_{j+\ell}}) = P(A_{n_j}) P(A_{n_{j+\ell}}) (1 - b_{n_{j+\ell}})^{n_{j+\ell} - k_{n_{j+\ell}} - n_j},$$

which is for large enough j,

$$\leq P(A_{n_j}) P(A_{n_{j+\ell}}) \exp(2 n_j b_{n_{j+\ell}}).$$

Recalling that $k_{n_{j+\ell}} \geq n_{j+\ell}/2$ for large j, we readily obtain that

$$2 b_{n_{j+\ell}} n_j \leq 5 (n_j / n_{j+1}) \log\log n_{j+1} =: \delta_j$$

for large j, where $\delta_j \to 0$ by definition of $\{n_j\}$. □

We are now ready to prove (1.15). As in the proof of (1.14), we will assume that

$$X_i = Q(U_i),\ i = 1, 2, \ldots \tag{3.9}$$

where $U_1, U_2, \ldots$ is a sequence of i.i.d. uniform $(0,1)$ random variables.

In view of Lemma 3.2. it is enough to prove,

$$\sum_{j=1}^{\infty} p_j(\varepsilon) < \infty,\ \varepsilon > 0, \tag{3.10}$$

where $p_j(\varepsilon) := P\left(\left\{\sum_{n_j - k_{n_j}+1}^{n_j} \left(Q(U_i) - \mu(b_{n_j})\right) \geq \varepsilon \beta_{n_j}^{-1}\right\} \cap A_{n_j}\right),\ j = 1, 2, \ldots.$

Set for $j \geq 1$,

$$Z_j := \max_{n_j - k_{n_j} < i \leq n_j} U_i,$$

and denote its distribution function by $F_j(x)$. Then we can rewrite $p_j(\varepsilon)$ as

$$\int_0^{1-b_{n_j}} P\left(\left\{\sum_{n_j-k_{n_j}+1}^{n_j} (Q(U_i)-\mu(b_{n_j})) \geq \varepsilon\beta_{n_j}^{-1}\right\} \big| Z_j = x\right) dF_j(x).$$

Arguing as in Mason (1994) we can infer that for large j

$$p_j(\varepsilon) \leq p_{j,1} + p_{j,2}(\varepsilon), \tag{3.11}$$

where $p_{j,1} := F_j(1-2b_{n_j})$,

$$p_{j,2}(\varepsilon) := \int_{1-2b_{n_j}}^{1-b_{n_j}} P\left\{\sum_{i=1}^{k_{n_j}-1} (Q(V_i(x)) - EQ(V_i(x))) \geq \frac{\varepsilon}{2}\beta_{n_j}^{-1}\right\} dF_j(x)$$

and $V_i(x),\ 1 \leq i < k_{n_j}$ are independent, uniform $(0,x)$ random variables.

Noting that for large enough j,

$$p_{j,1} \leq (1-2b_{n_j})^{k_{n_j}} \leq (k_{n_j}/n_j \log n_j)^{3/2},$$

we get from Lemma 3.1.b,

$$\sum_{j=1}^{\infty} p_{j,1} < \infty. \tag{3.12}$$

To bound $p_{j,2}(\varepsilon)$, we will use the Bernstein inequality (see Lemma 2.2. above). Using the fact that $Q(V_i(x)) \geq 0$ (since $X \geq 0$), it is easy to see that for $1-2b_{n_j} \leq x \leq 1-b_{n_j}$,

$$|Q(V_i(x)) - EQ(V_i(x))| \leq Q(1-b_{n_j}) \tag{3.13}$$

which in view of Fact 2.1.a is for large j,

$$\leq \frac{3}{2} b_{n_j}^{-1/2}\sigma(b_{n_j}).$$

Moreover, recalling Fact 2.1.b, we find that for $1-2b_{n_j} \leq x \leq 1-b_{n_j}$ and large j,

$$\mathrm{Var}(Q(V_i(x))) \leq 2\sigma^2(b_{n_j}),\ 1 \leq i < k_{n_j}. \tag{3.14}$$

We now can infer from Lemma 2.2. that for $1-2b_{n_j} \leq x \leq 1-b_{n_j}$,

$$P\left\{\sum_{i=1}^{k_{n_j}-1} (Q(V_i(x)) - EQ(V_i(x))) \geq (\varepsilon/2)\beta_{n_j}^{-1}\right\}$$

$$\leq (k_{n_j}/n_j \log n_j)^{\delta_\varepsilon},$$

where $\delta_\varepsilon > 0$.

Using the last inequality, we readily obtain that

$$p_{j,2}(\varepsilon) \le F_j(1-b_{n_j})(k_{n_j}/n_j \log n_j)^{\delta_\varepsilon},$$

which is for large j bounded above by

$$(k_{n_j}/n_j \log n_j)^{1+\delta_\varepsilon/2}$$

whence we have by Lemma 3.1.b,

$$\sum_{j=1}^{\infty} p_{j,2}(\varepsilon) < \infty. \tag{3.15}$$

Combining (3.11), (3.12) and (3.15), we get (3.10), and consequently the assertion.

4. Proof of Theorem 2. As in the previous sections we can and do assume that $X_i = Q(U_i)$, $i = 1, 2, \ldots$, where U_i, $i = 1, 2, \ldots$ is a sequence of i.i.d. uniform $(0,1)$ random variables.

We will make repeated use of the following fact, which can be inferred from relation (1.42c) in Csörgő, Häusler and Mason (1988a).

FACT 4.1. *Let $X \ge 0$ be a nondegenerate random variable in the Feller class. There exists a constant $K \ge 1$ such that*

$$\limsup_{s \downarrow 0} \sigma(s)/\tau(s) \le K.$$

The next lemma will be the crucial tool for the proof of Theorem 2.

LEMMA 4.1. *Let $X \ge 0$ be a nondegenerate random variable in the Feller class. There exist positive constants C, δ depending on the distribution of X such that for large enough n,*

$$P\{k_n\mu(b_n) - S_{k_n} \ge C\beta_n^{-1}\} \ge (k_n/(n \log n))^{1-\delta}.$$

Proof. (i) Consider the event $E_n := \{U_i \le 1 - \lambda b_n,\ 1 \le i \le k_n\}$, where $\lambda = 1 - 2\delta$ and $0 < \delta < 1/8$ will be specified later. We then obviously have,

$$P\{k_n\mu(b_n) - S_{k_n} \ge C\beta_n^{-1}\} \ge P(E_n)P\left(\{k_n\mu(b_n) - S_{k_n} > C\beta_n^{-1}\}|E_n\right) \tag{4.1}$$

Since for large n,

$$P(E_n) \geq (k_n/(n\log n))^{1-1.5\delta} \tag{4.2}$$

it is enough to show that if we choose δ and C small enough, we have for large n,

$$P\left(\{k_n\mu(b_n) - (S_n - S_{n-k_n}) > C\beta_n^{-1}\} \,|E_n\right) \geq (k_n/(n\log n))^{\delta/2} \tag{4.3}$$

(ii) To prove (4.3), we first note that the conditional probability in (4.3) is equal to

$$P\left\{k_n\mu(b_n) - \sum_{j=1}^{k_n} Q(V_{n,j}) > C\sqrt{2k_n\gamma_n}\sigma(b_n)\right\}, \tag{4.4}$$

where $V_{n,j},\ 1 \leq j \leq k_n$ are independent uniform $(0, 1-\lambda b_n)$ random variables.

Consequently, we have

$$\sum_{j=1}^{k_n} (EQ(V_{n,j}) - \mu(b_n)) = k_n\left(\mu(\lambda b_n)(1-\lambda b_n)^{-1} - \mu(b_n)\right),$$

which is for large n,

$$\leq 2k_n\left\{\mu(\lambda b_n) - \mu(b_n) + b_n\mu(b_n)\right\}.$$

Using Fact 2.1.a, we get for large n,

$$\begin{aligned} \mu(\lambda b_n) - \mu(b_n) &= \int_{\lambda b_n}^{b_n} Q(1-u)du \\ &\leq 1.5\int_{\lambda b_n}^{b_n} \sigma(u)/\sqrt{u}du \\ &\leq 3\delta\sigma(\lambda b_n)\sqrt{b_n/\lambda}, \end{aligned}$$

which since $\lambda \geq 3/4$, is bounded above by

$$4\delta\sqrt{\gamma_n/k_n}\sigma(\lambda b_n).$$

Moreover, on account of Facts 2.2 and 2.3.b, we have as $n \to \infty$,

$$b_n\mu(b_n) = O\left(b_n\sigma(b_n)\right) = o\left(\sqrt{\gamma_n/k_n}\sigma(\lambda b_n)\right).$$

We now can conclude that if n is large, the probability in (4.4) is bounded below by

$$\text{(4.5)} \qquad P\left\{\sum_{j=1}^{k_n} Z_{n,j} > (C+3\delta)\sqrt{2k_n\gamma_n}\sigma(\lambda b_n)\right\},$$

where $Z_{n,j} := EQ(V_{n,j}) - Q(V_{n,j}),\ 1 \le j \le k_n$.

Further letting

$$\tau_n^2 := \mathrm{Var}(Z_{n,1}),\ n \ge 1,$$

it is easy to see that

$$\text{(4.6)} \qquad \tau_n^2 \sim \tau^2(\lambda b_n) \text{ as } n \to \infty.$$

Recalling Fact 4.1 we find that if n is large, the probability in (4.5) is bounded below by

$$\text{(4.7)} \qquad P\left\{\sum_{j=1}^{k_n} Z_{n,j} > 2K(C+3\delta)\sqrt{k_n\gamma_n}\tau_n\right\}.$$

(iii) To find a lower bound for the last probability, we shall use a blocking argument in combination with the Berry-Esseen inequality.

Let $m_n := [\alpha k_n/\gamma_n] - 1$, where $\alpha > 0$ will be specified later.

Further, set $\ell_n := [k_n/m_n] - 1$. It is easy now to see that if n is large enough, the probability in (4.7) is bounded below by

$$\text{(4.8)} \qquad p_n(1)^{\ell_n} \cdot p_n(2),$$

where

$$p_n(1) := P\left\{\sum_{j=1}^{m_n} Z_{n,j} > 2K(C+3\delta)\sqrt{k_n\gamma_n}\tau_n/\ell_n\right\}$$

$$\ge P\left\{\sum_{j=1}^{m_n} Z_{n,j} > 3K\sqrt{\alpha m_n}(C+3\delta)\tau_n\right\},$$

and

$$p_n(2) := P\left\{\sum_{j=1}^{k_n-\ell_n m_n} Z_{n,j} > 0\right\}.$$

A straightforward application of the Berry-Esseen inequality yields,

$$p_n(1) \geq 1 - \Phi\left(3K\sqrt{\alpha}(C+3\delta)\right) - E|Z_{n,1}|^3/\sqrt{m_n}\tau_n^3. \tag{4.9}$$

Using Facts 2.1.a and 4.1 we get for large n,

$$E|Z_{n,1}|^3 \leq \tau_n^2 Q(1-\lambda b_n) \leq 2K\tau_n^3/\sqrt{b_n}.$$

Combining this bound with (4.9), we find that for large n,

$$p_n(1) \geq 1 - \Phi\left(3K\sqrt{\alpha}(C+3\delta)\right) - 3K/\sqrt{\alpha}. \tag{4.10}$$

Letting $C = \delta = 4\alpha^{-1}$, and using the trivial inequality $1-\Phi(t) \geq \frac{1}{2} - t,\ t > 0$, we readily obtain that the last term is

$$\geq \frac{1}{2} - 51K/\sqrt{\alpha}$$

which is equal to $\frac{1}{8}$ if we choose $\alpha = 136^2K^2$. A slight modification of the above argument shows that we also have with this choice of α,

$$p_n(2) \geq \frac{1}{8}, \tag{4.11}$$

and we can infer via (4.8) that for large n,

$$P\left\{\sum_{j=1}^{k_n} Z_{n,j} > 2K(C+3\delta)\sqrt{k_n\gamma_n}\tau_n\right\} \geq \exp\left(-\log(8)(\ell_n+1)\right) \tag{4.12}$$

By the definition of ℓ_n we have as $n \to \infty$,

$$\ell_n + 1 \sim \gamma_n/\alpha = \delta\gamma_n/4, \tag{4.13}$$

and we get for large n,

$$\exp\left(-\log(8)(\ell_n+1)\right) \geq \exp(-\gamma_n\delta/2),$$

which in combination with (4.5), (4.7) and (4.12) implies (4.3), and consequently the assertion of Lemma 4.1. $\square$

If $\rho := \lim_{n\to\infty} k_n/n < 1$, we can define $\{n_j\}$ as in Sect. 3. In this case, the events

$$A_j := \left\{k_{n_j}\mu(b_{n_j}) - (S_{n_j} - S_{n_j - k_{n_j}}) > C\beta_{n_j}^{-1}\right\},\ j = 1, 2, \ldots$$

are independent, and we obtain (1.16) from Lemmas 3.1 and 4.1 via a standard Borel-Cantelli argument.

We now turn to the more difficult case, when $\rho = 1$. In this case we set for $j \geq 1$,

$$n_j := \left[\exp(j^\lambda)\right],$$

where $1 < \lambda < (1-\delta)^{-1}$ and δ is as in Lemma 4.1.

Define for any $j \geq 1$, $m_j := (n_j - n_{j-1}) \wedge k_{n_j}$, and consider the events

$$B_j := \left\{m_j\mu(b_{n_j}) - (S_{n_j} - S_{n_j - m_j}) \geq \frac{1}{2}C\beta_{n_j}^{-1}\right\},$$

where C is as in Lemma 4.1.

We then obviously have,

$$\begin{aligned} P(B_j) &\geq P\left\{k_{n_j}\mu(b_{n_j}) - (S_{n_j} - S_{n_j - k_{n_j}}) \geq C\beta_{n_j}^{-1}\right\} \\ &\quad - P\left\{(k_{n_j} - m_j)\mu(b_{n_j}) - S_{k_{n_j} - m_j} \geq \frac{1}{2}C\beta_{n_j}^{-1}\right\} \\ &=: p_j(1) - p_j(2). \end{aligned}$$

It is easy to see that for large j,

$$p_j(2) \leq P\left\{(k_{n_j} - m_j)\mu(b_{n_j}) - \sum_{i=1}^{k_{n_j} - m_j} Q(U_i)1\{U_i \leq 1 - b_{n_j}\} \geq \frac{1}{2}C\beta_{n_j}^{-1}\right\}$$

which by Chebyshev's inequality and Fact 2.1.b is

$$\leq 3C^{-2}n_{j-1}/(n_j\gamma_{n_j}).$$

It is now evident that

$$\sum_{j=1}^{\infty} p_j(2) < \infty. \tag{4.14}$$

On the other hand, we obtain via Lemma 4.1,

$$\sum_{j=1}^{\infty} p_j(1) = \infty, \tag{4.15}$$

whence we have,

$$\sum_{j=1}^{\infty} P(B_j) = \infty, \tag{4.16}$$

which in turn implies by independence and the Borel-Cantelli lemma,

$$P\{B_j \text{ infinitely often}\} = 1. \tag{4.17}$$

We next claim that we have with probability one,

$$(k_{n_j} - m_j)\mu(b_{n_j}) - (S_{n_j - m_j} - S_{n_j - k_{n_j}}) = o(\beta_{n_j}^{-1}) \text{ as } j \to \infty. \tag{4.18}$$

To verify (4.18), we first note that we can show by an obvious modification of the proof of (4.14) for any $\varepsilon > 0$,

$$\sum_{j=1}^{\infty} P\left\{ (k_{n_j} - m_j)\mu(b_{n_j}) - \sum_{n_j - k_{n_j} + 1}^{n_j - m_j} Q(U_i) 1\{U_i \le 1 - b_{n_j}\} \ge \varepsilon \beta_{n_j}^{-1} \right\} < \infty \tag{4.19}$$

Moreover, we have for $j \ge 1$,

$$P\left\{ S_{n_j - m_j} - S_{n_j - k_{n_j}} \ne \sum_{n_j - k_{n_j} + 1}^{n_j - m_j} Q(U_i) 1\{U_i \le 1 - b_{n_j}\} \right\}$$

$$\le (k_{n_j} - m_j) b_{n_j} \le n_{j-1}\gamma_{n_j}/k_{n_j}.$$

Recalling that $k_{n_j} \sim n_j$ as $j \to \infty$, it is easy to see that

$$\sum_{j=1}^{\infty} n_{j-1}\gamma_{n_j}/k_{n_j} < \infty, \tag{4.20}$$

and we obtain (4.18) by combining (4.19) and (4.20) with the Borel-Cantelli lemma.

Statement (4.17) in conjunction with (4.18) now implies our assertion (1.16) with $C_1 = C/2$.

It remains to prove (1.18). It is enough to show that if κ_n satisfies condition (1.17), we have,

$$\sum_{n=1}^{\infty} P\left\{M_n(b_n, k_n) \leq C\beta_n^{-1}\right\} < \infty, \tag{4.21}$$

where $C > 0$ is the constant in Lemma 4.1. Then using once more the Borel-Cantelli lemma, we readily obtain (1.18) with $C_2 = C$.

To prove (4.21), set $\ell_n := [n/k_n]$, and note that

$$P\left\{M_n(b_n, k_n) \leq C\beta_n^{-1}\right\}$$

$$\leq P\left(\cap_{j=1}^{\ell_n}\left\{k_n\mu(b_n) - (S_{jk_n} - S_{(j-1)k_n}) \leq C\beta_n^{-1}\right\}\right)$$

which by Lemma 4.1 is

$$\leq \exp\left(\ell_n \log\left(1 - (k_n/(n\log n))^{1-\delta}\right)\right).$$

Using the inequality $\log(1-t) \leq -t,\ 0 < t < 1$, we see that the last term is for large n

$$\leq \exp\left(-\frac{1}{2}\left(\frac{n}{k_n}\right)^{\delta}(\log n)^{\delta-1}\right)$$

which under condition (1.17) is a summable sequence.

This establishes (4.21), thereby completing the proof of Theorem 2. □

5. Proof of the Corollary. Choose $K < 0$ small enough so that $P\{X \leq K\} \leq \frac{1}{3}$ and, consequently, $Q(1-b) > K,\ 0 < b < \frac{1}{2}$.

Set for any $i \geq 1$,

$$Y_i := Y_i(K) = X_i \wedge K, \text{ and}$$

$$Z_i := Z_i(K) = X_i - Y_i.$$

Let $\overline{Q}$ be the quantile function of the distribution function of Z_1, and let $\overline{\mu}(b)$ and $\overline{\sigma}(b)$ be defined as $\mu(b)$ and $\sigma(b)$ with $\overline{Q}$ replacing Q.

Then we have for any $0 < b < \frac{1}{2}$,

$$EY_1 + \overline{\mu}(b) = \mu(b) + K \cdot b \tag{5.1}$$

Next set for $0 \leq k \leq n$,

$$\Delta_n(k) := \max_{0 \leq i \leq n-k} \max_{0 \leq j \leq k} |jEY_1 - \sum_{\ell=i+1}^{i+j} Y_\ell|,$$

and let $\overline{M}_n(b,k)$ be defined as $M_n(b,k)$ with the Z_i's replacing the X_i's.

Using Theorem A and (1.14), respectively, we can infer that with probability one,

$$\limsup_{n\to\infty} \Delta_n(k_n)/\sqrt{2k_n\gamma_n} \leq \mathrm{Var}(Y_1)^{1/2} \tag{5.2}$$

and

$$\limsup_{n\to\infty} \overline{M}_n(b_n,k_n)/\sqrt{2k_n\gamma_n}\,\overline{\sigma}(b_n) \leq 1. \tag{5.3}$$

Further observe that on account of (5.1),

$$M_n(b_n,k_n) \leq \overline{M}_n(b_n,k_n) + \Delta_n(k_n) - Kk_nb_n. \tag{5.4}$$

After some work one can show that

$$\overline{\sigma}(b) \leq \sigma(b),\ \ 0 < b < 1. \tag{5.5}$$

Combining this with the fact that under condition (1.5),

$$k_nb_n/\sqrt{k_n\gamma_n} \to 0 \text{ as } n \to \infty, \tag{5.6}$$

we readily obtain from (5.2) - (5.5),

$$\limsup_{n\to\infty} M_n(b_n,k_n)/\sqrt{2k_n\gamma_n}\,\sigma(b_n) \leq 1+\eta \tag{5.7}$$

where $\eta = 0$ or $= \mathrm{Var}(Y_1)^{1/2}/\mathrm{Var}(X)^{1/2}$ according as $EX^2 = \infty$ or $< \infty$.

Noting that we can make η arbitrarily small in the second case by choosing K small enough, we obtain (1.14) for not necessarily positive random variables whose negative parts have finite moment generating functions.

The proofs of the general versions of (1.15) and Theorem 2 are very similar. We only mention that one also needs for the proof of Theorem 2 that

$$\lim_{K\to-\infty} \limsup_{n\to\infty} \sigma(b_n)/\overline{\sigma}(b_n) = 1, \tag{5.8}$$

which can be proven after some calculation.

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