DEFORMATIONS OF BIMODULE PROBLEMS

CHRISTOF GEISS (MEXICO D.F.)

Abstract. We prove that deformations of tame Krull-Schmidt bimodule problems with trivial differential are again tame. Here we understand deformations via the structure constants of the projective realization which may be considered as elements of an adequate variety. We present also some applications to the representation theory of vector space categories which are a special case of the above bimodule problems.

1. Introduction

Let k be an algebraically closed field. Consider the variety $\mathrm{alg}_V(\mathrm{k})$ of associative unitary k-algebra structures on a vector space V together with the operation of $\mathrm{Gl}_V(\mathrm{k})$ by transport of structure. In this context we say that an algebra Λ_1 is *deformation* of the algebra Λ_0 if the corresponding structures λ_1, λ_0 are elements of $\mathrm{alg}_V(\mathrm{k})$ and λ_0 lies in the closure of the $\mathrm{Gl}_V(\mathrm{k})$-orbit of λ_1. In [11] it was shown, using Drozd's tame-wild theorem, that deformations of tame algebras are tame.

Similar results may be expected for other classes of problems where Drozd's theorem is valid. In this paper we meet the case of bimodule problems with trivial differential (in the sense of [4]); here we interprete deformations via the structure constants of the respective projective realizations. Note, that the bimodule problems include as special cases the representation theory of finite dimensional algebras ([4, 2.2]), subspace problems (4.1) and prinjective modules ([16, 1]). We also discuss some examples concerning subspace problems.

We understand, that W.W. Crawley-Boevey has obtained similar results. P. Dräxler draw my attention to examples as in 4.3.

2. Bimodule Problems

2.1. Let us recall some basic definitions from [4] and [18]. A category is called k-*category*, if all morphism spaces are k-modules and the composition is k-bilinear; a functor between k-categories is called k-*functor*, if it is k-linear. As a general convention we will compose morphisms in k-categories from *left to right*. A k-category is a *Krull-Schmidt category*, if it has finite direct sums, split idempotents and the morphism spaces are all finite dimensional. If $\mathcal{K}$ is a k-category, a $\mathcal{K}$ left- (resp. right-) module is a contravariant (resp. covariant) k-functor $\mathcal{K} \to$ k-mod, and accordingly, a $\mathcal{K}$-bimodule is a k-functor $\mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to$ k-mod. If M is a $\mathcal{K}$-bimodule, we write conveniently $amb := mM(a,b) \in \mathcal{K}(X', Y')$ for $m \in M(X, Y)$, $a \in \mathcal{K}(X', X)$, and $b \in \mathcal{K}(Y, Y')$.

1991 *Mathematics Subject Classification.* 16G60, 16G20, 16E40.

Key words and phrases. bimodule problems, vector space categories, tame, wild, deformations, degenerations.

Part of this work was done while the author was supported by a catedra patrimonial of CONACYT at the Instituto de Matemáticas, UNAM (México).

Definition. A *Krull-Schmidt bimodule problem* is a pair $(\mathcal{K}, M)$, where $\mathcal{K}$ is a Krull-Schmidt k-category and M a $\mathcal{K}$-bimodule. The category $\mathrm{Mat}(\mathcal{K}, M)$ has as objects the pairs (X, m) with $X \in \mathrm{Obj}\,\mathcal{K}$, $m \in M(X, X)$ and $\mathrm{Mat}(\mathcal{K}, M)((X, m), (X', m')) := \{f \in \mathcal{K}(X, X') \mid mf = fm'\}$

Since in the following we will consider only bimodule problems which are Krull-Schmidt, we sometimes drop the word "Krull-Schmidt" for brevity.

Remark . The categories $\mathrm{Mat}(\mathcal{K}, M)$ are Krull-Schmidt categories, being a special case of the categories considered in [4, 2].

2.2. Let $(\mathcal{K}, M)$ be a bimodule problem where $\mathcal{K}$ has only a finite number of isoclasses of indecomposable objects. It is of *finite type*, if $\mathrm{Mat}(\mathcal{K}, M)$ admits only a finite number of isoclasses of indecomposable objects. Recall from [4] that the bimodule problem $(\mathcal{K}, M)$ may be formulated quite directly as normal free triangular linear bocs, thus if $(\mathcal{K}, M)$ is not of finite type then it is either tame or wild by the theorem of Drozd.

For convenience let us trace back the definitions of tame and wild for bocses from [2] through the constructions in [4]. In order to formulate the result, we need however some setup - essentially we adapt the corresponding notions from [8]. Let $X \in \mathrm{Obj}\,\mathcal{K}$, $\mathbf{b} := (b_1, \dots, b_n)$ some k-base of $M(X, X)$ and $\boldsymbol{\varphi} := (\varphi_1, \dots, \varphi)$ a sequence of elements of some k-algebra R, then we call the triple $(X, \mathbf{b}, \boldsymbol{\varphi})$ a *R-frame* for $(\mathcal{K}, M)$. Such a R-frame induces a functor

$$F_{X,\mathbf{b},\boldsymbol{\varphi}} \colon \mathrm{mod}\text{-}R \to \mathrm{Mat}(\mathcal{K}, M),\ N \mapsto \big(X \otimes N|_{\mathrm{k}}, \eta(\sum_{i=1}^{n} b_i \otimes N(\varphi_i))\big).$$

Here $X \otimes W$ represents the functor $\mathrm{Hom}_{\mathrm{k}}(W, \mathcal{K}(X, ?))$ (hence $X \otimes \mathrm{k}^n \cong X^n$, see [8, 2.1]); since we deal only with finite dimensional vector spaces there is a natural isomorphism $\eta \colon M(X, Y) \otimes \mathrm{Hom}_{\mathrm{k}}(V, W) \to M(X \otimes V, Y \otimes W)$. Finally $N(\varphi_i) \in \mathrm{End}_{\mathrm{k}}(N|_{\mathrm{k}})$ is induced by the module multiplication $n \mapsto n\varphi_i$.

Definition. The bimodule problem $(\mathcal{K}, M)$ is *wild*, if for some $X \in \mathrm{Obj}\,\mathcal{K}$ there exists a $\mathrm{k}\langle x, y\rangle$-frame $(X, \mathbf{b}, \boldsymbol{\varphi})$ such, that the induced functor $F_{X,\mathbf{b},\boldsymbol{\varphi}} \colon \mathrm{mod}\text{-}\mathrm{k}\langle x, y\rangle \to \mathrm{Mat}(\mathcal{K}, M)$ preserves indecomposability and isoclasses.

$(\mathcal{K}, M)$ is *tame*, if for every $X \in \mathrm{Obj}\,\mathcal{K}$ there exists a finite number of $\mathrm{k}[t]$-frames $(X, \mathbf{b}^{(j)}, \boldsymbol{\varphi}^{(j)})$ such, that for every indecomposable object (X, m) in $\mathrm{Mat}(\mathcal{K}, M)$ we have $(X, m) \cong F_{(X,\mathbf{b}^{(j)},\boldsymbol{\varphi}^{(j)})}(\mathrm{k}[t]/(t - \lambda))$ for some j and some $\lambda \in \mathrm{k}$.

Remark . If $(\mathcal{K}, M)$ corresponds to a wild subspace problem (see 4.1), we can find by the results of [8] for some X a $\mathrm{k}\langle x, y\rangle$-frame $(\mathbf{b} = (b_1, \dots, b_n), \boldsymbol{\varphi} = (1, x, y, 0, \dots, 0))$ such that the induced functor preserves indecomposability and respects isoclasses.

2.3. Let Λ be a finite dimensional associative k-algebra and M a finite dimensional Λ-Λ-bimodule. Consider $\mathcal{P}(\Lambda)$, the *category of finitely generated projective Λ-left modules* and the functor

$$\overline{M} \colon \mathcal{P}(\Lambda)^{\mathrm{op}} \times \mathcal{P}(\Lambda) \to \mathrm{k}\text{-mod}, (P_1, P_2) \mapsto \mathrm{Hom}_{\Lambda}(P_1, M \otimes_{\Lambda} P_2),$$

then $(\mathcal{P}(\Lambda), \overline{M})$ is a Krull-Schmidt bimodule problem, where $\mathcal{P}(\Lambda)$ has a finite number of isoclasses of indecomposable objects.

Remark . Every bimodule problem $(\mathcal{K}, M)$ where $\mathcal{K}$ has only a finite number of isoclasses of indecomposable objects is of the above form. Indeed, let $\{X_1, \dots X_n\}$

represent the indecomposable objects of $\mathcal{K}$ and set $\Lambda_{\mathcal{K}} := \oplus_{i,j=1}^{n} \mathcal{K}(X_i, X_j)$ which becomes with the obvious matrix multiplication a k-algebra; further set $\underline{M} := \oplus_{i,j=1}^{n} M(X_i, X_j)$ with the corresponding $\Lambda_{\mathcal{K}}$-$\Lambda_{\mathcal{K}}$-bimodule structure. Now it is not hard to see, that $\mathrm{Mat}(\mathcal{K}, M)$ and $\mathrm{Mat}(\mathcal{P}(\Lambda_{\mathcal{K}}), \underline{M})$ are equivalent categories. We call the above construction *projective realization* of $(\mathcal{K}, M)$, compare [18], [19].

2.4. For given vector spaces V, W consider the affine variety $\mathrm{algb}_V^W(\mathrm{k})$ which consists of the pairs $(\lambda, \mu) \in \mathrm{Hom}_{\mathrm{k}}(V \otimes V, V) \times \mathrm{Hom}_{\mathrm{k}}(V \otimes W \otimes V, V)$ where λ defines an associative unitary k-algebra structure Λ on V, and μ defines a Λ-bimodule structure M on W – compare [7],[11]. On $\mathrm{algb}_V^W(\mathrm{k})$ operates the algebraic group $\mathrm{Gl}_{V,W}(\mathrm{k}) := \mathrm{Gl}_V(\mathrm{k}) \times \mathrm{Gl}_W(\mathrm{k})$ by transport of structure, i.e. $(\lambda, \mu)^{g,h}(v_1 \otimes v_2, v_l \otimes w \otimes v_r) = ((v_1 g^{-1} \otimes v_2 g^{-1})\lambda g, (v_l g^{-1} \otimes w h^{-1} \otimes v_r g^{-1})\mu g)$.

Definition. The bimodule problem $(\mathcal{K}_1, M_1)$ is a *deformation* of the problem $(\mathcal{K}_0, M_0)$, if for the respective projective realizations $(\mathcal{P}(\Lambda_i), \underline{M}_i)$ the pairs $(\Lambda_i, \underline{M}_i)$ have the same underlying vector spaces (V, W), and if for the corresponding structures $(\lambda_i, \mu_i) \in \mathrm{algb}_V^W(\mathrm{k})$ (allways $i \in \{0, 1\}$) we find (λ_0, μ_0) in the closure of the $\mathrm{Gl}_{V,W}(\mathrm{k})$ orbit of (λ_1, μ_1).

Theorem. *Deformations of tame bimodule problems are tame.*

The proof will be given in 3.6.

3. Varieties

3.1. Let $(\mathcal{K}, M)$ be a bimodule problem and $X \in \mathrm{Obj}\,\mathcal{K}$ Set $\mathrm{mat}_{\mathcal{K},M}^X(\mathrm{k}) := M(X, X)$, we will consider however $\mathrm{mat}_{\mathcal{K},M}^X(\mathrm{k})$ as affine space in the sense of algebraic geometry. On $\mathrm{mat}_{\mathcal{K},M}^X(\mathrm{k})$ operates the algebraic group $\mathrm{Aut}_X(\mathrm{k}) := \{f \in \mathcal{K}(X, X) \mid f \text{ is invertible}\}$ by conjugation: $m^f := f^{-1} m f$. Thus the orbits of $\mathrm{Aut}_X(\mathrm{k})$ on $\mathrm{mat}_{\mathcal{K},M}^X(\mathrm{k})$ correspond bijectively to the isoclasses of representations of $(\mathcal{K}, M)$ of the form (X', m) with $X' \cong X$.

Next we consider the variety

$$\mathrm{emat}_{\mathcal{K},M}^X(\mathrm{k}) := \{(m, f) \in M(X, X) \times \mathcal{K}(X, X) \mid mf = fm\}$$

Remark . In general $\mathrm{emat}_{\mathcal{K},M}^X(\mathrm{k})$ is not irreducible, but it is connected. Indeed, consider the action of the multiplicative group $\mathrm{Gl}_1(\mathrm{k})$ on $\mathrm{emat}_{\mathcal{K},M}^X(\mathrm{k})$ by scalar multiplication, then we find that $(0, 0)$ lies in the closure of any orbit and consequently $(0, 0)$ lies in every irreducible component.

3.2. If $X \in \mathrm{Obj}\,\mathcal{K}$, we write $\#(X)$ for the number of direct summands of X.

Proposition. *Let $(\mathcal{K}, M)$ be a bimodule problem where $X_1, \ldots, X_n$ represent the isoclasses of indecomposable objects of $\mathcal{K}$ and $X_0 := \oplus_{i=1}^{n} X_i$. Then the following are equivalent:*

a) *$(\mathcal{K}, M)$ is tame.*
b) *$\dim \mathrm{emat}_{\mathcal{K},M}^X(\mathrm{k}) \leq \dim \mathrm{Aut}_X(\mathrm{k}) + \#(X)$ for all $X \in \mathrm{Obj}\,\mathcal{K}$*
c) *$\dim \mathrm{emat}_{\mathcal{K},M}^{X_0^m}(\mathrm{k}) \leq n^2 \dim \mathrm{Aut}_{X_0}(\mathrm{k}) + nm$ for all $m \in \mathbb{N}$*

Proof. a)$\Longrightarrow$ b) is Lemma 2 below ; b)$\Longrightarrow$ c) is trivial. c) $\Longrightarrow$ a) Follows from Drozd's Theorem and Lemma 1 below by trivial estimations. $\square$

Corollary. *Let Λ be a d-dimensional* k*-algebra and M a Λ-Λ bimodule. Then the bimodule problem $(\mathcal{P}(\Lambda), \overline{M})$ is tame iff* $\dim \operatorname{emat}^{\Lambda^m}_{\mathcal{P}(\Lambda),\overline{M}}(\mathrm{k}) \leq dm^2 + dm$ *for all* $m \in \mathbb{N}$.

Proof. Observe, that every finitely generated indecomposable projective Λ-module is a direct summand of ${}_\Lambda\Lambda \in \mathcal{P}(\Lambda)$. Moreover we trivially have $\#(\Lambda) \leq d$. Now, Lemma 2 implies "$\Longrightarrow$", while the other direction follows from Drozd's theorem and Lemma 1 by simple estimations. □

Lemma 1. *Suppose with the above notations that $(\mathcal{K}, M)$ is wild, then there exists $p \in \mathbb{N}$ such that the following inequality holds*

$$\dim \operatorname{emat}^{X_0^{pn}}_{\mathcal{K},M}(\mathrm{k}) \geq \dim \operatorname{Aut}_{X_0}(\mathrm{k})(pn)^2 + n^2 + 1$$

for all $n \in \mathbb{N}$.

Lemma 2. *Suppose with the above notations that $(\mathcal{K}, M)$ is tame, then the following inequality holds*

$$\dim \operatorname{emat}^{X}_{\mathcal{K},M}(\mathrm{k}) \leq \dim \operatorname{Aut}_{X}(\mathrm{k}) + \#(X)$$

for all objects X in $\mathcal{K}$.

3.3. **Proof of Lemma 1.** Consider the affine variety $\operatorname{mod}^n_{\mathrm{k}\langle x,y\rangle}(\mathrm{k}) := (\operatorname{End}_{\mathrm{k}}(\mathrm{k}^n))^2$ with the operation of $\mathrm{Gl}_d(\mathrm{k})$ by transport of structure, i.e. $(\mu_x, \mu_y)^g = (g^{-1}\mu_x g, g^{-1}\mu_y g)$; thus the $\mathrm{Gl}_d(\mathrm{k})$-orbits on $\operatorname{mod}^n_{\mathrm{k}\langle x,y\rangle}(\mathrm{k})$ correspond bijectively to the isoclasses of d-dimensional $\mathrm{k}\langle x,y\rangle$-(right-) modules. The elements of the locally closed subset $U_n \subset \operatorname{mod}^d_{\mathrm{k}\langle x,y\rangle}(\mathrm{k})$ are by definition of the form

$$\left(\begin{pmatrix} T_1 & 0 & \ldots\ldots & 0 \\ 0 & T_2 & 0 \ \ldots & 0 \\ \ldots\ldots\ldots\ldots & & \\ 0 & \ldots & 0 & T_{n-1} & 0 \\ 0 & \ldots\ldots & 0 & T_n \end{pmatrix}, \begin{pmatrix} T_{1,1} & \ldots\ldots\ldots\ldots & T_{1,n} \\ 1 & T_{2,2} & \ldots\ldots\ldots & T_{2,n} \\ T_{3,1} & 1 & T_{3,3} \ \ldots & T_{3,n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ T_{n,1} & \ldots & T_{n,n-2} & 1 & T_{n,n} \end{pmatrix} \right)$$

with $\prod_{i<j}(T_i - T_j) \neq 0$. The points of U_n correspond to pairwise non-isomorphic $\mathrm{k}\langle x,y\rangle$-modules with trivial endomorphism ring.

Now, let $(X, \mathbf{b}, \boldsymbol{\varphi})$ be a $\mathrm{k}\langle x,y\rangle$-frame for $(\mathcal{K}, M)$ such that the induced functor $F_{X,\mathbf{b},\boldsymbol{\varphi}}$ preserves indecomposability and respects isoclasses ($(\mathcal{K}, M)$ is wild!). Then $F_{X,\mathbf{b},\boldsymbol{\varphi}}$ gives rise to regular maps

$$F^n_{X,\mathbf{b},\boldsymbol{\varphi}} \colon \operatorname{mod}^n_{\mathrm{k}\langle x,y\rangle}(\mathrm{k}) \to \operatorname{mat}^{X\otimes \mathrm{k}^n}_{\mathcal{K},M}(\mathrm{k}), (\mu_x, \mu_y) \mapsto \sum_{i=1}^{n} b_i \otimes \varphi_i(\mu_x, \mu_y).$$

Here, for example is $xyx(\mu_x, \mu_y) := \mu_x\mu_y\mu_x$. Notice, that $F_{X,\mathbf{b},\boldsymbol{\varphi}}$ maps different $\mathrm{Gl}_n(\mathrm{k})$-orbits to different $\operatorname{Aut}_{X\otimes \mathrm{k}^n}(\mathrm{k})$-orbits, thus the restriction to U_n is injective.

Next, take $Y \in \operatorname{Obj}\mathcal{K}$ such that $X \oplus Y \cong X_0^p$ for some $p \in \mathbb{N}$ and consider the composition

$$\hat{F}_{X,\mathbf{b},\boldsymbol{\varphi}} \colon \operatorname{mod}^n_{\mathrm{k}\langle x,y\rangle}(\mathrm{k}) \xrightarrow{F_{X,\mathbf{b},\boldsymbol{\varphi}}} \operatorname{mat}^{X\otimes \mathrm{k}^n}_{\mathcal{K},M}(\mathrm{k}) \xrightarrow{\text{can.}} \operatorname{mat}^{(X\oplus Y)}_{\mathcal{K},M}(\mathrm{k}) \otimes \mathrm{k}^n.$$

Again, the restriction to U_n is injective and the constructible subset[1] $Z_n := \hat{F}_{X,\mathbf{b},\boldsymbol{\varphi}}(U_n) \subset \operatorname{mat}^{(X\oplus Y)}_{\mathcal{K},M}(\mathrm{k}) \otimes \mathrm{k}^n$ intersects any $\operatorname{Aut}_{(X\oplus Y)^n}(\mathrm{k})$ orbit at most in

[1] see [12, p. 91] for the definition and basic properties of constructible sets, compare [13] and [14] for former use of this concept in representation theory

one point. Standard calculations, using $\dim Z_n = \dim U_n = n^2+1$ show finally

$$\begin{aligned}\dim \operatorname{emat}^{X_0^{pn}}_{\mathcal{K},M}(\mathrm{k}) &= \dim \operatorname{emat}^{(X\oplus Y)^n}_{\mathcal{K},M}(\mathrm{k}) \geq \dim \operatorname{Aut}_{(X\oplus Y)^n}(\mathrm{k}) + n^2 + 1\\ &= \dim \operatorname{Aut}_{X_0}(\mathrm{k})(pn)^2 + n^2 + 1.\end{aligned}$$

3.4. **Proof of Lemma 2.** As in the case of varieties of modules, (see e.g. [6, 2] and also [15, 1.3]) we find in $\operatorname{mat}^X_{\mathcal{K},M}(\mathrm{k})$, using the fact that an object of $\operatorname{Mat}(\mathcal{K},M)$ of the form (X,m) has at most $\#(X)$ direct summands, a constructible subset C of dimension at most $\#(X)$, such that $C^{\operatorname{Aut}_X(\mathrm{k})} = \operatorname{mat}^X_{\mathcal{K},M}(\mathrm{k})$.

On the other hand, we may stratify $\operatorname{mat}^X_{\mathcal{K},M}(\mathrm{k})$ into the locally closed subsets

$$\operatorname{mat}^{X,t}_{\mathcal{K},M}(\mathrm{k}) := \{m \in \operatorname{mat}^X_{\mathcal{K},M}(\mathrm{k}) \mid \dim \operatorname{End}_{(\mathcal{K},M)}(X,m) = t\}$$

since $\dim C \leq \#(X)$ we get $\dim \operatorname{mat}^{X,t}_{\mathcal{K},M}(\mathrm{k}) \leq (\dim \operatorname{Aut}_X(\mathrm{k}) - t) + \#(X)$.

Next we consider the locally closed subsets

$$\operatorname{emat}^{X,t}_{\mathcal{K},M}(\mathrm{k}) := \{(m,f) \in \operatorname{emat}^X_{\mathcal{K},M}(\mathrm{k}) \mid m \in \operatorname{mat}^{X,t}_{\mathcal{K},M}(\mathrm{k})\}$$

of $\operatorname{emat}^{X,t}_{\mathcal{K},M}(\mathrm{k})$ and observe, that $\operatorname{emat}^{X,t}_{\mathcal{K},M}(\mathrm{k})$ is a vector bundle of rank t over $\operatorname{mat}^{X,t}_{\mathcal{K},M}(\mathrm{k})$ and therefore

$$\dim \operatorname{emat}^{X,t}_{\mathcal{K},M}(\mathrm{k}) \leq \dim \operatorname{Aut}_X(\mathrm{k}) + \#(X) \quad \forall t \in \{1,\dots,\dim \operatorname{Aut}_X(\mathrm{k})\},$$

while for all other t obviously $\operatorname{emat}^{X,t}_{\mathcal{K},M}(\mathrm{k}) = \emptyset$. This proves Lemma 2.

3.5. In this subsection we concentrate on bimodule problems of the form $(\mathcal{P}(\Lambda), \overline{M})$.

Proposition. *For given vector spaces V, W and $n \in \mathbb{N}$ the function*

$$\delta_n \colon \operatorname{algb}^W_V(\mathrm{k}) \to \mathbb{N}, (\lambda,\mu) \mapsto \dim \operatorname{emat}^{\Lambda^n}_{\mathcal{P}(\Lambda),\overline{M}}(\mathrm{k})$$

is upper semicontinous.

Proof. First, recall that for given $(\lambda,\mu) \in \operatorname{algb}^W_V(\mathrm{k})$ we write (Λ, M) for the corresponding pair of an algebra and a bimodule with underlying vectorspaces (V,W). Now, (λ,μ) induces canonically on $V^{n\times n}$ the algebra structure λ_n corresponding to $\operatorname{End}_\Lambda({}_\Lambda\Lambda^n)$ and on $W^{n\times n}$ the $\operatorname{End}_\Lambda({}_\Lambda\Lambda^n)$-bimodule structure μ_n corresponding to $\operatorname{Hom}_\Lambda(\Lambda^n, M\otimes_\Lambda \Lambda^n)$. With this setup we may consider the variety

$$\begin{aligned}\operatorname{algbemat}^{W,n}_V(\mathrm{k}) := \{((\lambda,\mu),m,f) \in \operatorname{algb}^W_V(\mathrm{k}) \times W^{n\times n} \times V^{n\times n} \mid \\ \mu_n(f\otimes m\otimes \mathrm{id}) = \mu_n(\mathrm{id}\otimes m\otimes f)\}\end{aligned}$$

and the canonical projection $\pi_n \colon \operatorname{algbemat}^{W,d}_V(\mathrm{k}) \to \operatorname{algb}^W_V(\mathrm{k})$. By construction we may identify the fibres $\pi_n^{-1}(\lambda,\mu)$ with $\operatorname{emat}^{\Lambda^n}_{\mathcal{P}(\Lambda),\overline{M}}(\mathrm{k})$.

Now, the dimension of the fibres of π_n is upper semicontinous by Chevalley's theorem since π_n admits the canonical section

$$\sigma_n \colon \operatorname{algb}^W_V(\mathrm{k}) \to \operatorname{algbemat}^{W,n}_V(\mathrm{k}),\ (\lambda,\mu) \mapsto ((\lambda,\mu),0,0)$$

which meets any component of every fibre of π_n, see remark 3.1. □

3.6. **Proof of Theorem 2.4.** Let the bimodule problem $(\mathcal{K}_1, M_1)$ be a deformation of the tame bimodule problem $(\mathcal{K}_0, M_0)$, and let $(\lambda_i, \mu_i) \in \mathrm{algb}_V^W(\mathrm{k})(i = 0, 1)$ be the structure constants of the corresponding projective realizations. Thus we have for all $m \in \mathbb{N}$

$$\dim \mathrm{emat}^{\Lambda_1^m}_{\mathcal{P}(\Lambda_1),\overline{M_1}}(\mathrm{k}) \leq \dim \mathrm{emat}^{\Lambda_0^m}_{\mathcal{P}(\Lambda_0),\overline{M_0}}(\mathrm{k}) \leq [V : k](m^2 + m)$$

Here, the first inequality comes from Proposition 3.5 (see our definition of deformations), while the second one is Corollary 3.2. On the other hand this chain of inequality shows the tameness of $(\mathcal{P}(\Lambda_1), \overline{M}_1)$ by the same corollary, and thus the tameness of $(\mathcal{K}_1, M_1)$.

4. Examples

4.1. The main application we have in mind is the case of vector space categories. Let us recall the definitions: If $\mathcal{K}$ is a Krull-Schmidt k-category together with a functor $|\cdot|\colon \mathcal{K} \to \text{k-mod}$, then the pair $(\mathcal{K}, |\cdot|)$ is called a vector space categor. The subspace category $\mathcal{U}(\mathcal{K}, |\cdot|)$ has as objects the triples of the form (X, V, φ) with $X \in \mathrm{Obj}\,\mathcal{K}$, $V \in \mathrm{Obj}(\text{k-mod})$ and $\varphi \in \mathrm{Hom}_{\mathrm{k}}(V, |X|)$ with the obvious morphisms.

$\mathcal{U}(\mathcal{K}, |\cdot|)$ is canonically equivalent to the representations of the bimodule problem $(\underline{\mathcal{K}}, M_{|\cdot|})$, where $\underline{\mathcal{K}} := \mathcal{K} \times \text{k-mod}$ and

$$M_{|\cdot|}\colon \underline{\mathcal{K}}^{\mathrm{op}} \times \underline{\mathcal{K}} \to \text{k-mod},\ \big((X_1, V_1), (X_2, V_2)\big) \mapsto \mathrm{Hom}_{\mathrm{k}}(V_1, |X_2|)$$

The equivalence is given by

$$\mathcal{U}(\mathcal{K}, |\cdot|) \to \mathrm{Mat}(\underline{\mathcal{K}}, M_{|\cdot|}),\ (X, V, \varphi) \mapsto ((X, V), \varphi)$$

Thus we can speak of deformations of vector space categories, meaning deformations of the corresponding bimodule problem.

For applications the following setup is more handy: Recall from 2.3 that $(\underline{\mathcal{K}}, M_{|\cdot|})$ is equivalent to $(\mathcal{P}(\underline{\Lambda}), \mathrm{Hom}_{\underline{\Lambda}}(-, \underline{M} \otimes_{\underline{\Lambda}} -)$ by a standard construction. Now, observe, $\underline{\Lambda} = \Lambda \times \mathrm{k}$ with $\mathcal{K} \cong \mathcal{P}(\Lambda)$. Moreover $\underline{M}$ inherits a natural k-Λ-bimodule stucture, and we find a canonical isomorphism $\mathcal{U}(\mathcal{K}, |\cdot|) \cong \mathcal{U}(\mathcal{P}(\Lambda), \underline{M} \otimes_\Lambda -)$, the *projective realization* of $(\mathcal{K}, |\cdot|)$. This prompts us to study for given vectorspaces V and W the variety $\mathrm{algmod}_V^W(\mathrm{k})$ which consists of the pairs $(\lambda, \mu) \in \mathrm{Hom}_{\mathrm{k}}(V \otimes V, V) \times \mathrm{Hom}_{\mathrm{k}}(W \otimes V, W)$, where λ defines an associative unitary k algebra Λ on V, and μ defines a Λ right module on W with the natural action of $\mathrm{Gl}_V(\mathrm{k}) \times \mathrm{Gl}_W(\mathrm{k})$ by transport of structure. This gives rise to an other obvious definition for deformations of vector space categories: $(\mathcal{K}_1, |\cdot|_1)$ is a deformation of $(\mathcal{K}_0, |\cdot|_0)$ if for the corresponding projective realizations $(\mathcal{P}(\Lambda_i), M_i \otimes_{\Lambda_i} -)$ ... (compare 2.3).

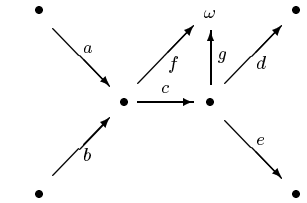
Remark . Deformations in this latter sense are special cases of deformations with respect to the first definition. In deed, there are natural embeddings $\mathrm{algmod}_V^W(\mathrm{k}) \to \mathrm{algb}_{V \oplus \mathrm{k}}^W(\mathrm{k})$ and $\mathrm{Gl}_V(\mathrm{k}) \times \mathrm{Gl}_W(\mathrm{k}) \to \mathrm{Gl}_{V \oplus \mathrm{k}}(\mathrm{k}) \times \mathrm{Gl}_W(\mathrm{k})$ compatible with the above constructions. This implis directly our affirmation.

Corollary. *Deformations of tame vector space categories are tame.*

4.2. Suppose for a moment, that $\operatorname{char} \mathrm{k} \neq 2$ and consider the algebraic family of vectorsppace categories $(\mathcal{P}(\Lambda_t), \Lambda_t \otimes_{\Lambda_t} -)_{t \in \mathrm{k}}$, where $\Lambda_t = \mathrm{k}[T]/(T^2 + tT + t) \times \mathrm{k}[S]/(S^2 + tS + t) \times \mathrm{k}$. Thus $\Lambda_t \cong \mathrm{k} \times \mathrm{k} \times \mathrm{k} \times \mathrm{k} \times \mathrm{k}$ for $t \neq 0$, while $\Lambda_0 = \mathrm{k}[T]/(T^2) \times \mathrm{k}[S]/(S^2) \times \mathrm{k}$. The case $t \neq 0$ represents the five-subspace problem which is well known to be wild, and by the corollary above we conclude, that also the case $t = 0$ represents a wild problem.

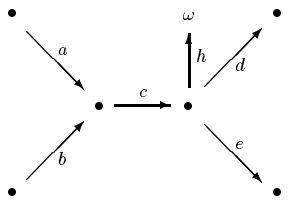
4.3. Consider the following two algebras which we define by presenting their quivers with relations.

Γ_0 :



$0 = af = bf = cg$

Γ_1 :

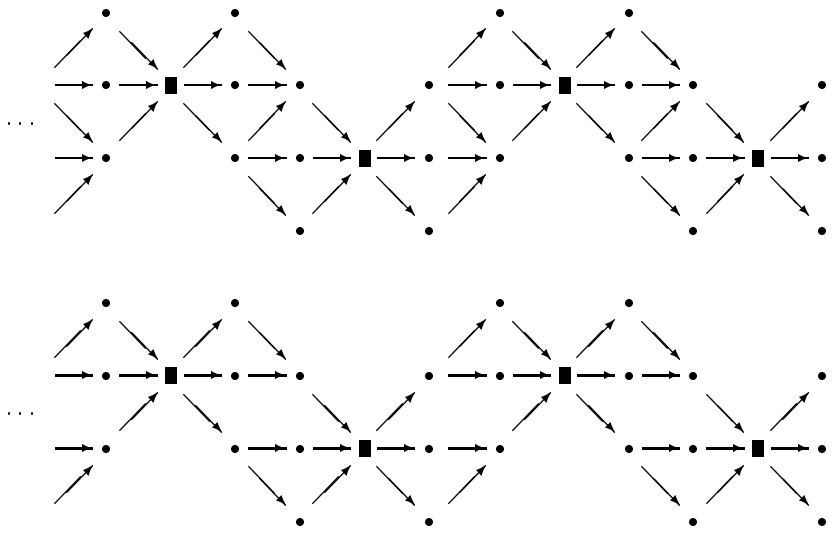


$0 = ach = bch$

Γ_0-mod can be described by the representations of a clannish algebra [3], this means especially that Γ_0 is tame. Γ_1 is a deformation of Γ_0 thus it is tame by [11]. We observe, that both algebras are one-point extensions of the hereditary algebra Δ of type $\widetilde{\mathbb{D}}_5$, i.e. $\Gamma_0 = \Delta[M_0]$ and $\Gamma_1 = \Delta[M_1]$ by regular modules. Moreover there exists an exact sequence of Δ-modules

$$0 \to \tau S \to M_1 \to S \to 0$$

where $M_0 = \tau S \oplus S$ thus M_1 is a deformation of the semisimple Δ-module M_0. Now let us consider the patterns (see [17] for definitions and conventions) for these one-point extensions:



The corresponding vector space categories must be tame since we allready know, that the Γ_i are tame. This is not new, indeed these patterns appear already in Ringel's list of tame patterns.

If we take only finite pieces of these two patterns and consider the corresponding vector space categories, say $(\mathcal{K}, |\cdot|_i)$ we find that $(\mathcal{K}, |\cdot|_1)$ is a deformation of $(\mathcal{K}, |\cdot|_0)$. Here, $\mathcal{K}$ is the Krull-Schmidt category generated by a finite number of indecomposable Δ-modules.

References

[1] K. Bongartz, *On degenerations and extensions of finite dimensional modules,* preprint (1991), To appear in Adv. Math.

[2] W.W. Crawley-Boevey, *On tame algebras and bocses,* Proc. London Math. Soc. **56** (1988), 451–483.

[3] W.W. Crawley-Boevey, *Functorial Filtrations* II*: Clans and the Gelfand problem,* J. London Math. Soc. **40** (1989), 9–30.

[4] W.W. Crawley-Boevey, *Matrix problems and Drozd's theorem,* in: Banach Center Publ. **26(1)**, Warsaw (1990), 199–222.

[5] Y.A. Drozd, *Tame and wild matrix problems,* in: Representation Theory II, Lecture Notes in Math. **832** (1980), 242–258.

[6] Y.A. Drozd, G.M. Greuel, *Tame-wild dichotonomy for Cohen-Macaulay modules,* Math. Ann. **294** (1992), 387–394.

[7] P. Gabriel, *Finite representation type is open,* in: Representations of Algebras, Lecture Notes in Math. **488** (1975) 132–155.

[8] P. Gabriel, L.A. Nazarova, A.V. Roiter, V.V. Sergejchuk, D. Vossieck, *Tame and wild subspace problems,* Ukrainian Math. Journal **45**, (1993), 313–352.

[9] P. Gabriel, A.V. Roiter, *Representations of finite-dimensional algebras,* Encyclopedia of Math. Sci. Vol. **73**, Algebra VIII, 1992.

[10] Ch. Geiß, *Tame distributive algebras and related topics.* Dissertation, Universität Bayreuth (1993)

[11] Ch. Geiß, *On degenerations of tame and wild algebras,* Arch. Math. **64** (1995), 11–16.

[12] R. Hartshorne, *Algebraic Geometry,* Springer, 1977

[13] H. Kraft, Ch. Riedtmann: *Geometry of Representations of Quivers,* in: Representations of Algebras, LMS Lecture Note Series **116**, Cambridge Univ. Press, 1985, 109–145.

[14] J.A. de la Peña, *On the dimension of module varieties of tame and wild algebras,* Comm. Alg.**19(6)** (1991), 1795–1805.

[15] J.A. de la Peña, *Functors preserving tameness,* Fund. Math. **137**, (1991) 77-185.

[16] J.A. de la Peña, D. Simson, *Prinjective Modules, Reflection Functors, Quadratic Forms, and Auslander-Reiten Sequences,* Trans. Am. Math. Soc. **329 (2)** (1992), 733–753.

[17] C.M. Ringel, *Tame algebras,* in: Representation Theory I. Lecture Notes in Math. **831** (1980), 134–287.

[18] C.M. Ringel, *Tame algebras and integral quadratic forms,* Lecture Notes in Math. **1099**, (1984).

[19] D. Simson, *Linear representations of partially ordered sets and vector space categories,* Gordon and Breach, 1992.

Instituto de Matemáticas, UNAM,04510 México D.F., MEXICO

E-mail address: Geiss@servidor.dgsca.unam.mx