HALL ALGEBRA IN A ROOT CATEGORY

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Abstract In order to realize the quantum group in a global way, the Hall algebra of a root category (which is a orbit category of the derived category of a hereditary algebra) is constructed. The PBW-basis, filtration structure and integral form are investigated. Their degeneration at $v = 1$ actually provide a realization of the universal enveloping algebra of semisimple Lie algebras.

1. Introduction

1.1 Let $C = (a_{ij})_{ij}$ be the Cartan matrix of a finite dimensional semisimple complex Lie algebra $\mathbf{g}$ of rank n, this is a symmetrizable matrix, we denote by $(\varepsilon_i)_i$ the minimal symmetrization, thus ε_i are integers without a proper common divisor such that $\varepsilon_i a_{ij} = \varepsilon_j a_{ji}$. Take a variable v, $\mathbf{Q}(v)$ is the function field over the field $\mathbf{Q}$ of rational numbers. The quantum group $\mathcal{U} = \mathcal{U}_q(\mathbf{g})$ introduced by Drinfeld and Jimbo is an associative $\mathbf{Q}(v)$-algebra on the generators: $F_1, \cdots, F_n, K_1, \cdots, K_n, K_1^{-1}, \cdots, K_n^{-1}, E_1, \cdots, E_n$ subject to the relations:

$$K_iK_j = K_jK_i,\ K_iK_i^{-1} = K_i^{-1}K_i = 1$$
$$K_iE_j = v_i^{a_{ij}}E_jK_i,\ K_iF_j = v_i^{-a_{ij}}F_jK_i$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{v_i - v_i^{-1}}$$
$$\sum_{t=0}^{1-a_{ij}}(-1)^t\left[\begin{array}{c}1-a_{ij}\\ t\end{array}\right]_{\varepsilon_i}E_i^tE_jE_i^{1-a_{ij}-t} = 0 \ \text{ for } \ i \neq j$$
$$\sum_{t=0}^{1-a_{ij}}(-1)^t\left[\begin{array}{c}1-a_{ij}\\ t\end{array}\right]_{\varepsilon_i}F_i^tF_jF_i^{1-a_{ij}-t} = 0 \ \text{ for } \ i \neq j$$

[1] Supported by the sonderforschungsbereich 343, Diskrete Strukturen in der Mathematik, Universität Bielefeld
Key words and phrases: quantum group,Hall algebra, Root category, Hereditary algebra
1991 Mathematics subject classification: primary 17B37, 16G20; secondary 17B35, 16S30

We admit the notations $q = v^2$, $v_i = v^{\varepsilon_i}$ and

$$[s] = \frac{v^s - v^{-s}}{v - v^{-1}} = v^{s-1} + v^{s-3} + \cdots + v^{-s+1}$$

$$[s]! = \prod_{r=1}^{s} [r], \quad \begin{bmatrix} s \\ r \end{bmatrix} = \frac{[s]!}{[r]![s-r]!}$$

and, for given a polynomial $f \in \mathbf{Z}[v, v^{-1}]$ and an integer a, we denote by f_a the polynomial obtained from f by replacing v by v^a. $\mathcal{U}$ has the triangular decomposition

$$\mathcal{U} \simeq \mathcal{U}^- \otimes \mathcal{T} \otimes \mathcal{U}^+$$

where $\mathcal{U}^+$ (resp. $\mathcal{U}^-$) is the subalgebra generated by E_i (resp. F_i), $i = 1, \cdots, n$ and $\mathcal{T}$ generatd by $K_i^{\pm}, i = 1, \cdots, n$.

1.2 Now several programs for the realization of quantum group have been raised. A successful model is the Hall algebra of hereditary algebra which is introduced by Ringel [15].

Let k be a finite field, Λ be a finite dimensional hereditary k-algebra of finite representation type. For a Λ-module M, denoted by $\underline{\dim} M$ its dimension vector, it is an element in the Grothendieck group $G_0(\Lambda)$. It is well-known from the Gabriel theorem (see [4] and [3]) that Λ is characterized by the Dynkin quivers and $\underline{\dim}$ furnishes a bijection between the isomorphism classes of the indecomposable Λ-modules and the positive roots of the corresponding simple Lie algebra $\mathbf{g}$. It is natural to ask whether it is possible to recover a Lie algebra structure in terms of the representation theory of Λ. This was successfully attained by Ringel ([16], [17]). Not only this, he actually found a remarkable algebra structure from $\operatorname{mod}\Lambda$ which gives a realization of the positive part $\mathcal{U}^+$ of the quantum group $\mathcal{U}$. This is the Hall algebra (see [15], [18]).

By the Krull-Schmidt theorem and Gabriel theorem, we can identify the isomorphism classes of finite dimensional Λ-modules with the set $\{\varphi : \Phi^+ \to \mathbf{N_0}\}$. The bijection is given by

$$M(\varphi) = \bigoplus_{\alpha \in \Phi^+} \varphi(\alpha) M(\alpha)$$

where $M(\alpha)$ is indecomposable Λ-module with $\underline{\dim} M(\alpha) = \alpha$.

Let I be the index set of simple Λ-modules , $S_i = M(e_i)$ is the simple Λ-module corresponding to $i \in I$. For $\varphi, \psi : \Phi^+ \to \mathbf{N_0}$, we write

$$\langle \varphi, \psi \rangle = \langle M(\varphi), M(\psi) \rangle$$

$$= \dim_k \operatorname{Hom}(M(\varphi), M(\psi)) - \dim_k \operatorname{Ext}(M(\varphi), M(\psi))$$

where $\langle -, - \rangle$ is the Euler form defined on $\mathbf{Z}[I]$.

Given $\varphi, \psi, \lambda : \Phi^+ \to \mathbf{N_0}$, the Hall polynomial $\phi^{\lambda}_{\varphi\psi}$ is a integral polynomial in $\mathbf{Z}[q]$ such that $\phi^{\lambda}_{\varphi\psi}(|E|)$ counts the number of submodules M of $M(\lambda)^E$ which are isomorphic to $M(\psi)^E$ such that $M(\lambda)^E/M$ is isomorphic to $M(\varphi)^E$ for a finite extension E of k, where $(\ )^E$ is $(\ )\otimes_k E$ as $\Lambda \otimes_k E$-module. The existence of Hall polynomials are guaranteed by [17]. We admit $q = v^2$. Now the (twisted) Hall algebra $\mathcal{H}(\Lambda)$ is defined to be a free $\mathbf{Q}(v)$-space with the basis $\{M(\varphi) \mid \varphi : \Phi^+ \to \mathbf{N_0}\}$ and the multiplication is given by

$$M(\varphi)M(\psi) = \sum_{\lambda:\Phi^+ \to \mathbf{N_0}} v^{\langle \varphi, \psi \rangle} \phi^{\lambda}_{\varphi\psi} M(\lambda).$$

1.2.1 Theorem There exists a natural isomorphism $\eta : \mathcal{U}^+ \to \mathcal{H}(\Lambda)$ of $\mathbf{N_0}[\mathbf{I}]$-graded $\mathbf{Q}(v)$-algebra such that $\eta(E_i) = M(e_i)$ for $1 \le i \le n$. The degeneration of $\mathcal{H}(\Lambda)$ at $v = 1$ gives a realization of the positive part of the universal enveloping algebra of $\mathbf{g}$ and its subspace with a basis indexed by indecomposable Λ-modules gives a realization of the positive part of $\mathbf{g}$.

So the next interesting question is, asked by Ringel in [15], to find a natural enlargement of $\mathcal{H}(\Lambda)$ in order to obtain the whole quantum group and the whole enveloping algebra (not only the positive parts).

1.3 The present work is devoted to this question. As we did in [13], the whole structure of $\mathbf{g}$ can be recovered from the derived category $\mathrm{D}^b(\Lambda)$ of $\mathrm{mod}\,\Lambda$, more precisely, from a orbit category of $\mathrm{D}^b(\Lambda)$ which is introduced by Happel [6]. This is the root category $\mathcal{R}$. Following a method in [19], which indicated that the directing property of $\mathrm{mod}\,\Lambda$ provide a good way to list a PBW-basis of $\mathcal{H}(\Lambda)$. We set up our algebra $\mathcal{H}(\mathcal{R})$ by taking the indecomposable objects of $\mathcal{R}$ as generators. By the property an admissible element can be written as a monomial of divided powers of indecomposable objects, we found there are sufficient (but not all) objects in $\mathcal{R}$ can be put into $\mathcal{H}(\mathcal{R})$. Although our definition seems to produce a lot of relations, the homological behavior of $\mathcal{R}$ permit us to reduce those relations to basic ones, which is very close to the Drinfeld-Jimbo relations in (1.1). Let us mention that the definition of $\mathcal{H}(\mathcal{R})$ is given in a global way, this means that the multiplication is given in terms of the root category and not depends on the particular choice of triangular decomposition and generators E_i. However the concrete computation can be furnished locally, for example, if we calculate a commutator $[M(\alpha), M(\beta)]_v$ for indecomposable objects $M(\alpha), M(\beta)$, except the case $\beta = -\alpha$ and then $[M(\alpha), M(-\alpha)] = \frac{K_\alpha - K_\alpha^{-1}}{v^{\varepsilon(\alpha)} - v^{-\varepsilon(\alpha)}}$, it coincides with that of $M(\alpha), M(\beta)$ as modules of a hereditary algebra. This implies that the results of Ringel in [16] and [19] is not only valid for the positive part $\mathcal{U}^+$, actually also for the whole $\mathcal{U}$. Choose any complete slice of $\mathcal{R}$, we get a triangular decomposition of $\mathcal{H}(\mathcal{R})$ and generators corresponding to simple modules, but the root vectors of $\mathcal{H}(\mathcal{R})$ are automatically provided by all indecomposable objects in $\mathcal{R}$. So the PBW-basis has a stronger form; the filtrations , integral form and etc. can be obtained naturally.

Our aim is still not reached completely. A big shortage in this article is that we are not able to give a Hopf algebra structure for $\mathcal{H}(\mathcal{R})$. Anyway, if we consider the degeneration of $\mathcal{H}(\mathcal{R})$ at $v = 1$, this exactly gives a complete realization of the universal enveloping algebra $\mathcal{U}(\mathbf{g})$, hence the structure of $\mathcal{U}(\mathbf{g})$ and $\mathbf{g}$, particularly, the Konstant $\mathbf{Z}$-form and Chevalley basis, can be recovered from $\mathcal{R}$.

Acknowledgement The work was done during my stay at the Department of Mathematics, University of Bielefeld. I am deeply indebted to Prof. C. M. Ringel for his kind hospitality and continuous encouragement. The cooperation with Prof. L. G. Peng in [13], [11] is also highly appreciated.

2. Root categories and their Hall algebras

2.1 Let k be a finite field, A be a finite dimensional hereditary k-algebra. In this paper we only consider the case A is representation-finite type. This means, according to the Gabriel theorem (see [4] and its generalization [3]), the ordinary quivers of A lies in the classes A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2.

Given an abelian category $\mathcal{A}$, denoted by $\mathrm{D}^b(\mathcal{A})$ the derived category of bounded complexes over $\mathcal{A}$. The category $\mathrm{D}^b(\mathcal{A})$ is a triangulated category with the translation functor T obtained from the category of bounded complexes over $\mathcal{A}$ by localizing with respect to the set of quasi-isomorphisms. Of course the situation we consider is $\mathcal{A} = \mathrm{mod}\, A$, we simply denote their derived category by $\mathrm{D}^b(A, k)$ or $\mathrm{D}^b(A)$. Denote by $\mathcal{R}$ the orbit category $\mathrm{D}^b(A)/T^2$ and let $F : \mathrm{D}^b(A) \to \mathcal{R}$ be the canonical covering functor. According to Happel [6], the isomorphism classes of indecomposable objects in $\mathcal{R}$ correspond bijectively to the root system of the corresponding simple Lie algebra $\mathbf{g}$. For this reasion $\mathcal{R}$ is called the root category.

2.2 We define the triangle set Ω of $\mathcal{R}$ to be all sixtuples isomorphic to

$$FM \xrightarrow{Ff} FN \xrightarrow{Fg} FL \xrightarrow{Fh} TFM$$

with

$$M \xrightarrow{f} N \xrightarrow{g} L \xrightarrow{h} TM$$

being a triangle in $\mathrm{D}^b(A)$. According to [13], $\mathcal{R}$ together with T and Ω is an 'almost' triangulated category.

2.2.1 Lemma Given a triangle of the form

$$M \xrightarrow{(f_1,f_2)} N_1 \oplus N_2 \xrightarrow{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} L \xrightarrow{h} TM$$

in $\mathcal{R}$. If $f_2 = 0$, then it is isomorphic to the triangle of the form

$$M \xrightarrow{(f_1,f_2)} N_1 \oplus N_2 \xrightarrow{\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}} L_1 \oplus L_2 \xrightarrow{\begin{pmatrix} h_1 \\ 0 \end{pmatrix}} TM,$$

where $g_{22} : N_2 \to L_2$ is an isomorphism and

$$M \xrightarrow{f_1} N_1 \xrightarrow{g_{11}} L_1 \xrightarrow{h_1} TM$$

is a triangle. In particular, if $h \neq 0$ and both M, L are indecomposible, then for any nontrivial component f_t of (f_1, f_2) and g_s of $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, $f_t \neq 0$, $g_s \neq 0$.

Proof See [13].

2.3 Given objects X, Y and Z in $\mathcal{R}$, set

$$W(X,Y;Z) = \{(f,g,h) \in \mathrm{Hom}_{\mathcal{R}}(X,Z) \times \mathrm{Hom}_{\mathcal{R}}(Z,Y) \times \mathrm{Hom}_{\mathcal{R}}(Y,TX) \mid X \xrightarrow{f} Z \xrightarrow{g} Y \xrightarrow{h} TX \text{ is a triangle}\}.$$

We can introduce the action of $\mathrm{Aut}\, X \times \mathrm{Aut}\, Y$ on $W(X,Y;Z)$ as follows

$$(\alpha,\beta)(f,g,h) = (\alpha f, g\beta, \beta^{-1}h(T\alpha)^{-1})$$

for any $(\alpha,\beta) \in \mathrm{Aut}\, X \times \mathrm{Aut}\, Y$, $(f,g,h) \in W(X,Y;Z)$ and define the orbit space to be

$$V(X,Y;Z) = W(X,Y;Z)/(\mathrm{Aut}\, X \times \mathrm{Aut}\, Y).$$

If a field E is a finite extension of k, remember from [20] that E is called conservative for $\mathcal{R}$ provided $(\mathrm{End}\, X/ \mathrm{rad}\, \mathrm{End}\, X)^E$ is a field for any indecomposable object $X \in \mathcal{R}$, where $(\ )^E$ means $(\ ) \otimes_k E$. It is easily seen that E is conservative for $\mathcal{R}$ if and only if so is E for $\mathrm{mod}\, A$. Let $\mathcal{E}$ be the set of isomorphism classes of finite extension of k which are conservative for $\mathcal{R}$. It follows that $\mathcal{E}$ is an infinite set from [20]. Following [20] we have the following definition.

For objects $X, Y, Z \in \mathcal{R}$, if there exists an integral polynomial $\phi^Z_{XY} \in \mathbf{Z}[q]$ such that

$$\phi^Z_{XY}(|E|) = |V(X^E, Y^E; Z^E)|$$

holds for any $E \in \mathcal{E}$, then we say that $\mathcal{R}$ has the Hall polynomial ϕ^Z_{XY}.

We have the following notations. A complete slice of $\mathrm{D}^b(A)$ is a subcategory generated by n indecomposable objects such that each one belongs to a distinct τ-orbit and they are connected with respect to irreducible maps, where τ is the Auslander-Reiten translation. For a complete slice $\mathcal{S}$ of $\mathrm{D}^b(A)$, there exists a unique hereditary subcategory $\mathcal{B}$ of $\mathrm{D}^b(A)$ which take $\mathcal{S}$ as its subcategory of projective objects (in $\mathcal{B}$), where the hereditary subcategory $\mathcal{B}$ means that $\mathcal{B} \simeq \mathrm{mod}\, B$ with B a hereditary algebra, in fact B enjoys the same type with A. Notice that both of the slice $\mathcal{S}$ and the hereditary subcategory $\mathcal{B}$ are totally preserved under the covering functor F. So we have the notations for complete slice and hereditary subcategory of $\mathcal{R}$ which are, respectively, the image of their correspondents of $\mathrm{D}^b(A)$ under F.

2.3.1 Lemma The root category $\mathcal{R}$ has the Hall polynomials ϕ^Z_{XY} and ϕ^Z_{YX} for any $X, Y, Z \in \mathcal{R}$ with Y indecomposable; moreover, if there is a hereditary subcategory $\mathcal{B}$ which contains X and Y, then the Hall polynomial ϕ^Z_{XY} used here is equal to that defined by Ringel [17] for the module category $\mathcal{B}$.

Proof This is just Lemma 3.1 and Proposition 3.2 in [13]

2.4 We can define the Grothendieck group $G_0(\mathcal{R})$ as usual, it is the quotient of a free group with a basis $\{[M]\}$ indexed by the isomorphism classes of all objects in $\mathcal{R}$ subject to the relations

$$[L] - [M] - [N]$$

provided their exists a triangle of form

$$M \to L \to N \to TM$$

By the Gabriel theorem and its generalization, we can understand as follows: for an indecomposable object M in $\mathcal{R}$, the dimension vector $\underline{\dim} M$ is just the natural image of M in $G_0(\mathcal{R})$ and the root system is identified with the set $\{\underline{\dim} M \mid M \text{ is indecomposable in } \mathcal{R}\}$. For

$\alpha \in \Phi$, we fix an indecomposable object $M(\alpha)$ with $\underline{\dim} M(\alpha) = \alpha$. Then the isomorphism classes of objects in $\mathcal{R}$ can be identified with the morphism set $\{\varphi : \Phi \to \mathbf{N_0}\}$ and the bijection is given by

$$M(\varphi) = \bigoplus_{\alpha\in\Phi} \varphi(\alpha) M(\alpha)$$

Because $\underline{\dim} M = -\underline{\dim} TM$, so we can require $TM(\alpha) = M(-\alpha)$ for any $\alpha \in \Phi$. Also we have the Euler form on $G_0(\mathcal{R})$:

$$\langle \underline{\dim} M, \underline{\dim} N\rangle = \dim_k \mathrm{Hom}_{\mathcal{R}}(M, N) - \dim_k \mathrm{Hom}_{\mathcal{R}}(M, TN)$$

$$(\underline{\dim} M, \underline{\dim} N) = \langle \underline{\dim} M, \underline{\dim} N\rangle + \langle \underline{\dim} N, \underline{\dim} M\rangle$$

we simply write

$$\langle \alpha, \beta\rangle = \langle \underline{\dim} M(\alpha), \underline{\dim} M(\beta)\rangle$$

and

$$(\alpha, \beta) = (\underline{\dim} M(\alpha), \underline{\dim} M(\beta))$$

A object $M(\varphi)$ is called admissible if there exists a hereditary subcategory $\mathcal{B}$ such that $M(\varphi) \in \mathcal{B}$. For an admissible object $M(\varphi)$,

$$M(\varphi) = \varphi(\alpha_1)M(\alpha_1) \oplus \cdots \oplus \varphi(\alpha_m)M(\alpha_m)$$

is called an admissible presentation if $\alpha_i \in \Phi$ and satisfy that $\varepsilon(\alpha_j, \alpha_i) = 0$ for $j > i$ and $\xi(\alpha_i, \alpha_j) = 0$ for $i \leq j$; where $\varepsilon(\alpha_j, \alpha_i) = \dim_k \mathrm{Hom}_{\mathcal{R}}(M(\alpha_j), M(\alpha_i))$ and $\xi(\alpha_i, \alpha_j) = \dim_k \mathrm{Hom}_{\mathcal{R}}(M(\alpha_i), TM(\alpha_j))$. By the directing property of an hereditary algebra, admissible objects always have admissible presentations.

Now we come to give the definition of the Hall algebra $\mathcal{H}(\mathcal{R})$.

2.5 Definition The Hall algebra $\mathcal{H}(\mathcal{R})$ is defined to be an associative $\mathbf{Q}(v)$-algebra generated by $\{M(\alpha) \mid \alpha \in \Phi\}$ and $\{K(\beta) \mid \beta \in G_0(\mathcal{R})\}$ subject to the following relations

(1) $K_\alpha K_\beta = K_{\alpha+\beta},\ K_0 = 1$

(2) $K_\beta M(\alpha) = v^{(\beta,\alpha)} M(\alpha) K_\beta$

(3) $M(\alpha)M(-\alpha) - M(-\alpha)M(\alpha) = \frac{K_\alpha - K_\alpha^{-1}}{v^{\varepsilon(\alpha)} - v^{-\varepsilon(\alpha)}}$

(4) For $\alpha, \gamma \in \Phi$ with $\alpha \neq -\gamma$

$$M(\alpha)M(\gamma) = \sum_{\varphi:\Phi\to\mathbf{N_0}} v^{(\sum_i \langle\alpha_i,\alpha_i\rangle\varphi(\alpha_i) - \sum_{i\leq j}\langle\alpha_i,\alpha_j\rangle\varphi(\alpha_i)\varphi(\alpha_j))} v^{\langle\alpha,\gamma\rangle} \times$$

$$\phi^{\varphi}_{\alpha\gamma} (M(\alpha_1))^{(\varphi(\alpha_1))} \cdots (M(\alpha_m))^{(\varphi(\alpha_m))}$$

where $(M(\alpha))^{(t)} = \frac{(M(\alpha))^t}{[t]!_{\varepsilon(\alpha)}}$, $\varepsilon(\alpha) = \varepsilon(\alpha, \alpha)$ and $M(\varphi) = \varphi(\alpha_1)M(\alpha_1) \oplus \cdots \oplus \varphi(\alpha_m)M(\alpha_m)$ is an admissible presentation.

Remark 1. If $\alpha \neq -\gamma$, there exists a hereditary subcategory $\mathcal{B}$ contains α and γ, so $M(\varphi)$ is always admissible in case $\phi^{\varphi}_{\alpha\gamma} \neq 0$. 2. The Hall algebra $\mathcal{H}(\mathcal{R})$ only depends on the corresponding Cartan matrix C, not depends on the orientations of its Dynkin graph and even the (different) realizations of $\mathcal{R}$. 3. $\mathcal{R}$ has the Auslander-Reiten quiver and the Auslander-Reiten translation τ and the translation functor T are isomorphisms of $\mathcal{R}$. So they induce the isomorphisms of $\mathcal{H}(\mathcal{R})$, respectively, according to Definition 2.5.

3. PBW-basis

3.1 Given $\alpha, \gamma \in \Phi$, if $\gamma \neq -\alpha$, there is a hereditary subcategory $\mathcal{B}$ which contains α and γ. Then one of the following cases should occur.

(1) $\varepsilon(\gamma, \alpha) = 0,\ \xi(\alpha, \gamma) = 0$

(2) $\varepsilon(\alpha, \gamma) = 0,\ \xi(\gamma, \alpha) = 0$

We have the following conclusion.

3.1.1 Proposition Let $\alpha, \gamma \in \Phi$ with $\gamma \neq -\alpha$ and $\varepsilon(\gamma, \alpha) = 0,\ \xi(\alpha, \gamma) = 0$. Then

$$M(\gamma)M(\alpha) = v^{(\alpha,\gamma)}M(\alpha)M(\gamma) + \sum_{\varphi \in J(\alpha,\gamma)} c_{\varphi}(M(\alpha_1))^{(\varphi(\alpha_1))} \cdots (M(\alpha_m))^{(\varphi(\alpha_m))}$$

with coefficients c_{φ} in $\mathbf{Z}[\mathbf{v}, \mathbf{v}^{-1}]$, where $J(\alpha, \gamma)$ is the set of maps $\varphi : \Phi \to \mathbf{N_0}$ with $\phi^{\varphi}_{\gamma\alpha} \neq 0$ but different from $\alpha \oplus \gamma$, and $M(\varphi) = \varphi(\alpha_1)M(\alpha_1) \oplus \cdots \oplus \varphi(\alpha_m)M(\alpha_m)$ is an admissible presentation.

Proof In view of Definition 2.5,(4), we only need to consider the term corresponding to $\varphi = \alpha \oplus \gamma$ in the expansion of $M(\gamma)M(\alpha)$. Now $M(\varphi) = M(\alpha) \oplus M(\gamma)$ is an admissible presentation; and the coefficient of $M(\alpha)M(\gamma)$ in the expansion is $v^{-\langle\alpha,\gamma\rangle}v^{\langle\gamma,\alpha\rangle}\phi^{\alpha\oplus\gamma}_{\gamma\alpha}$, it equals to $v^{(\alpha,\gamma)}$ since $\phi^{\alpha\oplus\gamma}_{\gamma\alpha} = |\operatorname{Hom}_{\mathcal{R}}(M(\alpha), M(\gamma))| = q^{\varepsilon(\alpha,\gamma)} = v^{2\varepsilon(\alpha,\gamma)} = v^{2\varepsilon\langle\alpha,\gamma\rangle}$. The proof is completed.

3.1.2 Take any complete slice $\mathcal{S}$ of $\mathcal{R}$, denote the corresponding hereditary subcategory by $\mathcal{B}$. Consider the set

$$\{M(\alpha) \mid \alpha \in \Phi,\ M(\alpha) \in \mathcal{B}\}.$$

The subalgebra of $\mathcal{H}(\mathcal{R})$ generated by this set is denoted by $\mathcal{H}(\mathcal{B})$. Since $\mathcal{B} = \operatorname{mod} B$ is a directing category, we can arrange a total ordering of indecomposable objects of $\mathcal{B}$, say $M(\alpha_1), M(\alpha_2), \cdots, M(\alpha_m)$, such that $\operatorname{Hom}_{\mathcal{R}}(M(\alpha_i), M(\alpha_j)) \neq 0$ implies $i \leq j$.

Lemma (Ringel [19]) Let $M(\alpha_1), M(\alpha_2), \cdots, M(\alpha_m)$ be a total ordering of indecomposable objects of $\mathcal{B}$, then $\langle\alpha_i, \alpha_j\rangle > 0$ implies $i \le j$ and $\langle\alpha_i, \alpha_j\rangle < 0$ implies $i > j$.

3.2 We may take the hereditary subcategory $\mathcal{B}$ as the module category of a hereditary algebra B, according to Ringel [18], we have the twisted Hall algebra of B, denoted by $\mathcal{H}(\mathcal{B}, *)$, where the multiplication is denoted by $*$ for a differentiate from that of $\mathcal{H}(\mathcal{R})$. But we will soon see that the two coincide.

For $M(\varphi) \in \mathcal{B}$, it is easy to see

$$M(\varphi) = v^{-\sum_{i<j}\langle\alpha_i,\alpha_j\rangle\varphi(\alpha_i)\varphi(\alpha_j)}(\varphi(\alpha_1)M(\alpha_1)) * \cdots * (\varphi(\alpha_m)M(\alpha_m))$$

and

$$(\varphi(\alpha_i)M(\alpha_i)) = v^{-\langle\alpha_i,\alpha_i\rangle\varphi(\alpha_i)(\varphi(\alpha_i)-1)}(M(\alpha_i))^{(*\varphi(\alpha_i))}$$

On the other hand, by Lemma 2.3.1,

$$M(\alpha) * M(\gamma) = \sum_{\varphi:\Phi\to\mathbf{N_0}} v^{\langle\alpha,\gamma\rangle}\phi^{\varphi}_{\alpha\gamma}M(\varphi)$$

(Note: if $M(\alpha), M(\gamma) \in \mathcal{B}$ and $\phi^{\varphi}_{\alpha\gamma} \neq 0$ then $M(\varphi) \in \mathcal{B}$.) Therefore

$$M(\alpha) * M(\gamma) = \sum_{\varphi:\Phi\to\mathbf{N_0}} v^{(\sum_i\langle\alpha_i,\alpha_i\rangle\varphi(\alpha_i)-\sum_{i\le j}\langle\alpha_i,\alpha_j\rangle\varphi(\alpha_i)\varphi(\alpha_j))}v^{\langle\alpha,\gamma\rangle}\phi^{\varphi}_{\alpha\gamma}\times$$

$$(M(\alpha_1))^{(*\varphi(\alpha_1))} * \cdots * (M(\alpha_m))^{(*\varphi(\alpha_m))}$$

This means that the relation (2.5)(4) is satisfied by $\mathcal{H}(\mathcal{B}, *)$ too.

3.3 Proposition Let $M(\alpha_1), M(\alpha_2), \cdots, M(\alpha_m)$ be a total ordering of $\mathcal{B}$, then $(M(\alpha_1))^{a_1}\cdots(M(\alpha_m))^{a_m}$ with $a_1, \cdots, a_m \in \mathbf{N_0}$ form a $\mathbf{Q}(v)$-basis of $\mathcal{H}(\mathcal{B})$ and for all $i < j$

$$M(\alpha_j)M(\alpha_i) = v^{(\alpha_i,\alpha_j)}M(\alpha_i)M(\alpha_j)$$

$$+\sum_{I(i,j)} c(\alpha_{i+1}, \cdots, \alpha_{j-1})(M(\alpha_{i+1}))^{a_{i+1}}\cdots(M(\alpha_{j-1}))^{a_{j-1}}$$

with coefficients $c(\alpha_{i+1}, \cdots, \alpha_{j-1})$ in $\mathbf{Q}(v)$, where the index set $I(i,j)$ is the set of sequences $(a_{i+1}, \cdots, a_{j-1})$ of natural numbers such that $\sum_{t=i+1}^{j-1} a_t\alpha_t = \alpha_i + \alpha_j$.

Proof We have seen that $\mathcal{H}(\mathcal{B}, *)$ satisfies the relation (2.5)(4). However the relation (2.5)(4) is the define relation of $\mathcal{H}(\mathcal{B})$ by (2.5) , so the canonical map $\eta : \mathcal{H}(\mathcal{B}) \to \mathcal{H}(\mathcal{B}, *)$ which maps $M(\alpha)$ to $M(\alpha)$ induces an algebra homomorphism. By [19], the monomials

$(M(\alpha_1))^{*a_1} * \cdots * (M(\alpha_m))^{*a_m}$ with $a_1, \cdots, a_m \in \mathbf{N_0}$ form a $\mathbf{Q}(v)$-basis of $\mathcal{H}(\mathcal{B}, *)$. Therefore the monomials $(M(\alpha_1))^{a_1} \cdots (M(\alpha_m))^{a_m}$ with $a_1, \cdots, a_m \in \mathbf{N_0}$ are, at least, independent over $\mathbf{Q}(v)$. By Proposition 3.1.1, the space $\mathcal{H}(\mathcal{B})$ is $\mathbf{Q}(v)$-spanned by these monomials. It follows that they form a $\mathbf{Q}(v)$-basis of $\mathcal{H}(\mathcal{B})$, and hence, η is an isomorphism.

Let $i < j$, we may apply Proposition 3.1.1 in this case. Consider $\varphi : \Phi \to \mathbf{N_0}$ with $\varphi \neq \alpha_i \oplus \alpha_j$, $\phi^{\varphi}_{\alpha_j \alpha_i} \neq 0$, there exists a non-split triangle

$$M(\alpha_i) \xrightarrow{f} \bigoplus_{t=1}^{m} \varphi(\alpha_t) M(\alpha_t) \xrightarrow{g} M(\alpha_j) \xrightarrow{h} TM(\alpha_i)$$

where $M(\varphi) = \varphi(\alpha_1)M(\alpha_1) \oplus \cdots \oplus \varphi(\alpha_m)M(\alpha_m)$ is an admissible presentation. If $\varphi(\alpha_t) \neq 0$, then the components $f_t \neq 0$ and $g_t \neq 0$ by Lemma 2.2.1, so $M(\alpha_i) < M(\alpha_t) < M(\alpha_j)$ in $\mathcal{B}$. The region of t runs from $i+1$ to $j-1$ and $M(\varphi) \in \mathcal{B}$, and moreover, $\sum\limits_{t=i+1}^{j-1} a_t \alpha_t = \alpha_i + \alpha_j$. Therefore, by Proposition 3.1.1, the relation in (3.3) holds.

3.3.1 Corollary The canonical map $\eta : \mathcal{H}(\mathcal{B}) \to \mathcal{H}(\mathcal{B}, *)$ is an algebra isomorphism.

This means that the restriction of $\mathcal{H}(\mathcal{R})$ to a hereditary subcategory of $\mathcal{R}$ is just the corresponding twisted Hall algebra introduced by Ringel [18].

3.4 Take a complete slice $\mathcal{S}$ and get the corresponding hereditary subcategory $\mathcal{B}$, then the root category $\mathcal{R}$ is divided into the two non-intersection parts $\mathcal{B}$ and $T\mathcal{B}$. Let $S_1, \cdots, S_n$ be a complete set of simple objects in $\mathcal{B}$ and take $e_i = \underline{\dim} S_i$, $E_i = S_i$, $F_i = TS_i$ and $K_i = K_{e_i}$ in $\mathcal{H}(\mathcal{R})$.

It is well-known from [15] that the Hall algebra of a representation finite type hereditary algebra coincides with its composition algebra. So $E_i, F_i, K_i^{\pm}, 1 \leq i \leq n$ are generators of the algebra $\mathcal{H}(\mathcal{R})$. We want to know what happens if we apply the relations in Definition 2.5 to these generators.

First it is easy to see

(1) $K_iK_j = K_jK_i,\ K_iK_i^{-1} = K_i^{-1}K_i = 1$

(2) $K_iE_j = v^{\varepsilon_i a_{ij}} E_jK_i,\ K_iF_j = v^{-\varepsilon_i a_{ij}} F_jK_i$

(3) $E_iF_i - F_iE_i = \frac{K_i - K_i^{-1}}{v^{\varepsilon_i} - v^{-\varepsilon_i}}$

for $1 \leq i, j \leq n$, where $\varepsilon_i = \dim_k \mathrm{Hom}_{\mathcal{R}}(S_i, S_i)$, $a_{ij} = \frac{(e_i, e_j)}{\varepsilon_i}$.

If $i \neq j$, then $\mathrm{Hom}_{\mathcal{R}}(S_i, S_j) = 0 = \mathrm{Hom}_{\mathcal{R}}(S_j, S_i)$. So the triangles with the following form

$$TS_i \to Y \to S_j \to S_i$$

or

$$TS_j \to Y \to S_i \to S_j$$

are always split. On the other side, they satisfy one of the following two possibilities: $\mathrm{Hom}_{\mathcal{R}}(S_i, TS_j) = 0$, which implies $\langle e_i, e_j\rangle = 0$ or $\mathrm{Hom}_{\mathcal{R}}(S_j, TS_i) = 0$, which implies $\langle e_j, e_i\rangle = 0$. If $\mathrm{Hom}_{\mathcal{R}}(S_i, TS_j) = 0$, since $\mathrm{Hom}_{\mathcal{R}}(S_i, S_j) = 0$, now $\langle e_i, e_j\rangle = 0$, by Proposition 3.1.1, $E_iF_j = v^{-\langle e_j, e_i\rangle}F_jE_i$. If $\mathrm{Hom}_{\mathcal{R}}(TS_j, S_i) = 0$, since $\mathrm{Hom}_{\mathcal{R}}(S_j, S_i) = 0$, now $\langle e_j, e_i\rangle = 0$, similarly we have, $F_jE_i = v^{-\langle e_i, e_j\rangle}E_iF_j$. Anyway we get

(4) $v^{-\langle e_i, e_j\rangle}E_iF_j - v^{-\langle e_j, e_i\rangle}F_jE_i = 0$ for $i \neq j$

It is well-known from Ringel [15] and [18], the (twisted) Hall algebra of a hereditary algebra satisfies the quantum Serre relations. According to Proposition 3.3 and Corollary 3.3.1 we have the following quantum Serre relations in $\mathcal{H}(\mathcal{R})$.

(5)

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{\varepsilon_i} E_i^t E_j E_i^{1-a_{ij}-t} = 0$$

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{bmatrix} 1-a_{ij} \\ t \end{bmatrix}_{\varepsilon_i} F_i^t F_j F_i^{1-a_{ij}-t} = 0$$

for all $i \neq j$.

3.5 We define the algebra $\mathcal{U}'$ to be an associative $\mathbf{Q}(v)$-algebra on the generators $E_1, \cdots, E_n,\ K_1, \cdots, K_n,\ K_1^{-1}, \cdots, K_n^{-1},\ F_1, \cdots, F_n$ subject to the relations (1),(2),(3),(4) and (5) in (3.4).

According to a standard method in Lie theory (see [21]) or the diamond lemma in ring theory (see [1]), it is not difficult to see that $\mathcal{U}'$ has the triangular decomposition

$$\mathcal{U}' \simeq \mathcal{U}'^- \otimes \mathcal{U}'^0 \otimes \mathcal{U}'^+$$

where $\mathcal{U}'^0$ is the subalgebra generated by $K_i^{\pm 1}, 1 \leq i \leq n$, in fact $\mathcal{U}'^0 = \mathbf{Q}(v)[K_i, K_i^{-1}]_{1\leq i\leq n}$; $\mathcal{U}'^+$ the subalgebra generated by $E_i, 1 \leq i \leq n$; $\mathcal{U}'^-$ generated by $F_i, 1 \leq i \leq n$.

We have the canonical epimorphism $\eta : \mathcal{U}' \to \mathcal{H}(\mathcal{R})$ which maps E_i to E_i, F_i to F_i and K_i to K_i. By Corollary3.3.1 and Theorem 1.2.1, we have known that $\eta\mid_{\mathcal{U}'^+}: \mathcal{U}'^+ \to \mathcal{H}(\mathcal{B})$, $\eta\mid_{\mathcal{U}'^-}: \mathcal{U}'^- \to \mathcal{H}(\mathcal{TB})$, and $\eta\mid_{\mathcal{U}'^0}: \mathcal{U}'^0 \to \mathcal{H}'$, are all isomorphisms.

3.6 We want to prove that η is an isomorphism itself. To this end, we consider the monomials of form

$$M_{b,c,a} := M(-\alpha_1)^{b_1} \cdots M(-\alpha_m)^{b_m} K_1^{c_1} \cdots K_n^{c_n} M(\alpha_1)^{a_1} \cdots M(\alpha_m)^{a_m}$$

with $b = (b_1, \cdots, b_m) \in \mathbf{N_0^m}$, $c = (c_1, \cdots, c_n) \in \mathbf{Z}^n$ and $a = (a_1, \cdots, a_m) \in \mathbf{N_0^m}$.

Lemma The monomials $M_{b,c,a}$ with $(b, c, a) \in \mathbf{N_0^m} \times \mathbf{Z^n} \times \mathbf{N_0^m}$ are linear independent over $\mathbf{Q}(v)$.

The proof is a hard part of our work, it will be given in Section 4.

3.7 Proposition The canonical epimorphism $\eta : \mathcal{U}' \to \mathcal{H}(\mathcal{R})$ is an isomorphism.

Proof Since $\eta\mid_{\mathcal{U}'^0}: \mathcal{U}'^0 \to \mathcal{H}^0$, $\eta\mid_{\mathcal{U}'^+}: \mathcal{U}'^+ \to \mathcal{H}(\mathcal{B})$ and $\eta\mid_{\mathcal{U}'^-}: \mathcal{U}'^- \to \mathcal{H}(T\mathcal{B})$ are all isomorphisms, so we can take $\overline{M}(\alpha_1), \cdots, \overline{M}(\alpha_m) \in \mathcal{U}'^+$ and $\overline{M}(-\alpha_1), \cdots, \overline{M}(-\alpha_m) \in \mathcal{U}'^-$ with $\eta(\overline{M}(\alpha_i)) = M(\alpha_i)$ and $\eta(\overline{M}(-\alpha_i)) = M(-\alpha_i)$. By Proposition 3.3 and the triangular decomposition of $\mathcal{U}'$, the monomials

$$\overline{M}(-\alpha_1)^{b_1} \cdots \overline{M}(-\alpha_m)^{b_m} K_1^{c_1} \cdots K_n^{c_n} \overline{M}(\alpha_1)^{a_1} \cdots \overline{M}(\alpha_m)^{a_m}$$

with $(b, c, a) \in \mathbf{N_0^m} \times \mathbf{Z^n} \times \mathbf{N_0^m}$ form a $\mathbf{Q}(v)$-basis of $\mathcal{U}'$. Since η is an epimorphism, it suffices to prove the injectivity of η. The later is equivalent to say that the monomials $M_{b,c,a}$ with $(b, c, a) \in \mathbf{N_0^m} \times \mathbf{Z^n} \times \mathbf{N_0^m}$ are independent over $\mathbf{Q}(v)$.

Now we arrive at a position to state the main result in this section, the PBW-theorem of $\mathcal{H}(\mathcal{R})$.

3.8 Theorem Let $\mathcal{R}$ be the root category, $\mathcal{H}(\mathcal{R})$ the Hall algebra of $\mathcal{R}$. Choose any complete slice $\mathcal{S}$ in $\mathcal{R}$, $\mathcal{B}$ the corresponding hereditary subcategory. Let $S_1, \cdots, S_n$ be the complete set of simple objects in $\mathcal{B}$, denoted by $E_i = S_i$, $F_i = TS_i$ and $K_i = K_{\underline{\dim} S_i}$ in $\mathcal{H}(\mathcal{R})$. Let $M(\alpha_1), M(\alpha_2), \cdots, M(\alpha_m)$ be a tortal ordering of indecomposable objects in $\mathcal{B}$. Then

(1) The relations (1),(2),(3),(4) and (5) in (3.4) are define relations of $\mathcal{H}(\mathcal{R})$.

(2) The monomials

$$M_{b,c,a} = M(-\alpha_1)^{b_1} \cdots M(-\alpha_m)^{b_m} K_1^{c_1} \cdots K_n^{c_n} M(\alpha_1)^{a_1} \cdots M(\alpha_m)^{a_m}$$

with $(b, c, a) \in \mathbf{N_0^m} \times \mathbf{Z^n} \times \mathbf{N_0^m}$ form a $\mathbf{Q}(v)$-basis of $\mathcal{H}(\mathcal{R})$.

(3) **a**. If $i < j$, then

$$M(\alpha_j)M(\alpha_i) = v^{(\alpha_i, \alpha_j)} M(\alpha_i)M(\alpha_j)$$

$$+\sum_{I(i,j)} c(\alpha_{i+1},\cdots,\alpha_{j-1})M(\alpha_{i+1})^{a_{i+1}}\cdots M(\alpha_{j-1})^{a_{j-1}}$$

with coefficients $c(\alpha_{i+1},\cdots,\alpha_{j-1})$ in $\mathbf{Q}(v)$, where the index set $I(i,j)$ is the set of sequences $(a_{i+1},a_{i+2},\cdots,a_{j-1})$ of natural numbers such that $\sum_{t=i+1}^{j-1} a_t\alpha_t=\alpha_i+\alpha_j$.

$$M(-\alpha_j)M(-\alpha_i)=v^{(\alpha_i,\alpha_j)}M(-\alpha_i)M(-\alpha_j)$$
$$+\sum_{I(i,j)} c(-\alpha_{i+1},\cdots,-\alpha_{j-1})M(-\alpha_{i+1})^{b_{i+1}}\cdots M(-\alpha_{j-1})^{b_{j-1}}$$

with coefficients $c(-\alpha_{i+1},\cdots,-\alpha_{j-1})$ in $\mathbf{Q}(v)$, where the index set $I(i,j)$ is the set of sequences $(b_{i+1},b_{i+2},\cdots,b_{j-1})$ of natural numbers such that $\sum_{t=i+1}^{j-1} b_t(-\alpha_t)=(-\alpha_i)+(-\alpha_j)$.

b. If $i<j$, then

$$M(\alpha_i)M(-\alpha_j)=v^{-(\alpha_j,\alpha_i)}M(-\alpha_j)M(\alpha_i)$$
$$+\sum_{I(j,i)} c(-\alpha_{j+1},\cdots,-\alpha_m,\alpha_1,\cdots,\alpha_{i-1})\times$$
$$M(-\alpha_{j+1})^{b_{j+1}}\cdots M(-\alpha_m)^{b_m}M(\alpha_1)^{a_1}\cdots M(\alpha_{i-1})^{a_{i-1}}$$

with coefficients $c(-\alpha_{j+1},\cdots,-\alpha_m,\alpha_1,\cdots,\alpha_{i-1})$ in $\mathbf{Q}(v)$, where the index set $I(j,i)$ is the set of sequences $(b_{j+1},\cdots,b_m,a_1,\cdots,a_{i-1})$ of natural numbers such that $\sum_{t=j+1}^{m} b_t(-\alpha_t)+\sum_{t=1}^{i-1} a_t\alpha_t=\alpha_i-\alpha_j$.

$$M(-\alpha_i)M(\alpha_j)=v^{-(\alpha_j,\alpha_i)}M(\alpha_j)M(-\alpha_i)$$
$$+\sum_{I(j,i)} c(\alpha_{j+1},\cdots,\alpha_m,-\alpha_1,\cdots,-\alpha_{i-1})\times$$
$$M(\alpha_{j+1})^{a_{j+1}}\cdots M(\alpha_m)^{a_m}M(-\alpha_1)^{b_1}\cdots M(-\alpha_{i-1})^{b_{i-1}}$$

with coefficients $c(\alpha_{j+1},\cdots,\alpha_m,-\alpha_1,\cdots,-\alpha_{i-1})$ in $\mathbf{Q}(v)$, where the index set $I(j,i)$ is the set of sequences $(a_{j+1},\cdots,a_m,b_1,\cdots,b_{i-1})$ of natural numbers such that $\sum_{t=j+1}^{m} a_t(\alpha_t)+\sum_{t=1}^{i-1} b_t(-\alpha_t)=\alpha_j-\alpha_i$.

Proof (1) is just Proposition 3.7. (2) is a direct consequence of the proof of Proposition 3.7. (3) (a) is just Proposition 3.3. Only (3) (b) needs a proof.

If $i<j$, then $\varepsilon(M(\alpha_i),M(-\alpha_j))=0$, $\xi(M(-\alpha_j),M(\alpha_i))=0$ because they amount to $\xi(M(\alpha_i),M(\alpha_j))=0$, $\varepsilon(M(\alpha_j),M(\alpha_i))=0$. By Proposition 3.1.1 we get

$$M(\alpha_i)M(-\alpha_j)=v^{-(\alpha_j,\alpha_i)}M(-\alpha_j)M(\alpha_i)$$
$$+\sum_{\varphi\in J(-\alpha_j,\alpha_i)} c_\varphi M(\beta)^{\varphi(\beta_1)}\cdots M(\beta_l)^{\varphi(\beta_l)}$$

with $M(\varphi) = \varphi(\beta_1)M(\beta_1) \oplus \cdots \oplus \varphi(\beta_l)M(\beta_l) \neq M(\alpha_i) \oplus M(-\alpha_j)$ such that $\phi^{M(\varphi)}_{M(\alpha_i)M(-\alpha_j)} \neq 0$. Therefore we have the non-split triangle

$$M(-\alpha_j) \xrightarrow{f} \bigoplus \varphi(\beta_t)M(\beta_t) \xrightarrow{g} M(\alpha_i) \xrightarrow{h} M(\alpha_j)$$

with every component $f_t \neq 0$ and $g_t \neq 0$ by Lemma 2.2.1. Then, $M(\beta_t) \in T\mathcal{B}$ implies that $\beta_t = -\alpha_{t'}$ for some t' with $j < t' \leq m$; or $M(\beta_t) \in \mathcal{B}$ implies $\beta_t = \alpha_{t'}$ with $1 \leq t' < i$. Of course $\sum \varphi(\beta_t)\beta_t = \alpha_i - \alpha_j$. Thus the first conclusion of (3) (b) is proved, the second is similar.

3.8.1 Remark Roughly speaking, the theorem tells us that a PBW-basis of $\mathcal{H}(\mathcal{R})$ is given by the monomials of indecomposable objects of $\mathcal{R}$, which are not arranged in a row order, but in a round way according to the structure of the Auslander-Reiten quiver of $\mathcal{R}$.

4. A $\mathcal{H}(\mathcal{R})$-module V

4.1 Let $\varphi : \Phi \to \mathbf{N_0}$, then $M(\varphi) = \bigoplus_{\alpha\in\Phi} \varphi(\alpha)M(\alpha)$. Let

$$\begin{aligned}\varepsilon(\varphi) &= \varepsilon(M(\varphi)) = \dim_k \operatorname{Hom}_{\mathcal{R}}(M(\varphi), M(\varphi)) \\ &= \sum_{\alpha_i,\alpha_j\in\Phi} \varphi(\alpha_i)\varphi(\alpha_j) \dim_k \operatorname{Hom}_{\mathcal{R}}(M(\alpha_i), M(\alpha_j))\end{aligned}$$

$$\begin{aligned}\xi(\varphi) &= \xi(M(\varphi)) = \dim_k \operatorname{Hom}_{\mathcal{R}}(M(\varphi), TM(\varphi)) \\ &= \sum_{\alpha_i,\alpha_j\in\Phi} \varphi(\alpha_i)\varphi(\alpha_j) \dim_k \operatorname{Hom}_{\mathcal{R}}(M(\alpha_i), TM(\alpha_j))\end{aligned}$$

$$d(\varphi) = \varepsilon(\varphi) + \xi(\varphi)$$

The following lemma is well-known in the representation theory of finite dimensional algebras, it seems to appear first in [8], see also [5].

4.1.1 Lemma Let

$$(*) : \quad X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} TX$$

be any triangle and Z any object in $\mathcal{R}$. Then

(1)

$$\dim_k \operatorname{Hom}_{\mathcal{R}}(Z, L) \leq \dim_k \operatorname{Hom}_{\mathcal{R}}(Z, X \oplus Y)$$
$$\dim_k \operatorname{Hom}_{\mathcal{R}}(Z, TL) \leq \dim_k \operatorname{Hom}_{\mathcal{R}}(Z, TX \oplus TY)$$

(2)

$$\dim_k \operatorname{Hom}_{\mathcal{R}}(L, Z) \leq \dim_k \operatorname{Hom}_{\mathcal{R}}(X \oplus Y, Z)$$
$$\dim_k \operatorname{Hom}_{\mathcal{R}}(TL, Z) \leq \dim_k \operatorname{Hom}_{\mathcal{R}}(TX \oplus TY, Z)$$

Proof Let $F : \mathrm{D}^b(B) \to \mathcal{R}$ be the natural covering functor induced by T^2, where B can be taken to be the hereditary algebra corresponding to any hereditary subcategory of $\mathcal{R}$; and suppose

$$(*') : \quad A \overset{a}{\to} C \overset{b}{\to} B \overset{c}{\to} TA$$

to be a triangular in $\mathrm{D}^b(B)$ which lies in $F^{-1}(*)$ and take $D \in F^{-1}(Z)$. Applying $\mathrm{Hom}_{\mathrm{D}^b(B)}(D, -)$ to $(*')$ we obtain the long exact sequence

$$\cdots \overset{c^*}{\to} \mathrm{Hom}(D, T^{-1}A) \to \mathrm{Hom}(D, T^{-1}C) \to \mathrm{Hom}(D, T^{-1}B)$$
$$\overset{c^*}{\to} \mathrm{Hom}(D, A) \to \mathrm{Hom}(D, C) \to \mathrm{Hom}(D, B)$$
$$\overset{c^*}{\to} \mathrm{Hom}(D, TA) \to \mathrm{Hom}(D, TC) \to \mathrm{Hom}(D, TB) \overset{c^*}{\to} \cdots$$

Therefore

$$\sum_{i\in\mathbf{Z}} \dim_k \mathrm{Hom}(D, T^{2i}C) \le \sum_{i\in\mathbf{Z}} \dim_k \mathrm{Hom}(D, T^{2i}(A \oplus B))$$
$$\sum_{i\in\mathbf{Z}} \dim_k \mathrm{Hom}(D, T^{2i+1}C) \le \sum_{i\in\mathbf{Z}} \dim_k \mathrm{Hom}(D, T^{2i+1}(A \oplus B))$$

Similarly, applying $\mathrm{Hom}_{\mathrm{D}^b(B)}(-, D)$ to $(*')$ we get

$$\cdots \overset{c^*}{\leftarrow} \mathrm{Hom}(A, D) \leftarrow \mathrm{Hom}(C, D) \leftarrow \mathrm{Hom}(B, D)$$
$$\overset{c^*}{\leftarrow} \mathrm{Hom}(TA, D) \leftarrow \mathrm{Hom}(TC, D) \leftarrow \mathrm{Hom}(TB, D)$$
$$\overset{c^*}{\leftarrow} \mathrm{Hom}(T^2A, D) \leftarrow \mathrm{Hom}(T^2C, D) \leftarrow \mathrm{Hom}(T^2B, D) \overset{c^*}{\leftarrow} \cdots$$

then

$$\sum_{i\in\mathbf{Z}} \dim_k \mathrm{Hom}(T^{2i}C, D) \le \sum_{i\in\mathbf{Z}} \dim_k \mathrm{Hom}(T^{2i}(A \oplus B), D)$$
$$\sum_{i\in\mathbf{Z}} \dim_k \mathrm{Hom}(T^{2i+1}C, D) \le \sum_{i\in\mathbf{Z}} \dim_k \mathrm{Hom}(T^{2i+1}(A \oplus B), D)$$

Since F is the covering functor, we have the inequalities in (4.1.1).

4.1.2 As a consequence of Lemma 4.1.1, the set $\{\varphi : \Phi \to \mathbf{N_0}\}$ is equiped with a partial order, that is, we define $\varphi_1 \le \varphi_2$ if for any $M \in \mathcal{R}$

$$\dim_k \mathrm{Hom}_{\mathcal{R}}(M(\varphi_1), M) \le \dim_k \mathrm{Hom}_{\mathcal{R}}(M(\varphi_2), M)$$

and

$$\dim_k \mathrm{Hom}_{\mathcal{R}}(M, M(\varphi_1)) \le \dim_k \mathrm{Hom}_{\mathcal{R}}(M, M(\varphi_2))$$

Thus, if there is a triangle of form

$$M(\varphi_1) \to M(\varphi) \to M(\varphi_2) \to TM(\varphi_1)$$

then $\varphi \le \varphi_1 \oplus \varphi_2$.

4.1.3 Corollary Let

$$(*): \quad X \xrightarrow{f} L \xrightarrow{g} Y \xrightarrow{h} TX$$

be a triangle in $\mathcal{R}$, then

$$\varepsilon(L) \le \varepsilon(X \oplus Y), \quad \xi(L) \le \xi(X \oplus Y)$$

So $d(L) \le d(X \oplus Y)$; moreover, the equality holds if and only if $L \simeq X \oplus Y$.

Proof In the proof of Lemma 4.1.1, if we take $D = C$ and $D = A \oplus B$ respectively, then we have $\varepsilon(L) \le \varepsilon(X \oplus Y)$, $\xi(L) \le \xi(X \oplus Y)$. If the equality holds, then every $c^* = 0$. However $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : TA \to TA \oplus TB$ is an embedding, then $\begin{pmatrix} c \\ 0 \end{pmatrix} : B \to TA \oplus TB$ is the zero map. Thus $c = 0$, hence $L \simeq X \oplus Y$.

4.2 In general, let $\{X_\alpha \mid \alpha \in \Phi\}$ be a family of variables indexed by the root system Φ, suppose X is a monomial on these variables, say $X = X_{\alpha_{i_1}} X_{\alpha_{i_2}} \cdots X_{\alpha_{i_t}}$. We define

$$\varphi_X = \alpha_{i_1} \oplus \alpha_{i_2} \oplus \cdots \oplus \alpha_{i_t}, \quad d(X) = d(\varphi_X) = d(M(\varphi_X))$$

¿From now on we fix a complete slice $\mathcal{S}$ of $\mathcal{R}$, $\mathcal{B}$ the hereditary subcategory corresponding to $\mathcal{S}$, B the hereditary algebra of $\mathcal{B}$ and $M(\alpha_1), M(\alpha_2), \cdots, M(\alpha_m)$ be a total ordering of indecomposable objects of $\mathcal{B}$.

Let R be a free $\mathbf{Q}(v)$-algebra on the generators $M(-\alpha_i), i = 1, \cdots, m$, $K_i^{\pm 1}, i = 1, \cdots, n$ and $M(\alpha_i), i = 1, \cdots, m$ subject to the only relations $K_i K_i^{-1} = K_i^{-1} K_i = 1$, for $i = 1, \cdots, n$. For any monomial M of R, we define φ_M as above, however we should have a new rule $\varphi_{K_i^{\pm 1}} = 0$. Also $d(M) = d(\varphi_M)$. Obviously $\mathcal{H}(\mathcal{R})$ is a quotient of R modulo the relations in (2.5).

4.3 We introduce the following notation in R.

For $\alpha, \beta \in \Phi$, we define

$$[M(\alpha), M(-\alpha)]_v = \frac{K_\alpha - K_\alpha^{-1}}{v^{\varepsilon(\alpha)} - v^{-\varepsilon(\alpha)}}$$

if $\beta \ne -\alpha$,

$$[M(\beta), M(\alpha)]_v = \sum_{\substack{\varphi: \Phi \to \mathbf{N_0} \\ \varphi \ne \beta \oplus \alpha}} v^{\left(\sum_i \langle \beta_i, \beta_i \rangle \varphi(\beta_i) - \sum_{i \le j} \langle \beta_i, \beta_j \rangle \varphi(\beta_i) \varphi(\beta_j)\right)} v^{\langle \beta, \alpha \rangle} \times$$

$$\phi^{\varphi}_{\beta\alpha}(M(\beta_1))^{(\varphi(\beta_1))}\cdots(M(\beta_t))^{(\varphi(\beta_t))}$$
$$-\sum_{\substack{\varphi:\Phi\to\mathbf{N_0}\\ \varphi\neq\beta\oplus\alpha}} v^{(\sum_i\langle\beta_i,\beta_i\rangle\varphi(\beta_i)-\sum_{i\leq j}\langle\beta_i,\beta_j\rangle\varphi(\beta_i)\varphi(\beta_j))}v^{-\langle\beta,\alpha\rangle}\times$$
$$\phi^{\varphi}_{\alpha\beta}(M(\beta_1))^{(\varphi(\beta_1))}\cdots(M(\beta_t))^{(\varphi(\beta_t))}$$

where $M(\varphi)=\varphi(\beta_1)M(\beta_1)\oplus\cdots\oplus\varphi(\beta_t)M(\beta_t)$ is an admissible presentation.

We have

$$[M(\alpha),M(-\alpha)]_v=-[M(-\alpha),M(\alpha)]_v$$

and if $\beta\neq-\alpha$ and $\varepsilon(\beta,\alpha)=0,\ \xi(\alpha,\beta)=0$ then

$$[M(\alpha),M(\beta)]_v=-v^{-(\alpha,\beta)}[M(\beta),M(\alpha)]_v.$$

4.4 We define V to be the $\mathbf{Q}(v)$-space with the basis

$$X_{b,c,a}=X(-\alpha_1)^{b_1}\cdots X(-\alpha_m)^{b_m}X_1^{c_1}\cdots X_n^{c_n}X(\alpha_1)^{a_1}\cdots X(\alpha_m)^{a_m}$$

where $(b,c,a)=(b_1,\cdots,b_m,c_1,\cdots,c_n,a_1,\cdots,a_m)\in\mathbf{N_0^m}\times\mathbf{Z^n}\times\mathbf{N_0^m}$. In particularly, 1 is a element of this basis.

For a monomial $X_{b,c,a}$ in V, we define $\varphi_{X_{b,c,a}}$ and $d(X_{b,c,a})$ as in (4.2), also we provide $\varphi_{X_i^{\pm1}}=0$ for $1\leq i\leq n$.

The initial of $X_{b,c,a}$ is defined to be

$$\begin{cases} X(-\alpha_t) & \text{if } b_t\neq0 \text{ and } b_i=0 \text{ for } 1\leq i<t\\ X_t^{c_t} & \text{if } b=0,\ c_t\neq0 \text{ and } c_i=0 \text{ for } 1\leq i<t\\ X(\alpha_t) & \text{if } b=0,\ c=0,\ a_t\neq0 \text{ and } a_i=0 \text{ for } 1\leq i<t\end{cases}$$

For $\lambda\in\mathbf{Z}$ we write

$$X_{b+\lambda i,c,a}=X(-\alpha_1)^{b_1}\cdots X(-\alpha_i)^{b_i+\lambda}\cdots X(-\alpha_m)^{b_m}\times$$
$$X_1^{c_1}\cdots X_n^{c_n}X(\alpha_1)^{a_1}\cdots X(\alpha_m)^{a_m}$$

Similarly for the definitions of $X_{b,c+\lambda i,a}$ and $X_{b,c,a+\lambda i}$.

First we define V to be a R-module. If r is a monomial in R and w a monomial in V, define

$$d(r,w):=d(\varphi_r\oplus\varphi_w)=d(M(\varphi_r)\oplus M(\varphi_w))$$

We are proceeding by induction on $d(r,w)$.

In general, (1) we assume that for monomials $r_1\in R$ and $w_1\in V$ with $d(r_1,w_1)<d(r,w)=d$, the action r_1w_1 is well-defined already such that r_1w_1 is a linear combination of monomials X satisfying $\varphi_X\leq\varphi_{r_1}\oplus\varphi_{w_1}$. Without loss of generality we may assume that $w=X_{b,c,a}$, $r=M(-\alpha_1),K_i$ or $M(\alpha_i)$.

We also provide the following order

$$M(-\alpha_1)\prec M(-\alpha_2)\prec\cdots\prec M(-\alpha_m)\prec M(\alpha_1)\prec\cdots\prec M(\alpha_m)$$

and (2) assume that if $d(r_1, w_1) = d(r, w)$, $\varphi_{r_1} \oplus \varphi_{w_1} \le \varphi_r \oplus \varphi_w$ and $r_1 \prec r$ then $r_1 w_1$ is well-defined already and $r_1 w_1$ is a linear combination of monomials X satisfying $\varphi_X \le \varphi_{r_1} \oplus \varphi_{w_1}$.

4.4.1 When $d(r, w) = 0$, then $r = K_1^{c_1} \cdots K_n^{c_n}$, $w = X_1^{d_1} \cdots X_n^{d_n}$, define $rw = X_1^{c_1+d_1} \cdots X_n^{c_n+d_n}$.

4.4.2 When $r = M(-\alpha_i)$.

(a) If $b = 0$, define $rw = M(-\alpha_i)X_{0,c,a} = X_{0+i,c,a}$.

(b) If the initial of $X_{b,c,a}$ is $X(-\alpha_j)$ with $j \ge i$, define $rw = M(-\alpha_i)X_{b,c,a} = X_{b+i,c,a}$.

(c) If the initial of $X_{b,c,a}$ is $X(-\alpha_j)$ with $j < i$, define

$$rw = M(-\alpha_i)X_{b,c,a}$$

$$= v^{(\alpha_j,\alpha_i)} M(-\alpha_j)(M(-\alpha_i)X_{b-j,c,a}) + [M(-\alpha_i), M(-\alpha_j)]_v X_{b-j,c,a}$$

Since $d(M(-\alpha_i), X_{b-j,c,a}) < d(M(-\alpha_i), X_{b,c,a})$, by the inductive assumption (1), $M(-\alpha_i)X_{b-j,c,a}$ is well-defined and can be written as a linear combination of monomials X with $\varphi_X \le (-\alpha_i) \oplus \varphi_{X_{b-j,c,a}}$; for these monomials X,

$$(-\alpha_j) \oplus \varphi_X \le (-\alpha_j) \oplus (-\alpha_i) \oplus \varphi_{X_{b-j,c,a}} = (-\alpha_i) \oplus \varphi_{X_{b,c,a}}$$

and

$$d(M(-\alpha_j), X) = d((-\alpha_j) \oplus \varphi_X) \le d((-\alpha_j) \oplus (-\alpha_i) \oplus \varphi_{X_{b-j,c,a}})$$

$$= d((-\alpha_i) \oplus \varphi_{X_{b,c,a}}) = d(M(-\alpha_i), X_{b,c,a})$$

Since $M(-\alpha_j) \prec M(-\alpha_i)$, then, by the inductive assumption (2), $M(-\alpha_j)X$ is well-defined and can be written as a linear combination of monomials Y with $\varphi_Y \le (-\alpha_j) \oplus \varphi_X \le (-\alpha_i) \oplus \varphi_{X_{b,c,a}}$. Therefore $M(-\alpha_j)(M(-\alpha_i)X_{b-j,c,a})$ is well-defined and can be written as a linear combination of monomials X with $\varphi_X \le (-\alpha_i) \oplus \varphi_{X_{b,c,a}}$.

By (4.1.2) and Corollary 4.1.3, $[M(-\alpha_i), M(-\alpha_j)]_v$ is a linear combination of monomials M with $\varphi_M \le (-\alpha_i) \oplus (-\alpha_j)$ and $d(M) < d((-\alpha_i) \oplus (-\alpha_j))$, then

$$d(M, X_{b-j,c,a}) < d((-\alpha_i) \oplus (-\alpha_j) \oplus \varphi_{X_{b-j,c,a}}) = d((-\alpha_i) \oplus \varphi_{X_{b,c,a}})$$

$$= d(M(-\alpha_i), X_{b,c,a})$$

By the inductive assumption (1), $MX_{b-j,c,a}$ is well-defined and can be written as a linear combination of monomials X with $\varphi_X \le \varphi_M \oplus \varphi_{X_{b-j,c,a}}$.

So by our definition in (c), rw is well-defined and can be written as a combination of monomials X with $\varphi_X \le \varphi_r \oplus \varphi_w$.

4.4.3 When $r = K_i^{c_i}$, it is easy to define

$$K_i^{c_i} X_{b,c,a} := v^{(-c_i \sum_{j=1}^m b_j(e_i,\alpha_j))} X_{b,c+c_i i,a}.$$

4.4.4 When $r = M(\alpha_i)$.

(a) If the initial of $X_{b,c,a}$ is $M(\alpha_j)$ with $j \geq i$, define

$$M(\alpha_i)X_{b,c,a} = M(\alpha_i)X_{0,0,a} = X_{0,0,a+i}$$

If the initial of $X_{b,c,a}$ is $M(\alpha_j)$ with $j < i$, define

$$M(\alpha_i)X_{b,c,a} = M(\alpha_i)X_{0,0,a}$$

$$= v^{(\alpha_j,\alpha_i)} M(\alpha_j)(M(\alpha_i)X_{0,0,a-j}) + [M(\alpha_i), M(\alpha_j)]_v X_{0,0,a-j}$$

Base upon the same discussion as in (4.4.2) (c), we have the well-defined action $M(\alpha_i)X_{b,c,a}$ and it can be written as a linear combination of monomials X with $\varphi_X \leq \alpha_i \oplus \varphi_{X_{b,c,a}}$.

(b) If the initial of $X_{b,c,a}$ is $X_j^{c_j}$, define

$$M(\alpha_i)X_{b,c,a} = M(\alpha_i)X_{0,c,a} = v^{(-\sum_{j=1}^n c_j(e_j,\alpha_i))} K_1^{c_1} \cdots K_n^{c_n} M(\alpha_i)X_{0,0,a}$$

Then the action $M(\alpha_i)X_{0,0,a}$ and $K_1^{c_1} \cdots K_n^{c_n}(M(\alpha_i)X_{0,0,a})$ is reduced to (4.4.4) (a) and (4.4.3).

(c) The last case, if the initial of $X_{b,c,a}$ is $X(-\alpha_j)$; if $i = j$, define

$$M(\alpha_i)X_{b,c,a} = M(-\alpha_j)(M(\alpha_i)X_{b-j,c,a}) + [M(\alpha_i), M(-\alpha_j)]_v X_{b-j,c,a}$$

if $i < j$, define

$$M(\alpha_i)X_{b,c,a}$$

$$= v^{-(\alpha_j,\alpha_i)} M(-\alpha_j)(M(\alpha_i)X_{b-j,c,a}) + [M(\alpha_i), M(-\alpha_j)]_v X_{b-j,c,a}$$

if $i > j$, define

$$M(\alpha_i)X_{b,c,a}$$

$$= v^{(\alpha_i,\alpha_j)} M(-\alpha_j)(M(\alpha_i)X_{b-j,c,a}) - v^{(\alpha_i,\alpha_j)}[M(-\alpha_j), M(\alpha_i)]_v X_{b-j,c,a}$$

Base upon the same discussion as in (4.4.2) (c), we have the well-defined action $M(\alpha_i)X_{b,c,a}$ and it can be written as a linear combination of monomials X with $\varphi_X \leq \alpha_i \oplus \varphi_{X_{b,c,a}}$.

By the induction principal we have constructed the R-module V. Furthermore, we claim that the R-module V is a $\mathcal{H}(\mathcal{R})$-module. For this aim, it suffices to prove

4.5 Proposition The action of R on V satisfy the following operator equations

(1) $M(\alpha)M(-\alpha) - M(-\alpha)M(\alpha) = \frac{K_\alpha - K_\alpha^{-1}}{v^{\varepsilon(\alpha)} - v^{-\varepsilon(\alpha)}}$ for any $\alpha \in \Phi$.

(2) For any $\alpha, \beta \in \Phi$ with $\varepsilon(\beta,\alpha) = 0$, $\xi(\alpha,\beta) = 0$, then

$$M(\beta)M(\alpha) - v^{(\alpha,\beta)} M(\alpha)M(\beta) - [M(\beta), M(\alpha)]_v = 0.$$

4.6 More general, for a monomial $M = M(\beta_1)\cdots M(\beta_t)$ in R, we define by induction

$$[M(\beta), M]_v = [M(\beta), M(\beta_1)]_v M(\beta_2)\cdots M(\beta_t)$$
$$+v^{f(\beta,\beta_1)} M(\beta_1)[M(\beta), M(\beta_2)\cdots M(\beta_t)]_v$$

where

$$f(\beta,\alpha) = \begin{cases} (\alpha,\beta) & \text{if } \varepsilon(\beta,\alpha)=0,\ \xi(\alpha,\beta)=0 \\ -(\beta,\alpha) & \text{if } \varepsilon(\alpha,\beta)=0,\ \xi(\beta,\alpha)=0 \end{cases}$$

and $\beta, \beta_i, i = 1, \cdots, t, \in \Phi$.

Additionally, we define $f(\alpha,-\alpha)=0,\ f(\alpha,\alpha)=0$ and $[K_i^{c_i}, K_j^{c_j}]_v = 0$, $[M(\beta), K_i^{c_i}]_v = 0$.

A generalization of Proposition 4.5 is the following

4.7 Proposition For any $\beta,\ \beta_i, i = 1, \cdots, s$ in Φ, the action of R on V satisfy the following operator equations

$$M(\beta)M(\beta_1)\cdots M(\beta_s)$$
$$= v^{(f(\beta,\beta_1)+\cdots f(\beta,\beta_s))} M(\beta_1)\cdots M(\beta_s)M(\beta) + [M(\beta), M(\beta_1)\cdots M(\beta_s)]_v$$

Proof Also we proceed by induction on $d(M(\beta)M(\beta_1)\cdots M(\beta_s), X_{b,c,a})$. Assume that (1) when

$$d(M(\beta')M(\beta'_1)\cdots M(\beta'_t), X_{b',c',a'}) < d(M(\beta)M(\beta_1)\cdots M(\beta_s), X_{b,c,a}) = d,$$

we have the equation

$$M(\beta')M(\beta'_1)\cdots M(\beta'_t)X_{b',c',a'}$$
$$= v^{(f(\beta',\beta'_1)+\cdots f(\beta',\beta'_t))} M(\beta'_1)\cdots M(\beta'_t)M(\beta')X_{b',c',a'}$$
$$+[M(\beta'), M(\beta'_1)\cdots M(\beta'_t)]_v X_{b',c',a'}$$

Since

$$[M(\beta), M(\beta_1)\cdots M(\beta_s)]_v X_{b,c,a}$$
$$= [M(\beta), M(\beta_1)]_v (M(\beta_2)\cdots M(\beta_s)X_{b,c,a})$$
$$+v^{f(\beta,\beta_1)} M(\beta_1)[M(\beta), M(\beta_2)\cdots M(\beta_s)]_v X_{b,c,a},$$

it suffices to prove that when $d(M(\alpha)M(\beta), X_{b,c,a}) = d$, we have

$$M(\alpha)M(\beta)X_{b,c,a} = v^{f(\alpha,\beta)} M(\beta)M(\alpha)X_{b,c,a} + [M(\alpha), M(\beta)]_v X_{b,c,a}$$

We assume also that (2) when

$$d(M(\alpha)M(\beta'), X_{b',c',a'}) = d(M(\alpha)M(\beta), X_{b,c,a}) = d$$

and

$$\beta' \oplus \varphi_{X_{b',c',a'}} \leq \beta \oplus \varphi_{X_{b,c,a}}$$

but $\beta' \prec \beta$, then

$M(\alpha)M(\beta')X_{b',c',a'} = v^{f(\alpha,\beta')}M(\beta')M(\alpha)X_{b',c',a'}+[M(\alpha),M(\beta')]_v X_{b',c',a'}$

We only check one situation here, the others are similar.

Let $i \leq j$, we come to show

(*1)

$$M(\alpha_i)M(-\alpha_j)X_{b,c,a}$$
$$= v^{f(\alpha_i,-\alpha_j)}M(-\alpha_j)M(\alpha_i)X_{b,c,a} + [M(\alpha_i),M(-\alpha_j)]_v X_{b,c,a}$$

(a) If the initial of $X_{b,c,a}$ is $X(\alpha_t)$ with $1 \leq t \leq m$, or $X_t^{c_i}$ with $1 \leq t \leq n$, or $X(-\alpha_t)$ with $j \leq t$. Then

$$M(\alpha_i)M(-\alpha_j)X_{b,c,a} = M(\alpha_i)(M(-\alpha_j)X_{b,c,a}) = M(\alpha_i)X_{b+j,c,a}$$

Now the initial of $X_{b+j,c,a}$ is $X(-\alpha_j)$. If $j = i$, then $f(\alpha_i,-\alpha_j) = 0$, by the action in (4.4.4) (c),

$$M(\alpha_i)X_{b+j,c,a} = M(-\alpha_j)(M(\alpha_i)X_{b,c,a}) + [M(\alpha_i),M(-\alpha_j)]_v X_{b,c,a}$$

If $i < j$, then $f(\alpha_i,-\alpha_j) = -(\alpha_j,\alpha_i)$, by the action in (4.4.4) (c),

$$M(\alpha_i)X_{b+j,c,a} = v^{-(\alpha_j,\alpha_i)}M(-\alpha_j)(M(\alpha_i)X_{b,c,a})+[M(\alpha_i),M(-\alpha_j)]_v X_{b,c,a}$$

We have (*1).

(b) If the initial of $X_{b,c,a}$ is $X(-\alpha_t)$ with $t < j$. By the inductive assumption we have the operator equations

$$M(-\alpha_j)M(-\alpha_t) = v^{f(\alpha_j,\alpha_t)}M(-\alpha_t)M(-\alpha_j) + [M(-\alpha_j),M(-\alpha_t)]_v$$

$$M(\alpha_i)M(-\alpha_t) = v^{f(\alpha_i,-\alpha_t)}M(-\alpha_t)M(\alpha_i) + [M(\alpha_i),M(-\alpha_t)]_v$$

where the second is only applied to the non-zero monomials of $M(-\alpha_j)X_{b-t,c,a}$, it is guaranteed by the inductive assumption (2); and

$$M(\alpha_i)M(-\alpha_j)X_{b-t,c,a} = v^{f(\alpha_i,-\alpha_j)}M(-\alpha_j)M(\alpha_i)X_{b-t,c,a}$$
$$+[M(\alpha_i),M(-\alpha_j)]_v X_{b-t,c,a}$$

By a computation we have

$$M(\alpha_i)M(-\alpha_j)X_{b,c,a} = M(\alpha_i)M(-\alpha_j)M(-\alpha_t)X_{b-t,c,a}$$
$$= v^{f(\alpha_i,-\alpha_j)}M(-\alpha_j)M(\alpha_i)X_{b,c,a}+v^{f(\alpha_j,\alpha_t)}[M(\alpha_i),M(-\alpha_t)]_v M(-\alpha_j)X_{b-t,c,a}$$
$$+M(\alpha_i)[M(-\alpha_j),M(-\alpha_t)]_v X_{b-t,c,a}$$
$$+v^{f(\alpha_j,\alpha_t)+f(\alpha_i,-\alpha_t)}M(-\alpha_t)[M(\alpha_i),M(-\alpha_j)]_v X_{b-t,c,a}$$
$$-v^{f(\alpha_i,-\alpha_t)+f(\alpha_i,-\alpha_j)}[M(-\alpha_j),M(-\alpha_t)]_v M(\alpha_i)X_{b-t,c,a}$$
$$-v^{f(\alpha_i,-\alpha_j)}M(-\alpha_j)[M(\alpha_i),M(-\alpha_t)]_v X_{b-t,c,a}$$

So the question (*1) is reduced to prove
(*2)

$$\begin{aligned} &[M(\alpha_i), M(-\alpha_j)]_v M(-\alpha_t) X_{b-t,c,a} \\ &= v^{f(\alpha_j,\alpha_t)}[M(\alpha_i), M(-\alpha_t)]_v M(-\alpha_j) X_{b-t,c,a} \\ &+M(\alpha_i)[M(-\alpha_j), M(-\alpha_t)]_v X_{b-t,c,a} \\ &+v^{f(\alpha_j,\alpha_t)+f(\alpha_i,-\alpha_t)} M(-\alpha_t)[M(\alpha_i), M(-\alpha_j)]_v X_{b-t,c,a} \\ &-v^{f(\alpha_i,-\alpha_t)+f(\alpha_i,-\alpha_j)}[M(-\alpha_j), M(-\alpha_t)]_v M(\alpha_i) X_{b-t,c,a} \\ &-v^{f(\alpha_i,-\alpha_j)} M(-\alpha_j)[M(\alpha_i), M(-\alpha_t)]_v X_{b-t,c,a} \end{aligned}$$

By the inductive assumption we knew that

$$\begin{aligned} &M(\alpha_i)[M(-\alpha_j), M(-\alpha_t)]_v X_{b-t,c,a} \\ &= v^{f(\alpha_i,-\alpha_t)+f(\alpha_i,-\alpha_j)}[M(-\alpha_j), M(-\alpha_t)]_v M(\alpha_i) X_{b-t,c,a} \\ &+[M(\alpha_i), [M(-\alpha_j), M(-\alpha_t)]_v]_v X_{b-t,c,a} \end{aligned}$$

Since

$$\begin{aligned} &[M(\alpha_i), M(-\alpha_j)M(-\alpha_t)]_v \\ &= [M(\alpha_i), M(-\alpha_j)]_v M(-\alpha_t) + v^{f(\alpha_i,-\alpha_j)} M(-\alpha_j)[M(\alpha_i), M(-\alpha_t)]_v \end{aligned}$$

and

$$\begin{aligned} &[M(\alpha_i), M(-\alpha_t)M(-\alpha_j)]_v \\ &= [M(\alpha_i), M(-\alpha_t)]_v M(-\alpha_j) + v^{f(\alpha_i,-\alpha_t)} M(-\alpha_t)[M(\alpha_i), M(-\alpha_j)]_v \end{aligned}$$

So the question (*2) is reduced to prove
(*3)

$$\begin{aligned} &[M(\alpha_i), M(-\alpha_j)M(-\alpha_t)]_v X_{b-t,c,a} \\ &= v^{f(\alpha_j,\alpha_t)}[M(\alpha_i), M(-\alpha_t)M(-\alpha_j)]_v X_{b-t,c,a} \\ &+[M(\alpha_i), [M(-\alpha_j), M(-\alpha_t)]_v]_v X_{b-t,c,a} \end{aligned}$$

However by the inductive assumption we have the operator equation

$$[M(-\alpha_j), M(-\alpha_t)]_v = M(-\alpha_j)M(-\alpha_t) - v^{f(\alpha_j,\alpha_t)} M(-\alpha_t)M(-\alpha_j)$$

So (*3) is valid. The proof is finished.

4.8 Corollary The R-module V is moreover a $\mathcal{H}(\mathcal{R})$-module, in particular, the monomials $M_{b,c,a}$ with $(b,c,a) \in \mathbf{N_0^m} \times \mathbf{Z^n} \times \mathbf{N_0^m}$ are linear independent over $\mathbf{Q}(v)$.

Proof The relations in Proposition 4.7 imply the relations in Proposition 4.5. So V satisfies the relations in Definition 2.5. Namely V is a $\mathcal{H}(\mathcal{R})$-module. Additionally, $X_{b,c,a}$ in V are obtained by applying $M_{b,c,a}$ to $1 \in V$, so $M_{b,c,a}$ are linear independent since $X_{b,c,a}$ constitute a basis.

4.9 Corollary For any $\alpha, \beta, \gamma \in \Phi$ we have the Jacobi-type identity in $\mathcal{H}(\mathcal{R})$:

$$v^{f(\gamma,\alpha)}[M(\alpha),[M(\beta),M(\gamma)]_v]_v + v^{f(\alpha,\beta)}[M(\beta),[M(\gamma),M(\alpha)]_v]_v \\ +v^{f(\beta,\gamma)}[M(\gamma),[M(\alpha),M(\beta)]_v]_v = 0$$

Proof Expand them according to Proposition 4.7 and then a straightforward computation.

5. A filtration and integral form of $\mathcal{H}(\mathcal{R})$

5.1 Lemma 4.1.1 provides us a way to give a filtration on $\mathcal{H}(\mathcal{R})$. For any $\alpha \in \Phi$, we define

$$o(M(\alpha)) = (\dim_k \operatorname{Hom}_{\mathcal{R}}(M(\alpha), M(\beta)))_{\beta\in\Phi} \in \mathbf{N}_\mathbf{0}^\Phi$$

Dually

$$p(M(\alpha)) = (\dim_k \operatorname{Hom}_{\mathcal{R}}(M(\beta), M(\alpha)))_{\beta\in\Phi} \in \mathbf{N}_\mathbf{0}^\Phi$$

In general, for a monomial $M(\alpha)\cdots M(\gamma)$, we define

$$o(M(\alpha)\cdots M(\gamma)) = o(M(\alpha)) + \cdots + o(M(\gamma))$$
$$p(M(\alpha)\cdots M(\gamma)) = p(M(\alpha)) + \cdots + p(M(\gamma))$$

Particularly for $K \in \mathcal{H}^0$ we take $o(K) = 0$ and $p(K) = 0$.

There is an ordinary partial order in $\mathbf{N}_\mathbf{0}^\Phi$: $a \geq b$ if $a - b \in \mathbf{N}_\mathbf{0}^\Phi$; $a > b$ if $a \geq b$ and $a - b \neq 0$.

By Lemma 4.1.1 and Corollary 4.1.3, we have

5.1.1 Lemma For any $\alpha, \beta \in \Phi$, then, in $\mathcal{H}(\mathcal{R})$,

$$[M(\alpha), M(\beta)]_v = M(\alpha)M(\beta) - v^{f(\alpha,\beta)}M(\beta)M(\alpha) = \sum_i c_i M_i$$

where $c_i \in \mathbf{Q}(v)$ and M_i are monomials such that $o(M_i) < o(M(\alpha)) + o(M(\beta))$, $p(M_i) < p(M(\alpha)) + p(M(\beta))$ whenever $c_i \neq 0$.

5.2 Set $\mathcal{H}^{(0)} = \mathcal{H}^0$, in general, for $s \in \mathbf{N}_\mathbf{0}^\Phi$, let $\mathcal{H}^{(s)}$ be the submodule of $\mathcal{H}(\mathcal{R})$ spanned by the monomials M with $o(M) \leq s$ over $\mathcal{H}^{(0)}$. Obviously

$$\mathcal{H}(\mathcal{R}) = \bigcup_{t\in\mathbf{N}_\mathbf{0}^\Phi} \mathcal{H}^{(t)}$$

and by Lemma 5.1.1

$$\mathcal{H}^{(s)}\mathcal{H}^{(t)} \subseteq \mathcal{H}^{(s+t)}$$

Therefore it is not difficult to see the following result.

Proposition The Hall algebra $\mathcal{H}(\mathcal{R})$ is a filtrated algebra over $\mathbf{Q}(v)$ in which the filtration is given by $\{\mathcal{H}^{(s)} \mid s \in \mathbf{N}_{\mathbf{0}}^{\Phi}\}$ with respect to $o(\ )$. The associated graded algebra $Gr\mathcal{H}(\mathcal{R})$ is an algebra over $\mathbf{Q}(v)$ on generators $M(\alpha)$ ($\alpha \in \Phi$) and K_β ($\beta \in G_0(\mathcal{R})$) subject to the following relations

$$K_\alpha K_\beta = K_{\alpha+\beta},\ K_0 = 1$$

$$K_\beta M(\alpha) = v^{(\beta,\alpha)} M(\alpha) K_\beta$$

$$M(\alpha)M(\beta) = v^{f(\alpha,\beta)} M(\beta)M(\alpha)$$

Dually we have the same conclusion respect to $p(\ )$ and its induced filtration. Also as a consequence of this proposition, $\mathcal{H}(\mathcal{R})$ is a noetherian domain (without zero divisor).

Remark A similar filtration for quantum group is well-known according to the investigation of de Concini-Kac [2], which depends on a lemma of Levendorskii-Soibelman in [10]. The filtration of $\mathcal{H}(\mathcal{R})$ is dominated by the homological behavior of $\mathcal{R}$, so we avoid a case by case examination in the proof and the filtration here is little bit more strict.

5.3 For $\alpha \in \Phi$, let $\underline{\dim} M(\alpha)$ be the natural image of the object $M(\alpha)$ in the Grothendick group $G_0(\mathcal{R})$, which is called the dimension vector of $M(\alpha)$. In general, for any monomial $M = M(\alpha) \cdots M(\gamma)$, define $\underline{\dim} M = \underline{\dim} M(\alpha) + \cdots + \underline{\dim} M(\gamma)$. Additionally, for any $K_\alpha \in \mathcal{H}^0$, define $\underline{\dim} K_\alpha = 0$. For any $s \in \mathbf{Z}^n$ set $\mathcal{H}_s$ to be the submodule of $\mathcal{H}(\mathcal{R})$ spanned by the monomials M with $\underline{\dim} M = s$ over $\mathcal{H}^0$. Particularly, $\mathcal{H}_0 = \mathcal{H}^0$.

Proposition $\mathcal{H}(\mathcal{R}) = \bigoplus_{t \in \mathbf{Z}^n} \mathcal{H}_t$ is a $G_0(\mathcal{R})$-graded algebra.

Proof We only have to notice that $\phi_{\alpha\beta}^{\varphi} \neq 0$ implies $\underline{\dim} M(\alpha) + \underline{\dim} M(\beta) = \underline{\dim} M(\varphi)$.

5.4 Choosing arbitrary complete slice of $\mathcal{R}$, we get the corresponding hereditary subcategory $\mathcal{B}$ of $\mathcal{R}$. Let $S_1, \cdots, S_n$ be a complete set of simple object in $\mathcal{B}$, denoted by $e_i = \underline{\dim} S_i$, $K_i = K_{e_i}$, $E_i = S_i$, $F_i = TS_i$. Set

$$M(\alpha)^{(t)} = \frac{(M(\alpha))^t}{[t]!_{\varepsilon(\alpha)}} \quad \text{for any } \alpha \in \Phi$$

This element is called the divided power of $M(\alpha)$. Set

$$\left[\begin{array}{c} K_i;c \\ t \end{array}\right] = \prod_{s=1}^{t} \frac{K_i v_i^{c-s+1} - K_i^{-1} v_i^{-c+s-1}}{v_i^s - v_i^{-s}}$$

We have the following two fact due to Lusztig [12] and Ringel [19] respectively.

(1)

$$E_i^{(a)} F_i^{(b)} = \sum_{t=0}^{min(a,b)} F_i^{(b-t)} \left[\begin{array}{c} K_i;2t-a-b \\ t \end{array}\right] E_i^{(a-t)}$$

and the $\mathbf{Z}[v,v^{-1}]$-submodule of $\mathcal{H}^0$ with basis

$$K_1^{\delta_1} \cdots K_n^{\delta_n} \left[\begin{array}{c} K_i;0 \\ c_1 \end{array}\right] \cdots \left[\begin{array}{c} K_i;0 \\ c_n \end{array}\right]$$

is a subring of $\mathcal{H}^0$, where $\delta_i \in \{0,1\}$ and $c_i \in \mathbf{N}$.

(2) For any indecomposable object $M(\alpha) \in \mathcal{B}$, the divided power $M(\alpha)^{(t)}$ belongs to the $\mathbf{Z}[v,v^{-1}]$-subalgebra of $\mathcal{H}(\mathcal{B})$ generated by $E_i^{(s)}$, $i = 1, \cdots, n, s \geq 1$.

The property (2) may be considered as a quantum form of the Kostant theorem for the positive part of the universal enveloping algebra of semisimple Lie algebra. For whole quantum group $\mathcal{U}$, this integral form is known from Lusztig [12], the property (1) is only an easy part of it.

5.5 Let $M(\alpha_1), \cdots, M(\alpha_m)$ be a total ordering of indecomposable objects in $\mathcal{B}$ (or any fixed ordering), set

$$M^{(b,\delta,c,a)} = M(-\alpha_1)^{(b_1)} \cdots M(-\alpha_m)^{(b_m)} K_1^{\delta_1} \cdots K_n^{\delta_n} \times$$
$$\left[\begin{array}{c} K_1;0 \\ c_1 \end{array}\right] \cdots \left[\begin{array}{c} K_n;0 \\ c_n \end{array}\right] M(\alpha_1)^{(a_1)} \cdots M(\alpha_m)^{(a_m)}$$

where $b = (b_1, \cdots, b_m) \in \mathbf{N_0^m}$, $\delta = (\delta_1, \cdots, \delta_n) \in \{0,1\}^n$, $c = (c_1, \cdots, c_n) \in \mathbf{N^n}$, $a = (a_1, \cdots, a_m) \in \mathbf{N_0^m}$.

5.5.1 Theorem The $\mathbf{Z}[v,v^{-1}]$-submodule $\mathcal{K}$ of $\mathcal{H}(\mathcal{R})$ with $\mathbf{Z}[v,v^{-1}]$-basis

$$\{M^{(b,\delta,c,a)} \mid (b,\delta,c,a) \in \mathbf{N_0^m} \times \mathbf{\{0,1\}^n} \times \mathbf{N^n} \times \mathbf{N_0^m}\}$$

is a subring of $\mathcal{H}(\mathcal{R})$.

Proof By Theorem 3.8 and (5.4)(1), it is clear that $M^{(b,\delta,c,a)}$ are linear independent over $\mathbf{Q}(v)$. Base on (5.4)(1), we only need to prove that any monomial of divided powers belongs to $\mathcal{K}$. Let $M = M(\alpha)^{(a)} M(\beta)^{(b)} \cdots$

be a monomial of divided powers with $\alpha,\beta \in \Phi$, $a,b \in \mathbf{N_0}$. We are proceeding by induction on $o(M)$.

If $\alpha = -\beta$, without loss of generality, we may assume $M(\alpha) \in \mathcal{B}$. By (5.4)(2), $M(\alpha)^{(a)}$ is in the $\mathbf{Z}[v,v^{-1}]$-subalgebra of $\mathcal{H}(\mathcal{B})$ generated by $E_i^{(t)}$. From the formula in (5.4) (1) and the relation in (3.4)(4), it follows that $M = M(\alpha)^{(a)}M(-\alpha)^{(b)}\cdots$ belongs to $\mathcal{K}$ if and only if $M' = M(-\alpha)^{(b)}M(\alpha)^{(a)}\cdots$ belongs to $\mathcal{K}$, where we use the inductive assumption .

If $\alpha \neq -\beta$. There exists a hereditary subcategory $\mathcal{B}'$ such that $M(\alpha),\ M(\beta) \in \mathcal{B}'$. We have the notation $M(a\alpha)$ and $M(b\beta)$ in $\mathcal{H}(\mathcal{B}')$ in the sense of Ringel [19], where $M(a\alpha) = v^{a\varepsilon(\alpha)(1-a)}M(\alpha)^{(a)}$, $M(b\beta) = v^{b\varepsilon(\beta)(1-b)}M(\beta)^{(b)}$ without loss of generality we may assume $\beta < \alpha$ in $\mathcal{B}'$, then

$$\begin{aligned} M(a\alpha)M(b\beta) &= \sum_{\varphi:\Phi\to\mathbf{N_0}} \phi^{\varphi}_{a\alpha\ b\beta}M(\varphi) \\ &= v^{ab(\beta,\alpha)}M(b\beta)M(a\alpha) + \sum_{\substack{\varphi:\Phi\to\mathbf{N_0}\\ \varphi\neq a\alpha\oplus b\beta}} M(\varphi) \end{aligned}$$

where $M(\varphi)$, when $\varphi \neq a\alpha \oplus b\beta$, is a monomial of divided powers up to a power of v as coefficient. So by the inductive assumption, $M = M(\alpha)^{(a)}M(-\alpha)^{(b)}\cdots$ belongs to $\mathcal{K}$ if and only if $M' = M(-\alpha)^{(b)}M(\alpha)^{(a)}\cdots$ belongs to $\mathcal{K}$.

To sum up, any monomial of divided powers can be reduced to one of form $M^{(b,\delta,c,a)}$. The proof is finished.

6. Degeneration of $\mathcal{H}(\mathcal{R})$ to the universal enveloping algebras

6.1 We may choose a family of new operators $\{H_\beta \mid \beta \in G_0(\mathcal{R})\}$ which correspond to $\{K_\beta \mid \beta \in G_0(\mathcal{R})\}$ according to the relations $K_\beta = v^{H_\beta}$ or $H_\beta = \log_v K_\beta$. Also set $v = e^x$. In view of the formula:

$$e^{xA}Be^{-xA} = \sum_{n=0}^{\infty}\frac{x^n}{n!}(adA)^n(B)$$

where $(adA)(B) = [A,B] = AB - BA$, we have

$$\begin{aligned} K_\beta M(\alpha)K_\beta^{-1} &= v^{H_\beta}M(\alpha)v^{-H_\beta} = e^{xH_\beta}M(\alpha)e^{-xH_\beta} \\ &= M(\alpha) + x[H_\beta, M(\alpha)] + \frac{x^2}{2!}[H_\beta,[H_\beta,M(\alpha)]] + \cdots \end{aligned}$$

If we set $[H_\beta, M(\alpha)] = (\beta,\alpha)M(\alpha)$, then

$$K_\beta M(\alpha)K_\beta^{-1} = M(\alpha) + x(\beta,\alpha)M(\alpha) + \frac{(x(\beta,\alpha))^2}{2!}M(\alpha) + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(x(\beta,\alpha))^n}{n!} M(\alpha) = e^{x(\beta,\alpha)} M(\alpha) = v^{(\beta,\alpha)} M(\alpha)$$

So the relations (1) and (2) in Definition (2.5) can be replaced by
(1) $H_\alpha + H_\beta = H_{\alpha+\beta},\ H_0 = 0$
(2) $H_\beta M(\alpha) - M(\alpha) H_\beta = (\beta,\alpha) M(\alpha)$
Obviously $\frac{v^{H_\alpha} - v^{-H_\alpha}}{v^{\varepsilon(\alpha)} - v^{-\varepsilon(\alpha)}} \to \frac{H_\alpha}{\varepsilon(\alpha)}$ as $v \to 1$, then the relation (3) in Definition (2.5) is degenerated into
(3) $M(\alpha)M(-\alpha) - M(-\alpha)M(\alpha) = \frac{H_\alpha}{\varepsilon(\alpha)}$

Lemma For $\alpha,\ \gamma \in \Phi$ with $\alpha \neq -\gamma$ and $\varphi : \Phi \to \mathbf{N_0}$ with $\varphi \neq \alpha \oplus \gamma$. If $M(\varphi)$ is decomposable, then $\phi^{\varphi}_{\alpha\gamma} = 0$.

Proof A direct consequence of Theorem 2 of Ringel [17] and Lemma 2.3.1.

By this lemma, the relation (4) in Definition 2.5 is degenerated into
(4) For $\alpha,\ \gamma \in \Phi$ with $\alpha \neq -\gamma$ and $\varepsilon(\gamma,\alpha) = 0$, $\xi(\alpha,\gamma) = 0$,

$$M(\gamma)M(\alpha) - M(\alpha)M(\gamma) = \sum_{\lambda \in \Phi} \phi^{\lambda}_{\gamma\alpha}(1) M(\lambda)$$

Since indecomposable object in $\mathcal{R}$ is determined by its dimension vector, if $M(\gamma)M(\alpha) - M(\alpha)M(\gamma) \neq 0$, then there exists a unique $\lambda \in \Phi$ such that $M(\gamma)M(\alpha) - M(\alpha)M(\gamma) = \phi^{\lambda}_{\gamma\alpha}(1)M(\lambda)$

We are able to define $\mathcal{H}_1(\mathcal{R})$: the degeneration of $\mathcal{H}(\mathcal{R})$ at $v = 1$.

6.2 Definition $\mathcal{H}_1(\mathcal{R})$ is defined to be an associative $\mathbf{Q}$-algebra generated by $\{M(\alpha) \mid \alpha \in \Phi\}$ and $\{H_\beta \mid \beta \in G_0(\mathcal{R})\}$ subject to the relations (1), (2), (3) and (4) in (6.1).

As same as above, take any complete slice $\mathcal{S}$ of $\mathcal{R}$ and get the corresponding hereditary subcategory $\mathcal{B}$. Let $S_1, \cdots, S_n$ be a complete set of simple objects in $\mathcal{B}$, let $e_i = \underline{\dim} S_i$, $E_i = S_i$, $F_i = TS_i$ and $H_i = H_{e_i}$. Then relations (1), (2), (3), (4) and (5) in (3.4) are degenerated into, respectively,
(1) $[H_i, H_j] = 0$
(2) $[H_i, E_j] = a_{ij} E_j,\quad [H_i, F_j] = -a_{ij} F_j$
(3) $E_i F_j - F_j E_i = \delta_{ij} \frac{H_i}{\varepsilon_i}$
(4)

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \begin{pmatrix} 1 - a_{ij} \\ t \end{pmatrix} E_i^t E_j E_i^{1-a_{ij}-t} = 0 \quad \text{for } i \neq j$$

$$\sum_{t=0}^{1-a_{ij}} (-1)^t \left(\begin{array}{c} 1-a_{ij} \\ t \end{array} \right) F_i^t F_j F_i^{1-a_{ij}-t} = 0 \quad \text{for } i \neq j$$

where $1 \leq i,j \leq n$, $a_{ij} = \frac{(e_i,e_j)}{\varepsilon_i}$, $\varepsilon_i = \dim_k \mathrm{Hom}_{\mathcal{R}}(S_i, S_i)$. Of course, $\mathcal{H}_1(\mathcal{R})$ satisfies these relations.

6.3 Let $M(\alpha_1), M(\alpha_2), \cdots, M(\alpha_m)$ be a tortal ordering of indecomposable objects in $\mathcal{B}$. Put

$$N_{b,c,a} := M(-\alpha_1)^{b_1} \cdots M(-\alpha_m)^{b_m} H_1^{c_1} \cdots H_n^{c_n} M(\alpha_1)^{a_1} \cdots M(\alpha_m)^{a_m}$$

with $(b,c,a) \in \mathbf{N_0^m} \times \mathbf{N_0^n} \times \mathbf{N_0^m}$. The degeneration form of Lemma 3.6 is the following , but the proof is much easier.

Lemma The monomials $N_{b,c,a}$ with $(b,c,a) \in \mathbf{N_0^m} \times \mathbf{N_0^n} \times \mathbf{N_0^m}$ are linear independent over $\mathbf{Q}$.

Base on this fact we have the degeneration form of Theorem 3.8.

Theorem Let $\mathcal{R}$ be the root category, $\mathcal{H}_1(\mathcal{R})$ the degenerated Hall algebra of $\mathcal{R}$. Choose any complete slice $\mathcal{S}$ in $\mathcal{R}$. $\mathcal{B}$ the corresponding hereditary subcategory. Let $S_1, \cdots, S_n$ be the complete set of simple objects in $\mathcal{B}$, denoted by $e_i = \underline{\dim} S_i$ and $E_i = S_i$, $F_i = TS_i$, $H_i = H_{e_i}$ in $\mathcal{H}_1(\mathcal{R})$. Then

(1) The relations (1), (2), (3) and (4) in (6.2) are define relations of $\mathcal{H}_1(\mathcal{R})$, that is, $\mathcal{H}_1(\mathcal{R})$ is the universal enveloping algebra of the corresponding simple Lie algebra.

(2) Let $M(\alpha_1), M(\alpha_2), \cdots, M(\alpha_m)$ be a tortal ordering of indecomposable objects in $\mathcal{B}$, then the monomials $N_{b,c,a}$ with $(b,c,a) \in \mathbf{N_0^m} \times \mathbf{N_0^n} \times \mathbf{N_0^m}$ form a $\mathbf{Q}$-basis of $\mathcal{H}_(\mathcal{R})$.

6.4 Let $\mathbf{g}$ be the $\mathbf{Q}$-subspace of $\mathcal{H}_1(\mathcal{R})$ with basis $\{H_i,\ M(\alpha) \mid i = 1, \cdots, n, \alpha \in \Phi\}$ Obviously $\mathbf{g}$ is closed under the operation $[-,-]$, so $\mathbf{g}$ is a Lie subalgebra. It is well-known that the relation (4) in (6.2) is nothing else but

$$(adE_i)^{1-a_{ij}}(E_j) = 0, \quad (adF_i)^{1-a_{ij}}(F_j) = 0$$

for $i \neq j$. Therefore we get

Corollary $\mathbf{g}$ is the simple Lie algebra corresponding to $\mathcal{R}$ and $\mathcal{H}_1(\mathcal{R})$ is the universal enveloping algebra of $\mathbf{g}$.

6.5 We have pointed out in [13] that the indecomposable objects in $\mathcal{R}$ provide a Chevalley basis of $\mathbf{g}$. Analogously, the construction of $\mathcal{H}_1(\mathcal{R})$ provides a Kostant $\mathbf{Z}$-form for the universal enveloping algebra, which is exactly the degeneration of that in Theorem 5.5.1.

Let $h_i = \frac{H_i}{\varepsilon_i}$ and

$$\begin{pmatrix} h_i \\ c_i \end{pmatrix} = \frac{h_i(h_i-1)\cdots(h_i-c_i+1)}{c_i!}$$

for $c_i \in \mathbf{N_0}$. Let $M(\alpha_1), \cdots, M(\alpha_m)$ be a total ordering of indecomposable objects in $\mathcal{B}$ (or any fixed ordering). Set

$$N^{(b,c,a)}$$

$$= \frac{M(-\alpha_1)^{b_1}}{b_1!} \cdots \frac{M(-\alpha_m)^{b_m}}{b_m!} \begin{pmatrix} h_1 \\ c_1 \end{pmatrix} \cdots \begin{pmatrix} h_n \\ c_n \end{pmatrix} \frac{M(\alpha_1)^{a_1}}{a_1!} \cdots \frac{M(\alpha_m)^{a_m}}{a_m!}$$

for $(b,c,a) \in \mathbf{N_0^m} \times \mathbf{N_0^n} \times \mathbf{N_0^m}$.

Theorem The $\mathbf{Z}$-submodule of $\mathcal{H}_1(\mathcal{R})$ with the $\mathbf{Z}$-basis $N^{(b,c,a)}$, $(b,c,a) \in \mathbf{N_0^m} \times \mathbf{N_0^n} \times \mathbf{N_0^m}$, is a subring of $\mathcal{H}_1(\mathcal{R})$.

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