A local–global principle for finiteness properties of S–arithmetic groups over number fields

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1 Introduction

In [AT] compactness properties C_n for locally compact groups were introduced. In this paper the following theorem is proved:: Let Γ be an S–arithmetic subgroup of a linear algebraic group G over a number field. Then Γ is of type F_n if and only if for every non–Archimedian $p \in S$ the p–adic completion G_p is of type C_n. This generalizes a result of Kneser ([Kn]), who proved it for $n = 1,\ 2$. An immediate consequence is that one only has to deal with one prime at a time in the investigation of finiteness properties of S–arihmetic groups. Also, one can use the strong results on algebraic groups over local fields.

Furthermore it is shown that reductive groups over locally compact fields of characteristic 0 are of type C_∞. So the local–global principle gives a new proof of a result of Borel and Serre ([BS]), stating that S–arithmetic subgroups of reductive groups have all finiteness properties. The new proof, however, is much easier.

We begin with summarizing the basic definitions of [AT]:
Let G be a locally compact group and suppose X is a G–space. The free simplicial set on X, denoted by EX, is the simplicial set whose set of n–simplices is

$$(EX)_n = X^{n+1}.$$

The face maps are given by cancelling, the degeneracy maps by doubling one factor. Then EX is contractible and G acts on the geometric realization $|EX|$. For $C \subseteq X$ denote by $G \cdot EC$ the subcomplex of EX defined as follows: $\sigma = (x_0, \ldots, x_n)$ is a simplex of $G{\cdot}EC$ if there is a $g \in G$ such that $\{x_0, \ldots, x_n\} \subseteq gC$.

Definition 1.1 *We say that the G–space X satisfies the condition P_n (or, more briefly, that $P_n(X, G)$ holds) if for every compact $C \subseteq X$ there is a compact $D \subseteq X$ with $C \subseteq D$, such that the inclusion*

$$i : G{\cdot}EC \to G{\cdot}ED$$

induces the zero map on π_i for all $i < n$.
The group G is said to be of type C_n, if the G–space G satisfies the condition P_n.

The property C_n is a generalization of the finiteness property F_n. By using homology groups instead of homotopy groups one obtains the property CP_n, which generalizes the property FP_n. In this paper we will only be concerned with the properties C_n. However, all the results are also valid for CP_n by almost literally the same proofs.

2 Borel's finiteness theorem

To begin with we establish some notation which will remain fixed throughout the rest of the paper. It is mostly adopted form [Bo, chapter 1]. Denote by

k	an algebraic number field of finite degree
o	an ring of integers of k
V	an set of primes of k
P	an set of non–Archimedian primes
k_v	an completion of k with respect to the valuation v
o_p	an ring of p–adic integers of k_p for $p \in P$

For $v \in V - P$ let $o_v = k_v$ and for $S \subseteq V$ let $o(S)$ be the ring of S–integers of k, i.e. $o(S) = \{x \in k : x \in o_v \text{ for } v \in V - S\}$. If $S \subseteq V - P$, then $o(S) := o$.

Let G be a linear algebraic group, which we will assume to be embedded in a general linear group. There is no loss of generality in this assumption, because the local–global principle we are going to proove (theorem 3.1) does not depend on the chosen embedding: The groups on its local side are k_v–points of G, these do not at all depend on the embedding. On the global side of the principle we have the S–arithmetic group $G_{o(S)}$, which depends on the embedding, but only up to commensurability, so the embedding has no effect on its finiteness properties.

For a subring of an overfield of k let G_B be the group of elements of G with coefficients in B and with determinant a unit of B. We write G_v instead of G_{k_v} and G_A for the adele group of G. The group G_v is locally compact and for $p \in P$ the group G_{o_p} is open and compact in G_p. For $S \subseteq V$ let

$$G_S := \prod_{v \in S} G_v \ , \quad G_{A(S)} := \prod_{v \in S} G_v \times \prod_{v \in V-S} G_{o_v} \ .$$

If $S \subseteq V$ is finite, then G_S and $G_{A(S)}$ are locally compact. Let $\pi_S : G_A \to G_S$ be the canonical projection and let $j_S : G_S \to G_A$ be the injection given by the condition that the components of the image of j_S outside S are equal to the neutral element.

In case $S = V - P$ we write G_o, G_∞, G_A^∞, π_∞ and j_∞ instead of $G_{o(S)}$, G_S, $G_{A(S)}$, π_S and j_S. With every $a \in G_k$ there is associated an element of G_A, the so called principle adele of a. It is the element, whose components are all equal to a. In this way, we can identify G_k, $G_{o(S)}$ and G_o with subgroups of G_A. They are discrete, $G_{o(S)} = G_{A(S)} \cap G_k$ and $G_o = G_A^\infty \cap G_k$. A subgroup Γ of G is called arithmetic, if it is commensurable with G_o, and is called S–arithmetic if it is commensurable with $G_{o(S)}$.

Now we fix a linear algebraic group G and a finite subset $S \subseteq V$ which contains the Archimedian primes. The finiteness theorem of Borel ([Bo, theorem 5.1]) states

Theorem 2.1 *The number of double cosets $G_A^\infty \cdot x \cdot G_k$ in G_A is finite.*

The next corollaries show, that this finiteness is inherited by some subquotient groups of G_A.

Corollary 2.2 *The number of double cosets $G_A^\infty \cdot x \cdot G_{o(S)}$ in $G_{A(S)}$ is finite.*
Proof: By theorem 2.1 there are $x_1, \dots, x_n$ such that

$$G_{A(S)} \subseteq \bigcup_{i=1}^{n} G_A^\infty x_i G_k \ .$$

The x_i can be chosen to be in $G_A(S)$. Then

$$\begin{array}{rll} & G_{A(S)} \cap G_A^\infty x_i G_k & \\ = & G_A^\infty (G_{A(S)} \cap x_i G_k) & \text{because } G_A^\infty \subseteq G_{A(S)} \\ = & G_A^\infty x_i (G_{A(S)} \cap G_k) & \text{because } x_i \in G_{A(S)} \\ = & G_A^\infty x_i G_{o(S)} \ . & \end{array}$$

Hence

$$G_{A(S)} = \bigcup_{i=1}^{n} G_A^\infty x_i G_{o(S)}.$$

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In the next corollary we will use the observation: Suppose that for $i = 1,\ 2$ we are given groups F_i, G_i and H_i with $F_i < H_i$ and $G_i < H_i$. If the number of double cosets $(G_1 \times G_2) \cdot x \cdot (F_1 \times F_2)$ in $H_1 \times H_2$ is finite, then so is the number of double cosets $G_1 \cdot x \cdot F_1$ in H_1.

Now define

$$K_S := \prod_{v \in P \cap S} G_{o_v} \text{ and } G'_{o(S)} := \pi_{P \cap S}(G_{o(S)}).$$

Corollary 2.3 *The number of double cosets $K_S \cdot x \cdot G'_{o(S)}$ in $G_{P \cap S}$ is finite.*
Proof: We have

$$\begin{array}{rcl} G_{A(S)} & = & G_{P \cap S} \times G_\infty \times K_{P-S} \\ G_A^\infty & = & K_S \times G_\infty \times K_{P-S} \\ G_{o(S)} & \subseteq & G'_{o(S)} \times \pi_{V-P}(G_{o(S)}) \times \pi_{P-S}(G_{o(S)}) \end{array}$$

So the corollary is an immediate consequence of corollary 2.2 and the preceding observation. •

3 A local–global principle

We retain the terminology of the preceeding chapter. Let $S \subseteq V$ be a finite set of primes containing the Archimedian primes and let $X := G_{P\cap S}/K_S$. For $s \in P \cap S$ each of the groups G_{o_s} is open in G_s, so K_S is open in $G_{P\cap S}$ and hence X is discrete.

Consider the action of $\Gamma := G_{o(S)}$ on X. For $x = gK_S \in X$ let $Stab(x)$ be the stabilizer of x in Γ. Then $Stab(x) = \Gamma \cap gK_Sg^{-1}$. Suppose $g = (g_s)_{s\in P\cap S}$, then

$$gK_Sg^{-1} = \prod_{s\in P\cap S} g_sG_{o_s}g_s^{-1} \ .$$

Now $g_sG_{o_s}g_s^{-1}$ is open and compact in G_s, hence it is commensurable with G_{o_s}. It follows that $Stab(x)$ is commensurable with $G_{o(S)} \cap K_S = G_o$, hence it is arithmetic. So for the action of Γ on EX, the free simplicial set on X, the stabilizers of the simplices are arithmetic.

Theorem 3.1 *The group $G_{o(S)}$ is of type F_n if and only if G_s is of type C_n for all $s \in S \cap P$.*
Proof: By [AT, lemma 3.1.1] each of the groups G_s is of type C_n if and only if $G_{P\cap S}$ is so. Let $C := G_{P\cap S}$ and let as above $\Gamma := G_{o(S)}$ and $X := C/K_S$. The structure of the proof is the following: First we show that Γ is of type F_n if and only if $P_n(X, \Gamma)$ holds (see definition 1.1), then that C is of type C_n if and only if $P_n(X, C)$ holds, and finally that $P_n(X, \Gamma)$ and $P_n(X, C)$ are equivalent.

For the action of Γ on EX we have already seen that the stabilizers of the simplices are arithmetic, hence of type F_∞ ([Ra]). So the first claim follows from [Br, theorem 3.2] together with the "Hurewicz–theorem" for ind–spaces [AT, theorem 1.1.5].

Since K_S is compact, C acts properly on X, so the second claim is just the statement of [AT, theorem 3.2.2].

Now let Σ be the set of compact (i.e. finite) subsets of X. For $K \in \Sigma$ we obviously have $\Gamma{\cdot}EK \subseteq C{\cdot}EK$, so it remains to prove that there is an $L \subseteq \Sigma$, such that $C{\cdot}EK \subseteq \Gamma{\cdot}EL$.

We know that X/Γ is finite by corollary 2.3. Let x_1 be the coset of the neutral element in X and choose $x_2, \ldots, x_m$ such that

$$X = \bigcup_{i=1}^{m} \Gamma x_i \ .$$

For $1 \leq i \leq m$ there is a $t_i \in C$ such that $t_i x_1 = x_i$. Define

$$L := \bigcup_{i=1}^{m} t_i K_S K \ ,$$

then L is a compact subset of X. Let $g \in G$, then $g x_i = \gamma t_i x_1$ for suitable $i \in \{1, \ldots, m\}$ and $\gamma \in \Gamma$, hence $g \in \gamma t_i K_S$. It follows that $gK \subseteq \gamma t_i K_S K \subseteq \gamma L$, and therefore $C \cdot EK \subseteq \Gamma \cdot EL$. •

Nothing has been said so far about the group completions with respect to the Archimedian valuations. In this case, the completed group G_v is a Lie–group with finitely many connected components. It has been proved in [Ti, Kapitel 5], that these groups are always of type C_∞.

4 Horospherical subgroups

In this section we want to start investigating the compactness properties of split solvable groups. In this investigation the so called horospherical subgroups turn out to be important. For example, Abels proved in [Ab], that they are compactly presentable, and this was one of the starting points for proving his criterion for compact presentability of solvable groups.

We will show in this section, that horospherical groups are even of type C_∞. The application of this result to reductive groups gives a new proof for a theorem of Borel and Serre on finiteness properties of S–arithmetic subgroups of reductive groups ([BS]).

Definition 4.1 *Let G be a locally compact group. An automorphism $\alpha : G \to G$ is called contracting, if the sequence $\{\alpha^n\}_{n \in N}$ converges uniformly on compact subsets to the neutral element.*

Now let $S = T \ltimes U$ be a k–split solvable group over a field k. As usual, T denotes a maximal k–split torus and U the unipotent radical of S. For $t \in T$ let $\alpha_t : U \to U$ be the conjugation map $u \mapsto tut^{-1}$.

Definition 4.2 *With the notation as above $t \in T$ is called a contracting element of S, if α_t is contracting. The group S is called horospherical, if T contains a contracting element.*

As claimed above we have

Theorem 4.3 *Let $S = T \ltimes U$ be a horospherical group over a local field of characteristic 0. Then S is of type C_∞.*
Proof: To begin with, T is of type C_∞. In fact, it is the direct product of copies of k^*, and k^* contains a finitely generated abelian cocompact subgroup. Now let $K \subseteq S$ be compact, let $K_U := K \cap U$ and $K_T := K \cap T$. We may assume that $K = K_U \cdot K_T$. Let $\nu \in \boldsymbol{N}$, suppose A is a pointed simplicial set homeomorphic to the ν–sphere and $\varphi : A \to S \cdot EK$ is a pointed simplicial map, where $S \cdot EK$ is the subcomplex of the free simplicial set ES described in section 1. It follows from the simplicial approximation theorem ([Cu, thm 11.6]), that every element of $\pi_\nu(S \cdot EK)$ can be represented by such a map φ. So we have to show, that there is a compact subset $K' \subseteq S$ with $K \subseteq K'$, not depending on φ and such that φ regarded as a map to $S \cdot EK'$ is nullhomotopic.

We will make use of the following fact: Let P and Q be subcomplexes of ES and let $f, g : P \to Q$ be two simplicial maps. Suppose that for every simplex $\sigma = (s_o, \ldots, s_n)$ of P and for every $0 \leq i \leq n$ the tupel $(f(s_0), \ldots, f(s_i), g(s_i), \ldots, g(s_n))$ is a simplex of Q. Then f is homotopic to g.

Let $x \in T$ be a contracting element of S and let $\alpha : U \to U$ be the conjugation map $u \mapsto xux^{-1}$. After replacing x by a power of x if necessary we may assume that $\alpha(K_U) \subseteq K_U$. The map $s \mapsto sx^{-1}$ induces an S–equivariant automorphism on ES. Under this automorphism, $S \cdot EK$ is mapped on itself: Let σ be a simplex of $S \cdot EK$, suppose $\sigma = g(t_o u_o, \ldots, t_n u_n)$ with $g \in S$, $t_i \in K_T$ and $u_i \in K_U$. Then

$$\sigma x^{-1} = g(t_0 x^{-1} \alpha(u_0), \ldots, t_n x^{-1} \alpha(u_n)) = g x^{-1}(t_0 \alpha(u_0), \ldots, t_n \alpha(u_n)).$$

Because of $\alpha(K_U) \subseteq K_U$ we have $t_i \alpha(u_i) \in K$, hence $\sigma x^{-1} \in S \cdot EK$.

Let φx^{-1} be the composition of φ and multiplication on the right by x^{-1}. Define $L := K \cup K x^{-1}$, then φ and φx^{-1} are homotopic as maps from A to $S \cdot EL$. For the explicit description of σx^{-1} given above shows that the set of vertices of $\sigma \cup \sigma x^{-1}$ is a simplex of $S \cdot EL$. It follows by induction, that for every $c \in \boldsymbol{N}$ the maps φ and φx^{-c}, regarded as maps to $S \cdot EL$, are homotopic.

Now let $\Sigma \subseteq S$ be the set of vertices of the image of φ and let Σ_U and Σ_T be the components of the elements of Σ in U and T. Then Σ_U and Σ_T are finite and $\Sigma \subseteq \Sigma_U \cdot \Sigma_T$. Choose $c \in \boldsymbol{N}$ such that $\alpha^c(\Sigma_U) \subseteq K_U$ and define $\phi := \varphi x^{-c}$.

In the next step we want to construct a map $\psi : A \to S \cdot EK$, such that

1. ψ is homotopic to $\phi : A \to S \cdot EK$.

2. The image of ψ is contained in ET, regarded as a subcomplex of ES.

Let $p : ES \to ET$ and $i : ET \to ES$ be the maps induced by the projection of S onto T and by the inclusion of T into S. It is easy to see that $p(S{\cdot}EK) \subseteq T{\cdot}EK_T$. Now take an arbitrary simplex σ of the image of φ, say $\sigma = (t_0u_0, \dots t_nu_n)$ with $t_i \in \Sigma_T$ and $u_i \in \Sigma_U$. Then

$$\sigma x^{-c} = x^{-c}(t_0\alpha^c(u_0), \dots, t_n\alpha^c(u_n)).$$

Furthermore $p(\sigma x^{-c}) = x^{-c}(t_0, \dots, t_n)$ is a simplex of $T{\cdot}EK_T$, so there is a $t \in T$ and there are $k_i \in K_T$ for $0 \leq i \leq n$, such that $t_i = tk_i$ for all i. So finally we get

$$\sigma x^{-c} = tx^{-c}(k_0\alpha^c(u_0), \dots, k_n\alpha^c(u_n)).$$

Since by assumption $\alpha^c(u_i) \in K_U$, this shows that the simplices of the image of ϕ are of the form $\mu = t(k_ol_o, \dots, k_nl_n)$ with $t \in T$, $k_i \in K_T$ and $l_i \in K_U$. Then the set of vertices of $\mu \cup i(p(\mu))$ is a simplex of $S{\cdot}EK$, so $\psi := i \circ p \circ \phi$ fulfills the conditions 1 and 2 described above.

As mentioned in the beginning of the proof T is of type C_∞. Hence there is a compact subset L' of T with $T_K \subseteq L'$, such that the inclusion $j : T{\cdot}EK_T \to T{\cdot}EL'$ induces the zero map on π_ν. In particular, if we regard $p \circ \phi$ as a map from A to $T{\cdot}EL'$, it is nullhomotopic.

Now define $K' := L \cup L'$ and consider φ, ϕ and ψ as maps to $S{\cdot}EK'$. Then

$$\varphi \simeq \phi \simeq \psi = i \circ p \circ \phi,$$

so φ is nullhomotopic.

An analysis of the proof shows, that K' does not depend on φ or A. •

Now we are going to look at reductive groups in characteristic zero. The reference for the used terminology and results concerning these groups is always [BT].

Theorem 4.4 *Let G be a reductive group over a locally compact field k of characteristic zero. Then G_k is of type C_∞. If P is a parabolic subgroup of G, then P_k is also of type C_∞.*
Proof: Since Lie groups are always of type C_∞ ([Ti, Kapitel 5]), we may assume that k is a field with a non Archimedian valuation v.
We describe a maximal k–split solvable subgroup S of G such that S_k contains a contracting element. Then S_k is of type C_∞, and by [BT, proposition 9.3] S_k is cocompact in G_k, so G_k is of type C_∞, too ([AT, corollary 3.2.3]). Then P_k is also C_∞, because it is cocompact in G_k.

Let T be a maximal k–split torus of G. By $X^+(T)$ we denote the set of algebraic characters of T and by $X_+(T)$ its algebraic one–parameter subgroups. Let $\Phi := \Phi(T, G)$ be the set of roots of the adjoint representations of T on the Lie–algebra g of G. For every $\alpha \in \Phi$ let V_α be the subgroup of G which is generated by all those one–parameter subgroups U_α such that $t \in T$ acts on the Lie–algebra u_α of U_α by multiplication with $\alpha(t)$.

Now regard Φ as contained in the real vector space $E := X^+(T) \otimes_{\mathbf{Z}} \mathbf{R}$. Choose a Weyl chamber C of Φ and let Φ_+ be the set of positive roots with respect to C. Recall that C is contained in the dual space E^* of E and that $\phi \in \Phi_+$ if and only if $c \circ \phi > 0$ for all $c \in C$. Denote by V_+ the subgroup of G generated by all V_α with $\alpha \in \Phi_+$. Then V_+ is unipotent ([BT, 3.8 iv]) and $T \ltimes V_+$ is a maximal k–split solvable subgroup of G.
An element $t \in T$ acts contracting on V_+, if for every $\alpha \in \Phi_+$ we have $v(\alpha(t)) > 0$, where v is the valuation on k. Now $X_+(T)$ is a lattice and C is an open cone in E^*. Therefore there is a $\tau \in C \cap X_+(T)$, say τ is given by $x \mapsto (x^{\tau_1}, \ldots, x^{\tau_n})$ with $\tau_i \in \boldsymbol{Z}$. Because of $\tau \in C$ we have $\alpha(\tau(x)) = x^{s_\alpha}$ for a positive integer s_α for all $\alpha \in \Phi_+$. Choose $\pi \in k$ with $v(\pi) = 1$ and let $t := \tau(\pi)$. Then t is k–rational and $v(\alpha(t)) = v(\pi^{s_\alpha}) > 0$ for all $\alpha \in \Phi_+$. •

Corollary 4.5 *Let G and P be as in theoren 4.4 and Γ be an S–arithmetic subgroup of G or P. Then Γ is of type F_∞.*
Proof: By the local–global principle (theorem 3.1) it is sufficient to show that G_p (resp. P_p) is of type C_∞ for all $p \in S$. This has just been proved. •

Remark: The reductive case of the last corollary had been proved in [BT], the parabolic case was very recently shown by Abramenko. In both cases, the authors proved even more, namely that the S–arithmetic groups have virtually finite cohomological dimension.

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