DRINFELD DOUBLE AND GREEN-RINGEL THEORY OF HALL ALGEBRAS

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Abstract As a continuation to the work of Ringel and Green on Hall algebras, the Hopf algebra structure of a (extended twisted) Hall algebra is presented here. By introducing the Ringel pairing of Hall algebras, the Drinfeld double of a Hall algebra is constructed. The reduced form of this Drinfeld double is appliable to the generic composition algebra, which gives a complete realization of quantum group. This Green-Ringel theory make us possible give the formulae for the computation of commutators in quantum group in terms of Hall algebra.

1. Introduction

1.1 Let $C=(a_{ij})_{ij}$ be a symmetrizable generalized Cartan matrix, we denote by $(\epsilon_i)_i$ the minimal symmetrization, thus ϵ_i are integers without a proper common divisor such that $\epsilon_i a_{ij}=\epsilon_j a_{ji}$. Also, according to Lusztig [8], the so-called Cartan datum is an equivalent object with C (see (2.5)). Let $\mathbf{Q}(v)$ be the function field in one variable v over the field $\mathbf{Q}$ of rational numbers. The quantum group $\mathbf{U}=\mathbf{U}_q(\mathbf{g})$ introduced by Drinfeld and Jimbo is an associative $\mathbf{Q}(v)$-algebra on the generators: $F_1,\cdots,F_n,K_1,\cdots,$ $K_n,K_1^{-1},\cdots,K_n^{-1},E_1,\cdots,E_n$ subject to the relations:

$$K_iK_j=K_jK_i,\ K_iK_i^{-1}=K_i^{-1}K_i=1$$

$$K_iE_j=v_i^{a_{ij}}E_jK_i,\ K_iF_j=v_i^{-a_{ij}}F_jK_i$$

$$E_iF_j-F_jE_i=\delta_{ij}\frac{K_i-K_i^{-1}}{v_i-v_i^{-1}}$$

$$\sum_{t=0}^{1-a_{ij}}(-1)^t\left[\begin{array}{c}1-a_{ij}\\ t\end{array}\right]_{\epsilon_i}E_i^tE_jE_i^{1-a_{ij}-t}=0\ \text{ for }\ i\neq j$$

$$\sum_{t=0}^{1-a_{ij}}(-1)^t\left[\begin{array}{c}1-a_{ij}\\ t\end{array}\right]_{\epsilon_i}F_i^tF_jF_i^{1-a_{ij}-t}=0\ \text{ for }\ i\neq j$$

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We admit the notations $q=v^2$, $v_i=v^{\epsilon_i}$ and

$$[s]=\frac{v^s-v^{-s}}{v-v^{-1}}=v^{s-1}+v^{s-3}+\cdots+v^{-s+1}$$

$$[s]!=\prod_{r=1}^{s}[r],\quad \begin{bmatrix} s \\ r \end{bmatrix}=\frac{[s]!}{[r]![s-r]!}$$

and, for given a polynomial $f\in \mathbf{Z}[v,v^{-1}]$ and an integer a, we denote by f_a the polynomial obtained from f by replacing v by v^a. $\mathbf{U}$ has the triangular decomposition

$$\mathbf{U}\simeq \mathbf{U}^-\otimes \mathbf{T}\otimes \mathbf{U}^+$$

where $\mathbf{U}^+$ (resp. $\mathbf{U}^-$) is the subalgebra generated by E_i (resp. F_i), $i=1,\cdots,n$ and $\mathbf{T}$ generatd by $K_i^{\pm}, i=1,\cdots,n$.

A strong feature of $\mathbf{U}$ is its Hopf algebra structure, however the Hopf operations is not closed in the positive part $\mathbf{U}^+$ (resp. $\mathbf{U}^-$), but it works for the Borel part $\mathbf{U}_q(\mathbf{b}^+)$ (resp. $\mathbf{U}_q(\mathbf{b}^-)$) of $\mathbf{U}$. Drinfeld found a general construction, the Drinfeld double, mirroring a Lie bialgebra. Applying the Drinfeld construction to the Borel parts $\mathbf{U}_q(\mathbf{b}^+)$ and $\mathbf{U}_q(\mathbf{b}^-)$, this gives $\mathbf{U}=\mathbf{U}_q(\mathbf{g})$ up to passage to a quotient (modulo a trivial Hopf ideal). We get this idea from [7].

1.2 Now several programs for the realization of quantum group have been raised. A successful model is the Hall algebra of hereditary algebra which is introduced by Ringel [13].

Let k be a finite field, Λ a finite dimensional hereditary k-algebra, $\mathcal{P}$ the set of isomorphism classes of finite dimensional Λ-modules. For every $\alpha\in\mathcal{P}$, choose u_α to be a representative in α, for $\alpha,\beta,\lambda\in\mathcal{P}$, let $g^\lambda_{\alpha\beta}$ be the number of submodules B of u_λ such that $B\simeq u_\beta$ and $u_\lambda/B\simeq u_\alpha$. The bilinear form $\langle -,-\rangle$ is given by $\langle\alpha,\beta\rangle=\dim_k \mathrm{Hom}_\Lambda(u_\alpha,u_\beta)-\dim_k \mathrm{Ext}_\Lambda(u_\alpha,u_\beta)$. The (twisted) Hall algebra $\mathbf{H}(\Lambda)$ is defined to be a $\mathbf{Q}$-space with the basis $\{u_\alpha\mid \alpha\in\mathcal{P}\}$, which multiplication is given by

$$u_\alpha u_\beta=\sum_{\lambda\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta} u_\lambda$$

Ringel [13] has proved that the quantum serre relation is a zero element in Hall algebra and, if Λ and $\mathbf{g}$ enjoy a common Cartan datum of finite type, the generic form of $\mathbf{H}(\Lambda)$ gives a realization of $\mathbf{U}^+$ of $\mathbf{U}_q(\mathbf{g})$. Recently Green [5] found a remarkable formula about the filtration numbers of modules over a hereditary algebra, this led to a formulation for the comultiplication of $\mathbf{H}(\Lambda)$, he proved (depends on Lusztig [8]) that the positive part $\mathbf{U}^+$ of $\mathbf{U}_q(\mathbf{g})$ is canonically isomorphic to the generic composition algebra of Λ if $\mathbf{g}$ and Λ enjoy a common Cartan datum (of any type), where composition algebra is the subalgebra of the Hall algebra generated by simple Λ-modules.

1.3 Combine the Ringel′s multiplication and Green′s comultiplication, Hall algebra becomes nearly a bialgebra. It is natural (as did for quantum group) to add the torus algebra $\mathbf{T}$ to $\mathbf{H}$. We presented below (4.5), as an easy consequence of the Green-Ringel theory, this extended Hall algebra becomes really a Hopf algebra, and the antipode can be formulated in a global way too. Due to Ringel [16], we can define the Ringel pairing between positive and negative extended Hall algebras. The (reduced) Drinfeld double is obtained from the Ringel pairing, so according to the philosophy of Drinfeld, Hall algebra also join the Hopf algebra classes which fit to the solutions of the quantum Yang-Baxter equation. We can restrict the construction to the generic composition algebra, it really gives a realization of the whole quantum group $\mathbf{U}$.

It is well-known quantum groups are Hopf algebras, but the operations (multiplication, comultiplication and antipode) can only be defined on the generators: E_i, F_i, K_i. An obvious advantage of the Hall algebra approach is that these operations are globally defined over a basis. Elements in the basis can be interpreted as modules over Λ and these operations have some interpretations from the representation theory of Λ. So the Drinfeld double structure provides us a chance to write down a formula for the commutators $u^-_{\lambda'}u^+_\lambda - u^+_\lambda u^-_{\lambda'}$ for any $\lambda, \lambda' \in \mathcal{P}$.

1.4 In Section 2, we give the definitions of Cartan datum, skew-Hopf pairing and Drinfeld double. Our concern is devoted to the skew-Hopf pairing beloning to a Cartan datum C, this class is denoted by $\mathcal{L}(C)$. We review the construction of Lusztig and Joseph in the framework of skew-Hopf pairing, both are non-degenerate members of $\mathcal{L}(C)$. Section 3 is a review of Green′s result only with a slight modification. For $(A^+, A^-, \varphi) \in \mathcal{L}(C)$, there is an algorithm to calculate φ which only depends on the Cartan datum. This implies that any two non-degenerate members are canonically isomorphic, so the construction of Lusztig and Joseph are canonically isomorphic to each other. In Section 4, one of our main results is Theorem 4.5, which gives the Hopf algebra structure of the (extended twisted) Hall algebra. Since the Hall algebra is naturally a graded and filtrated algebra, although the antipode σ is quite complicated, yet σ^2 is only a v-deformation of the ide! ntity map. This lead to a family o f orthogonal relations for the filtration numbers. In the last section, the positive and negative Hall algebras and their Ringel pairing are formulated precisely, so we have the reduced Drinfeld double of Hall algebras, which restriction to the generic composition algebras give a realization of quantum groups (5.8). The formula for $u^-_{\lambda'}u^+_\lambda - u^+_\lambda u^-_{\lambda'}$ is given, if we take one to be simple Λ-module u_i or take C to be finite type, the formula has some simplified form.

1.5 In this paper it is always to be that; $\mathbf{N}$: the set of non-negative integers, $\mathbf{Z}$: the set of integers, $\mathbf{Q}$: the set of rational numbers. We should make clear that, in the following, Hall algebra always means extended twisted Hall algebra (see (4.5)).

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2. (Skew-)Hopf pairing of a Cartan datum

2.1 In this paper, we choose R to be a commutative integral domain of characteristic zero (we need $\mathbf{Z} \subset R$), and v an invertible element of R. A special case is $\mathbf{Q}(v)$, the rational functional field over $\mathbf{Q}$ in one variable v. In the following the tensor product $\otimes$ means $\otimes_R$.

A R-module H is called a *Hopf algebra,* if H possess the R-linear operations (1) multiplication $\mu : H \otimes H \to H$, (2) unit $\eta : R \to H$, (3) comultiplication $\Delta : H \to H \otimes H$, (4) counit $\varepsilon : H \to R$ and (5) antipode $\sigma : H \to H$ which satisfy the conditions:

(a) (H, μ, η) is an associative R-algebra.
(b) (H, Δ, ε) is an associative R-coalgebra.
(c) Δ, ε are algebra morphisms.
(d) $\mu(\sigma \otimes 1)\Delta = \mu(1 \otimes \sigma)\Delta = \eta\varepsilon$.

2.2 Given Hopf algebra A and B, a *Hopf pairing* of A and B is a R- bilinear function $\varphi : A \times B \to R$ satisfying

(a) $\varphi(1, b) = \varepsilon_B(b), \varphi(a, 1) = \varepsilon_A(a)$
(b) $\varphi(a, bb') = \varphi(\Delta_A(a), b \otimes b')$
(c) $\varphi(aa', b) = \varphi(a \otimes a', \Delta_B(b))$
(d) $\varphi(\sigma_A(a), b) = \varphi(a, \sigma_B(b))$

for all $a, a' \in A$ and $b, b' \in B$, where $\varphi(a \otimes a', b \otimes b') = \varphi(a, b)\varphi(a', b')$.

We define Δ_B^{opp} as $\Delta_B^{opp}(b) = \Sigma b_2 \otimes b_1$ where $\Delta_B(b) = \Sigma b_1 \otimes b_2$, and assume that σ_B is invertable. Then the conditions (c),(d) of (2.2) can be replaced by

(c′) $\varphi(aa', b) = \varphi(a \otimes a', \Delta_B^{opp}(b))$
(d′) $\varphi(\sigma_A(a), b) = \varphi(a, \sigma_B^{-1}(b))$

So for given Hopf algebras A and B, a *skew-Hopf pairing* of A and B is a R-bilinear function $\varphi : A \times B \to R$ which satisfies the conditions (a), (b), (c′) and (d′).

A (skew-)Hopf pairing $\varphi : A \times B \to R$ is called (two-sided) *non-degenerate* if $\varphi(a, B) = 0$ implies $a = 0$ and $\varphi(A, b) = 0$ implies $b = 0$ for any $a \in A, b \in B$.

2.3 If φ is a skew-Hopf pairing, it is well-know that $A \otimes B$ has the Hopf algebra structure which is the so-called *Drinfeld double* of (A, B, φ).

The algebra structure of $A \otimes B$ is determined by

(a) $(a \otimes 1)(a' \otimes 1) = aa' \otimes 1$

(b) $(1 \otimes b)(1 \otimes b') = 1 \otimes bb'$

(c) $(a \otimes 1)(1 \otimes b) = a \otimes b$

(d) $(1 \otimes b)(a \otimes 1) = \sum \varphi(a_1, \sigma_B(b_1))(a_2 \otimes b_2)\varphi(a_3, b_3)$

for all $a, a' \in A, b, b' \in B$, where $\Delta_A^2(a) = \sum a_1 \otimes a_2 \otimes a_3, \Delta_B^2(b) = \sum b_1 \otimes b_2 \otimes b_3$. (Note:$\Delta^2 = (1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$).

We may identify A and B as the subalgebras of $A \otimes B$ by taking $a \mapsto a \otimes 1, b \mapsto 1 \otimes b$. $A \otimes B$ has the other operations: unit $1 \otimes 1$, counit $\varepsilon_A \otimes \varepsilon_B$, comultiplication $\Delta_{A \otimes B} = \tau_{23}(\Delta_A \otimes \Delta_B)$, i.e. $\Delta_{A \otimes B}(a \otimes b) = \Sigma(a_1 \otimes b_3) \otimes (a_2 \otimes b_4)$ where $\Delta_A(a) = \Sigma a_1 \otimes a_2, \Delta_B(b) = \Sigma b_3 \otimes b_4$ and antipode $\sigma_{A \otimes B}$ with $\sigma_{A \otimes B}(a \otimes b) = (1 \otimes \sigma_B(b))(\sigma_A(a) \otimes 1)$.

Retain the operations of $A \otimes B$ in (2.3), we have

2.4 Proposition *Let (A, B, φ) be a skew-Hopf pairing, then $A \otimes B$ is a Hopf algebra.*

For a proof, see [7].

2.5 According to Lusztig [8], a Cartan datum is a pair $(I, (,))$ consisting of a finite set I and a symmetric bilinear form $(-,-)$ on the free abelian group $\mathbf{Z}[I]$ with values in $\mathbf{Z}$ such that the following conditions are satisfied

(a) $(i, i) \in \{2, 4, 6, \cdots\}$ for any $i \in I$,

(b) $\frac{2(i,j)}{(i,i)} \in \{0, -1, -2, \cdots\}$ for any $i \neq j$ in I.

Then $C = (a_{ij})_{i,j \in I}$ is a symmetrizable Cartan matrix where $a_{ij} = \frac{2(i,j)}{(i,i)}$ for all $i, j \in I$. Without loss of generality, we identify $C = (a_{ij})_{i,j \in I}$ to $(I, (,))$.

2.6 The torus of C is the associative R-algebra $\mathbf{T}$ defined by the generators $K_\alpha, \alpha \in \mathbf{Z}[I]$ subject to the relations $K_0 = 1, K_\alpha K_\beta = K_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbf{Z}[I]$. $\mathbf{T}$ is a commutative R-algebra, in fact, $\mathbf{T} = R[K_i, K_i^{-1}]_{i \in I}$.

2.7 We say that the skew-Hopf pairing (A^+, A^-, φ) belongs to the Cartan class $C = (I, (,))$, or (A^+, A^-, φ) is a member of $\mathcal{L}(C)$ if the following conditions are satisfied,

(A^+1) $A^+ = \bigoplus_{\nu \in \mathbf{N}[I]} A_\nu^+$ is an $\mathbf{N}[I]$-graded, associative R-algebra generated by elements $x_i^+ \in A_i^+$ $(i \in I)$ and $A_0^+ = \mathbf{T}$ such that

$$K_\alpha x_i^+ = v^{(\alpha, i)} x_i^+ K_\alpha$$

for all $i \in I, \ \alpha \in \mathbf{Z}[I]$.

(A^-1) $A^- = \bigoplus_{\nu \in \mathbf{N}[I]} A_\nu^-$ is an $\mathbf{N}[I]$-graded, associative R-algebra generated by elements $x_i^- \in A_i^-$ $(i \in I)$ and $A_0^- = \mathbf{T}$ such that

$$K_\alpha x_i^- = v^{-(\alpha, i)} x_i^- K_\alpha$$

for all $i \in I,\ \alpha \in \mathbf{Z}[I]$.

$(A2)$ $\Delta_{A^+}(x_i^+) = x_i^+ \otimes 1 + K_i \otimes x_i^+, \Delta_{A^+}(K_\alpha) = K_\alpha \otimes K_\alpha$

$$\Delta_{A^-}(x_i^-) = x_i^- \otimes K_{-i} + 1 \otimes x_i^-, \Delta_{A^-}(K_\alpha) = K_\alpha \otimes K_\alpha$$

for all $i \in I, \alpha \in \mathbf{Z}[I]$.

$(A3)$ $\varphi(x_i^+, x_i^-) \neq 0$ for all $i \in I$.

$$\varphi(K_\alpha, K_\beta) = v^{-(\alpha,\beta)}, \varphi(x_i^+, K_\beta) = \varphi(K_\alpha, x_i^-) = 0$$

for all $i \in I, \alpha, \beta \in \mathbf{Z}[I]$.

Given any Cartan datum $C = (I, (,))$, we list a few examples which belong to $\mathcal{L}(C)$.

2.8 Example 1 (Lusztig). The free object of $\mathcal{L}(C)$.

2.8.1 Let $\tilde{f}^+$ be the free R-algebra on free generators $\theta_i^+, i \in I$ and $\tilde{F}^+$ be the R-algebra generated by $\tilde{f}^+$ and $\mathbf{T}$ subject to the relations

$$K_\alpha \theta_i^+ = v^{(\alpha,i)} \theta_i^+ K_\alpha$$

for all $i \in I,\ \alpha \in \mathbf{Z}[I]$.

$\tilde{F}^+$ has a natural $\mathbf{N}[I]$-gradation, for any $\nu = \sum_{i \in I} \nu_i i \in \mathbf{N}[I]$, we denote by $\tilde{F}_\nu^+$ the $\mathbf{T}$-submodule of $\tilde{F}^+$ spanned by the monomials $\theta_{i_1}^+ \theta_{i_2}^+ \cdots \theta_{i_r}^+$ such that for any $i \in I$, the number of occurrences of i in the sequence $(i_1, i_2, \cdots, i_r)$ is equal to ν_i. Then each $\tilde{F}_\nu^+$ is a free $\mathbf{T}$-module of finite rank and $\tilde{F}^+ = \bigoplus_{\nu \in \mathbf{N}[I]} \tilde{F}_\nu^+$. We have $\tilde{F}_\nu^+ \tilde{F}_{\nu'}^+ \subset \tilde{F}_{\nu+\nu'}^+, \theta_i^+ \in \tilde{F}_i^+$ for $i \in I$ and $\tilde{F}_0^+ = \mathbf{T}$. The Hopf algebra structure of $\tilde{F}^+$ is given by

$$\Delta(\theta_i^+) = \theta_i^+ \otimes 1 + K_i \otimes \theta_i^+, \Delta(K_\alpha) = K_\alpha \otimes K_\alpha$$

$$\varepsilon(\theta_i^+) = 0, \varepsilon(K_\alpha) = 1$$

$$\sigma(\theta_i^+) = -K_{-i} \theta_i^+, \sigma(K_\alpha) = K_{-\alpha}$$

for all $i \in I, \alpha \in \mathbf{Z}[I]$. It is easy to verify that $\tilde{F}^+$ is a Hopf algebra.

2.8.2 Similarly,let $\tilde{f}^-$ be the free R-algebra on free generators $\theta_i^-, i \in I$ and $\tilde{F}^-$ be the R-algebra generated by $\tilde{f}^-$ and $\mathbf{T}$ subject to the relations

$$K_\alpha \theta_i^- = v^{-(\alpha,i)} \theta_i^- K_\alpha$$

for all $i \in I, \alpha \in \mathbf{Z}[I]$. $\tilde{F}^-$ has a natural $\mathbf{N}[I]$-gradation, for any $\nu = \sum_{i \in I} \nu_i i \in \mathbf{N}[I]$, we denote by $\tilde{F}_\nu^-$ the $\mathbf{T}$-submodule of $\tilde{F}^-$ spanned by the monomials $\theta_{i_1}^- \theta_{i_2}^- \cdots \theta_{i_r}^-$ such that for any $i \in I$, the number of occurrences of i in the sequence $(i_1, i_2, \cdots, i_r)$ is equal to ν_i. Then each $\tilde{F}_\nu^-$ is a free $\mathbf{T}$-module of finite rank and $\tilde{F}^- = \bigoplus_{\nu \in \mathbf{N}[I]} \tilde{F}_\nu^-$. We have $\tilde{F}_\nu^- \tilde{F}_{\nu'}^- \subset \tilde{F}_{\nu+\nu'}^-, \theta_i^- \in \tilde{F}_i^-$ for $i \in I$ and $\tilde{F}_0^- = \mathbf{T}$. The Hopf algebra structure of $\tilde{F}^-$ is given by

$$\Delta(\theta_i^-) = \theta_i^- \otimes K_{-i} + 1 \otimes \theta_i^-, \Delta(K_\alpha) = K_\alpha \otimes K_\alpha$$

$$\varepsilon(\theta_i^-) = 0, \varepsilon(K_\alpha) = 1$$
$$\sigma(\theta_i^-) = -\theta_i^- K_i, \sigma(K_\alpha) = K_{-\alpha}$$

for all $i \in I, \alpha \in \mathbf{Z}[I]$. Now Δ^{opp} and σ^{-1} in $\tilde{F}^-$ exist and are given by

$$\Delta^{opp}(\theta_i^-) = \theta_i^- \otimes 1 + K_{-i} \otimes \theta_i^-, \Delta^{opp}(K_\alpha) = K_\alpha \otimes K_\alpha$$
$$\sigma^{-1}(\theta_i^-) = -K_i\theta_i^-, \sigma^{-1}(K_\alpha) = K_{-\alpha}.$$

2.8.3 Proposition (Lusztig) *There exists a unique pairing $\varphi : \tilde{F}^+ \times \tilde{F}^- \to R$ satisfying $\varphi(1,1) = 1$ and*

(a) $\varphi(\theta_i^+, \theta_j^-) = \delta_{ij}(1 - v^{-(i,i)})^{-1}$, $\varphi(K_\alpha, K_\beta) = v^{-(\alpha,\beta)}$ for all $i, j \in I, \alpha, \beta \in \mathbf{Z}[I]$.

(b) $\varphi(x, y'y'') = \varphi(\Delta(x), y' \otimes y'')$ for all $x \in \tilde{F}^+, y', y'' \in \tilde{F}^-$.

(c) $\varphi(xx', y'') = \varphi(x \otimes x', \Delta^{opp}(y''))$ for all $x, x' \in \tilde{F}^+, y'' \in \tilde{F}^-$.

Proof The comultiplication Δ of $\tilde{F}^+$ induces a multiplication in its dual space $\sum_{\nu \in I} \tilde{F}_\nu^{+*}$, $\tilde{F}_\nu^{+*} \otimes \tilde{F}_{\nu'}^{+*} \to \tilde{F}_{\nu+\nu'}^{+*}$, $(g * h)(x) := \mu(g \otimes h)(\Delta(x))$. Let $\xi_i \in \tilde{F}_i^{+*}$ be the linear function given by $\xi_i(K_\alpha\theta_i^+) = v^{(\alpha,i)}(1 - v^{-(i,i)})^{-1}$, $T_\alpha \in \tilde{F}_0^{+*}$ be given by $T_\alpha(K_\beta) = v^{-(\beta,\alpha)}$ for all $i \in I, \alpha, \beta \in \mathbf{Z}[I]$. It is easy to see that $T_\alpha, \alpha \in \mathbf{Z}[I]$ are one family of commutative linear functions and satisfy that $T_\alpha T_\beta = T_{\alpha+\beta}, T_\alpha\xi_i = v^{-(i,\alpha)}\xi_i T_\alpha$, for example, since $\Delta(K_\beta\theta_i^+) = K_\beta\theta_i^+ \otimes K_\beta + K_\beta K_i \otimes K_\beta\theta_i^+$, then $T_\alpha\xi_i(K_\beta\theta_i^+) = T_\alpha(K_\beta K_i)\xi_i(K_\beta\theta_i^+) = v^{-(\beta+i,\alpha)}v^{(\beta,i)}(1-v^{-(i,i)})^{-1}$ and $\xi_i T_\alpha(K_\beta\theta_i^+) = \xi_i(K_\beta\theta_i^+)T_\alpha(K_\beta) = v^{(\beta,i)}(1-v^{-(i,i)})^{-1}v^{-(\beta,\alpha)}$. So $T_\alpha\xi_i = v^{-(i,\alpha)}\xi_i T_\alpha$.

$\tilde{F}^-$ is an associative R-algebra with free generators $\theta_i^-, i \in I$ and generators $K_\alpha, \alpha \in \mathbf{Z}[I]$ subject to the relations

$$K_\alpha\theta_i^- = v^{-(\alpha,i)}\theta_i^- K_\alpha, K_\alpha K_\beta = K_{\alpha+\beta}$$

Then there exists an algebra morphism $\phi : \tilde{F}^- \to \sum_{\nu \in \mathbf{N}[I]} \tilde{F}_\nu^{+*}$ such that

$$\phi(\theta_i^-) = \xi_i, \phi(K_\alpha) = T_\alpha$$
$$\phi(y'y'')(x) = \sum \phi(y')(x_1)\phi(y'')(x_2)$$

where $\Delta(x) = \sum(x_1 \otimes x_2)$. We define $\varphi : \tilde{F}^+ \times \tilde{F}^- \to R$ by $\varphi(x,y) = \phi(y)(x)$. So the conditions (a), (b) of (2.8.3) are satisfied, and clearly

(d) $\varphi(x,y) = 0$ if x, y homogenous with different weights.

We prove (c) by induction. First we take $y'' = \theta_i^-, x = \theta_i^+$ and $x' = K_\alpha$, or $x = K_\alpha, x' = \theta_i^+$. Note that $\Delta^{opp}(\theta_i^-) = \theta_i^- \otimes 1 + K_{-i} \otimes \theta_i^-$.

$$\varphi(\theta_i^+K_\alpha, \theta_i^-) = \xi_i(\theta_i^+K_\alpha) = v^{-(i,\alpha)}\xi_i(K_\alpha\theta_i^+)$$
$$= v^{-(i,\alpha)}v^{(\alpha,i)}(1 - v^{-(i,i)})^{-1} = (1 - v^{-(i,i)})^{-1}$$
$$\varphi(\theta_i^+ \otimes K_\alpha, \theta_i^- \otimes 1 + K_{-i} \otimes \theta_i^-) = \varphi(\theta_i^+, \theta_i^-)\varphi(K_\alpha, 1) = (1 - v^{-(i,i)})^{-1}$$
$$\varphi(K_\alpha\theta_i^+, \theta_i^-) = \xi_i(K_\alpha\theta_i^+) = v^{(\alpha,i)}(1 - v^{-(i,i)})^{-1}$$
$$\varphi(K_\alpha \otimes \theta_i^+, \theta_i^- \otimes 1 + K_{-i} \otimes \theta_i^-) = \varphi(K_\alpha, K_{-i})\varphi(\theta_i^+, \theta_i^-)$$

$$= v^{-(-i,\alpha)}(1-v^{-(i,i)})^{-1} = v^{(\alpha,i)}(1-v^{-(i,i)})^{-1}$$

Let x, x' and y, y' are homogeneous elements of $\tilde{F}^+$ and $\tilde{F}^-$ respectively and assume that

$$\Delta(x) = \sum x_1 \otimes x_2, \Delta(x') = \sum x'_1 \otimes x'_2$$

$$\Delta(y) = \sum y_1 \otimes y_2, \Delta(y') = \sum y'_1 \otimes y'_2$$

then

$$\Delta(xx') = \sum x_1x'_1 \otimes x_1x'_2, \Delta(yy') = \sum y_1y'_1 \otimes y_2y'_2$$

We want to show (c) holds for $y'' = yy'$.

$$\varphi(xx', yy') = \phi(yy')(xx') = (\phi(y)\phi(y'))(xx')$$

$$(\phi(y) \otimes \phi(y'))(\Delta(xx')) = \sum \phi(y)(x_1x'_1)\phi(y')(x_2x'_2)$$

$$= \sum \varphi(x_1x'_1, y)\varphi(x_2x'_2, y')$$

(by the inductive assumption)

$$= \sum \varphi(x_1 \otimes x'_1, \Delta^{opp}(y))\varphi(x_2 \otimes x'_2, \Delta^{opp}(y'))$$

$$= \sum \varphi(x_1, y_2)\varphi(x'_1, y_1)\varphi(x_2, y'_2)\varphi(x'_2, y'_1)$$

(according to (b) of (2.8.3))

$$= \sum \varphi(x, y_2y'_2)\varphi(x', y_1y'_1) = \varphi(x \otimes x', \Delta^{opp}(yy'))$$

Therefore $\varphi(xx', y'') = \varphi(x \otimes x', \Delta^{opp}(y''))$. The uniqueness of φ is immediate.

2.8.4 Lemma *$(\tilde{F}^+, \tilde{F}^-, \varphi)$ is a skew-Hopf pairing which belongs to the Cartan class $C = (I, (,))$.*

Proof It remains to show $\varphi(\sigma(a), b) = \varphi(a, \sigma^{-1}(b))$ for all $a \in \tilde{F}^+, b \in \tilde{F}^-$. We may assume that a, b are homgeneous elements and induct on the weights of a and b.

(1) $\varphi(\sigma(K_\alpha), K_\beta) = \varphi(K_\alpha, \sigma^{-1}(K_\beta) = v^{(\beta,\alpha)}$.

(2) If $a = K_\alpha, b = b_1b_2$ is a homogeneous element of $\tilde{F}^-$, then

$$\varphi(\sigma(K_\alpha), b_1b_2) = \varphi(K_{-\alpha}, b_1b_2) = \varphi(K_{-\alpha}, b_1)\varphi(K_{-\alpha}, b_2)$$

$$= \varphi(\sigma(K_\alpha), b_1)\varphi(\sigma(K_\alpha), b_2)$$

(by the inductive assumption)

$$= \varphi(K_\alpha, \sigma^{-1}(b_1))\varphi(K_\alpha, \sigma^{-1}(b_2)) = \varphi(\Delta(K_\alpha), \sigma^{-1}(b_2) \otimes \sigma^{-1}(b_1))$$

$$= \varphi(K_\alpha, \sigma^{-1}(b_2)\sigma^{-1}(b_1)) = \varphi(K_\alpha, \sigma^{-1}(b_1b_2))$$

where one should notice that σ^{-1} is an anti-automorphism.

(3) $a = \theta_i^+, b = \theta_i^-$,

$$\varphi(\sigma(\theta_i^+), \theta_i^-) = \varphi(-K_{-i}\theta_i^+, \theta_i^-) = \xi_i(-K_{-i}\theta_i^+) = -v^{-(i,i)}(1-v^{-(i,i)})^{-1}$$

$$\varphi(\theta_i^+, \sigma^{-1}(\theta_i^-)) = \varphi(\theta_i^+, -K_i\theta_i^-) = \varphi(\Delta(\theta_i^+), -K_i \otimes \theta_i^-)$$

$= \varphi(\theta_i^+ \otimes 1 + K_i \otimes \theta_i^+, -K_i \otimes \theta_i^-) = -\varphi(K_i, K_i)\varphi(\theta_i^+, \theta_i^-) = -v^{-(i,i)}(1-v^{-(i,i)})^{-1}$.

(4) If $a = \theta_i^+$, $b = b_1 b_2$ homogeneous element of $\tilde{F}^-$, then

$$\varphi(\sigma(\theta_i^+), b_1 b_2) = \varphi(-K_{-i}\theta_i^+, b_1 b_2) = -\varphi(\Delta(K_{-i}\theta_i^+), b_1 \otimes b_2)$$
$$= -\varphi(K_{-i}\theta_i^+ \otimes K_{-i} + 1 \otimes K_{-i}\theta_i^+, b_1 \otimes b_2)$$
$$= -(\varphi(K_{-i}\theta_i^+, b_1)\varphi(K_{-i}, b_2) + \varphi(1, b_1)\varphi(K_{-i}\theta_i^+, b_2))$$
$$= \varphi(\sigma(\theta_i^+), b_1)\varphi(K_{-i}, b_2) + \varphi(1, b_1)\varphi(\sigma(\theta_i^+), b_2)$$

(by the inductive assumption)

$$= \varphi(\theta_i^+, \sigma^{-1}(b_1))\varphi(K_{-i}, b_2) + \varphi(1, b_1)\varphi(\theta_i^+, \sigma^{-1}(b_2))$$

and

$$\varphi(\theta_i^+, \sigma^{-1}(b_1 b_2)) = \varphi(\theta_i^+, \sigma^{-1}(b_2)\sigma^{-1}(b_1))$$
$$= \varphi(\theta_i^+ \otimes 1 + K_i \otimes \theta_i^+, \sigma^{-1}(b_2) \otimes \sigma^{-1}(b_1))$$
$$= \varphi(\theta_i^+, \sigma^{-1}(b_2))\varphi(1, \sigma^{-1}(b_1)) + \varphi(K_i, \sigma^{-1}(b_2))\varphi(\theta_i^+, \sigma^{-1}(b_1))$$
$$= \varphi(\theta_i^+, \sigma^{-1}(b_1))\varphi(K_{-i}, b_2) + \varphi(1, b_1)\varphi(\theta_i^+, \sigma^{-1}(b_2))$$

So $\varphi(\sigma(\theta_i^+), b_1 b_2) = \varphi(\theta_i^+, \sigma^{-1}(b_1 b_2))$.

(5) In general let $a = a_1 a_2, b = b_1 b_2$ be homegeneous elements of $\tilde{F}^+$ and $\tilde{F}^-$ respectively, $\Delta(a_1) = \sum a_{11} \otimes a_{12}, \Delta(a_2) = \sum a_{21} \otimes a_{22}$. By using the identity $\Delta\sigma = \tau_{12}(\sigma \otimes \sigma)\Delta$ we have

$$\varphi(\sigma(a_1 a_2), b_1 b_2) = \varphi(\Delta\sigma(a_1 a_2), b_1 \otimes b_2) = \varphi(\tau_{12}(\sigma \otimes \sigma)\Delta(a_1 a_2), b_1 \otimes b_2)$$
$$= \varphi(\tau_{12}(\sigma \otimes \sigma)(\sum a_{11}a_{21} \otimes a_{12}a_{22}), b_1 \otimes b_2) = \sum \varphi(\sigma(a_{12}a_{22}), b_1)\varphi(\sigma(a_{11}a_{21}), b_2)$$

(by the inductive assumption)

$$= \sum \varphi(a_{12}a_{22}, \sigma^{-1}(b_1))\varphi(a_{11}a_{21}, \sigma^{-1}(b_2))$$
$$= \sum \varphi(a_{11}a_{21} \otimes a_{12}a_{22}, \sigma^{-1}(b_2) \otimes \sigma^{-1}(b_1))$$
$$= \varphi(\Delta(a_1 a_2), \sigma^{-1}(b_2) \otimes \sigma^{-1}(b_1)) = \varphi(a_1 a_2, \sigma^{-1}(b_2)\sigma^{-1}(b_1)) = \varphi(a_1 a_2, \sigma^{-1}(b_1 b_2))$$

The proof is finished.

2.9 Example 2 (Lusztig) We continue Example 1. Let $\mathcal{I}^+ = \{x \in \tilde{F}^+ \mid \varphi(x, \tilde{F}^-) = 0\}$. If $x \in \mathcal{I}^+, y \in \tilde{F}^+$, for any $z \in \tilde{F}^-$ with $\Delta(z) = \sum z_1 \otimes z_2$ we have $\varphi(xy, z) = \varphi(x \otimes y, \Delta^{opp}(z)) = \sum \varphi(x, z_2)\varphi(y, z_1) = 0$ and $\varphi(yx, z) = \sum \varphi(y, z_2)\varphi(x, z_1) = 0$. So $xy, yx \in \mathcal{I}^+$, $\mathcal{I}^+$ is a two-sided ideal of $\tilde{F}^+$. It is also easy to see $\mathcal{I}^+ \simeq \mathbf{T} \otimes \mathcal{I}_0^+$, where $\mathcal{I}_0^+ = \mathcal{I}^+ \cap \tilde{f}^+$.

Dually let $\mathcal{I}^- = \{y \in \tilde{F}^- \mid \varphi(\tilde{F}^+, y) = 0\}$. $\mathcal{I}^-$ is a two-sided ideal of $\tilde{F}^-$. $\mathcal{I}^- \simeq \mathbf{T} \otimes \mathcal{I}_0^-$, where $\mathcal{I}_0^- = \mathcal{I}^- \cap \tilde{f}^-$.

Let $f^+ = \tilde{f}^+/\mathcal{I}_0^+, F^+ = \tilde{F}^+/\mathcal{I}^+, f^- = \tilde{f}^-/\mathcal{I}_0^-, F^- = \tilde{F}^-/\mathcal{I}^-$. Since $\mathcal{I}_0^\pm, \mathcal{I}^\pm$ are compatible with the weight decomposition of $\tilde{f}^\pm$ and $\tilde{F}^\pm$ respectively, we have the weight decomposition $f^\pm = \bigoplus_{\nu \in \mathbf{N}[I]} f_\nu^\pm$ where $f_\nu^\pm$ is the image of $\tilde{f}_\nu^\pm$ under the natural maps $\tilde{f}^\pm \to f^\pm$ and $F^\pm = \bigoplus_{\nu \in \mathbf{N}[I]} F_\nu^\pm$

where $F^{\pm}_\nu$ is the image of $\tilde{F}^{\pm}_\nu$ under the natural maps $\tilde{F}^{\pm} \to F^{\pm}$. Also we have $F^{\pm}_\nu \simeq \mathbf{T} \otimes f^{\pm}_\nu$, so $F^{\pm} \simeq \mathbf{T} \otimes f^{\pm}$.

If $x \in \mathcal{I}^+$, for any $y, z \in \tilde{F}^-$, then $\varphi(\Delta(x), y \otimes z) = \varphi(x, yz) = 0$. Therefore $\Delta(x)$ is contained in the (left) radical of the pairing φ defined on $\tilde{F}^+ \otimes \tilde{F}^+ \times \tilde{F}^- \otimes \tilde{F}^-$. This (left) radical equals to $\mathcal{I}^+ \otimes \tilde{F}^+ + \tilde{F}^+ \otimes \mathcal{I}^+$. Thus $\Delta(\mathcal{I}^+) \subset \mathcal{I}^+ \otimes \tilde{F}^+ + \tilde{F}^+ \otimes \mathcal{I}^+$. Then Δ induces an algebra homomorphism $F^+ \to F^+ \otimes F^+$, denote it again Δ. Same for $\Delta : F^- \to F^- \otimes F^-$ and $\sigma : F^+ \to F^+$, $\sigma : F^- \to F^-$. The pairing $\varphi : \tilde{F}^+ \times \tilde{F}^- \to R$ factor to a pairing $F^+ \times F^- \to R$, again by $\varphi : F^+ \times F^- \to R$.

2.9.1 Proposition *The induced skew-Hopf pairing (F^+, F^-, φ) is a non-degenerate member of $\mathcal{L}(C)$, where $C = (I, (,))$.*

2.10 Example 3 (Joseph)

2.10.1 Given a Cartan datum $(I, (,))$, let $\tilde{\mathbf{U}}^+$ be a free R-algebra with generators $x^+_i, i \in I$, $\tilde{\mathbf{U}}^-$ a free R-algebra with generators $x^-_i, i \in I$. Same as before they have the natural gradation

$$\tilde{\mathbf{U}}^+ = \bigoplus_{\nu \in \mathbf{N}[I]} \tilde{\mathbf{U}}^+_\nu, \tilde{\mathbf{U}}^- = \bigoplus_{\nu \in \mathbf{N}[I]} \tilde{\mathbf{U}}^-_\nu$$

with $\tilde{\mathbf{U}}^+_0 = \tilde{\mathbf{U}}^-_0 = R1$.

Let T_λ denote the automorphism of $\tilde{\mathbf{U}}^-$ defined by $T_\lambda(x^-_i) = v^{(i,\lambda)} x^-_i$ for any $\lambda \in \mathbf{Z}[I]$ and define the T_j-leftskew derivation δ_j of $\tilde{\mathbf{U}}^-$ for any $j \in I$ through

$$\delta_j(x^-_i) = \delta_{ij}$$
$$\delta_j(a^-_\nu b^-_\mu) = \delta_j(a^-_\nu) b^-_\mu + T_j(a^-_\nu)\delta_j(b^-_\mu)$$

for any $i \in I$ and any homogeneous elements $a^-_\nu \in \tilde{\mathbf{U}}^-_\nu$ and $b^-_\mu \in \tilde{\mathbf{U}}^-_\mu$, where δ_{ij} is the Kronecker sign. Particularly $\delta_j(R1) = 0$. Note that δ_j is not an algebra endomorphism of $\tilde{\mathbf{U}}^-$ and $T_\lambda(a^-_\nu) = v^{(\nu,\lambda)} a^-_\nu$ for any $a^-_\nu \in \tilde{\mathbf{U}}^-_\nu$. One can verify the relation

$$\delta_j x^-_i - v^{(i,j)} x^-_i \delta_j = \delta_{ij}$$

as operations acting on $\tilde{\mathbf{U}}^-$.

2.10.2 Since $\tilde{\mathbf{U}}^+$ is a free R-algebra with generators $x^+_i, i \in I$, there is a unique homomorphism $\varphi^+ : \tilde{\mathbf{U}}^+ \to \mathrm{Hom}_R(\tilde{\mathbf{U}}^-, \tilde{\mathbf{U}}^-)$ by taking $\varphi^+(x^+_j) = \delta_j$ for any $j \in I$. Since the decomposition $\tilde{\mathbf{U}}^- = \bigoplus_{\nu > 0} \tilde{\mathbf{U}}^-_\nu \oplus R1$, we have the canonical projection $\varepsilon^- : \tilde{\mathbf{U}}^- \to R$. ε^- is an algebra homomorphism. So we can define a pairing $\varphi : \tilde{\mathbf{U}}^+ \times \tilde{\mathbf{U}}^- \to R$ by $\varphi(x, y) = \varepsilon^-(\varphi^+(x)y)$.

2.10.3 We need to add the torus algebra $\mathbf{T}$ to $\tilde{\mathbf{U}}^+$ and $\tilde{\mathbf{U}}^-$ respectively. $\tilde{\mathbf{V}}^+ = \mathbf{T}\tilde{\mathbf{U}}^+$ subject to the relations $K_\alpha a^+_\beta = v^{(\alpha,\beta)} a^+_\beta K_\alpha$ where $\alpha \in \mathbf{Z}[I]$

and $a^+_\beta \in \tilde{\mathbf{U}}^+_\beta$ for any $\beta \in \mathbf{N}[I]$. $\tilde{\mathbf{V}}^- = \mathbf{T}\tilde{\mathbf{U}}^-$ subject to the relations $K_\alpha b^-_\beta = v^{-(\alpha,\beta)} b^-_\beta K_\alpha$ where $\alpha \in \mathbf{Z}[I]$ and $b^-_\beta \in \tilde{\mathbf{U}}^-_\beta$ for any $\beta \in \mathbf{N}[I]$. The action φ^+ of $\tilde{\mathbf{U}}^+$ on $\tilde{\mathbf{U}}^-$ extends to an action of $\tilde{\mathbf{V}}^+$ on $\tilde{\mathbf{V}}^-$ by setting $T_\beta(K_\alpha) = v^{(\alpha,\beta)} K_\alpha$, $\varphi^+(K_\beta) = T_\beta$ and $\delta_j(Y) = 0$ for all $j \in I$ and $\alpha, \beta \in \mathbf{Z}[I]$. For example we check that

$$T_\beta \delta_j(K_\alpha x^-_j) = T_\beta(T_j(K_\alpha)) = v^{(\alpha,j)+(\alpha,\beta)}$$

$$\delta_j T_\beta(K_\alpha x^-_j) = \delta_j(v^{(\alpha,\beta)+(j,\beta)} K_\alpha x^-_j) = v^{(\alpha,\beta)+(j,\beta)+(\alpha,j)}$$

So $T_\beta \delta_j = v^{(j,\beta)} \delta_j T_\beta$. Then $\varphi^+ : \tilde{\mathbf{V}}^+ \to \mathrm{Hom}_R(\tilde{\mathbf{V}}^-, \tilde{\mathbf{V}}^-)$ is a homomorphism. Now $\tilde{\mathbf{V}}^+, \tilde{\mathbf{V}}^-$ has the graded decomposition too,

$$\tilde{\mathbf{V}}^+ = \mathbf{T}\tilde{\mathbf{U}}^+ = \bigoplus_{\nu > 0} \mathbf{T}\tilde{\mathbf{U}}^+_\nu \oplus \mathbf{T}$$

$$\tilde{\mathbf{V}}^- = \mathbf{T}\tilde{\mathbf{U}}^- = \bigoplus_{\nu > 0} \mathbf{T}\tilde{\mathbf{U}}^-_\nu \oplus \mathbf{T}$$

Let $\varepsilon^-(K_\alpha) = 1$ for any $\alpha \in \mathbf{Z}[I]$, then $\varepsilon^- : \tilde{\mathbf{V}}^- \to R$ is a well-defined projection.

Let $\varphi : \tilde{\mathbf{V}}^+ \times \tilde{\mathbf{V}}^- \to R$ be given by $\varphi(x, y) = \varepsilon^-(\varphi^+(x)y)$, then φ is a well-defined pairing.

2.10.4 As same as before, there are induced Hopf algebra structure on $\tilde{\mathbf{V}}^+$ and $\tilde{\mathbf{V}}^-$. The Hopf algebra $\tilde{\mathbf{V}}^+$ is given by

(a) $\Delta(x^+_i) = x^+_i \otimes 1 + K_i \otimes x^+_i, \Delta(K_\alpha) = K_\alpha \otimes K_\alpha$
(b) $\varepsilon(x^+_i) = 0, \varepsilon(K_\alpha) = 1$
(c) $\sigma(x^+_i) = -K_{-i} x^+_i, \sigma(K_\alpha) = K_{-\alpha}$

for all $i \in I, \alpha \in \mathbf{Z}[I]$.

The Hopf algebra $\tilde{\mathbf{V}}^-$ is given by

(a′) $\Delta(x^-_i) = x^-_i \otimes K_{-i} + 1 \otimes x^-_i, \Delta(K_\alpha) = K_\alpha \otimes K_\alpha$
(b′) $\varepsilon(x^-_i) = 0, \varepsilon(K_\alpha) = 1$
(c′) $\sigma(x^-_i) = x^-_i K_\alpha, \sigma(K_\alpha) = K_{-\alpha}$

for all $i \in I, \alpha \in \mathbf{Z}[I]$.

The operations Δ^{opp} and σ^{-1} for $\tilde{\mathbf{V}}^-$ is

(d′) $\Delta^{opp}(x^-_i) = x^-_i \otimes 1 + K_i \otimes x^-_i, \Delta^{opp}(K_\alpha) = K_\alpha \otimes K_\alpha$
(e′) $\sigma^{-1}(x^-_i) = -K_i x^-_i, \sigma^{-1}(K_\alpha) = K_{-\alpha}$

2.10.5 Proposition *The pairing $\varphi : \tilde{\mathbf{V}}^+ \times \tilde{\mathbf{V}}^- \to R$ defined in (2.10.3) satisfies*

(a) $\varphi(1, y) = \varepsilon(y), \varphi(x, 1) = \varepsilon(x)$
(b) $\varphi(x, yy') = \varphi(\Delta(x), y \otimes y')$
(c) $\varphi(xx', y) = \varphi(x \otimes x', \Delta^{opp}(y))$
(d) $\varphi(\sigma(x), y) = \varphi(x, \sigma^{-1}(y))$

for all $x, x' \in \tilde{\mathbf{V}}^+$ and $y, y' \in \tilde{\mathbf{V}}^-$.

Proof As same as in (2.8), or see [7].

2.10.6 We come back $\tilde{\mathbf{U}}^+$ and $\tilde{\mathbf{U}}^-$ now. If $\mathcal{I}$ is a subspace of $\tilde{\mathbf{U}}^-$ stable by the derivations $\delta_i, i \in I$, then so are its graded components. It follows easily that there is a unique maximal proper ideal $\mathcal{I}^-$ of $\tilde{\mathbf{U}}^-$ contained in the graded complement of $R1$, i.e, contained in $\oplus_{\nu>0}\tilde{\mathbf{U}}^-_\nu$ and stable by $\delta_i, i \in I$. Moreover $\mathcal{I}^-$ is graded. Set $\mathbf{U}^- = \tilde{\mathbf{U}}^-/\mathcal{I}^-$. By the construction, the derivations $\delta_i, i \in I$ factor to acting on $\mathbf{U}^-$. Because $\tilde{\mathbf{U}}^+$ is a free algebra on generators $x_i^+, i \in I$, so we have the canonical homomorphism $\varphi^+ : \tilde{\mathbf{U}}^+ \to \mathrm{Hom}_R(\mathbf{U}^-, \mathbf{U}^-)$ by taking $\varphi^+(x_i^+) = \delta_i, i \in I$. Obviously $\mathcal{I}^+ = \ker\varphi^+$ is a graded ideal of $\tilde{\mathbf{U}}^+$. let $\mathbf{U}^+ = \tilde{\mathbf{U}}^+/\mathcal{I}^+$, so φ^+ factor to the algebra morphism $\mathbf{U}^+ \to \mathrm{Hom}_R(\mathbf{U}^-, \mathbf{U}^-)$, still denoted by φ^+. Since $\mathbf{U}^- = \bigoplus_{\nu>0}\mathbf{U}^-_\nu \oplus R1$, $\varepsilon^- : \mathbf{U}^- \to R$ is a well-defined projection too. Define the pairing $\varphi : \mathbf{U}^+ \times \mathbf{U}^- \to R$ by $\varphi(a, b) = \varepsilon^-(\varphi^+(a)(b))$. Obviously $\varphi(\mathbf{U}^+_\mu, \mathbf{U}^-_\nu) = 0$ unless $\mu = \nu$.

2.10.7 Lemma *The pairing $\varphi : \mathbf{U}^+ \times \mathbf{U}^- \to R$ is non-degenerate*

Proof Take $y_\mu \in \mathbf{U}^-_\mu$ non-zero, show that $\varphi(\mathbf{U}^+_\mu, y_\mu) \neq 0$ by induction on $|\mu|$. It is clear if $\mu = 0$. If $|\mu| > 0$, by the maximality of $\mathcal{I}^-$, there is $i \in I$ such that $\varphi^+(x_i^+)(y_\mu) \neq 0$. Then there exists $x_{\mu-i} \in \mathbf{U}^+_{\mu-i}$ such that $\varphi(x_{\mu-i}, \varphi^+(x_i^+)(y_\mu)) \neq 0$ by the inductive hypothesis. Thus $\varphi(x_{\mu-i}x_i^+, y_\mu) \neq 0$ as required.

Take $x_\mu \in \mathbf{U}^+_\mu$ non-zero. Because $\varphi^+(x_\mu)$ is a derivation of degree μ, then $\varphi^+(x_\mu)\mathbf{U}^-_\mu \subset R1$. Since $\varphi(x_\mu, \mathbf{U}^-_\nu) = 0$ if $\nu \neq \mu$, then $\varphi(x_\mu, \mathbf{U}^-_\mu) \neq 0$ by the definition of $\mathcal{I}^+$.

2.10.8 Let $\mathbf{V}^+ = \mathbf{T}\mathbf{U}^+$ by the relations $K_\alpha x_\beta = v^{(\alpha,\beta)}x_\beta K_\alpha$, $\mathbf{V}^- = \mathbf{T}\mathbf{U}^-$ by the relation $K_\alpha y_\beta = v^{-(\alpha,\beta)}y_\beta K_\alpha$ where $\alpha \in \mathbf{Z}[I], \beta \in \mathbf{N}[I], x_\beta \in \mathbf{U}^+_\beta, y_\beta \in \mathbf{U}^-_\beta$.

$\mathbf{V}^+$ and $\mathbf{V}^-$ are Hopf algebras under the operations induced from those of $\tilde{\mathbf{V}}^+$ and $\tilde{\mathbf{V}}^-$ respectively (2.10.4). The action φ^+ of $\mathbf{U}^+$ on $\mathbf{U}^-$ extends to an action of $\mathbf{V}^+$ on $\mathbf{V}^-$ by setting $\varphi^+(K_\beta) = T_\beta, \delta_i(\mathbf{T}) = 0, T_\beta(K_\alpha) = v^{(\alpha,\beta)}K_\alpha$ for all $i \in I, \alpha, \beta \in \mathbf{Z}[I]$. Let $\varepsilon^-(K_\alpha) = 1, \alpha \in \mathbf{Z}[I]$. Then $\varepsilon^- : \mathbf{V}^- \to R$ and the pairing $\varphi : \mathbf{V}^+ \times \mathbf{V}^- \to R$ with $\varphi(x, y) = \varepsilon^-(\varphi^+(x)y)$. We get

2.10.9 Proposition *The pairing $\varphi : \mathbf{V}^+ \times \mathbf{V}^- \to R$ is a non-degenerate skew-Hopf pairing in $\mathcal{L}(C)$.*

3. Uniqueness of non-degenerate pairing

3.1 Given a Cartan datum $C = (I, (,))$, consider the class of skew-Hopf pairing (A^+, A^-, φ) in $\mathcal{L}(C)$. If we take $(A^+, A^-, \varphi) \in \mathcal{L}(C)$, it is easy to see that φ is determined by (2.2) (a), (b) and (c$'$) as soon as the elements $\varphi(x_i^+, x_i^-), i \in I$ are given. However, according to Green [5], there is an algorithm to calculate φ which only depends on the Cartan datum. This implies that any two non-degenerate skew-Hopf pairing in $\mathcal{L}(C)$ are canonically isomorphic.

If $\nu = \sum_{i\in I} \nu_i i \in \mathbf{N}[I]$, let $tr\nu = \sum_{i\in I} \nu_i$ and write $p = tr\nu$. Define $I(\nu)$ to be the set of all vectors $a = (a_1, a_2, \cdots, a_p) \in I^p$ which have weight ν, i.e, which satisfy

$$\nu_i = |\{k \in \mathbf{N} \mid 1 \leq k \leq p, a_k = i\}|$$

for all $i \in I$. Define $I(0)$ to consist of a single element $\emptyset$. Let $I(\infty)$ denote the disjoint union $\bigcup_{\nu\in\mathbf{N}[I]} I(\nu)$. Then A_ν^+ (resp. A_ν^-) for any $\nu \in \mathbf{N}[I]$ is the $\mathbf{T}$-spain of the monomials $x_a^+ = x_{a_1}^+ \cdots x_{a_p}^+$ (resp. $x_a^- = x_{a_1}^- \cdots x_{a_p}^-$), $a \in I(\nu)$, in case $\nu = 0$ we define $x_\emptyset^\pm = 1$.

3.2 Lemma *Let $(A^+, A^-, \varphi) \in \mathcal{L}(C)$, then*

(1) $\varphi(A_\mu^+, A_\nu^-) = 0$ if $\mu \neq \nu$ in $\mathbf{N}[I]$.

(2) For any $\alpha, \beta \in \mathbf{Z}[I], \nu \in \mathbf{N}[I]$ and $a, b \in I(\nu)$, then

$$\varphi(K_\alpha x_a^+, K_\beta x_b^-) = v^{-(\alpha,\beta)-(\nu,\beta)+(\alpha,\nu)} \varphi(x_a^+, x_b^-).$$

Proof (1) It follows from (A2), (A3) of (2.7) and (b), (c$'$) of (2.2) by induction.

(2) Let $a = (a_1, \cdots, a_p), b = (b_1, \cdots, b_p)$, then $x_a^+ = x_{a_1}^+ \cdots x_{a_p}^+$, $x_b^- = x_{b_1}^- \cdots x_{b_p}^-$.

$$\varphi(K_\alpha x_a^+, K_\beta x_b^-) = \varphi(\Delta(K_\alpha x_a^+), K_\beta \otimes x_b^-)$$

$$= \varphi((K_\alpha \otimes K_\alpha)(x_{a_1}^+ \otimes 1 + K_{\alpha_1} \otimes x_{a_1}^+) \cdots (x_{a_p}^+ \otimes 1 + K_{\alpha_p} \otimes x_{a_p}^+), K_\beta \otimes x_b^-)$$

(by (1))

$$= \varphi(K_\alpha K_{\alpha_1} \cdots K_{\alpha_p} \otimes K_\alpha x_{a_1}^+ \cdots x_{a_p}^+, K_\beta \otimes x_b^-)$$

$$= \varphi(K_\alpha K_{nu}, K_\beta) \varphi(K_\alpha x_a^+, x_b^-)$$

$$= v^{-(\alpha,\beta)-(\nu,\beta)} \varphi(K_\alpha \otimes x_a^+, \Delta^{opp}(x_b^-))$$

(by (A2) of (2.7) and (1))

$$= v^{-(\alpha,\beta)-(\nu,\beta)+(\alpha,\nu)} \varphi(x_a^+, x_b^-).$$

3.3 Proposition (Green) *Given a Cartan datum $C = (I, (,))$, for any $\nu \in \mathbf{N}[I]$ and $a, b \in I(\nu)$, then there exists an element $M_{a,b}(t) \in \mathbf{Z}[t, t^{-1}]$ (t indeterminter) such that for any $(A^+, A^-, \varphi) \in \mathcal{L}(C)$, there holds*

$$\varphi(x_a^+, x_b^-) = M_{a,b}(v) \prod_{i\in I} \varphi(x_i^+, x_i^-)^{\nu_i},$$

moreover, $M_{a,b}(t) = M_{b,a}(t)$, therefore $\varphi(x_a^+, x_b^-) = \varphi(x_b^+, x_a^-)$.

Proof This is by induction on $p = tr\nu = \sum_{i\in I}\nu_i$. If $p=0$, then $a=b=\emptyset$ and $\varphi(x_a^+, x_b^-) = \varphi(1,1) = 1$ by (2.2) (a). So the statement is satisfied by taking $M_{\emptyset,\emptyset}(t) = 1$. Assume now that $p \geq 1$ and that the **Notation**: If $a \in I^p$ and S is a subset of $\{1,\cdots,p\}$, we denote by $a(S)$ the vector of length $p-|S|$ obtained by removing from $a=(a_1,\cdots,a_p)$ all the terms $a_i, i\in S$. In case $S=\{\pi\}$ for some $\pi\in\{1,\cdots,p\}$, write $a(\pi)$ for $a(S)$.

Now let $a,b\in I(\nu)$, we have $x_b^- = x_{b(p)}^- x_{b_p}^-$. By (2.2) (b)

$$\varphi(x_a^+, x_b^-) = \varphi(\Delta(x_a^+), x_{b(p)}^- \otimes x_{b_p}^-)$$

$$= \varphi(\prod_{\pi\in\{1,\cdots,p\}}(x_{a_\pi}^+\otimes 1 + K_{a_\pi}\otimes x_{a_\pi}^+), x_{b(p)}^-\otimes x_{b_p}^-)$$

Expand the product $\prod_{\pi\in\{1,\cdots,p\}}(x_{a_\pi}^+\otimes 1 + K_{a_\pi}\otimes x_{a_\pi}^+)$ as $\bigoplus_{P\subset\{1,\cdots,p\}} z(P)$ where for each subset $P\subset\{1,\cdots,p\}$, $z(P) = z_1\cdots z_p$ such that $z_\pi = x_{a_\pi}^+\otimes 1$ if $\pi\in P$ and $z_{\pi'} = K_{a_{\pi'}}\otimes x_{a_{\pi'}}^+$ if $\pi'\in P' = \{1,\cdots,p\}\setminus P$. Multiplying out this product according to the formula in (2.7) (A^+1), we get

$$z(P) = v^{(a:P)} K_{a(P)} x_{a(P')}^+ \otimes x_{a(P)}^+$$

where

$$(a:P) = -\sum_{\substack{\pi'\in P',\pi\in P\\ \pi<\pi'}} (a_{\pi'}, a_\pi).$$

But by Lemma 3.2, $\varphi(z(P), x_{b(p)}^-\otimes x_{b_p}^-) = 0$ unless $P' = \{\pi\}$ for some $\pi\in\{1,\cdots,p\}$, and in this case we have

$$\varphi(z(P), x_{b(p)}^-\otimes x_{b_p}^-) = v^{(a:P)}\varphi(K_{a_\pi}x_{a(P')}^+, x_{b(p)}^-)\varphi(x_{a_\pi}^+, x_{b_p}^-)$$

(by Lemma 3.2)

$$= v^{(a:P)} v^{(a_\pi, b(p))}\varphi(x_{a(P')}^+, x_{b(p)}^-)\varphi(x_{a_\pi}^+, x_{b_p}^-)$$

where $(a:P) = -(a_\pi, a_1+\cdots+a_{\pi-1})$, $(a_\pi, b(p)) = (a_\pi, b_1+\cdots b_{p-1})$. By the inductive assumption we have

$$\varphi(x_a^+, x_b^-) = \sum_{\substack{\pi\in\{1,\cdots,p\}\\ a_\pi=b_p}} v^{(a_\pi,b_1+\cdots b_{p-1})-(a_\pi,a_1+\cdots+a_{\pi-1})} M_{a(\pi),b(p)}(v)\prod_{i\in I}\varphi(x_i^+, x_i^-)^{\nu_i}.$$

Since $\nu = b_1+\cdots+b_{p-1}+b_p = a_1+\cdots+a_{\pi-1}+a_\pi+a_{\pi+1}+\cdots+a_p$, then if $a_\pi = b_p$, $v^{(a_\pi,b_1+\cdots b_{p-1})-(a_\pi,a_1+\cdots+a_{\pi-1})} = v^{(a_\pi,a_{\pi+1}+\cdots+a_p)}$. Hence

$$\varphi(x_a^+, x_b^-) = \sum_{\substack{\pi\in\{1,\cdots,p\}\\ a_\pi=b_p}} v^{(a_\pi,a_{\pi+1}+\cdots+a_p)} M_{a(\pi),b(p)}(v)\prod_{i\in I}\varphi(x_i^+, x_i^-)^{\nu_i}.$$

We get

$$M_{a,b}(t) = \sum_{\substack{\pi\in\{1,\cdots,p\}\\ a_\pi=b_p}} v^{(a_\pi,a_{\pi+1}+\cdots+a_p)} M_{a(\pi),b(p)}(t)$$

Let S_p be the permutation group of $\{1,\cdots,p\}$ and

$$G_{ab} = \{s\in S_p \mid a_{s(\pi)} = b_\pi \ \text{ for all } \pi\in\{1,\cdots,p\}\}.$$

By induction one can prove

$$M_{a,b}(t) = \sum_{s\in G_{ab}} t^{(s:a)}$$

where

$$(s:a) = \sum_{\substack{\pi'<\pi \\ s^{-1}(\pi')>s^{-1}(\pi)}} (a_{\pi'}, a_{\pi}).$$

Since $G_{ab} = G_{ba}$ and $(s : a) = (s^{-1} : b)$, we have $M_{a,b}(t) = M_{b,a}(t)$. The proof is finished.

3.4 Given a Cartan datum $C = (I, (,))$, let $(A^+, A^-, \varphi) \in \mathcal{L}(C)$. As we did as in Section 2, it follows that $\mathcal{I}^+ = \{x \in A^+ \mid \varphi(x, A^-) = 0\}$ and $\mathcal{I}^- = \{y \in A^- \mid \varphi(A^+, y) = 0\}$ are two sided ideals of A^+ and A^- respectively. It is easy to see that

(1)

$$\mathcal{I}^+ = \sum_{\nu\in\mathbf{N}[I]} (A^+_\nu \bigcap \mathcal{I}^+),\ A^+_0 \bigcap \mathcal{I}^+ = 0$$

$$\mathcal{I}^- = \sum_{\nu\in\mathbf{N}[I]} (A^-_\nu \bigcap \mathcal{I}^-),\ A^-_0 \bigcap \mathcal{I}^- = 0$$

(2)

$$\Delta(\mathcal{I}^+) \subset A^+ \otimes \mathcal{I}^+ + \mathcal{I}^+ \otimes A^+, \Delta(\mathcal{I}^-) \subset A^- \otimes \mathcal{I}^- + \mathcal{I}^- \otimes A^-.$$

Set $\overline{A}^+ = A^+/\mathcal{I}^+$, which is generated by $\overline{x}_i^+ = x_i^+ + \mathcal{I}^+, i \in I$ and $\overline{A}_0^+ = A_0^+ = \mathbf{T}$. $\overline{A}^- = A^-/\mathcal{I}^-$, which is generated by $\overline{x}_i^- = x_i^- + \mathcal{I}^-, i \in I$ and $\overline{A}_0^- = A_0^- = \mathbf{T}$. Then $(\overline{A}^+, \overline{A}^-, \overline{\varphi})$ is again a skew-Hopf pairing in $\mathcal{L}(C)$, where the Hopf algebra structure of $\overline{A}^+$ and $\overline{A}^-$ and $\overline{\varphi}$ inherited in obvious ways from A^+, A^- and φ respectively. Of course $\overline{\varphi}$ is non-degenerate.

3.5 Let (A^+, A^-, φ) and (A'^+, A'^-, φ') be two skew-Hopf pairing in $\mathcal{L}(C)$, we say a canonical isomorphism $f : (A^+, A^-, \varphi) \to (A'^+, A'^-, \varphi')$, it means that we have the isomorphisms $f : A^+ \to A'^+$ and $f : A^- \to A'^-$ as Hopf algebras, moreover $f(x_i^+) = x'^+_i, f(x_i^-) = x'^-_i$ for all $i \in I$ and f preserves $\mathbf{T} = A_0 = A'_0$ fixed.

According to Proposition 3.3, the relation between φ and φ' is also clear, precisely, for any $\nu \in \mathbf{N}[I]$ and any $a, b \in I(\nu)$, $\varphi'(x'^+_a, x'^-_b) = c(\nu)\varphi(x_a^+, x_b^-)$ where $c(\nu)$ is non-zero in R which only depends on ν.

3.6 Theorem (Green) *Given a Cartan datum $C = (I, (,))$, any two non-degenerate skew-Hopf pairings in $\mathcal{L}(C)$ are canonical isomorphic.*

Proof Let (A^+, A^-, φ) and (A'^+, A'^-, φ') be two non-degenerate members of $\mathcal{L}(C)$. Then any element of A^+ can be writen as $x = \sum_{a\in I(\infty)} c_a x_a^+$ where $c_a \in \mathbf{T}$ and $x_a^+ = x_{a_1}^+ \cdots x_{a_p}^+$ if $a = (a_1, \cdots, a_p) \in I^p$, in partular $x_\emptyset^+ = 1$.

We define the morphism $f : A^+ \to A'^+$ by sending $x = \sum_{a\in I(\infty)} c_a x_a^+$ to $x' = \sum_{a\in I(\infty)} c_a {x'}_a^+$. We show that f is a well-defined morphism. In fact $x = 0$ if and only if $\varphi(x, A^-) = 0$ since φ is non-degenerate. By Lemma 3.2 (2) it is equivalent to $\varphi(x, x_b^-) = 0$ for any $b \in I(\infty)$, that is $\sum_{a\in I(\infty)} c_a \varphi(x_a^+, x_b^-) = 0$ for any $b \in I(\infty)$. Since $\varphi(x_a^+, x_b^-) = 0$ if a, b are different weights by Lemma 3.2 (1), it is equivalent to

$$\sum_{a\in I(\nu)} c_a \varphi(x_a^+, x_b^-) = 0, \quad \text{for all } b \in I(\nu) \text{ and all } \nu \in \mathbf{N}[I].$$

By Proposition 3.3, this is equivalent to

$$\sum_{a\in I(\nu)} c_a M_{a,b}(v) = 0, \quad \text{for all } b \in I(\nu) \text{ and all } \nu \in \mathbf{N}[I]$$

since $\prod_{i\in I} \varphi(x_i^+, x_i^-)^{\nu_i} \neq 0$ by (A3) of (2.7). But this condition is independent of A^+ according to Proposition 3.3, we obtain that $x = \sum_{a\in I(\infty)} c_a x_a^+ = 0$ if and only if $x' = \sum_{a\in I(\infty)} c_a {x'}_a^+ = 0$. So $f : A^+ \to A'^+$ is well-defined. it is easily seen that f is a canonical algebra isomorphism and f is admissible with the comultiplication and counit. Although there is no hypothesis on antipodes in (2.7), by $\mu(\sigma\otimes 1)\Delta = \delta\varepsilon$, $\sigma(K_\alpha)K_\alpha = 1$ and $\sigma(x_i^+) + \sigma(K_i)x_i^+ = 0$, hence $\sigma(K_\alpha) = K_{-\alpha}, \sigma(x_i^+) = -K_{-i}x_i^+$ for all $\alpha \in \mathbf{Z}[I]$, $i \in I$. Thus f is admissible with antipodes. Therefore f is an isomorphism of Hopf algebras. Similar for $f : A^- \to A'^-$.

3.6.1 Remark The construction of non-degenerate skew-Hopf pairings of Lusztig and Joseph as did as in Section 2 are canonical isomorphic to each other, by Theorem 3.6.

4. Hall and composition algebras

4.1 ¿From now on, we come to consider the Hall algebras over hereditary algebras. We fix some notations in this section. Let k be a finite field, $v = \sqrt{q}$, $q = |k|$; Λ be an associative hereditary k-algebra, assume also that Λ is finitary in the sense that $|\operatorname{Ext}^1_R(S, S')| < \infty$ for all simple Λ-modules S, S' which are finite as sets. This condition is met if Λ is finitely generated as k-algebra [12].

Let $\mathcal{P}$ be the set of isomorphism classes of Λ-modules of finite cardinality. We denote by 0 both the zero module and its isomorphism class. Let $\mathcal{P}_1 = \mathcal{P} \setminus \{0\}$. For every $\alpha \in \mathcal{P}$ let u_α be a representative in α. Given $\alpha \in \mathcal{P}$, let a_α be the order of the automorphism group of u_α. Given $\alpha, \beta, \lambda \in \mathcal{P}$, let $g^\lambda_{\alpha\beta}$ be the number of submodules B of u_λ such that $B \simeq u_\beta$ and $u_\lambda / B \simeq u_\alpha$.

J.A.Green gave us the following remarkble formula.

4.2 Theorem (Green) *Assume that Λ is hereditary and finitary. Given any $\alpha,\beta,\alpha',\beta'\in\mathcal{P}$, then*

$$a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_{\lambda\in\mathcal{P}} g^\lambda_{\alpha\beta} g^\lambda_{\alpha'\beta'} a_\lambda^{-1} = \sum_{\rho,\sigma,\sigma',\tau\in\mathcal{P}} \frac{|\operatorname{Ext}^1(u_\rho,u_\tau)|}{|\operatorname{Hom}(u_\rho,u_\tau)|} g^\alpha_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^\beta_{\sigma'\tau} g^{\beta'}_{\sigma\tau} a_\rho a_\sigma a_{\sigma'} a_\tau.$$

The Green formula reflects the symmetricity of the module category of a hereditary algebra deeply. For its proofs we referee to [5] and [16].

More general, given $\alpha_1,\cdots,\alpha_t,\lambda\in\mathcal{P}$, let $\mathcal{F}^{u_\lambda}_{u_{\alpha_1},\cdots,u_{\alpha_t}}$ be the set of filtrations

$$u_\lambda = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0$$

such that M_{i-1}/M_i is isomorphic to u_{α_i} for all $1\le i\le t$. The cardinality of $\mathcal{F}^{u_\lambda}_{u_{\alpha_1},\cdots,u_{\alpha_t}}$ will be denoted by $g^\lambda_{\alpha_1\cdots\alpha_t}$. A routine generalization of the Green formula is the following.

4.3 Corollary *Let Λ be hereditary and finitary. Given $\{\alpha_i\}_{i=1}^n,\ \{\beta_j\}_{j=1}^m\subset\mathcal{P}$, then*

$$\prod_{i=1}^n a_{\alpha_i}\prod_{j=1}^m a_{\beta_j}\sum_{\lambda\in\mathcal{P}} g^\lambda_{\alpha_1\cdots\alpha_n} g^\lambda_{\beta_1\cdots\beta_m} a_\lambda^{-1}$$

$$= \sum_{(\rho_{ij})_{\substack{1\le i\le n\\ 1\le j\le m}}} \prod_{\substack{1\le i\le n\\ 1\le j\le m}} a_{\rho_{ij}} \prod_{\substack{i<k\\ j<l}} \frac{|\operatorname{Ext}(u_{\rho_{ij}},u_{\rho_{kl}})|}{|\operatorname{Hom}(u_{\rho_{ij}},u_{\rho_{kl}})|} \prod_{i=1}^n g^{\alpha_i}_{\rho_{i1}\rho_{i2}\cdots\rho_{im}} \prod_{j=1}^m g^{\beta_j}_{\rho_{1j}\rho_{2j}\cdots\rho_{nj}}$$

where the sum is taken over $\mathcal{P}$.

4.4 Given Λ-modules $M,\ N$, let

$$\langle M,N\rangle_R = \dim_k \operatorname{Hom}(M,N) - \dim_k \operatorname{Ext}^1(M,N).$$

Since Λ is hereditary, $\langle M,N\rangle_R$ depends only on the dimension vectors $\underline{\dim}M$ and $\underline{\dim}N$. For $\alpha,\ \beta\in\mathcal{P}$, we write

$$\langle\alpha,\beta\rangle_R = \langle u_\alpha,u_\beta\rangle_R$$

So the Ringel form $\langle -,-\rangle_R$ is defined on $\mathbf{Z}[I]$, where I is the set of isomorphism classes of simple Λ-modules. The Ringel symmetric form $(-,-)_R$ is given by

$$(\alpha,\beta)_R = \langle\alpha,\beta\rangle_R + \langle\beta,\alpha\rangle_R$$

on $\mathbf{Z}[I]$. The following two lemmas are due to Ringel (see [16]).

Lemma 1 *Let Λ be a finite dimensional hereditary k-algebra, then the Ringel symmetric form $(I,(-,-)_R)$ is a Cartan datum.*

We denote by $\overline{k}$ an algebraic closure of k. For any $n\in\mathbf{N}$, let $F(n)$ be the subfield of $\overline{k}$ with $[F(n):k]=n$.

Lemma 2 *Let $(I, (,))$ be any Cartan datum, k any finite field. Fix some total ordering $\preceq$ on the set I. Then there exists a k-species $\mathcal{S} = (F_i, {}_i M_j)_{i,j\in I}$ defined as follows.*

$$F_i = F(\frac{(i,i)}{2}) \quad \textit{for any} \quad i \in I$$

$${}_iM_j = F(-(i,j)) \quad \textit{if } i \prec j \textit{ or } 0 \textit{ otherwise.}$$

such that the tensor algebra $T(\mathcal{S})$ is a finite dimensional hereditary k-algebra, the simple $T(\mathcal{S})$-modules are indexes by I and the Ringel symmetric form $(-,-)_R$ of $T(\mathcal{S})$ is equal to $(-,-)$ of $(I,(,))$.

4.4.1 Remark This means that any Cartan datum can be realized by the Ringel symmetric form of some finite dimensional hereditary k-algebra, moreover, if the Cartan datum $(I, (,))$ is simply-laced, one may take $T(\mathcal{S})$ to be the path algebra kQ, where the quiver Q is made by orienting the Dynkin graph of $(I, (,))$ in any way which gives no oriented cycles.

R is a commutative integral domain containing the integers $\mathbf{Z}$ and containing also elements $v,\ v^{-1}$. $\Lambda,\ \mathcal{P},\ \mathcal{P}_1,\ I$ is given as above. Let $\mathbf{H}(\Lambda)$ be the free R-module with the basis

$$\{K_\alpha u_\lambda \mid \alpha \in \mathbf{Z}[I],\ \lambda \in \mathcal{P}\}.$$

With these notations we have

4.5 Theorem $\mathbf{H}(\Lambda)$ *is endowed with a Hopf algebra structure under the following operations.*
(a) (Ringel) Multiplication:

$$u_\alpha u_\beta = \sum_{\lambda\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta} u_\lambda \quad \textit{for all } \alpha, \beta \in \mathcal{P}$$

$$K_\alpha u_\beta = v^{(\alpha,\beta)} u_\beta K_\alpha \quad \textit{for all } \alpha \in \mathbf{Z}[I], \beta \in \mathcal{P}$$

$$K_\alpha K_\beta = K_{\alpha+\beta} \quad \textit{for all } \alpha, \beta \in \mathbf{Z}[I]$$

with unit $1 = u_0 = K_0$.
(b) (Green) Comultiplication:

$$\Delta(u_\lambda) = \sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} \frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha\beta} u_\alpha K_\beta \otimes u_\beta \quad \textit{for all } \lambda \in \mathcal{P}$$

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha \quad \textit{for all } \alpha \in \mathbf{Z}[I]$$

with counit: $\varepsilon u_\lambda = 0$ for $\lambda \neq 0$ in $\mathcal{P}$ and $\varepsilon K_\alpha = 1$ for all $\alpha \in \mathbf{Z}[I]$.
(c) Antipode:

$$\sigma(u_\lambda) = \delta_{\lambda 0} + \sum_{m\geq 1} (-1)^m \sum_{\substack{\pi\in\mathcal{P} \\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m} K_{-\lambda} u_\pi$$

for all $\lambda \in \mathcal{P}$

$$\sigma(K_\alpha) = K_{-\alpha} \quad \textit{for all } \alpha \in \mathbf{Z}[I]$$

where $\langle -,- \rangle = \langle -,- \rangle_R$, $(-,-) = (-,-)_R$ and $\delta_{\lambda 0}$ is the Kronecker sign.

Proof (a) The associativity of the multiplication follows from the fact that

$$(u_\alpha u_\beta)u_\gamma = \sum_{\lambda\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+\langle\alpha,\gamma\rangle+\langle\beta,\gamma\rangle} g^{\lambda}_{\alpha\beta\gamma} u_\lambda = u_\alpha(u_\beta u_\gamma).$$

Obviously the unit element is just $u_0 = K_0 = 1$.

(b) By [9] or [16], it is known that $\frac{a_\alpha a_\beta}{a_\lambda} g^{\lambda}_{\alpha\beta}$ belongs to $\mathbf{Z}[v^{-1}]$, so $\Delta : \mathbf{H}(\Lambda) \to \mathbf{H}(\Lambda)\otimes\mathbf{H}(\Lambda)$ is R-linear. Obviously $\Delta(K_\alpha K_{\alpha'}) = \Delta(K_\alpha)\Delta(K_{\alpha'})$ for all $\alpha,\ \alpha' \in \mathbf{Z}[I]$. It will be enough to prove that

$$\Delta(u_{\alpha'}u_{\beta'}) = \Delta(u_{\alpha'})\Delta(u_{\beta'}) \quad \text{for all } \alpha',\ \beta' \in \mathcal{P}.$$

Since

$$u_{\alpha'}u_{\beta'} = \sum_{\lambda\in\mathcal{P}} v^{\langle\alpha',\beta'\rangle} g^{\lambda}_{\alpha'\beta'} u_\lambda,$$

then

$$(*)\quad \Delta(u_{\alpha'}u_{\beta'}) = \sum_{\lambda\in\mathcal{P}} v^{\langle\alpha',\beta'\rangle} g^{\lambda}_{\alpha'\beta'}\Delta(u_\lambda) = \sum_{\lambda,\alpha,\beta\in\mathcal{P}} g^{\lambda}_{\alpha'\beta'} g^{\lambda}_{\alpha\beta}\frac{a_\alpha a_\beta}{a_\lambda} u_\alpha K_\beta \otimes u_\beta$$

and

$$\Delta(u_{\alpha'})\Delta(u_{\beta'}) = \Big(\sum_{\rho,\sigma\in\mathcal{P}} v^{\langle\rho,\sigma\rangle}\frac{a_\rho a_\sigma}{a_{\alpha'}} g^{\alpha'}_{\rho\sigma} u_\rho K_\sigma \otimes u_\sigma\Big)\Big(\sum_{\rho',\sigma'\in\mathcal{P}} v^{\langle\rho',\sigma'\rangle}\frac{a_{\rho'} a_{\sigma'}}{a_{\beta'}} g^{\beta'}_{\rho'\sigma'} u_{\rho'} K_{\sigma'} \otimes u_{\sigma'}\Big)$$

$$= \sum_{\rho,\sigma,\rho',\sigma'\in\mathcal{P}} v^{\langle\rho,\sigma\rangle+\langle\rho',\sigma'\rangle}\frac{a_\rho a_\sigma a_{\rho'} a_{\sigma'}}{a_{\alpha'} a_{\beta'}} g^{\alpha'}_{\rho\sigma} g^{\beta'}_{\rho'\sigma'} u_\rho K_\sigma u_{\rho'} K_{\sigma'} \otimes u_\sigma u_{\sigma'}$$

By (a) we have

$$(*')\quad \Delta(u_{\alpha'})\Delta(u_{\beta'}) = \sum_{\substack{\rho,\sigma,\rho',\sigma'\in\mathcal{P}\\ \alpha,\beta\in\mathcal{P}}} v^{D}\frac{a_\rho a_\sigma a_{\rho'} a_{\sigma'}}{a_{\alpha'} a_{\beta'}} g^{\alpha'}_{\rho\sigma} g^{\beta'}_{\rho'\sigma'} g^{\alpha}_{\rho\rho'} g^{\beta}_{\sigma\sigma'} u_\alpha K_\beta \otimes u_\beta$$

where $D = \langle\rho,\sigma\rangle + \langle\rho',\sigma'\rangle + \langle\rho,\rho'\rangle + \langle\sigma,\sigma'\rangle + (\sigma,\rho')$.

¿From the definition of $g^{\lambda}_{\alpha\beta}$ it follows that $g^{\lambda}_{\alpha\beta} \neq 0$ implies that $\underline{\dim} u_\lambda = \underline{\dim} u_\alpha + \underline{\dim} u_\beta$ for any $\alpha,\ \beta,\ \lambda \in \mathcal{P}$. So we may assume here that

$$\underline{\dim} u_\alpha + \underline{\dim} u_\beta = \underline{\dim} u_\lambda = \underline{\dim} u_{\alpha'} + \underline{\dim} u_{\beta'}$$

and that

$$\underline{\dim} u_\rho + \underline{\dim} u_\sigma = \underline{\dim} u_{\alpha'},\ \underline{\dim} u_{\rho'} + \underline{\dim} u_{\sigma'} = \underline{\dim} u_{\beta'}$$

$$\underline{\dim} u_\rho + \underline{\dim} u_{\rho'} = \underline{\dim} u_{\alpha},\ \underline{\dim} u_{\sigma} + \underline{\dim} u_{\sigma'} = \underline{\dim} u_{\beta}$$

So $K_\beta = K_\sigma K_{\sigma'}$. Therefore we have

$$\langle\alpha',\beta'\rangle = \langle\rho,\rho'\rangle + \langle\rho,\sigma'\rangle + \langle\sigma,\rho'\rangle + \langle\sigma,\sigma'\rangle$$

$$\langle\alpha,\beta\rangle = \langle\rho,\sigma\rangle + \langle\rho,\sigma'\rangle + \langle\rho',\sigma\rangle + \langle\rho',\sigma'\rangle$$

and hence

$$D = \langle\alpha,\beta\rangle + \langle\alpha',\beta'\rangle - 2\langle\rho,\sigma'\rangle.$$

Comparing with the coefficients of $u_\alpha K_\beta \otimes u_\beta$ in $(*)$ and $(*')$, we see that the validity of $\Delta(u_{\alpha'}u_{\beta'}) = \Delta(u_{\alpha'})\Delta(u_{\beta'})$ reduces to that of the following equation.

$$\sum_{\lambda\in\mathcal{P}} g^\lambda_{\alpha\beta} g^\lambda_{\alpha'\beta'} \frac{a_\alpha a_\beta a_{\alpha'} a_{\beta'}}{a_\lambda} = \sum_{\rho,\sigma,\rho',\sigma'\in\mathcal{P}} v^{-2\langle\rho,\sigma'\rangle} g^\alpha_{\rho\rho'} g^\beta_{\sigma\sigma'} g^{\alpha'}_{\rho\sigma} g^{\beta'}_{\rho'\sigma'} a_\rho a_\sigma a_{\rho'} a_{\sigma'}$$

Since $v^{-2\langle\rho,\sigma'\rangle} = q^{-\langle\rho,\sigma'\rangle} = q^{(\dim_k \mathrm{Ext}^1(u_\rho,u_{\sigma'}) - \dim_k \mathrm{Hom}(u_\rho,u_{\sigma'}))} = \frac{|\mathrm{Ext}^1(u_\rho,u_{\sigma'})|}{|\mathrm{Hom}(u_\rho,u_{\sigma'})|}$, this equation is exactly the Green formula (4.2).

It is easy to see ε given in (b) is the counit.

(c) It is also known that $\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m}$ belongs to $\mathbf{Z}[v^{-1}]$. According to the definition of Hopf algebra, we only need to verify that

$$\mu(\sigma\otimes 1)\Delta(u_\lambda) = \delta_{\lambda 0} \quad\text{and}\quad \mu(1\otimes\sigma)\Delta(u_\lambda) = \delta_{\lambda 0}$$

for all $\lambda\in\mathcal{P}$. Since

$$\Delta(u_\lambda) = \sum_{\lambda',\lambda_{m+1}\in\mathcal{P}} v^{\langle\lambda',\lambda_{m+1}\rangle} \frac{a_{\lambda'} a_{\lambda_{m+1}}}{a_\lambda} g^\lambda_{\lambda'\lambda_{m+1}} u_{\lambda'} K_{\lambda_{m+1}} \otimes u_{\lambda_{m+1}},$$

then

$$\mu(\sigma\otimes 1)\Delta(u_\lambda) = \sum_{\lambda',\lambda_{m+1}\in\mathcal{P}} v^{\langle\lambda',\lambda_{m+1}\rangle} \frac{a_{\lambda'} a_{\lambda_{m+1}}}{a_\lambda} g^\lambda_{\lambda'\lambda_{m+1}} K_{-\lambda_{m+1}} \sigma(u_{\lambda'}) u_{\lambda_{m+1}}$$

where we use $\sigma(u_{\lambda'}K_{\lambda_{m+1}}) = \sigma(K_{\lambda_{m+1}})\sigma(u_{\lambda'})$, this is easy to check according to the rule $\sigma(K_{\lambda_{m+1}}u_{\lambda'}) = \sigma(u_{\lambda'})\sigma(K_{\lambda_{m+1}})$. So we deduce to prove

$$(**)\quad \sigma(u_\lambda) = \delta_{\lambda 0} - \sum_{\lambda'\in\mathcal{P},\lambda_{m+1}\in\mathcal{P}_1} v^{\langle\lambda',\lambda_{m+1}\rangle} \frac{a_{\lambda'} a_{\lambda_{m+1}}}{a_\lambda} g^\lambda_{\lambda'\lambda_{m+1}} K_{-\lambda_{m+1}} \sigma(u_{\lambda'}) u_{\lambda_{m+1}}.$$

Because

$$\sigma(u_{\lambda'}) = \delta_{\lambda' 0} + \sum_{m\geq 1} (-1)^m \sum_{\substack{\pi'\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_{\lambda'}} g^{\lambda'}_{\lambda_1\cdots\lambda_m} g^{\pi'}_{\lambda_1\cdots\lambda_m} K_{-\lambda'} u_{\pi'},$$

the right side of $(**)$ is

$$= \delta_{\lambda 0} - K_{-\lambda}u_\lambda - \sum_{\lambda',\lambda_{m+1}\in\mathcal{P}_1} v^{\langle\lambda',\lambda_{m+1}\rangle} \frac{a_{\lambda'} a_{\lambda_{m+1}}}{a_\lambda} g^\lambda_{\lambda'\lambda_{m+1}} K_{-\lambda_{m+1}} \times$$

$$\sum_{m\geq 1} (-1)^m \sum_{\substack{\pi'\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_{\lambda'}} g^{\lambda'}_{\lambda_1\cdots\lambda_m} g^{\pi'}_{\lambda_1\cdots\lambda_m} K_{-\lambda'} u_{\pi'} u_{\lambda_{m+1}}$$

$$= \delta_{\lambda 0} - K_{-\lambda}u_\lambda - \sum_{m\geq 1} (-1)^m \sum_{\substack{\pi'\in\mathcal{P}\\ \lambda',\lambda_1,\cdots,\lambda_m,\lambda_{m+1}\in\mathcal{P}_1}} v^{2\sum_{i<j\leq m}\langle\lambda_i,\lambda_j\rangle} v^{\langle\lambda',\lambda_{m+1}\rangle} \times$$

$$\frac{a_{\lambda_1}\cdots a_{\lambda_m} a_{\lambda_{m+1}}}{a_\lambda} g^{\lambda'}_{\lambda_1\cdots\lambda_m} g^\lambda_{\lambda'\lambda_{m+1}} g^{\pi'}_{\lambda_1\cdots\lambda_m} K_{-\lambda_{m+1}} K_{\lambda'} \sum_{\pi\in\mathcal{P}} v^{\langle\pi',\lambda_{m+1}\rangle} g^\pi_{\pi'\lambda_{m+1}} u_\pi$$

$$= \delta_{\lambda 0} - K_{-\lambda}u_\lambda - \sum_{m\geq 1} (-1)^m \sum_{\substack{\pi\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m,\lambda_{m+1}\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \times$$

$$\frac{a_{\lambda_1}\cdots a_{\lambda_m}a_{\lambda_{m+1}}}{a_\lambda}g^{\lambda}_{\lambda_1\cdots\lambda_m\lambda_{m+1}}g^{\pi}_{\lambda_1\cdots\lambda_m\lambda_{m+1}}K_{-\lambda}u_\pi,$$

because that $g^{\lambda}_{\alpha_1\cdots\alpha_t}\neq 0$ implies that $\underline{\dim}u_\lambda=\underline{\dim}u_{\alpha_1}+\cdots+\underline{\dim}u_{\alpha_t}$, so where we may assume that

$$\underline{\dim}u_{\lambda'}=\underline{\dim}u_{\lambda_1}+\cdots+\underline{\dim}u_{\lambda_m},\quad \underline{\dim}u_{\pi'}=\underline{\dim}u_{\lambda_1}+\cdots+\underline{\dim}u_{\lambda_m}$$

$$\underline{\dim}u_{\lambda'}+\underline{\dim}u_{\lambda_{m+1}}=\underline{\dim}u_\lambda,\quad \underline{\dim}u_{\pi'}+\underline{\dim}u_{\lambda_{m+1}}=\underline{\dim}u_\pi.$$

Then the right side of $(**)$ is

$$=\delta_{\lambda 0}-K_{-\lambda}u_\lambda+\sum_{m\geq 2}(-1)^m\sum_{\substack{\pi\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}}v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda}g^{\lambda}_{\lambda_1\cdots\lambda_m}g^{\pi}_{\lambda_1\cdots\lambda_m}K_{-\lambda}u_\pi$$

$$=\delta_{\lambda 0}+\sum_{m\geq 1}(-1)^m\sum_{\substack{\pi\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}}v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda}g^{\lambda}_{\lambda_1\cdots\lambda_m}g^{\pi}_{\lambda_1\cdots\lambda_m}K_{-\lambda}u_\pi.$$

According to the definition of σ in (c), this is exactly equal to $\sigma(u_\lambda)$. It is same to verify that $\mu(1\otimes\sigma)\Delta(u_\lambda)=\delta_{\lambda 0}$. The proof of Theorem 4.5 is completed.

4.5.1 Remark 1. Let us mention the following special case of Theorem 4.5; if $u_i,\ i\in I$ are simple Λ-modules, we have

$$\Delta(u_i)=u_i\otimes 1+K_i\otimes u_i,\quad \sigma(u_i)=-K_{-i}u_i.$$

2. The subalgebra of $\mathbf{H}(\Lambda)$ generated by $\{K_\alpha\mid\alpha\in\mathbf{Z}[I]\}$ is a torus defined on $\mathbf{Z}[I]$, also it is isomorphic to the group algebra of the Grothendieck group of $\operatorname{mod}\Lambda$.

4.5.2 The Hopf algebra $\mathbf{H}(\Lambda)$ given in (4.5) is called the (extended twisted) *Hall algebra* of Λ.

4.6 Proof of Corollary 4.3 In view of (a) and (b) of Theorem 4.5

$$u_{\beta_1}u_{\beta_2}\cdots u_{\beta_m}=\sum_{\lambda\in\mathcal{P}}v^{\sum_{i<j}\langle\beta_i,\beta_j\rangle}g^{\lambda}_{\beta_1\cdots\beta_m}u_\lambda,$$

$$\Delta^n(u_\lambda)=\sum_{\alpha_1,\cdots,\alpha_n\in\mathcal{P}}v^{\sum_{i<j}\langle\alpha_i,\alpha_j\rangle}g^{\lambda}_{\alpha_1\cdots\alpha_n}\frac{a_{\alpha_1}\cdots a_{\alpha_n}}{a_\lambda}\times$$
$$u_{\alpha_1}K_{\alpha_2+\cdots+\alpha_n}\otimes u_{\alpha_2}K_{\alpha_3+\cdots+\alpha_n}\otimes\cdots\otimes u_{\alpha_{n-1}}K_{\alpha_n}\otimes u_{\alpha_n}$$

where $\Delta^n=(1\otimes\cdots\otimes 1\otimes\Delta)\cdots(1\otimes\Delta)\Delta$. By Theorem 4.5 (b),

$$(\star)\quad \Delta^n(u_{\beta_1}u_{\beta_2}\cdots u_{\beta_m})=\Delta^n(u_{\beta_1})\Delta^n(u_{\beta_2})\cdots\Delta^n(u_{\beta_m}).$$

The left side of $(\star)$ is

$$=\sum_{\lambda\in\mathcal{P}}v^{\sum_{i<j}\langle\beta_i,\beta_j\rangle}g^{\lambda}_{\beta_1\cdots\beta_m}\sum_{\alpha_1,\cdots,\alpha_n\in\mathcal{P}}v^{\sum_{i<j}\langle\alpha_i,\alpha_j\rangle}g^{\lambda}_{\alpha_1\cdots\alpha_n}\frac{a_{\alpha_1}\cdots a_{\alpha_n}}{a_\lambda}\times$$
$$u_{\alpha_1}K_{\alpha_2+\cdots+\alpha_n}\otimes u_{\alpha_2}K_{\alpha_3+\cdots+\alpha_n}\otimes\cdots\otimes u_{\alpha_{n-1}}K_{\alpha_n}\otimes u_{\alpha_n}$$

$$= \sum_{\alpha_1,\cdots,\alpha_n\in\mathcal{P}} v^{\sum_{i<j}\langle\beta_i,\beta_j\rangle+\sum_{i<j}\langle\alpha_i,\alpha_j\rangle}\prod_{i=1}^{n}a_{\alpha_i}\times$$

$$\sum_{\lambda\in\mathcal{P}} g^{\lambda}_{\alpha_1\cdots\alpha_n}g^{\lambda}_{\beta_1\cdots\beta_m}a_\lambda^{-1}u_{\alpha_1}K_{\alpha_2+\cdots+\alpha_n}\otimes u_{\alpha_2}K_{\alpha_3+\cdots+\alpha_n}\otimes\cdots\otimes u_{\alpha_n}.$$

The right side of $(\star)$ is

$$= \prod_{j=1}^{m}\Delta^n(u_{\beta_j}) = \prod_{j=1}^{m}\sum_{\rho_{1j}\cdots\rho_{nj}} v^{\sum_{i<k}\langle\rho_{ij},\rho_{kj}\rangle}g^{\beta_j}_{\rho_{1j}\cdots\rho_{nj}}\frac{a_{\rho_{1j}}\cdots a_{\rho_{nj}}}{a_{\beta_j}}\times$$

$$u_{\rho_{1j}}K_{\rho_{2j}+\cdots+\rho_{nj}}\otimes u_{\rho_{2j}}K_{\rho_{3j}+\cdots+\rho_{nj}}\otimes\cdots\otimes u_{\rho_{nj}}$$

$$= \sum_{\substack{\rho_{ij}\in\mathcal{P}\\ 1\le i\le n,1\le j\le m}} v^{(\sum_{1\le j\le m}\sum_{i<k}\langle\rho_{ij},\rho_{kj}\rangle)}\prod_{j=1}^{m}g^{\beta_j}_{\rho_{1j}\cdots\rho_{nj}}\prod_{\substack{1\le i\le n\\ 1\le j\le m}}a_{\rho_{ij}}(\prod_{j=1}^{m}a_{\beta_j})^{-1}\times$$

$$\prod_{j=1}^{m}u_{\rho_{1j}}K_{\rho_{2j}+\cdots+\rho_{nj}}\otimes\prod_{j=1}^{m}u_{\rho_{2j}}K_{\rho_{3j}+\cdots+\rho_{nj}}\otimes\cdots\otimes\prod_{j=1}^{m}u_{\rho_{nj}}$$

$$= \sum_{\substack{\rho_{ij}\in\mathcal{P}\\ 1\le i\le n,1\le j\le m}} v^{(\sum_{1\le j\le m}\sum_{i<k}\langle\rho_{ij},\rho_{kj}\rangle)}\prod_{j=1}^{m}g^{\beta_j}_{\rho_{1j}\cdots\rho_{nj}}\prod_{\substack{1\le i\le n\\ 1\le j\le m}}a_{\rho_{ij}}(\prod_{j=1}^{m}a_{\beta_j})^{-1}v^{(\sum_{\substack{i>k\\ j<l}}(\rho_{ij},\rho_{kl}))}\times$$

$$(\prod_{j=1}^{m}u_{\rho_{1j}})K_{(\sum_{1\le j\le m}\rho_{2j}+\cdots+\rho_{nj})}\otimes(\prod_{j=1}^{m}u_{\rho_{2j}})K_{(\sum_{1\le j\le m}\rho_{3j}+\cdots+\rho_{nj})}\otimes\cdots\otimes\prod_{j=1}^{m}u_{\rho_{nj}}$$

$$= \sum_{\alpha_1,\cdots,\alpha_n\in\mathcal{P}}\sum_{\substack{\rho_{ij}\in\mathcal{P}\\ 1\le i\le n,1\le j\le m}} v^{(\sum_{1\le j\le m}\sum_{i<k}\langle\rho_{ij},\rho_{kj}\rangle+\sum_{1\le i\le n}\sum_{j<l}\langle\rho_{ij},\rho_{il}\rangle+\sum_{\substack{i>k\\ j<l}}(\rho_{ij},\rho_{kl}))}\times$$

$$\prod_{j=1}^{m}g^{\beta_j}_{\rho_{1j}\cdots\rho_{nj}}\prod_{i=1}^{n}g^{\alpha_i}_{\rho_{i1}\cdots\rho_{im}}\prod_{\substack{1\le i\le n\\ 1\le j\le m}}a_{\rho_{ij}}(\prod_{j=1}^{m}a_{\beta_j})^{-1}\times$$

$$u_{\alpha_1}K_{(\alpha_2+\cdots+\alpha_n)}\otimes u_{\alpha_2}K_{(\alpha_3+\cdots+\alpha_n)}\otimes\cdots\otimes u_{\alpha_n}$$

Then we have

$$v^{\sum_{j<l}\langle\beta_j,\beta_l\rangle+\sum_{i<k}\langle\alpha_i,\alpha_k\rangle}\prod_{i=1}^{n}a_{\alpha_i}\sum_{\lambda\in\mathcal{P}}g^{\lambda}_{\alpha_1\cdots\alpha_n}g^{\lambda}_{\beta_1\cdots\beta_m}a_\lambda^{-1}$$

$$= (\prod_{j=1}^{m}a_{\beta_j})^{-1}\sum_{\substack{\rho_{ij}\in\mathcal{P}\\ 1\le i\le n,1\le j\le m}}\prod_{\substack{1\le i\le n\\ 1\le j\le m}}a_{\rho_{ij}}\times$$

$$v^{(\sum_{1\le j\le m}\sum_{i<k}\langle\rho_{ij},\rho_{kj}\rangle+\sum_{1\le i\le n}\sum_{j<l}\langle\rho_{ij},\rho_{il}\rangle+\sum_{\substack{i>k\\ j<l}}(\rho_{ij},\rho_{kl}))}\prod_{i=1}^{n}g^{\alpha_i}_{\rho_{i1}\cdots\rho_{im}}\prod_{j=1}^{m}g^{\beta_j}_{\rho_{1j}\cdots\rho_{nj}}$$

Since

$$\sum_{j<l}\langle\beta_j,\beta_l\rangle = \sum_{\substack{i,j,k,l\\ j<l}}\langle\rho_{ij},\rho_{kl}\rangle$$

$$\sum_{i<k}\langle\alpha_i,\alpha_k\rangle = \sum_{\substack{i,j,k,l\\ i<k}}\langle\rho_{ij},\rho_{kl}\rangle$$

$$\sum_{\substack{i>k\\ j<l}}(\rho_{ij},\rho_{kl}) = \sum_{\substack{i>k\\ j<l}}\langle\rho_{ij},\rho_{kl}\rangle + \sum_{\substack{i<k\\ j>l}}\langle\rho_{ij},\rho_{kl}\rangle$$

So we obtain th formula

$$\prod_{i=1}^{n} a_{\alpha_i} \prod_{j=1}^{m} a_{\beta_j} \sum_{\lambda\in\mathcal{P}} g^{\lambda}_{\alpha_1\cdots\alpha_n} g^{\lambda}_{\beta_1\cdots\beta_m} a_{\lambda}^{-1}$$

$$= \sum_{\substack{\rho_{ij}\in\mathcal{P}\\ 1\le i\le n, 1\le j\le m}} \prod_{\substack{1\le i\le n\\ 1\le j\le m}} a_{\rho_{ij}} v^{-2\sum_{\substack{i<k\\ j<l}}\langle\rho_{ij},\rho_{kl}\rangle} \prod_{i=1}^{n} g^{\alpha_i}_{\rho_{i1}\cdots\rho_{im}} \prod_{j=1}^{m} g^{\beta_j}_{\rho_{1j}\cdots\rho_{nj}}$$

This is just the formula in Corollary 4.3, the proof is finished.

4.7 Let $\mathbf{H}_\nu$ be the R-submodule of $\mathbf{H}(\Lambda)$ with the basis $\{K_\alpha u_\lambda \mid \alpha \in \mathbf{Z}[I], \underline{\dim} u_\lambda = \nu\}$ for any $\nu \in \mathbf{N}[I]$. Then

Proposition 1 $\mathbf{H}(\Lambda) = \bigoplus_{\nu\in\mathbf{N}[I]} \mathbf{H}_\nu$ *is a* $\mathbf{N}[I]$*-graded algebra.*

Proof We only have to observe that $g^{\lambda}_{\alpha\beta} \neq 0$ implies that $\underline{\dim} u_\lambda = \underline{\dim} u_\alpha + \underline{\dim} u_\beta$.

On the other hand, $\mathbf{H}(\Lambda)$ is also a filtrated algebra. Let $\operatorname{ind}\Lambda = \{u_\alpha \mid \alpha \in \mathcal{P},\ u_\alpha \text{ indecomposable}\}$. For any $u_\lambda, \lambda \in \mathcal{P}$, define $deg u_\lambda = (\dim_k \operatorname{Hom}(u_\lambda, u_\gamma))_{u_\gamma\in\operatorname{ind}\Lambda}$; of course, let $deg K_\alpha = 0$, $deg K_\alpha u_\lambda = deg u_\lambda$ for $\alpha \in \mathbf{Z}[I], \lambda \in \mathcal{P}$. $\mathbf{H}^r$ is defined, for any $r \in \mathbf{N}[\operatorname{ind}\Lambda]$, as the R-spain of $K_\alpha u_\lambda$ with $deg u_\lambda \le r$. Particularly $\mathbf{H}^0 = \mathbf{H}_0 = \mathbf{T}$.

Proposition 2 $\{\mathbf{H}^r \mid r \in \mathbf{N}[\operatorname{ind}\Lambda]\}$ *is a filtration of* $\mathbf{H}(\Lambda)$.

Proof Only need to show that $g^{\lambda}_{\alpha\beta} \neq 0$ implies that $deg u_\lambda \le deg u_\alpha + deg u_\beta$, this is equivalent to say

$$\dim_k \operatorname{Hom}(u_\lambda, u_\gamma) \le \dim_k \operatorname{Hom}(u_\alpha, u_\gamma) + \dim_k \operatorname{Hom}(u_\beta, u_\gamma)$$

for any $\gamma \in \mathcal{P}$. It can be observed by applying $\operatorname{Hom}(-, u_\gamma)$ to the exact sequence $0 \to u_\beta \to u_\lambda \to u_\alpha \to 0$ and from the induced long exact sequence.

4.7.1 Remark The Levendorkkii-Soibelman Lemma in [2] is a consequence of this simple observation.

4.8 Proposition *The antipode* σ *of* $\mathbf{H}(\Lambda)$ *satisfies that* $\sigma^2(x_\nu) = v^{-(\nu,\nu)} x_\nu$ *for any* $x_\nu \in \mathbf{H}_\nu$, $\nu \in \mathbf{N}[I]$.

Proof We consider $\mathbf{H}'$ which satisfies $\mathbf{H}' \equiv \mathbf{H}(\Lambda)$ as associative algebras but is equipped with its opposed comultiplication $\Delta' = \Delta^{opp}$, that is

$$\Delta'(u_\lambda) = \sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} \frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha\beta} u_\beta \otimes u_\alpha K_\beta$$

$$\Delta'(K_\alpha) = K_\alpha \otimes K_\alpha$$

for all $\lambda \in \mathcal{P}$ and $\alpha \in \mathbf{Z}[I]$, and the same counit as in $\mathbf{H}(\Lambda)$. As same as in the proof of Theorem 4.5, the following formulae

$$\sigma'(u_\lambda) = \delta_{\lambda 0} + \sum_{m\geq 1} (-1)^m \sum_{\substack{\pi\in\mathcal{P} \\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m} u_\pi K_{-\lambda}$$

$$\sigma'(K_\alpha) = K_{-\alpha}$$

for all $\lambda \in \mathcal{P}$, $\alpha \in \mathbf{Z}[I]$, define the antipode of $\mathbf{H}'$. It is easy to see $\sigma(x_\nu) = v^{-(\nu,\nu)}\sigma'(x_\nu)$ for any $x_\nu \in \mathbf{H}_\nu$, $\nu \in \mathbf{N}[I]$. Set $\phi : \mathbf{H}(\Lambda) \to \mathbf{H}(\Lambda)$ by taking $\phi(x_\nu) = v^{-(\nu,\nu)} x_\nu$ for any $x_\nu \in \mathbf{H}_\nu$, $\nu \in \mathbf{N}[I]$. ϕ is a bijection. We have $\sigma = \phi\sigma' = \sigma'\phi$, $\sigma' = \phi^{-1}\sigma = \sigma\phi^{-1}$, for σ and σ' are graded-morphisms. Since $\mathbf{H}(\Lambda)$ is a Hopf algebra, $\mathrm{End}_R \mathbf{H}(\Lambda)$ is an associative algebra under the multiplication: $f * g = \mu(f \otimes g)\Delta$ and the unit $\eta\varepsilon$. For any $h \in \mathbf{H}(\Lambda)$, assume that $\Delta(h) = \sum h_1 \otimes h_2$, since $\mu(\sigma' \otimes 1)\Delta' = \eta\varepsilon$ then $\sum \sigma'(h_2)h_1 = \eta\varepsilon(h) = \varepsilon(h)$, hence $(\sigma * (\sigma\sigma'))(h) = \sum \sigma(h_1)\sigma\sigma'(h_2) = \sigma(\sum \sigma'(h_2)h_1) = \sigma(\varepsilon(h)) = \varepsilon(h)$, where we admit that σ is an anti-homorphism of $\mathbf{H}(\Lambda)$. So $\sigma * (\sigma\sigma') = \eta\varepsilon$. However also $\sigma * 1 = \eta\varepsilon$, then $\sigma\sigma' = 1$. Therefore $\sigma^2 = \phi$ by $\sigma' = \phi^{-1}\sigma$. The proof is completed.

It follows from Proposition 4.8 that $\mathbf{H}(\Lambda)$ is a non-commutative and non-cocommutative Hopf algebra excepts Λ is semisimple or their degeneration at $v = 1$.

4.9 Let $\lambda,\ \pi \in \mathcal{P}$, $m \in \mathbf{N}$. Set

$$B^{\lambda\pi}_m = \sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m}$$

$$A^{\lambda\pi}_m = \sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m}$$

It is easily seen that

$$A^{\lambda\pi}_m = \sum_{i=0}^{m-1} (-1)^{m-i} \begin{pmatrix} m \\ i \end{pmatrix} B^{\lambda\pi}_{m-i}$$

Set $A^{\lambda\pi} = \sum_{m\geq 1}(-1)^m A^{\lambda\pi}_m$. Note that $A^{\lambda\pi}_m = 0$ for almost all m, $A^{\lambda\pi}$ is well-defined. By Theorem 4.5,

$$\sigma(u_\lambda) = \sum_{\pi\in\mathcal{P}} A^{\lambda\pi} K_{-\lambda} u_\pi$$

$$\sigma(u_\pi) = \sum_{\gamma\in\mathcal{P}} A^{\pi\gamma} K_{-\pi} u_\gamma$$

then

$$\sigma^2(u_\lambda) = \sum_{\pi\in\mathcal{P}} A^{\lambda\pi} \sum_{\gamma\in\mathcal{P}} A^{\pi\gamma} K_{-\pi} u_\gamma K_\lambda.$$

Since σ is a graded-morphism, we may assume that $\underline{\dim} u_\lambda = \underline{\dim} u_\pi = \underline{\dim} u_\gamma$, then

$$\sigma^2(u_\lambda) = v^{-(\lambda,\lambda)} \sum_{\gamma\in\mathcal{P}} (\sum_{\pi\in\mathcal{P}} A^{\lambda\pi} A^{\pi\gamma}) u_\gamma.$$

Therefore Proposition 4.8 is equivalent to the following orthogonal relations

4.9.1

$$\sum_{\pi\in\mathcal{P}} A^{\lambda\pi} A^{\pi\gamma} = \delta_{\lambda\gamma}$$

for any $\lambda,\ \gamma \in \mathcal{P}$. By Corollary 4.3, the orthogonal relation (4.9.1) can be expanded into

4.9.2

$$\sum_{m,n\geq 1} (-1)^{m+n} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (-1)^{m+n-(i+j)} \left(\begin{array}{c} m \\ i \end{array}\right) \left(\begin{array}{c} n \\ j \end{array}\right) \times$$

$$\sum_{\substack{\rho_{st}\in\mathcal{P} \\ 1\leq s\leq n-j, 1\leq t\leq m-i}} v^{2\sum_{\substack{1\leq s,k\leq n-j \\ 1\leq t,l\leq m-i}} \langle \rho_{st}, \rho_{kl}\rangle + 2\sum_{\substack{s<k \\ t<l}} \langle \rho_{st}, \rho_{kl}\rangle} \times$$

$$\prod_{\substack{1\leq s\leq n-j \\ 1\leq t\leq m-i}} a_{\rho_{st}} a_\lambda^{-1} g^{\lambda}_{\rho_{11}\cdots\rho_{n-j1}\cdots\rho_{1m-i}\cdots\rho_{n-jm-i}} g^{\gamma}_{\rho_{11}\cdots\rho_{1m-i}\cdots\rho_{n-j1}\cdots\rho_{n-jm-i}} = \delta_{\lambda\gamma}$$

4.10 The subalgebra of $\mathbf{H}(\Lambda, R)$ generated by the element u_i with $i \in I$ and K_α with $\alpha \in \mathbf{Z}[I]$ is called the composition algebra of Λ and denoted by $\mathbf{C}(\Lambda, R)$ or simply $\mathbf{C}(\Lambda)$. It is easy to see that $\mathbf{C}(\Lambda)$ is a Hopf subalgebra of $\mathbf{H}(\Lambda)$ which is still $\mathbf{N}[I]$-graded.

4.11 Given any Cartan datum $C = (I, (,))$, by Lemma 2 of (4.4), we have a finite dimensional hereditary k-algebra Λ such that the simple Λ-modules u_i are indexed by $i \in I$ and the Ringel symmetric form $(-,-)_R$ is $(-,-)$ of $C = (I, (,))$. We denote $\overline{k}$ an algebraic closure of k. For any $n \in \mathbf{N}$, let $F(n)$ be the subfield of $\overline{k}$ such that $[F(n) : k] = n$. Let $\Lambda(n) = \Lambda \otimes_k F(n)$, then $\Lambda(n)$ is a hereditary $F(n)$-algebra corresponding to the Cartan datum $C = (I, (,))$. We have $\Lambda(m) \simeq \Lambda(n) \otimes_{F(n)} F(m)$ if $n|m$. Let $\mathcal{P}(n)$ be the set of isomorphism classes of $\Lambda(n)$-modules which are finite dimensional over $F(n)$ and $\mathcal{P}^*$ the set of

$$\alpha = \{\alpha(n) \in \mathcal{P}(n) \mid \text{there exists } m \in \mathbf{N},\ u_{\alpha(n)} \sim u_{\alpha(m)} \text{ for all } n \geq m\}.$$

where $u_{\alpha(n)} \sim u_{\alpha(m)}$ means that there is $l \in \mathbf{N}$ such that $n|l,\ m|l$ and $u_{\alpha(n)} \otimes_{F(n)} F(l) \simeq u_{\alpha(m)} \otimes_{F(m)} F(l) \simeq u_{\alpha(l)}$. Now $\alpha = \beta \in \mathcal{P}^*$ means that $\alpha(n) = \beta(n)$ for almost all $n \in \mathbf{N}$. For $\alpha,\ \beta \in \mathcal{P}^*$, since $\langle\alpha(n), \beta(n)\rangle$

depends only on the dimension vectors of $\alpha(n)$ and $\beta(n)$, there is $m \in \mathbf{N}$ such that $\langle\alpha(n),\beta(n)\rangle = \langle\alpha(m),\beta(m)\rangle$ for all $n \geq m$. So we can take $\langle\alpha,\beta\rangle = \lim\limits_{n\to\infty}\langle\alpha(n),\beta(n)\rangle$.

Take $v_n = \sqrt{|F(n)|}$, we have the Hall algebra $\mathbf{H}(\Lambda(n),\mathbf{Z}[v_n,v_n^{-1}])$ according to Theorem 4.5. Let

$$\phi_{ml} : \prod_{n\geq m}\mathbf{H}(\Lambda(n),\mathbf{Z}[v_n,v_n^{-1}]) \to \prod_{n\geq l}\mathbf{H}(\Lambda(n),\mathbf{Z}[v_n,v_n^{-1}])$$

be the canonical projection for $l \geq m$. Then $\{\phi_{ml} \mid m,l \in \mathbf{N}\}$ is a directed system since $\phi_{ml}\phi_{lp} = \phi_{mp}$. We consider the directed limit

$$\mathcal{L} = \lim_{m\to\infty}\prod_{n\geq m}\mathbf{H}(\Lambda(n),\mathbf{Z}[v_n,v_n^{-1}])$$

We define a new tensor product for $\mathcal{L}$ as

$$\mathcal{L}\otimes\mathcal{L} = \lim_{m\to\infty}\prod_{n\geq m}(\mathbf{H}(\Lambda(n),\mathbf{Z}[v_n,v_n^{-1}])\otimes\mathbf{H}(\Lambda(n),\mathbf{Z}[v_n,v_n^{-1}]))$$

where each $\otimes$ in the right hand side is over $\mathbf{Z}[v_n,v_n^{-1}]$. With this $\otimes$, $\mathcal{L}$ become a Hopf algebra which is induced from $\mathbf{H}(\Lambda(n),\mathbf{Z}[v_n,v_n^{-1}])$. Let $u_\alpha = \lim\limits_{m\to\infty}(u_{\alpha(n)})_{n\geq m}$ for $\alpha \in \mathcal{P}^*$ and $K_\alpha = K_\alpha 1$ for $\alpha \in \mathbf{Z}[I]$, where 1 is the unit of $\mathcal{L}$. Define $v = \lim\limits_{m\to\infty}(v_n)_{n\geq m}$, $a_\alpha = \lim\limits_{m\to\infty}(a_{\alpha(n)})_{n\geq m}$ and $g^\lambda_{\alpha\beta} = \lim\limits_{m\to\infty}(g^{\lambda(n)}_{\alpha(n)\beta(n)})_{n\geq m}$ for $\alpha,\beta,\lambda \in \mathcal{P}^*$. Define $\mathbf{H}^*(\Lambda)$ to be the subalgebra of $\mathcal{L}$ generated by $\{K_\alpha u_\beta \mid \alpha \in \mathbf{Z}[I], \beta \in \mathcal{P}^*\}$. It is easy to see that

4.12 Proposition $\mathbf{H}^*(\Lambda)$ *is a Hopf subalgebra of $\mathcal{L}$ satisfying the following relations*

(a)

$$u_\alpha u_\beta = \sum_{\lambda\in\mathcal{P}^*} v^{\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta} u_\lambda \quad \textit{for all } \alpha,\beta \in \mathcal{P}^*$$

$$K_\alpha u_\beta = v^{(\alpha,\beta)} u_\beta K_\alpha \quad \textit{for all } \alpha \in \mathbf{Z}[I], \beta \in \mathcal{P}^*$$

$$K_\alpha K_\beta = K_{\alpha+\beta} \quad \textit{for all } \alpha,\beta \in \mathbf{Z}[I]$$

with unit $1 = u_0 = K_0$.

(b)

$$\Delta(u_\lambda) = \sum_{\alpha,\beta\in\mathcal{P}^*} v^{\langle\alpha,\beta\rangle}\frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha\beta} u_\alpha K_\beta \otimes u_\beta \quad \textit{for all } \lambda \in \mathcal{P}^*$$

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha \quad \textit{for all } \alpha \in \mathbf{Z}[I]$$

with counit: $\varepsilon u_\lambda = 0$ *for* $\lambda \neq 0$ *in* $\mathcal{P}^*$ *and* $\varepsilon K_\alpha = 1$ *for all* $\alpha \in \mathbf{Z}[I]$.

(c)

$$\sigma(u_\lambda) = \delta_{\lambda 0} + \sum_{m\geq 1}(-1)^m \sum_{\substack{\pi\in\mathcal{P}^* \\ \lambda_1,\cdots,\lambda_m\in{\mathcal{P}_1}^*}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m} K_{-\lambda} u_\pi$$

for all $\lambda \in \mathcal{P}^*$.

$$\sigma(K_\alpha) = K_{-\alpha} \quad \textit{for all } \alpha \in \mathbf{Z}[I]$$

4.13 Of course v is a central element of $\mathbf{H}^*(\Lambda)$ and transcendental over $\mathbf{Q}$. Since $\mathbf{H}^*(\Lambda)$ is torsionfree as $\mathbf{Z}[v,v^{-1}]$-module, if we tensor $\mathbf{H}^*(\Lambda)$ with the quotient field $\mathbf{Q}(v)$ of $\mathbf{Z}[v,v^{-1}]$, we obtain an embedding of $\mathbf{H}^*(\Lambda)$ into $\mathbf{H}^*(\Lambda)\otimes_{\mathbf{Z}[v,v^{-1}]}\mathbf{Q}(v)$. We call $\mathbf{H}^*(\Lambda)\otimes_{\mathbf{Z}[v,v^{-1}]}\mathbf{Q}(v)$ to be the (extended twisted) generic Hall algebra of Λ, it is a Hopf algebra over $\mathbf{Q}(v)$, denote it as $\mathbf{H}^*(\Lambda,\mathbf{Q}(v))$.

The subalgebra $\mathbf{C}^*(\Lambda,\mathbf{Q}(v))$ of $\mathbf{H}^*(\Lambda,\mathbf{Q}(v))$ generated by $u_i, i\in I$ and $K_\alpha, \alpha\in\mathbf{Z}[I]$ is called the *(extended twisted) generic composition algebra* of Λ. It is a Hopf subalgebra too. Since

$$\Delta(u_i)=u_i\otimes 1+K_i\otimes u_i,\ \ \Delta(K_\alpha)=K_\alpha\otimes K_\alpha,$$

so the tensor product for $\mathbf{C}^*(\Lambda,\mathbf{Q}(v))$ concides with the ordinary tensor product over $\mathbf{Q}(v)$. Hence $\mathbf{C}^*(\Lambda,\mathbf{Q}(v))$ has the Hopf algebra structure as $\mathbf{Q}(v)$-algebra.

Note that $\mathbf{C}^*(\Lambda,\mathbf{Q}(v))=\mathbf{H}^*(\Lambda,\mathbf{Q}(v))$ when Λ is finite representation type, i.e, the Cartan datum $C=(I,(,))$ is finite type. In this situation a more pleasant treatment for $g^\lambda_{\alpha\beta}$ is the replacement by the Hall polynomial corresponding to it (see [11],[12]).

5. Drinfeld double of Hall algebras

5.1 Given a Cartan datum $C=(I,(,))$, let Λ be a finite dimensional hereditary k-algebra corresponding to it. Let R be a commutative integral domain containing the integers $\mathbf{Z}$ and elements v,v^{-1}, where $v^2=q=|k|$.

5.1.1 Let $\mathbf{H}^+(\Lambda)$ be the Hall algebra defined as in Theorem 4.5, that is, $\mathbf{H}^+(\Lambda)$ is the free R-module with the basis $\{K_\alpha u^+_\lambda \mid \alpha\in\mathbf{Z}[I],\ \lambda\in\mathcal{P}\}$. The Hopf algebra of $\mathbf{H}^+(\Lambda)$ is given as follows.

(a)

$$u^+_\alpha u^+_\beta=\sum_{\lambda\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta}u^+_\lambda \ \text{ for all } \alpha,\beta\in\mathcal{P}$$

$$K_\alpha u^+_\beta=v^{(\alpha,\beta)}u^+_\beta K_\alpha \ \text{ for all } \alpha\in\mathbf{Z}[I],\beta\in\mathcal{P}$$

$$K_\alpha K_\beta=K_{\alpha+\beta} \ \text{ for all } \alpha,\beta\in\mathbf{Z}[I]$$

with unit $1=u^+_0=K_0$.

(b)

$$\Delta(u^+_\lambda)=\sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle}\frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha\beta}u^+_\alpha K_\beta\otimes u^+_\beta \ \text{ for all } \lambda\in\mathcal{P}$$

$$\Delta(K_\alpha)=K_\alpha\otimes K_\alpha \ \text{ for all } \alpha\in\mathbf{Z}[I]$$

with counit: $\varepsilon u_\lambda^+ = 0$ for $\lambda \neq 0$ in $\mathcal{P}$ and $\varepsilon K_\alpha = 1$ for all $\alpha \in \mathbf{Z}[I]$.
(c)

$$\sigma(u_\lambda^+) = \delta_{\lambda 0} + \sum_{m\geq 1}(-1)^m \sum_{\substack{\pi\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m} K_{-\lambda} u_\pi^+$$

for all $\lambda \in \mathcal{P}$.

$$\sigma(K_\alpha) = K_{-\alpha} \quad \text{for all } \alpha \in \mathbf{Z}[I]$$

5.1.2 Dually, let $\mathbf{H}^-(\Lambda)$ be the free R-module with the basis $\{K_\alpha u_\lambda^- \mid \alpha \in \mathbf{Z}[I],\ \lambda \in \mathcal{P}\}$. The Hopf algebra of $\mathbf{H}^-(\Lambda)$ is given as follows.
(a)

$$u_\alpha^- u_\beta^- = \sum_{\lambda\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta} u_\lambda^- \quad \text{for all } \alpha,\beta\in\mathcal{P}$$

$$K_\alpha u_\beta^- = v^{-(\alpha,\beta)} u_\beta^- K_\alpha \quad \text{for all } \alpha \in \mathbf{Z}[I], \beta\in\mathcal{P}$$

$$K_\alpha K_\beta = K_{\alpha+\beta} \quad \text{for all } \alpha,\beta \in \mathbf{Z}[I]$$

with unit $1 = u_0^- = K_0$.
(b)

$$\Delta(u_\lambda^-) = \sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} \frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha\beta} u_\beta^- \otimes u_\alpha^- K_{-\beta} \quad \text{for all } \lambda\in\mathcal{P}$$

$$\Delta(K_\alpha) = K_\alpha \otimes K_\alpha \quad \text{for all } \alpha \in \mathbf{Z}[I]$$

with counit: $\varepsilon u_\lambda^- = 0$ for $\lambda \neq 0$ in $\mathcal{P}$ and $\varepsilon K_\alpha = 1$ for all $\alpha \in \mathbf{Z}[I]$.
(c)

$$\sigma(u_\lambda^-) = \delta_{\lambda 0} + \sum_{m\geq 1}(-1)^m \sum_{\substack{\pi\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m} u_\pi^- K_\lambda$$

for all $\lambda \in \mathcal{P}$.

$$\sigma(K_\alpha) = K_{-\alpha} \quad \text{for all } \alpha \in \mathbf{Z}[I]$$

In $\mathbf{H}^-(\Lambda)$, Δ^{opp} and σ^{-1} exist and satisfy that
(d)

$$\Delta^{opp}(u_\lambda^-) = \sum_{\alpha,\beta\in\mathcal{P}} v^{\langle\alpha,\beta\rangle} \frac{a_\alpha a_\beta}{a_\lambda} g^\lambda_{\alpha\beta} u_\alpha^- K_{-\beta} \otimes u_\beta^- \quad \text{for all } \lambda\in\mathcal{P}$$

$$\Delta^{opp}(K_\alpha) = K_\alpha \otimes K_\alpha \quad \text{for all } \alpha \in \mathbf{Z}[I]$$

(e)

$$\sigma^{-1}(u_\lambda^-) = \delta_{\lambda 0} + \sum_{m\geq 1}(-1)^m \sum_{\substack{\pi\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda} g^\lambda_{\lambda_1\cdots\lambda_m} g^\pi_{\lambda_1\cdots\lambda_m} K_\lambda u_\pi^-$$

for all $\lambda \in \mathcal{P}$.

$$\sigma^{-1}(K_\alpha) = K_{-\alpha} \quad \text{for all } \alpha \in \mathbf{Z}[I]$$

It is a same work to prove that $\mathbf{H}^-(\Lambda)$ is a Hopf algebra as did for $\mathbf{H}^+(\Lambda)$ as in Theorem 4.5.

5.2 Following Ringel [16], we associate a bilinear form $\varphi : \mathbf{H}^+(\Lambda)\times\mathbf{H}^-(\Lambda) \to R$ which is defined as follows.

$$\varphi(K_\alpha u_\beta^+, K_{\alpha'} u_{\beta'}^-) = v^{-(\alpha,\alpha')-(\beta,\alpha')+(\alpha,\beta')}|u_\beta| a_\beta^{-1}\delta_{\beta\beta'}$$

where $|u_\beta|$ is the ordinary cardinality of the module u_β.

5.3 Proposition *The bilinear form $\varphi : \mathbf{H}^+(\Lambda)\times\mathbf{H}^-(\Lambda) \to R$ given as in (5.2) is a skew-Hopf pairing.*

Proof We need to check the conditions (a), (b), (c′) and (d′) in (2.2). (a) is trivial to do. For (b) we should prove

$$\varphi(K_{\alpha_1}u_{\beta_1}^+, K_{\alpha_2}u_{\beta_2}^- K_{\alpha_3}u_{\beta_3}^-) = \varphi(\Delta(K_{\alpha_1}u_{\beta_1}^+), K_{\alpha_2}u_{\beta_2}^- \otimes K_{\alpha_3}u_{\beta_3}^-)$$

Indeed we have

$$\varphi(K_{\alpha_1}u_{\beta_1}^+, K_{\alpha_2}u_{\beta_2}^- K_{\alpha_3}u_{\beta_3}^-) = v^{(\alpha_3,\beta_2)}\varphi(K_{\alpha_1}u_{\beta_1}^+, K_{\alpha_2}K_{\alpha_3}u_{\beta_2}^- u_{\beta_3}^-)$$

$$= v^{(\alpha_3,\beta_2)}\varphi(K_{\alpha_1}u_{\beta_1}^+, K_{\alpha_2}K_{\alpha_3}\sum_{\lambda\in\mathcal{P}} v^{\langle\beta_2,\beta_3\rangle} g_{\beta_2\beta_3}^{\lambda} u_\lambda^-) = v^{D_1} g_{\beta_2\beta_3}^{\beta_1}|u_{\beta_1}|a_{\beta_1}^{-1}$$

where $D_1 = (\alpha_3,\beta_2) + \langle\beta_2,\beta_3\rangle - (\alpha_1,\alpha_2) - (\alpha_1,\alpha_3) - (\beta_1,\alpha_2) - (\beta_1,\alpha_3) + (\alpha_1,\beta_1)$, and

$$\varphi(\Delta(K_{\alpha_1}u_{\beta_1}^+), K_{\alpha_2}u_{\beta_2}^- \otimes K_{\alpha_3}u_{\beta_3}^-)$$

$$= \varphi(\sum_{\rho_1,\rho_2\in\mathcal{P}} v^{\langle\rho_1,\rho_2\rangle} g_{\rho_1\rho_2}^{\beta_1} a_{\rho_1}a_{\rho_2}a_{\beta_1}^{-1} K_{\alpha_1}u_{\rho_1}^+ K_{\rho_2} \otimes K_{\alpha_1}u_{\rho_2}^+, K_{\alpha_2}u_{\beta_2}^- \otimes K_{\alpha_3}u_{\beta_3}^-)$$

$$= v^{D_2} g_{\beta_2\beta_3}^{\beta_1}|u_{\beta_2}||u_{\beta_3}|a_{\beta_1}^{-1}$$

where $D_2 = \langle\beta_2,\beta_3\rangle + (\beta_2,\beta_3) - (\alpha_1,\alpha_2) - (\beta_3,\alpha_2) - (\beta_2,\alpha_2) + (\alpha_1,\beta_2) + (\beta_3,\beta_2) - (\alpha_1,\alpha_3) - (\beta_3,\alpha_3) + (\alpha_1,\beta_3)$. If $g_{\beta_2\beta_3}^{\beta_1} \neq 0$ then

$$(\beta_1,\alpha_2) = (\beta_3,\alpha_2) + (\beta_2,\alpha_2),\ (\alpha_1,\beta_1)$$

$$= (\alpha_1,\beta_2) + (\alpha_1,\beta_3),\ (\alpha_3,\beta_2) - (\beta_1,\alpha_3) = -(\beta_3,\alpha_3)$$

and $|u_{\beta_1}| = |u_{\beta_2}||u_{\beta_3}|$. Therefore $D_1 = D_2$, and (b) is verified.

For (c′) we should show that

$$\varphi(K_{\alpha_1}u_{\beta_1}^+ K_{\alpha_2}u_{\beta_2}^+, K_{\alpha_3}u_{\beta_3}^-) = \varphi(K_{\alpha_1}u_{\beta_1}^+ \otimes K_{\alpha_2}u_{\beta_2}^+, \Delta^{opp}(K_{\alpha_3}u_{\beta_3}^-))$$

We have

$$\varphi(K_{\alpha_1}u_{\beta_1}^+ K_{\alpha_2}u_{\beta_2}^+, K_{\alpha_3}u_{\beta_3}^-) = v^{-(\beta_1,\alpha_2)}\varphi(K_{\alpha_1}K_{\alpha_2}u_{\beta_1}^+ u_{\beta_2}^+, K_{\alpha_3}u_{\beta_3}^-)$$

$$= v^{-(\beta_1,\alpha_2)}\varphi(K_{\alpha_1}K_{\alpha_2}\sum_{\lambda\in\mathcal{P}} v^{\langle\beta_1,\beta_2\rangle} g_{\beta_1\beta_2}^{\lambda} u_\lambda^+, K_{\alpha_3}u_{\beta_3}^-) = v^{D_1} g_{\beta_1\beta_2}^{\beta_3}|u_{\beta_3}|a_{\beta_3}^{-1}$$

where $D_1 = -(\beta_1,\alpha_2) + \langle\beta_1,\beta_2\rangle - (\alpha_1,\alpha_3) - (\alpha_2,\alpha_3) - (\beta_3,\alpha_3) + (\alpha_1,\beta_3) + (\alpha_2,\beta_3)$ and

$$\varphi(K_{\alpha_1}u_{\beta_1}^+ \otimes K_{\alpha_2}u_{\beta_2}^+, \Delta^{opp}(K_{\alpha_3}u_{\beta_3}^-))$$

$$= \varphi(K_{\alpha_1}u_{\beta_1}^+ \otimes K_{\alpha_2}u_{\beta_2}^+, \sum_{\rho_1,\rho_2\in\mathcal{P}} v^{\langle\rho_1,\rho_2\rangle} a_{\rho_1}a_{\rho_2}a_{\beta_3}^{-1} g_{\rho_1\rho_2}^{\beta_3} K_{\alpha_3}u_{\rho_1}^- K_{-\rho_2} \otimes K_{\alpha_3}u_{\rho_2}^-)$$

$$= \varphi(K_{\alpha_1}u^+_{\beta_1} \otimes K_{\alpha_2}u^+_{\beta_2}, v^{\langle\beta_1,\beta_2\rangle}a_{\beta_1}a_{\beta_2}a^{-1}_{\beta_3}g^{\beta_3}_{\beta_1\beta_2}K_{\alpha_3}u^-_{\beta_1}K_{-\beta_2} \otimes K_{\alpha_3}u^-_{\beta_2})$$
$$= v^{D_2}g^{\beta_3}_{\beta_1\beta_2}|u_{\beta_1}||u_{\beta_2}|a^{-1}_{\beta_3}$$

where $D_2 = \langle\beta_1,\beta_2\rangle - (\beta_1,\beta_2) - (\alpha_1,\alpha_3) + (\alpha_1,\beta_2) - (\beta_1,\alpha_3) + (\beta_1,\beta_2) + (\alpha_1,\beta_1) - (\alpha_2,\alpha_3) - (\beta_2,\alpha_3) + (\alpha_2,\beta_2)$. If $g^{\beta_3}_{\beta_1\beta_2} \neq 0$ then

$$(\beta_3,\alpha_3) = (\beta_1,\alpha_3) + (\beta_2,\alpha_3),\ (\alpha_1,\beta_3)$$
$$= (\alpha_1,\beta_1) + (\alpha_1,\beta_2),\ -(\beta_1,\alpha_2) + (\alpha_2,\beta_3) = (\alpha_2,\beta_2)$$

and $|u_{\beta_1}||u_{\beta_2}| = |u_{\beta_3}|$. (c$'$) is proved.

It remains to show (d$'$)

$$\varphi(\sigma(K_\alpha u^+_\lambda), K_{\alpha'}u^-_{\lambda'}) = \varphi(K_\alpha u^+_\lambda, \sigma^{-1}(K_{\alpha'}u^-_{\lambda'}))$$

It is trivial if $\lambda = 0$ or $\lambda' = 0$. So we assume that $\lambda \neq 0 \neq \lambda'$ in $\mathcal{P}$. Then

$$\varphi(\sigma(K_\alpha u^+_\lambda), K_{\alpha'}u^-_{\lambda'})$$
$$= \varphi(\sum_{m\geq 1}(-1)^m \sum_{\substack{\pi\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda}g^\lambda_{\lambda_1\cdots\lambda_m}g^\pi_{\lambda_1\cdots\lambda_m}K_{-\lambda}u^+_\pi K_{-\alpha}, K_{\alpha'}u^-_{\lambda'})$$

$$= v^{(\alpha,\lambda')}\varphi(\sum_{m\geq 1}(-1)^m \sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_\lambda}g^\lambda_{\lambda_1\cdots\lambda_m}g^{\lambda'}_{\lambda_1\cdots\lambda_m}K_{-\lambda}K_{-\alpha}u^+_{\lambda'}, K_{\alpha'}u^-_{\lambda'})$$

$$= v^{D_1}\sum_{m\geq 1}(-1)^m \sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}a_{\lambda_1}\cdots a_{\lambda_m}g^\lambda_{\lambda_1\cdots\lambda_m}g^{\lambda'}_{\lambda_1\cdots\lambda_m}|u_{\lambda'}|a^{-1}_\lambda a^{-1}_{\lambda'}$$

where $D_1 = (\alpha,\lambda') + (\lambda,\alpha') + (\alpha,\alpha') - (\lambda',\alpha') - (\lambda,\lambda') - (\alpha,\lambda')$, if $g^\lambda_{\lambda_1\cdots\lambda_m}g^{\lambda'}_{\lambda_1\cdots\lambda_m} \neq 0$ then $D_1 = (\alpha,\alpha') - (\lambda,\lambda')$.

$$\varphi(K_\alpha u^+_\lambda, \sigma^{-1}(K_{\alpha'}u^-_{\lambda'}))$$
$$= \varphi(K_\alpha u^+_\lambda, \sum_{m\geq 1}(-1)^m \sum_{\substack{\pi\in\mathcal{P}\\ \lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_{\lambda'}}g^{\lambda'}_{\lambda_1\cdots\lambda_m}g^\pi_{\lambda_1\cdots\lambda_m}K_{\lambda'}u^-_\pi K_{-\alpha'})$$
$$= \varphi(K_\alpha u^+_\lambda, \sum_{m\geq 1}(-1)^m \sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_{\lambda'}}g^{\lambda'}_{\lambda_1\cdots\lambda_m}g^\lambda_{\lambda_1\cdots\lambda_m}K_{\lambda'}u^-_\lambda K_{-\alpha'})$$

$$= v^{D_2}\sum_{m\geq 1}(-1)^m \sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}a_{\lambda_1}\cdots a_{\lambda_m}g^{\lambda'}_{\lambda_1\cdots\lambda_m}g^\lambda_{\lambda_1\cdots\lambda_m}|u_\lambda|a^{-1}_{\lambda'}a^{-1}_\lambda$$

where $D_2 = -(\lambda,\alpha') - (\alpha,\lambda') + (\alpha,\alpha') - (\lambda,\lambda') + (\lambda,\alpha') - (\alpha,\lambda)$, if $g^{\lambda'}_{\lambda_1\cdots\lambda_m}g^\lambda_{\lambda_1\cdots\lambda_m} \neq 0$ then $D_2 = (\alpha,\alpha') - (\lambda,\lambda')$ and $|u_{\lambda'}| = |u_\lambda|$. So (d$'$) is proved. We complete the proof.

5.3.1 The skew-Hopf pairing $\varphi : \mathbf{H}^+(\Lambda) \times \mathbf{H}^-(\Lambda) \to R$ defined in (5.2) may be called the *Ringel pairing,* which is easily seen to be non-degenerate. We remark here that, if we restrict φ to their composition algebras, this restriction of φ coincide with the Lusztig pairing given in Proposition 2.8.3, because $\varphi(u_i^+, u_i^-) = |u_i| a_i^{-1} = v^{(i,i)}(v^{(i,i)} - 1)^{-1} = (1 - v^{-(i,i)})^{-1}$.

5.4 By the Ringel pairing φ, we have the Drinfeld double structure on $\mathbf{H}^+(\Lambda) \otimes \mathbf{H}^-(\Lambda)$, according to (2.3), which we denote by $\mathcal{D}_0(\Lambda)$. It is clear that the ideal of $\mathcal{D}_0(\Lambda)$ generated by $K_\alpha \otimes K_{-\alpha} - 1$, or equivalently, by $K_\alpha \otimes 1 - 1 \otimes K_\alpha$ for any $\alpha \in \mathbf{Z}[I]$ is a Hopf ideal. The corresponding quotient inherits a Hopf algebra structure, which is called the *reduced Drinfeld double* of Λ and denoted by $\mathcal{D}(\Lambda)$.

5.5 Proposition *Let Λ be a finite dimensional hereditary k-algebra, $\mathcal{D}(\Lambda)$ the reduced Drinfeld double of the Ringel pairing $(\mathbf{H}^+(\Lambda), \mathbf{H}^-(\Lambda), \varphi)$. For any $\lambda, \lambda' \in \mathcal{P}$, we have the following formula in $\mathcal{D}(\Lambda)$.*

$$u_{\lambda'}^- u_\lambda^+ - u_\lambda^+ u_{\lambda'}^- = \sum_{\substack{\beta,\beta' \in \mathcal{P} \\ \gamma \in \mathcal{P}_1}} v^{\langle \beta,\gamma\rangle - \langle \beta',\gamma\rangle} g_{\beta\gamma}^{\lambda} g_{\gamma\beta'}^{\lambda'} a_\beta a_\gamma a_{\beta'} a_\lambda^{-1} a_{\lambda'}^{-1} |u_\gamma| u_\beta^+ u_{\beta'}^- K_\gamma$$

$$+ \sum_{\substack{\beta,\gamma,\beta' \in \mathcal{P} \\ \alpha,\gamma' \in \mathcal{P}_1}} v^{\langle \beta,\gamma\rangle - \langle \beta,\alpha\rangle + \langle \beta',\alpha\rangle - \langle \beta',\gamma\rangle} g_{\alpha\beta\gamma}^{\lambda} g_{\gamma\beta'\gamma'}^{\lambda'} a_\beta a_\gamma a_{\beta'} a_\lambda^{-1} a_{\lambda'}^{-1} |u_\alpha||u_\gamma| \times$$

$$\sum_{m\geq 1} (-1)^m \sum_{\lambda_1,\cdots,\lambda_m \in \mathcal{P}_1} v^{2\sum_{i<j}\langle \lambda_i,\lambda_j\rangle} a_{\lambda_1}\cdots a_{\lambda_m} g_{\lambda_1\cdots\lambda_m}^{\gamma'} g_{\lambda_1\cdots\lambda_m}^{\alpha} u_\beta^+ u_{\beta'}^- K_\gamma K_{-\gamma'}$$

Proof Since

$$\Delta^2(u_\lambda^+) = \sum_{\alpha,\beta,\gamma \in \mathcal{P}} v^{\langle\alpha,\beta\rangle + \langle\alpha,\gamma\rangle + \langle\beta,\gamma\rangle} g_{\alpha\beta\gamma}^{\lambda} a_\alpha a_\beta a_\gamma a_\lambda^{-1} u_\alpha^+ K_{\beta+\gamma} \otimes u_\beta^+ K_\gamma \otimes u_\gamma^+$$

and

$$\Delta^2(u_{\lambda'}^-) = \sum_{\alpha',\beta',\gamma' \in \mathcal{P}} v^{\langle\alpha',\beta'\rangle + \langle\alpha',\gamma'\rangle + \langle\beta',\gamma'\rangle} g_{\alpha'\beta'\gamma'}^{\lambda'} a_{\alpha'} a_{\beta'} a_{\gamma'} a_{\lambda'}^{-1} u_{\gamma'}^- \otimes u_{\beta'}^- K_{-\gamma'} \otimes u_{\alpha'}^- K_{-(\beta'+\gamma')}$$

then, according to (2.3) (d), we have

$$u_{\lambda'}^- u_\lambda^+ = \sum_{\substack{\alpha,\beta,\gamma \in \mathcal{P} \\ \alpha',\beta',\gamma' \in \mathcal{P}}} v^{\langle\alpha,\beta\rangle + \langle\alpha,\gamma\rangle + \langle\beta,\gamma\rangle + \langle\alpha',\beta'\rangle + \langle\alpha',\gamma'\rangle + \langle\beta',\gamma'\rangle} g_{\alpha\beta\gamma}^{\lambda} g_{\alpha'\beta'\gamma'}^{\lambda'} \times$$

$$a_\alpha a_\beta a_\gamma a_{\alpha'} a_{\beta'} a_{\gamma'} a_\lambda^{-1} a_{\lambda'}^{-1} \varphi(u_\alpha^+ K_{\beta+\gamma}, \sigma(u_{\gamma'}^-)) u_\beta^+ K_\gamma u_{\beta'}^- K_{-\gamma'} \varphi(u_\gamma^+, u_{\alpha'}^- K_{-(\beta'+\gamma')})$$

Since that

$$\sigma(u_{\gamma'}^-) = \delta_{\gamma' 0} + \sum_{m\geq 1} (-1)^m \sum_{\substack{\pi \in \mathcal{P} \\ \lambda_1,\cdots,\lambda_m \in \mathcal{P}_1}} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_{\gamma'}} g_{\lambda_1\cdots\lambda_m}^{\gamma'} g_{\lambda_1\cdots\lambda_m}^{\pi} u_\pi^- K_{\gamma'}$$

and

$$u_\alpha^+ K_{\beta+\gamma} = v^{-(\alpha,\beta)-(\alpha,\gamma)} K_{\beta+\gamma} u_\alpha^+, \ u_\pi^- K_{\gamma'} = v^{(\gamma',\gamma')} K_{\gamma'} u_\pi^-$$

where we may assume that $\underline{\dim} u_\pi = \underline{\dim} u_{\gamma'}$, then

$$\varphi(u_\alpha^+ K_{\beta+\gamma}, \sigma(u_{\gamma'}^-)) = v^{-(\alpha,\beta)-(\alpha,\gamma)}(\delta_{\gamma' 0}\delta_{\alpha 0} + v^{-(\beta,\gamma')-(\gamma,\gamma')+(\beta,\alpha)+(\gamma,\alpha)-(\alpha,\gamma')+(\gamma',\gamma')} \times$$

$$\sum_{m\geq 1}(-1)^m \sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_{\gamma'}} g^{\gamma'}_{\lambda_1\cdots\lambda_m} g^{\alpha}_{\lambda_1\cdots\lambda_m}|u_\alpha|a_\alpha^{-1})$$

Since $\underline{\dim} u_\alpha = \underline{\dim} u_{\gamma'}$, then

$$\varphi(u_\alpha^+K_{\beta+\gamma},\sigma(u_{\gamma'}^-)) = v^{-(\alpha,\beta)-(\alpha,\gamma)}\delta_{\gamma' 0}\delta_{\alpha 0} + v^{-(\beta,\gamma')-(\gamma,\gamma')}\sum_{m\geq 1}(-1)^m\times$$

$$\sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_{\gamma'}} g^{\gamma'}_{\lambda_1\cdots\lambda_m} g^{\alpha}_{\lambda_1\cdots\lambda_m}|u_\alpha|a_\alpha^{-1}$$

Also

$$u_\beta^+K_\gamma u_{\beta'}^-K_{-\gamma'} = v^{-(\gamma,\beta')}u_\beta^+u_{\beta'}^-K_\gamma K_{-\gamma'}$$

and

$$\begin{aligned}\varphi(u_\gamma^+,u_{\alpha'}^-K_{-(\beta'+\gamma')}) &= v^{-(\alpha',\beta')-(\alpha',\gamma')}\varphi(u_\gamma^+,K_{-(\beta'+\gamma')}u_{\alpha'}^-)\\ &= v^{-(\alpha',\beta')-(\alpha',\gamma')+(\gamma,\beta')+(\gamma,\gamma')}|u_\gamma|a_\gamma^{-1}\delta_{\gamma\alpha'}\end{aligned}$$

Then

$$u_{\lambda'}^-u_\lambda^+ = \sum_{\alpha,\beta,\gamma,\beta',\gamma'\in\mathcal{P}} v^{\langle\alpha,\beta\rangle+\langle\alpha,\gamma\rangle+\langle\beta,\gamma\rangle+\langle\gamma,\beta'\rangle+\langle\gamma,\gamma'\rangle+\langle\beta',\gamma'\rangle} g^{\lambda}_{\alpha\beta\gamma}g^{\lambda'}_{\gamma\beta'\gamma'}\times$$

$$a_\alpha a_\beta a_\gamma a_\gamma a_{\beta'}a_{\gamma'}a_\lambda^{-1}a_{\lambda'}^{-1}(v^{-(\alpha,\beta)-(\alpha,\gamma)}\delta_{\gamma' 0}\delta_{\alpha 0} + v^{-(\beta,\gamma')-(\gamma,\gamma')}\sum_{m\geq 1}(-1)^m\times$$

$$\sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_{\gamma'}} g^{\gamma'}_{\lambda_1\cdots\lambda_m} g^{\alpha}_{\lambda_1\cdots\lambda_m}|u_\alpha|a_\alpha^{-1})v^{-(\gamma,\beta')}u_\beta^+u_{\beta'}^-K_\gamma K_{-\gamma'}|u_\gamma|a_\gamma^{-1}$$

Since that

$$\sum_{\substack{\beta,\gamma,\beta'\in\mathcal{P}\\ \alpha=0,\gamma'=0}} v^{\langle\alpha,\beta\rangle+\langle\alpha,\gamma\rangle+\langle\beta,\gamma\rangle+\langle\gamma,\beta'\rangle+\langle\gamma,\gamma'\rangle+\langle\beta',\gamma'\rangle} g^{\lambda}_{\alpha\beta\gamma}g^{\lambda'}_{\gamma\beta'\gamma'}\times$$

$$\begin{aligned}&a_\alpha a_\beta a_\gamma a_\gamma a_{\beta'}a_{\gamma'}a_\lambda^{-1}a_{\lambda'}^{-1}v^{-(\alpha,\beta)-(\alpha,\gamma)}v^{-(\gamma,\beta')}u_\beta^+u_{\beta'}^-K_\gamma K_{-\gamma'}|u_\gamma|a_\gamma^{-1}\\ &= \sum_{\beta,\gamma,\beta'\in\mathcal{P}} v^{\langle\beta,\gamma\rangle-\langle\beta',\gamma\rangle}g^{\lambda}_{\beta\gamma}g^{\lambda'}_{\gamma\beta'}a_\beta a_\gamma a_{\beta'}a_\lambda^{-1}a_{\lambda'}^{-1}|u_\gamma|u_\beta^+u_{\beta'}^-K_\gamma\\ &= u_\lambda^+u_{\lambda'}^- + \sum_{\substack{\beta,\beta'\in\mathcal{P}\\ \gamma\in\mathcal{P}_1}} v^{\langle\beta,\gamma\rangle-\langle\beta',\gamma\rangle}g^{\lambda}_{\beta\gamma}g^{\lambda'}_{\gamma\beta'}a_\beta a_\gamma a_{\beta'}a_\lambda^{-1}a_{\lambda'}^{-1}|u_\gamma|u_\beta^+u_{\beta'}^-K_\gamma\end{aligned}$$

and

$$\sum_{\substack{\beta,\gamma,\beta'\in\mathcal{P}\\ \alpha,\gamma'\in\mathcal{P}_1}} v^{\langle\alpha,\beta\rangle+\langle\alpha,\gamma\rangle+\langle\beta,\gamma\rangle+\langle\gamma,\beta'\rangle+\langle\gamma,\gamma'\rangle+\langle\beta',\gamma'\rangle} g^{\lambda}_{\alpha\beta\gamma}g^{\lambda'}_{\gamma\beta'\gamma'}a_\alpha a_\beta a_\gamma a_\gamma a_{\beta'}a_{\gamma'}a_\lambda^{-1}a_{\lambda'}^{-1}\times$$

$$\begin{aligned}&v^{-(\beta,\gamma')-(\gamma,\gamma')}\sum_{m\geq 1}(-1)^m\sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}\frac{a_{\lambda_1}\cdots a_{\lambda_m}}{a_{\gamma'}}\times\\ &\quad g^{\gamma'}_{\lambda_1\cdots\lambda_m} g^{\alpha}_{\lambda_1\cdots\lambda_m}|u_\alpha|a_\alpha^{-1}v^{-(\gamma,\beta')}u_\beta^+u_{\beta'}^-K_\gamma K_{-\gamma'}|u_\gamma|a_\gamma^{-1}\\ &= \sum_{\substack{\beta,\gamma,\beta'\in\mathcal{P}\\ \alpha,\gamma'\in\mathcal{P}_1}} v^{\langle\beta,\gamma\rangle-\langle\beta,\alpha\rangle+\langle\beta',\alpha\rangle-\langle\beta',\gamma\rangle}g^{\lambda}_{\alpha\beta\gamma}g^{\lambda'}_{\gamma\beta'\gamma'}a_\beta a_\gamma a_{\beta'}a_\lambda^{-1}a_{\lambda'}^{-1}\times\end{aligned}$$

$$|u_\alpha||u_\gamma|\sum_{m\geq 1}(-1)^m\sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1}v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle}a_{\lambda_1}\cdots a_{\lambda_m}g^{\gamma'}_{\lambda_1\cdots\lambda_m}g^{\alpha}_{\lambda_1\cdots\lambda_m}u^+_\beta u^-_{\beta'}K_\gamma K_{-\gamma'}$$

So, in $\mathcal{D}(\Lambda)$, we have our formula. The proof is completed.

5.5.1 Remark A special case of Proposition 5.5 is

$$u_j^- u_i^+ - u_i^+ u_j^- = \delta_{ij}\frac{|u_i|}{a_i}(K_i - K_i^{-1})$$

for any $i,j\in I$ where $u_i, i\in I$ are the representatives of simple Λ-modules.

Moreover, the following situation is easily checked from Proposition 5.5.

5.5.2 Corollary *For any $\lambda\in\mathcal{P}$ and $i\in I$, we have the formulae in $\mathcal{D}(\Lambda)$,*

$$u_i^- u_\lambda^+ - u_\lambda^+ u_i^- = \frac{|u_i|}{a_\lambda}\sum_{\beta\in\mathcal{P}} a_\beta u_\beta^+(v^{\langle\beta,i\rangle}g^\lambda_{\beta i}K_i - v^{-\langle\beta,i\rangle}g^\lambda_{i\beta}K_{-i})$$

$$u_\lambda^- u_i^+ - u_i^+ u_\lambda^- = \frac{|u_i|}{a_\lambda}\sum_{\beta\in\mathcal{P}} a_\beta u_\beta^-(v^{-\langle\beta,i\rangle}g^\lambda_{i\beta}K_i - v^{\langle\beta,i\rangle}g^\lambda_{\beta i}K_{-i})$$

5.6 Under the assumption of R being a principal ideal domain (only within (5.6)), we consider the composition algebra $\mathbf{C}^+(\Lambda,R)$, which is the subalgebra of $\mathbf{H}^+(\Lambda,R)$ generated by the elements $u_i^+, i\in I$ and $\mathbf{T}$. Dually, the composition algebra $\mathbf{C}^-(\Lambda,R)$ is the subalgebra of $\mathbf{H}^-(\Lambda,R)$ generated by the elements $u_i^-, i\in I$ and $\mathbf{T}$. Obviously they are Hopf subalgebras of $\mathbf{H}^+(\Lambda,R)$ and $\mathbf{H}^-(\Lambda,R)$ respectively. We can restrict the Ringel pairing

$$\varphi:\mathbf{H}^+(\Lambda,R)\times\mathbf{H}^-(\Lambda,R)\to R$$

to their composition algebras, the restriction

$$\varphi:\mathbf{C}^+(\Lambda,R)\times\mathbf{C}^-(\Lambda,R)\to R$$

is easily seen to be a member of $\mathcal{L}(C)$, where $C=(I,(,))$ is the Cartan datum corresponding to Λ. Therefore we have the reduced Drinfeld double of the skew-Hopf pairing $(\mathbf{C}^+(\Lambda,R),\mathbf{C}^-(\Lambda,R),\varphi)$, which we denote by $\mathcal{D}_c(\Lambda)$. Obviously $\mathcal{D}_c(\Lambda)$ is a Hopf subalgebra of $\mathcal{D}(\Lambda)$. By construction one has the triangular decomposition

$$\mathcal{D}_c(\Lambda)=\mathbf{C}^<(\Lambda,R)\otimes\mathbf{T}\otimes\mathbf{C}^>(\Lambda,R)$$

where $\mathbf{C}^<(\Lambda,R)$ is the subalgebra generated by $u_i^-, i\in I$, $\mathbf{C}^>(\Lambda,R)$ generated by $u_i^+, i\in I$.

5.7 Of course the skew-Hopf pairing φ in (5.6) induces a skew-Hopf pairing

$$\varphi^*:\mathbf{C}^{*+}(\Lambda,\mathbf{Q}(v))\times\mathbf{C}^{*-}(\Lambda,\mathbf{Q}(v))\to\mathbf{Q}(v)$$

where $\mathbf{C}^{*+}(\Lambda,\mathbf{Q}(v))$ (resp. $\mathbf{C}^{*-}(\Lambda,\mathbf{Q}(v))$) is the generic composition algebra corresponding to $\mathbf{H}^+(\Lambda)$ (resp. $\mathbf{H}^-(\Lambda)$) (see (4.13)). φ^* is also a

member of $\mathcal{L}(C)$ over $\mathbf{Q}(v)$. Once more we have the reduced Drinfeld double of $(\mathbf{C}^{*+}(\Lambda, \mathbf{Q}(v)), \mathbf{C}^{*-}(\Lambda, \mathbf{Q}(v)),$
$\varphi^*)$ which we denote by $\mathcal{D}^*(\Lambda)$ and the triangular decomposition

$$\mathcal{D}^*(\Lambda) = \mathbf{C}^{*<}(\Lambda, \mathbf{Q}(v)) \otimes \mathbf{T} \otimes \mathbf{C}^{*>}(\Lambda, \mathbf{Q}(v))$$

where $\mathbf{C}^{*<}(\Lambda, \mathbf{Q}(v))$ (resp. $\mathbf{C}^{*>}(\Lambda, \mathbf{Q}(v))$) is the subalgebra of $\mathbf{C}^{*-}(\Lambda, \mathbf{Q}(v))$ (resp. $\mathbf{C}^{*+}(\Lambda, \mathbf{Q}(v))$) generated by u_i^- (resp. u_i^+) for $i \in I$.

One of the significant applications of Green-Ringel theory of Hall algebras is the following

5.8 Theorem (Green-Ringel) *Let $C = (I, (,))$ be a Cartan datum, $\mathbf{U}$ the Drinfeld-Jimbo quantum group over $\mathbf{Q}(v)$ of type $(I, (,))$. Let Λ be a finite dimensional hereditary algebra over a finite field k of type $(I, (,))$, $\mathbf{C}^*(\Lambda, \mathbf{Q}(v))$ the generic composition algebra of Λ, $\mathcal{D}^*(\Lambda)$ the reduced Drinfeld double of $(\mathbf{C}^{*+}(\Lambda, \mathbf{Q}(v)), \mathbf{C}^{*-}(\Lambda, \mathbf{Q}(v)), \varphi^*)$ defined in (5.7). Then the map $\mathcal{D}^*(\Lambda) \to \mathbf{U}$ by taking $u_i^+ \mapsto E_i$, $u_i^- \mapsto -\frac{|u_i|}{a_i}(v_i - v_i^{-1})F_i$, $K_i \mapsto K_i$ for all $i \in I$ induces an isomorphism as Hopf $\mathbf{Q}(v)$-algebras.*

Proof Since the skew-Hopf pairing φ^* is a member of $\mathcal{L}(C)$, in view of (3.4) and Theorem 3.6, we have the canonical homomorphisms of Hopf algebras $\phi^+ : \mathbf{C}^{*+}(\Lambda, \mathbf{Q}(v)) \to F^+$ and $\phi^- : \mathbf{C}^{*-}(\Lambda, \mathbf{Q}(v)) \to F^-$ where $F^\pm$ are the Lusztig algebras defined in (2.9). By [13] ,[14] or to see [16] , the quantum serre relations are zero elements in $\mathbf{C}^*(\Lambda)$. However $F^\pm = \tilde{F}^\pm/\mathcal{I}^\pm$, according to Lusztig [8], Theorem 33.1.3, the quantum Serre relations are just the generators of $\mathcal{I}^\pm$. So we have the canonical homomorphisms $\psi^+ : F^+ \to \mathbf{C}^{*+}(\Lambda, \mathbf{Q}(v))$ and $\psi^- : F^- \to \mathbf{C}^{*-}(\Lambda, \mathbf{Q}(v))$. Clearly the composition of ϕ^+ and ψ^+ (resp. ϕ^- and ψ^-) is the identity map, So $\phi^\pm$ are isomorphisms of Hopf algebras. Again from Lusztig [8], Theorem 33.1.3 it follows that we have the canonical isomorphisms $\mathbf{C}^{*+}(\Lambda, \mathbf{Q}(v)) \simeq \mathbf{U}(\mathbf{b}^+)$ and $\mathbf{C}^{*-}(\Lambda, !\mathbf{Q}(v)) \simeq \mathbf{U}(\mathbf{b}^-)$, w here $\mathbf{U}(\mathbf{b}^+)$ (resp.$\mathbf{U}(\mathbf{b}^-)$) is the quantized enveloping algebra of the positive (resp. negative) Borel subalgebra of the corresponding Kac-Moody Lie algebra, of course, they are subalgebras of $\mathbf{U}$. We have the following relations in $\mathcal{D}^*(\Lambda)$ and $\mathbf{U}$ respectively

$$u_j^- u_i^+ - u_i^+ u_j^- = \delta_{ij} \frac{|u_i|}{a_i}(K_i - K_i^{-1})$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}}$$

for all $i, j \in I$. Then

$$(-|u_j|a_j^{-1}(v_j - v_j^{-1}))F_j E_i - E_i(-|u_j|a_j^{-1}(v_j - v_j^{-1}))F_j = \delta_{ij}|u_i|a_i^{-1}(K_i - K_i^{-1})$$

The relations involved in K_i are easy to be confirmed. So we have the morphism $\phi : \mathcal{D}^*(\Lambda) \to \mathbf{U}$ sending u_i^+ to E_i, u_i^- to $-|u_i|a_i^{-1}(v_i - v_i^{-1})F_i$ and

K_i to K_i for all $i \in I$, which is comparible with the triangular decompositions (moreover, with the weight space decomposition). It must be an isomorphism.

5.8.1 Corollary *The skew-Hopf pairing* $(\mathbf{C}^{*+}(\Lambda, \mathbf{Q}(v)), \mathbf{C}^{*-}(\Lambda, \mathbf{Q}(v)), \varphi^*)$ *is a non-degenerate member of* $\mathcal{L}(C)$ *over* $\mathbf{Q}(v)$.

Proof It is immediate from Theorem 5.8 and Theorem 3.6.

5.9 In the rest of this section, we concentrate to the case $C = (I, (,))$ is a finite type Cartan datum, that is, the Dynkin graphs of $(I, (,))$ are classified into the types $A_n,\ B_n,\ C_n,\ D_n,\ E_i\ (i = 6, 7, 8),\ F_4,\ G_2$ and the form $(-,-)$ is positive definite. In this case Λ is a finite representation type hereditary algebra and the Hall algebra $\mathbf{H}(\Lambda)$ coincide with its composition algebra $\mathbf{C}(\Lambda)$. So we have

5.9.1 Proposition *Let* $C = (I, (,))$ *be a finite type Cartan datum,* Λ *a finite dimensional hereditary* k*-algebra of type* $(I, (,))$*. Then the Ringel pairing* $(\mathbf{H}^+(\Lambda), \mathbf{H}^-(\Lambda), \varphi)$ *is a non-degenerate member of* $\mathcal{L}(C)$ *and hence canonical isomorphic to the skew-Hopf pairing* (F^+, F^-, φ) *of the Lusztig algebra of type* $(I, (,))$ *defined in (2.9).*

Proof Since The Ringel pairing of a Hall algebra is clearly non-degenerate, the assertion is immediate from Proposition 2.9.1 and Theorem 3.6.

This result means that the reduced Drinfeld double $\mathcal{D}(\Lambda)$ of $\mathbf{H}(\Lambda)$ give us a complete realization of the quantum group of type $(I, (,))$.

5.9.2 Set $\operatorname{ind}\mathcal{P} = \{\alpha \in \mathcal{P} \mid u_\alpha \text{ is an indecomposable } \Lambda\text{-module }\}$, this just corresponds to the positive root system of $(I, (,))$ according to the well-known Gabriel theorem (see [4] and its generalization in [3]). There exists a total ordering in $\operatorname{ind}\mathcal{P}$, say $\alpha^1, \alpha^2, \cdots, \alpha^m$ such that $\operatorname{Hom}_\Lambda(u_{\alpha^i}, u_{\alpha^j}) \neq 0$ implies that $i \leq j$, in this case it holds that $\operatorname{Ext}_\Lambda(u_{\alpha^i}, u_{\alpha^j}) = 0$. For any $\alpha \in \mathcal{P}$, we may write $\alpha = \alpha(1)\alpha^1 \oplus \cdots \oplus \alpha(m)\alpha^m$ which corresponds to the Krull-Schmidt decompositions of u_α into direct sum of indecomposables. Set $(u_\alpha)^{(n)} = \frac{1}{[n]!_{\epsilon(\alpha)}}(u_\alpha)^n$ where $\epsilon(\alpha) = \dim_k \operatorname{End}_\Lambda u_\alpha$. The following proposition together with Proposition 5.5 gives an answer for the computation of commutators of root vectors in terms of Hall algebra.

5.9.3 Proposition (Ringel [15]) *Let* $\beta, \gamma \in \operatorname{ind}\mathcal{P}$ *with* $\operatorname{Hom}_\Lambda(u_\gamma, u_\beta) = 0$, $\operatorname{Ext}_\Lambda(u_\beta, u_\gamma) = 0$, *then*

$$u_\gamma^+ u_\beta^+ - v^{(\beta,\gamma)} u_\beta^+ u_\gamma^+ = v^{\langle\gamma,\beta\rangle} \sum_{\substack{\alpha \in \mathcal{P} \\ \alpha \neq \gamma \oplus \beta}} v^{\left(\sum_{1 \leq i \leq m} \alpha(i)\langle\alpha^i, \alpha^i\rangle - \sum_{1 \leq i \leq j \leq m} \alpha(i)\alpha(j)\langle\alpha^i, \alpha^j\rangle\right)} \times$$
$$g_{\gamma\beta}^\alpha (u_{\alpha^1}^+)^{(\alpha(1))} \cdots (u_{\alpha^m}^+)^{(\alpha(m))}$$

$$u_\gamma^- u_\beta^- - v^{(\beta,\gamma)} u_\beta^- u_\gamma^- = v^{\langle\gamma,\beta\rangle} \sum_{\substack{\alpha\in\mathcal{P}\\ \alpha\neq\gamma\oplus\beta}} v^{\left(\sum_{1\le i\le m}\alpha(i)\langle\alpha^i,\alpha^i\rangle - \sum_{1\le i\le j\le m}\alpha(i)\alpha(j)\langle\alpha^i,\alpha^j\rangle\right)} \times$$
$$g_{\gamma\beta}^{\alpha} (u_{\alpha^1}^-)^{(\alpha(1))} \cdots (u_{\alpha^m}^-)^{(\alpha(m))}$$

5.9.4 Lemma *Let M be an indecomposable Λ-module. If there are exact sequences of Λ-modules $0 \to A \to M \to B \to 0$ and $0 \to B' \to M \to A' \to 0$ with $\underline{\dim} A = \underline{\dim} A'$ and $\underline{\dim} B = \underline{\dim} B'$, then $A = A' = 0$ or $B = B' = 0$.*

Proof If $A \neq 0$ and $B \neq 0$, then $A' \neq 0$ and $B' \neq 0$. Let $\alpha = \underline{\dim} A = \underline{\dim} A'$. The sequences are non-split since M is indecomposable; we have $\mathrm{Hom}_\Lambda(A', A) = 0$ since M is a directing module. Then

$$\langle\alpha,\alpha\rangle = \langle \underline{\dim} A', \underline{\dim} A\rangle = \dim_k \mathrm{Hom}_\Lambda(A', A) - \dim_k \mathrm{Ext}_\Lambda(A', A) \le 0$$

a contradiction to the positivity of $\langle -,-\rangle$. The proof is finished.

5.9.5 Corollary *For any $\lambda \in \mathrm{ind}\,\mathcal{P}$, in $\mathcal{D}(\Lambda)$ we have*

$$u_\lambda^- u_\lambda^+ - u_\lambda^+ u_\lambda^- = \frac{|u_\lambda|}{a_\lambda}(K_\lambda + \sum_{m\ge 1}(-1)^m \sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} \times$$
$$a_{\lambda_1}\cdots a_{\lambda_m} a_\lambda^{-1} (g_{\lambda_1\cdots\lambda_m}^{\lambda})^2 K_{-\lambda})$$

More general we have

$$(u_\lambda^-)^{(N)}(u_\lambda^+)^{(M)} = \frac{|u_\lambda|}{a_\lambda}\sum_{t\ge 0}(u_\lambda^+)^{(M-t)}\prod_{s=1}^{t}(v^{-2t+N+M+s-1}K_\lambda - v^{-2t-N-M-s+1}\times$$
$$\sum_{m\ge 1}(-1)^m \sum_{\lambda_1,\cdots,\lambda_m\in\mathcal{P}_1} v^{2\sum_{i<j}\langle\lambda_i,\lambda_j\rangle} a_{\lambda_1}\cdots a_{\lambda_m} a_\lambda^{-1}(g_{\lambda_1\cdots\lambda_m}^{\lambda})^2 K_{-\lambda})(u_\lambda^-)^{(N-t)}$$

for any natural numbers $M, N \ge 1$.

Proof The first formula is immediate from Proposition 5.5 and Lemma 5.9.4. The second can be proved directly from the first by induction.

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