TRIANGULAR DECOMPOSITION OF THE COMPOSITION ALGEBRA OF THE KRONECKER ALGEBRA

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Abstract. Let A be a finite dimensional hereditary algebra over a finite field, $\mathcal{H}(A)$ and $\mathcal{C}(A)$ be respectively the Ringel-Hall algebra and the composition algebra of A. Define $r_{\mathbf{d}}$ to be the element $\sum[M] \in \mathcal{H}(A)$, where $[M]$ runs over the isomorphism classes of the regular $A-$modules with dimension vector $\mathbf{d}$. We prove that $r_{\mathbf{d}}$ and the exceptional $A-$modules all lie in $\mathcal{C}(A)$. Let K be the Kronecker algebra, $\mathcal{P}$ (resp. $\mathcal{I}$) the subalgebra of $\mathcal{C}(K)$ generated by the preprojective (resp. preinjective) $K-$modules, and $\mathcal{T}$ the subalgebra generated by $r_{(n,n)}$ for $n \geq 0$, then we prove that $\mathcal{C}(K) = \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$ and then $\mathcal{T}$ is just the subalgebra of $\mathcal{C}(K)$ generated by all regular elements.

0. Introduction and Preliminaries.

Throughout this paper A is assumed to be a finite dimensional hereditary k-algebra, where k is a finite field with q elements. All modules are finite dimensional over k (hence finite as sets). For the representation theory of finite dimensional hereditary algebras we refer to [R1].

By the recent works of C.M.Ringel [R3-9] and J.A.Green [G, Theorem 2] we know that Ringel's twisted generic composition algebra $\mathcal{C}_*(A)$ is isomorphic to G. Lusztig's algebra $\mathbf{f}$ (and hence isomorphic to the positive part U^+ of the Drinfeld-Jimbo quantization $U = U^- \otimes U^0 \otimes U^+$ associated to a Kac-Moody Lie algebra of type $(I, \cdot)$ ([L], chap.33), where $(I, \cdot)$ is the Cartan datum associated to A). This remarkable result provides the Ringel-Hall algebra approach to quantum group and makes it possible that results in the representations of quiver have applications or interpretations in quantum group.

To be simple, we consider the untwisted case (all considerations remain true under the twisted multiplication introduced in [R7]). Let $\mathcal{H}(A)$ and $\mathcal{C}(A)$ denote the Ringel-Hall algebra and Ringel's composition algebra of A respectively. By definition $\mathcal{H}(A)$ is the $\mathbb{Q}$-vector space with basis the set of isomorphism classes $[X]$ of finite modules, with the multiplication

$$[N_1] \cdot [N_2] := \sum_{[M]} F^M_{N_1,N_2}[M]$$

where $F^M_{N_1,N_2}$ is the number of the submodule N of M with $N \cong N_2$ and $M/N \cong N_1$. Then $\mathcal{H}(A)$ is an associative algebra with the identity $[0]$. And $\mathcal{C}(A)$ is by definition the subalgebra of $\mathcal{H}(A)$ generated by the elements $[S_i]$ for $1 \leq i \leq n$, where $S_i (1 \leq i \leq n)$ are all simple $A-$modules (up to isomorphism).

Given an isomorphism class $[M]$, we assign to its degree $\mathbf{dim}M \in \mathbb{Z}^n$, where $(\mathbf{dim}M)_i$ is the Jordan-Hölder multiplicity of S_i in M. Let $\mathcal{H}(A)_{\mathbf{d}}$ be the free abelian group with basis the set of isomorphism class $[M]$ with $\mathbf{dim}M = \mathbf{d}$, and $\mathcal{C}(A)_{\mathbf{d}} = \mathcal{H}(A)_{\mathbf{d}} \cap \mathcal{C}(A)$, then $\mathcal{H}(A) = \bigoplus_{\mathbf{d}} \mathcal{H}(A)_{\mathbf{d}}$ and $\mathcal{C}(A) = \bigoplus_{\mathbf{d}} \mathcal{C}(A)_{\mathbf{d}}$ are both $\mathbb{Z}^n$-graded ring. Let $\mathbf{d} = (d_1, \cdots, d_n) \in \mathbb{Z}^n$ and $d := \sum_i d_i$, then $\mathcal{C}(A)_{\mathbf{d}}$ is the $\mathbb{Q}$-subspace of $\mathcal{H}(A)$ spanned by all monomials $[S_{i_1}] \cdots [S_{i_d}]$, such that the number of occurrences of i in the sequence $i_1, \cdots, i_n$ is just d_i for $1 \leq i \leq n$. In this way we get all elements in $\mathcal{C}(A)$.

If A is representation-finite, Ringel has proved that $\mathcal{C}(A)$ coincides with $\mathcal{H}(A)$ ([R4], Prop.6); otherwise, $\mathcal{C}(A)$ is a proper subalgebra of $\mathcal{H}(A)$. In this case it is then of interest to ask which kinds of isomorphism

This work was done in the Universität Bielefeld, and supported by the Alexander von Humboldt Foundation

classes $[M]$, or their combinations, lie in the composition algebra: one of the advantages of the Ringel-Hall algebra approach to quantum group is that it has a natural basis. We present two results towards this direction in Section 1, see Theorem 1 and 2 below.

Since the preprojective and preinjective modules lie in $\mathcal{C}(A)$, it is natural to consider the regular modules. In Section 2 we use the notation of the regular part to obtain more regular elements in $\mathcal{C}(K_m)$ (Theorem 3), where K_m is the generalized Kronecker algebra.

Finally we apply these results to determine the structure of $\mathcal{C}(K)$ (Theorem 4), where K is the Kronecker algebra. It turns out that $\mathcal{C}(K)$ shares a triangular decomposition (this may be an analogy to the one of the Kac-Moody Lie algebra $A_2^{(1)}$ and the quantum group $U(A_2^{(1)})$). From this a PBW-basis of $\mathcal{C}(K)$ can be explicitly written out.

I would like to thank Professor C. M. Ringel for his helpful conversations and hospitality. I have benefitted from his ideas, in particular, Theorem 2 is due to him. I am indebted to the Alexander von Humboldt Foundation for the support.

1. Exceptional modules.

An indecomposable $A-$module M is said to be preprojective (resp. preinjective; regular) provided that $\tau^m M = 0$ for some $m > 0$ (resp. $\tau^m M = 0$ for some $m < 0$; $\tau^m M \neq 0$ for all $m \in \mathbb{Z}$), where τ is the Auslander-Reiten translate. An $A-$module is said to be preprojective (resp. preinjective; regular) provided that each indecomposable direct summand is so.

For $\mathbf{d} \in \mathbb{N}_0^n$, denote by $r_{\mathbf{d}}$ the element $\sum_{[M]} [M] \in \mathcal{H}(A)$, where $[M]$ runs over the isomorphism classes of the regular modules with $\mathbf{dim} M = \mathbf{d}$ (since A is finitary (see [R2]), this sum is of course a finite sum). Note that M may be decomposable; and if there exists no regular module M with $\mathbf{dim} M = \mathbf{d}$, then $r_{\mathbf{d}} := [0]$.

Theorem 1. (i) The preprojective and preinjective $A-$modules lie in the composition algebra $\mathcal{C}(A)$.

(ii) For $\mathbf{d} \in \mathbb{N}_0^n$, $r_{\mathbf{d}} \in \mathcal{C}(A)$.

In order to prove Theorem 1 we need the following two lemmas, which will also be used later.

Lemma 1. Let $\mathbf{d} \in \mathbb{N}_0^n$.

(i) If $\mathbf{d} = \mathbf{dim} P$ with P an indecomposable preprojective A-modules, then $r_{\mathbf{d}} = [0]$, i.e. there is no regular module M with $\mathbf{dim} M = \mathbf{d}$.

(ii) If $\mathbf{d} = \mathbf{dim} I$ with I an indecomposable preinjective A-modules, then $r_{\mathbf{d}} = [0]$.

Proof. (i) Let $< -, - >$ be the Ringel form, i.e. $< \mathbf{dim} X, \mathbf{dim} Y >= dim_k Hom_A(X, Y) - dim_k Ext^1_A(X, Y)$ for $X, Y \in A-mod$. If there exists a regular module M with $\mathbf{dim} M = \mathbf{d}$, then we get a contradiction:

$$\begin{aligned} 1 &\leq < \mathbf{dim} P, \mathbf{dim} P > = < \mathbf{dim} M, \mathbf{dim} P > \\ &= dim_k Hom(M, P) - dim_k Ext^1(M, P) \\ &= -dim_k Ext^1(M, P). \end{aligned}$$

(ii) The dual of (i). $\square$

Let $\mathbf{x} = (x_1, \cdots, x_n)$, $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{N}_0^n$, we define $\mathbf{x} \prec \mathbf{y}$ if and only if $x_i \leq y_i$, $\sum_{i=1}^n x_i < \sum_{i=1}^n y_i$. This gives a partial order in $\mathbb{N}_0^n$. For $\mathbf{d} = (d_1, \cdots, d_n) \in \mathbb{N}_0^n$, denote by $\prec_{\mathbf{d}}$ the subset of $\mathbb{N}_0^n$ of elements $\mathbf{x} \prec \mathbf{d}$.

We fix an orientation of the simple A-modules such that $Ext_A^1(S_i, S_j) = 0$ for $1 \le i \le j \le n$. Note that $End_A S_i$ is again a field with q_i elements, where $q_i = q^{dim_k End_A S_i}$, $1 \le i \le n$. Let $\psi_m(T)$ be the polynomial $\frac{(1-T)\cdots(1-T^m)}{(1-T)^m}$.

Lemma 2. Let $\mathbf{d} = (d_1, \cdots, d_n) \in \mathbb{N}_0^n$.
(i) If $\mathbf{d} = \mathbf{dim}P$ with P an indecomposable preprojective A-modules, then

$$[S_n]^{d_n} \cdots [S_1]^{d_1} = \psi_{d_n}(q_n) \cdots \psi_{d_1}(q_1)([P] + \sum_{[P'],[I'],\mathbf{d}'} [P'] \cdot r_{\mathbf{d}'} \cdot [I']) \tag{1}$$

where P' and I' run over the isomorphism classes of preprojective and preinjective modules respectively (one of P' and I' may be zero), and $\mathbf{d}'$ runs over the set $\prec_{\mathbf{d}}$ such that

$$\mathbf{dim}P' + \mathbf{dim}I' + \mathbf{d}' = \mathbf{d}. \tag{2}$$

(ii) If $\mathbf{d} = \mathbf{dim}I$ with I an indecomposable preinjective A-modules, then

$$[S_n]^{d_n} \cdots [S_1]^{d_1} = \psi_{d_n}(q_n) \cdots \psi_{d_1}(q_1)([I] + \sum_{[P'],[I'],\mathbf{d}'} [P'] \cdot r_{\mathbf{d}'} \cdot [I']) \tag{1'}$$

where P', I', $\mathbf{d}'$ are same as in (i) and satisfy the dimensional equality (2).
(iii) If $\mathbf{d} = \mathbf{dim}M$ with M a regular A-module (may be decomposable), then

$$[S_n]^{d_n} \cdots [S_1]^{d_1} = \psi_{d_n}(q_n) \cdots \psi_{d_1}(q_1)(r_{\mathbf{d}} + \sum_{[P'],[I'],\mathbf{d}'} [P'] \cdot r_{\mathbf{d}'} \cdot [I']) \tag{3}$$

where P', I', $\mathbf{d}'$ are same as in (i) and satisfy the dimensional equality (2).

Proof. Write $[S_n]^{d_n} \cdots [S_1]^{d_1}$ as the combination of the natural basis $[N]$ where N runs over the isomorphism classes of finite A-modules, say,

$$[S_n]^{d_n} \cdots [S_1]^{d_1} = \psi_{d_n}(q_n) \cdots \psi_{d_1}(q_1)[d_n S_n] \cdots [d_1 S_1] = \sum_{[N]} a_N [N].$$

Note that $[S_i]^m = \psi_m(q_i)[mS_i]$, see [R4]. Then we have the following two observations: (a) if $a_N \neq 0$, then it is $\psi_{d_n}(q_n) \cdots \psi_{d_1}(q_1)$, since $Hom(d_i S_i, d_j S_j) = 0$; (b) $a_N \neq 0$ if and only if $\mathbf{dim}N = \mathbf{d}$, since by assumption $Ext^1(S_i, S_j) = 0$ for $i \le j$. It follows that

$$[S_n]^{d_n} \cdots [S_1]^{d_1} = \psi_{d_n}(q_n) \cdots \psi_{d_1}(q_1) \sum_{\mathbf{dim}N = \mathbf{d}} [N].$$

Now, if $\mathbf{d} = \mathbf{dim}P$ with P an indecomposable preprojective A-modules, rewriting the sum in the right hand of the equality above, we then obtain the formula (1). By Lemma 1(i) we see that $\mathbf{d}' \prec \mathbf{d}$ in (1). This proves the case (i), and the others are similarly proved. □

Proof of Theorem 1.
Use induction. Let $\mathbf{d} \in \mathbb{N}_0^n$. Note that in order to prove (i), it suffices to consider the indecomposables.
If $\mathbf{d}$ is the dimension vector of an indecomposable preprojective module P. Applying the formula (1) in Lemma 2. Since $\mathbf{d}' \prec \mathbf{d}$ in (1), we have $r_{\mathbf{d}'} \in \mathcal{C}(A)$ by induction. If all $\mathbf{dim}P', \mathbf{dim}I' \prec \mathbf{d}$, then by induction we see $[P] \in \mathcal{C}(A)$. If there exist some P' or I' with $\mathbf{dim}P' = \mathbf{d}$ or $\mathbf{dim}I' = \mathbf{d}$, then such P' or I' must be decomposable since P is uniquely determined by its dimension vector, and then again by induction $[P] \in \mathcal{C}(A)$.
If $\mathbf{d}$ is the dimension vector of an indecomposable preinjective module, then by the dual arguments the assertion is proved.

Finally, if $r_{\mathbf{d}} \neq [0]$, then use the formula (3) in Lemma 2 and the similar arguments as above we see that $r_{\mathbf{d}} \in \mathcal{C}(A)$. This completes the proof. $\square$

Recall that an indecomposable $A-$module M is said to be exceptional provided $Ext^1_A(M,M) = 0$. If $E = (M_1, \cdots, M_m)$ is an exceptional sequence (i.e. every M_i is exceptional and $Hom_A(M_i, M_j) = 0 = Ext^1_A(M_i, M_j)$ for $i > j$.), let $\mathcal{C}(E)$ be the smallest full subcategory of $A - mod$ which contains all M_i and is closed under extensions, kernels of epis, and cokernels of monos, then $\mathcal{C}(E)$ is equivalent to the representation category of a finite dimensional hereditary algebra, see Crawley-Boevey [CB].

Now we present a proof of Ringel's theorem (in private communication).

Theorem 2. (C.M.Ringel) Assume that M is an exceptional $A-$module, then $[M] \in \mathcal{C}(A)$.

Proof. Assume that M is a non-simple exceptional $A-$module. Then by Lemma 4 in [R8], there exists an exceptional sequence (M, N) such that M is not simple in $\mathcal{C}(M,N)$. Assume that $\mathcal{C}(M,N) \simeq B - mod$, where B is a finite dimensional hereditary k-algebra with two simple modules. Note that M is preprojective or preinjective as a B-module, so, by Theorem 1(i) we see that $[M]$ lies in the composition algebra of $\mathcal{C}(M,N)$. Let M_1 and M_2 be the two simple objects in $\mathcal{C}(M,N)$, then M_1, M_2 are both exceptional A-modules, note that M_1 and M_2 both have smaller lengths than M, by induction $[M_1]$, $[M_2] \in \mathcal{C}(A)$, so does $[M]$. This completes the proof. $\square$

2. Regular elements in $\mathcal{C}(K_m)$.

We first introduce the following

Definition. Let $X = \sum\limits_{[M]} a_M[M] \in \mathcal{H}(A)$. Then $r(X) := \sum\limits_{\substack{[M] is \\ regular}} a_M[M]$ is called the regular part of X. If $X = r(X)$, then X is called a regular element in $\mathcal{H}(A)$.

For example, $r([P] \cdot X) = 0 = r(X \cdot [I])$ for (non-zero) preprojective (resp. preinjective) A-module P (resp. I), and arbitrary element $X \in \mathcal{H}(A)$. It follows from Lemma 2(iii) that

Proposition 1. With the same assumptions as in Lemma 2(iii), we have

$$r([S_n]^{d_n} \cdots [S_1]^{d_1}) = \psi_{d_n}(q_n) \cdots \psi_{d_1}(q_1) r_{\mathbf{d}}.$$

In the following we consider the generalized Kronecker algebra K_m with $m \geq 2$ (i.e. K_m is the path algebra of the quiver with two vertices, say 1 and 2, and with m arrows from 2 to 1), and in particular the Kronecker algebra $K = K_2$. However, all considerations can be automatically generalized to the case of the bimodule algebras.

We denote by P_i (resp. I_i) be the i-th indecomposable preprojective (resp. preinjective) module in the preprojective (resp. preinjective) component of the Auslander-Reiten quiver of K_m (thus, P_1 is the simple projective module S_1, and I_1 is the simple injective module S_2). For the structure of the Kronecker modules (i.e. the modules over the Kronecker algebra) we refer to [R1, pp.122-126]; we shall often use the fact that the types of the dimension vectors of the indecomposable preprojective (resp. preinjective, regular) K-modules are $(i, i-1)$ (resp. $(i-1, i)$, (i,i)) for $i > 1$.

We denote the element $r_{(n,n)} \in \mathcal{C}(K)$ (in the sense of Section 1) by r_n. Let $\mathcal{T}$ be the subalgebra of $\mathcal{C}(K)$ generated by $r_n, n \geq 0$ ($r_0 := [0]$); then $\mathcal{T}$ is commutative (this follows a result due to P.H.Hall (see [M]): the Hall algebra of the polynomial algebra $k[x]$ is commutative, or equivalently, the Hall algebra of a homogeneous tube is commutative). Let $\mathcal{T}_n$ be the $\mathbb{Q}$-subspace of $\mathcal{T}$ spanned by all the monmials $r_{i_1}^{t_1} \cdots r_{i_s}^{t_s}$ with $1 \leq i_1 < \cdots < i_s \leq n$ and $i_1 t_1 + \cdots + i_s t_s = n$. Then $\mathcal{T} = \bigoplus\limits_n \mathcal{T}_n$ is a $\mathbb{N}_0$-graded commutative algebra. Note that every element in $\mathcal{T}$ is a regular element of $\mathcal{C}(K)$, moreover, we shall see later that $\mathcal{T}$ is exactly the subalgebra of $\mathcal{C}(K)$ generated by all regular elements in $\mathcal{C}(K)$.

Given elements x, y, c_1, c_2 in a ring R with c_1, c_2 central in R, we call the element $c_1xy - c_2yx$ a skew commutator of x and y (if $c_1 = c_2 = 1$, it is usually written as $[x, y]$, the commutator of x and y).

Theorem 3. Let K_m be the generalized Kronecker algebra with $m \geq 2$.

(i) We have

$$[I_j] \cdot [P_i] = r([I_j] \cdot [P_i]) + q^{d_{ij}}[P_i] \cdot [I_j] \tag{7}$$

where $d_{ij} :=< \mathbf{dim}P_i, \mathbf{dim}I_j >= dim_k Hom_{K_m}(P_i, I_j)$. (Thus, the regular part of the product $[I_j] \cdot [P_i]$ is a skew commutator of $[P_i]$ and $[I_j]$).

(ii) If $m = 2$ (i.e. for the Kronecker algebra K, in this case $d_{ij} = i + j - 2$), let $i + j = n + 2$. Then

$$\begin{aligned} r([I_j] \cdot [P_i]) &= r([I_{n+1}] \cdot [S_1]) \\ &= \frac{1 - q^{n+1}}{1 - q} r_{n+1} + \sum_{1 \leq t \leq n} (-1)^t \sum_{1 \leq m \leq n+1-t} \frac{1 - q^m}{1 - q} r_m \cdot_t r_{n+1-m} \end{aligned} \tag{8}$$

with ${}_tr_m := \sum\limits_{i_1, \cdots, i_t} r_{i_1} \cdot r_{i_2} \cdots r_{i_t}$ where $(i_1, i_2, \cdots, i_t) \in \mathbb{N}^t$ takes over the following possibilities: $1 \leq i_1, i_2, \cdots, i_t \leq n$; $i_1 + i_2 + \cdots i_t = m$.

For examples, we have
$r([I_2] \cdot [S_1]) = (1 + q)r_2 - r_1^2$; $r([I_3] \cdot [S_1]) = (1 + q + q^2)r_3 - (q + 2)r_1 \cdot r_2 + r_1^3$; $r([I_4] \cdot [S_1]) = (1 + q + q^2 + q^3)r_4 - (2 + q + q^2)r_1 \cdot r_3 + (3 + q)r_1^2 \cdot r_2 - (1 + q)r_2^2 - r_1^4$, etc.

The proof of Theorem 3 will occupies the rest of this section. First, we collect some counting formulae from Green [G].

Let Λ be arbitrary finitary ([R2]) k-algebra, N_1, N_2, M be finite Λ-modules. Denote by $\mathcal{E}^M_{N_1,N_2}$ the set of pairs (f, g) of Λ-homomorphisms such that $0 \longrightarrow N_2 \xrightarrow{f} M \xrightarrow{g} N_1 \longrightarrow 0$ is an exact sequence; and by a_X the order of automorphism group of Λ-module X.

Proposition 2. ([G, 2.3(a)]) We have $F^M_{N_1,N_2} = \frac{|\mathcal{E}^M_{N_1,N_2}|}{a_{N_1}a_{N_2}}$.

By (2.7(a)) and (2.3(b)) in [G] we get

Proposition 3. Let $\eta = 0 \longrightarrow N_2 \longrightarrow M \longrightarrow N_1 \longrightarrow 0$ be an extension class in $Ext^1_\Lambda(N_1, N_2)$, then the extension class η contains $\frac{a_M}{|Hom_\Lambda(N_1,N_2)|}$ exact sequences.

Denote by $Ext^1_\Lambda(N_1, N_2)_M$ the set of all extension classes in $Ext^1_\Lambda(N_1, N_2)$ with middle term M, see [Rm], then we get the following direct consequence of Proposition 2 and 3 (see also [P]).

Corollary 1. We have $F^M_{N_1,N_2} = \frac{|Ext^1_\Lambda(N_1,N_2)_M| a_M}{a_{N_1}a_{N_2}|Hom_\Lambda(N_1,N_2)|}$.

Corollary 2. Assume that $dim_k Ext^1_\Lambda(N_1, N_2) = 1$, and that $\eta = 0 \longrightarrow N_2 \xrightarrow{f} M \xrightarrow{g} N_1 \longrightarrow 0$ is a non-split exact sequence. Then

$$[N_1] \cdot [N_2] = \frac{(q - 1)a_M}{a_{N_1}a_{N_2}|Hom_\Lambda(N_1, N_2)|}[M] + \frac{a_{N_1 \oplus N_2}}{a_{N_1}a_{N_2}|Hom_\Lambda(N_1, N_2)|}[N_1 \oplus N_2].$$

Proof. By definition we see that for $0 \neq c \in k$

$$c\eta = 0 \longrightarrow N_2 \xrightarrow{c^{-1}f} M \xrightarrow{g} N_1 \longrightarrow 0.$$

First, let $M \not\cong N_1 \oplus N_2$, then $|Ext^1_\Lambda(N_1, N_2)_M| = q - 1$, $|Ext^1_\Lambda(N_1, N_2)_{N_1 \oplus N_2}| = 1$, and the assertion follows from Corollary 1. Note that in case $M \cong N_1 \oplus N_2$, the equality also holds. $\square$

The following Lemma is well-known (see e.g. [R2, 4]).

Lemma 3. (i) Let M be an indecomposable Λ-module with $dim_k End_\Lambda M = n, dim_k rad End_\Lambda M = m$. Then $a_M = (q^{n-m} - 1)q^m$.

(ii) Let $M = c_1 M_1 \oplus \cdots \oplus c_t M_t$ such that $M_i \not\cong M_j$ for $i \neq j$. Then $a_M = q^m a_{c_1 M_1} \cdots a_{c_t M_t}$, where $m = \sum\limits_{i \neq j} c_i c_j dim_k Hom_\Lambda(M_i, M_j)$.

(iii) Let $M = cN$ with $End_\Lambda N = k'$ a field. Then $a_M = |GL_c(k')| = \prod\limits_{1 \leq i \leq c} (d^c - d^{i-1})$, where $d = |k'| = q^{[k':k]}$.

For the homogeneous tube $T(\rho)$ of K, where $\rho \in \mathbb{P}_1 k$, the projective line, let $X_i(\rho)$, or briefly X_i be the indecomposable module with quasi-length i in $T(\rho)$. Then $r_1 = \sum\limits_{\rho \in \mathbb{P}_1 k} X_1(\rho)$. The following Lemma also means that the indecomposable preprojective and preinjective K-modules can be written as iterated skew commutators of the simples.

Lemma 4. For the Kronecker modules, we have

(i) $r_1 = [[S_2], [S_1]]$.

(ii) $[P_{m+1}] = \frac{1}{q+1}(r_1 \cdot [P_m] - q[P_m] \cdot r_1)$.

(iii) $[I_{n+1}] = \frac{1}{q+1}([I_n] \cdot r_1 - q r_1 \cdot [I_n])$.

Proof. (i) $\forall \rho \in \mathbb{P}_1 k$, there exists an exact sequence

$$0 \longrightarrow S_1 \longrightarrow X_1(\rho) \longrightarrow S_2 \longrightarrow 0$$

note also that $F_{S_2, S_1}^{X_1(\rho)} = 1 = F_{S_2, S_1}^{S_1 \oplus S_2}$ since $Hom_K(S_1, S_2) = 0$. It follows that $[S_2] \cdot [S_1] = r_1 + [S_1] \cdot [S_2]$.

(ii) Assume that $m \geq 1$. Let f be a nonzero element in $Hom_K(P_{m+1}, X_1(\rho))$, then f is epic (otherwise, the length $l(Imf) = 1$, this forces Imf to be simple, but Imf can not be the simple injective module S_2 since $Hom_k(S_2, X_1(\rho)) = 0$ and also Imf can not be the simple projective module S_1 since $Hom_K(P_{m+1}, S_1) = 0$); note that $kerf$ should be preprojective with dimension vector $(m, m-1)$, so we have an exact sequence

$$0 \longrightarrow P_m \longrightarrow P_{m+1} \longrightarrow X_1(\rho) \longrightarrow 0$$

it follows from Corollary 2 and Lemma 3 that for every $\rho \in \mathbb{P}_1 k$ there holds

$$[X_1(\rho)] \cdot [P_m] = [P_{m+1}] + q[P_m] \cdot [X_1(\rho)].$$

By taking sum for ρ over $\mathbb{P}_1 k$ we then get (ii).

(iii) The dual of (ii). $\square$

Remark. From the proof of Theorem 2 and Lemma 4 above we see that exceptional modules over a tame hereditary algebra can be written as iterated skew commutators of the simples.

Lemma 5. Assume that $0 \longrightarrow P_i \xrightarrow{\sigma} X \longrightarrow I_j \longrightarrow 0$ be a non-split exact sequence in $K_m - mod$. Then we have

(i) The middle term X is regular; moreover,

$$[I_j] \cdot [P_i] = r([I_j] \cdot [P_i]) + q^{d_{ij}} [P_i] \cdot [I_j].$$

(ii) If $m = 2$, then for each $\rho \in \mathbb{P}_1 k$, X has at most one indecomposable direct summand belonging to $T(\rho)$ (in particular, X is multiplicity-free).

Proof. (i) Let $X = P \oplus R \oplus I$, where P is preprojective, R regular and I preinjective. If $P \neq 0$, then we may assume that $X = (\bigoplus\limits_{t \geq 0} a_t P_{i+t}) \oplus R \oplus I$ since $Hom_{K_m}(P_i, P_{i-t}) = 0$ for $t > 0$. Applying $Hom_{K_m}(I_j, -)$ to the above exact sequence we get an epic morphism

$$Ext^1_{K_m}(I_j, P_i) \longrightarrow Ext^1_{K_m}(I_j, X) \longrightarrow 0$$

then we obtain the inequality

$$dim_k Ext^1_{K_m}(I_j, P_i) \geq \sum_{t\geq 0} a_t \cdot dim_k Ext^1_{K_m}(I_j, P_{i+t}) + dim_k Ext^1_{K_m}(I_j, R).$$

Since $dim_k Ext^1_{K_m}(I_j, P_i) < dim_k Ext^1_{K_m}(I_j, P_{i+t})$ for $t > 0$, we have $a_t = 0$ for $t > 0$, $a_0 = 1$ and $R = 0$. So, $X = P_i \oplus I_j$. Since $End_{K_m} P_i = k$, we see that the given exact sequence splits which contradicts the assumption. This proves that X is regular. It follows that

$$[I_j] \cdot [P_i] = r([I_j] \cdot [P_i]) + F^{P_i \oplus I_j}_{I_j, P_i}[P_i \oplus I_j] = r([I_j] \cdot [P_i]) + q^{d_{ij}}[P_i] \cdot [I_j].$$

(ii) Assume that $m = 2$. Let $\rho \in \mathbb{P}_1 k$. Applying $Hom_K(X_1(\rho), -)$ to the given exact sequence we then get a monic morphism

$$0 \longrightarrow Hom_K(X_1(\rho), X) \longrightarrow Hom_K(X_1(\rho), I_j)$$

Let $X = (\bigoplus_t a_{t\rho} X_t(\rho)) \oplus X'$, where X' has no summand lying in $T(\rho)$. Compare the $k-$dimensions we get $\sum_t a_{t\rho} \leq 1$, which means that there is at most one $a_{t\rho} = 1$, and $a_{s\rho} = 0$ for $s \neq t$. This completes the proof. □

Proof of Theorem 3.

It remains to prove (8) in (ii). So, in the following we always deal with the Kronecker modules. Assume that the assertion holds for $i + j \leq n + 1$. Let $i + j = n + 2$. First, we consider $r([I_{n+1}] \cdot [S_1])$. By the formula (1') in Lemma 2 we have

$$[S_2]^{n+1} \cdot [S_1]^n = \psi_{n+1}(q)\psi_n(q)([I_{n+1}] + \sum_{1\leq l\leq n} r_l \cdot [I_{n+1-l}] + \sum_{P,l,I} [P] \cdot r_l \cdot [I])$$

where in the last sum P, I run over the (non-zero) preprojective and preinjective K-modules respectively, and l runs over the set $\{0, \cdots, n-1\}$ such that $\mathbf{dim}P + \mathbf{dim}I + (l, l) = (n, n+1)$. Taking the regular part of the both sides of $[S_2]^{n+1} \cdot [S_1]^{n+1} = ([S_2]^{n+1} \cdot [S_1]^n) \cdot [S_1]$ we get

$$\begin{aligned}\psi^2_{n+1}(q) r_{n+1} &= r([S_2]^{n+1} \cdot [S_1]^{n+1}) = r(([S_2]^{n+1} \cdot [S_1]^n) \cdot [S_1]) \\ &= \psi_{n+1}(q)\psi_n(q)(r([I_{n+1}] \cdot [S_1]) + \sum_{1\leq l\leq n} r(r_l \cdot [I_{n+1-l}] \cdot [S_1])).\end{aligned}$$

It follows from Lemma 5 that

$$\begin{aligned}&r(r_l \cdot [I_{n+1-l}] \cdot [S_1]) \\ &= r(r_l \cdot r([I_{n+1-l}] \cdot [S_1]) + q^{n-l} r_l \cdot [S_1] \cdot [I_{n+1-l}]) \\ &= r_l \cdot r([I_{n+1-l}] \cdot [S_1]).\end{aligned}$$

Thus, using induction we get

$$\begin{aligned}
r([I_{n+1}]\cdot[S_1]) &= \frac{1-q^{n+1}}{1-q}r_{n+1} - \sum_{1\le l\le n} r_l\cdot r([I_{n+1-l}]\cdot[S_1]) \\
&= \frac{1-q^{n+1}}{1-q}r_{n+1} - \sum_{1\le l\le n}\frac{1-q^{n-l+1}}{1-q}r_l\cdot r_{n-l+1} \\
&\quad - \sum_{1\le l\le n} r_l \sum_{1\le t\le n-l}(-1)^t\sum_{m=1}^{n+1-l-t}\frac{1-q^m}{1-q}r_m\cdot_t r_{n+1-m-l} \\
&= \frac{1-q^{n+1}}{1-q}r_{n+1} - \sum_{1\le m\le n}\frac{1-q^m}{1-q}r_m\cdot r_{n-m+1} \\
&\quad + \sum_{1\le t\le n-1}(-1)^{t+1}\sum_{1\le m\le n-t}\frac{1-q^m}{1-q}r_m\sum_{l=1}^{n+1-m-t} r_l\cdot_t r_{n+1-m-l} \\
&= \frac{1-q^{n+1}}{1-q}r_{n+1} - \sum_{1\le m\le n}\frac{1-q^m}{1-q}r_m\cdot r_{n-m+1} \\
&\quad + \sum_{2\le t\le n}(-1)^t\sum_{1\le m\le n+1-t}\frac{1-q^m}{1-q}r_m(\sum_{l=1}^{n+2-m-t} r_l\cdot_{t-1} r_{n+1-m-l}) \\
&= \frac{1-q^{n+1}}{1-q}r_{n+1} + \sum_{1\le t\le n}(-1)^t\sum_{1\le m\le n+1-t}\frac{1-q^m}{1-q}r_m\cdot_t r_{n+1-m}.
\end{aligned}$$

It remains to prove that for $i+j=n+2$ there holds

$$r([I_j]\cdot[P_i]) = r([I_{n+1}]\cdot[S_1]).$$

Using Lemma 4 we have

$$\begin{aligned}
[I_j]\cdot[P_i] &= [I_j](\frac{1}{q+1}r_1\cdot[P_{i-1}] - \frac{q}{q+1}[P_{i-1}]\cdot r_1) \\
&= \frac{1}{q+1}((q+1)[I_{j+1}] + qr_1\cdot[I_j])[P_{i-1}] - \frac{q}{q+1}[I_j]\cdot[P_{i-1}]\cdot r_1 \\
&= [I_{j+1}]\cdot[P_{i-1}] + \frac{q}{q+1}[r_1,[I_j]\cdot[P_{i-1}]].
\end{aligned}$$

Applying Lemma 5 and comparing the regular parts of the above equality, we get (note that $r(X\cdot[I]) = 0 = r([P]\cdot X)$ for any $X\in\mathcal{H}(K)$, P preprojective and I preinjective)

$$r([I_j]\cdot[P_i]) = r([I_{j+1}]\cdot[P_{i-1}]) + \frac{q}{q+1}[r_1, r([I_j]\cdot[P_{i-1}])].$$

By induction $r([I_j]\cdot[P_{i-1}]) = r([I_n]\cdot[S_1])\in\mathcal{T}_n$. Since $\mathcal{T}$ is commutative we get $r([I_j]\cdot[P_i]) = r([I_{j+1}]\cdot[P_{i-1}])$. Repeat this process we have $r([I_j]\cdot[P_i]) = r([I_{n+1}]\cdot[S_1])\in\mathcal{T}_{n+1}$. This completes the proof. □

3. Triangular decomposition for $\mathcal{C}(K)$.

Let $\mathcal{P}$ (resp. $\mathcal{I}$) the subalgebra of $\mathcal{C}(K)$ generated by all isomorphism classes of the finite indecomposable preprojective (resp. preinjective) $K-$modules. Denote by $\mathcal{P}\cdot\mathcal{T}\cdot\mathcal{I}$ the set of elements which are $\mathbb{Q}$-span of $[P]\cdot T\cdot[I]$ with P (resp. I) preprojective (resp. preinjective), $T\in\mathcal{T}$. The aim of this section is to complete the proof of the following theorem.

Theorem 4. Let K be the Kronecker algebra, then $\mathcal{C}(K) = \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$.

Lemma 6. For $m, n \geq 1$, we have

$$[I_m] \cdot r_n = \frac{1-q^{n+1}}{1-q} \sum_{0 \leq i \leq n} r_i \cdot [I_{m+n-i}] - \sum_{i,j} r([S_2] \cdot [P_i]) \cdot r_j \cdot [I_{m+n-i-j}] \tag{9}$$

$$r_n \cdot [P_m] = \frac{1-q^{n+1}}{1-q} \sum_{0 \leq i \leq n} [P_{m+n-i}] \cdot r_i - \sum_{i,j} [P_{m+n-i-j}] \cdot r_j \cdot r([I_i] \cdot [S_1]) \tag{9'}$$

where i, j in the last sum of (9) and (9') run over $i \geq 1, j \geq 0, i+j \leq n$, and $r([S_2] \cdot [P_i]) = r([I_i] \cdot [S_1])$ is given by (8).

Proof. By duality we only prove (9). Use induction on m. First, let $m = 1$. Use the similar arguments in the proof of Lemma 2 to expand $[S_2]^{n+1} \cdot [S_1]^n = [S_2] \cdot ([S_2]^n \cdot [S_1]^n)$, we then get

$$\begin{aligned} [S_2] \cdot r_n &= \frac{\psi_{n+1}(q)}{\psi_n(q)} \sum_{P,i,I} [P] \cdot r_i \cdot [I] - \sum_{P',j,I'} [S_2] \cdot [P'] \cdot r_j \cdot [I'] \\ &= \frac{1-q^{n+1}}{1-q} \sum_{0 \leq i \leq n} r_i \cdot [I_{n-i+1}] - \sum_{i,j} [S_2] \cdot [P_i] \cdot r_j \cdot [I_{n-i-j+1}] + X \end{aligned}$$

where i, j in the last sum run over $i \geq 1, j \geq 0, i+j \leq n$, and

$$X = \frac{1-q^{n+1}}{1-q} \sum_{P,i,I} [P] \cdot r_i \cdot [I] - \sum_{P',j,I'} [S_2] \cdot [P'] \cdot r_j \cdot [I']$$

where P, P' (resp. I, I') run over the non-zero preprojective (resp. preinjective) modules, P' is in addition decomposable, and i, j run over the set $\{0, 1, \cdots, n-1\}$, such that

$$\mathbf{dim}P + \mathbf{dim}I + (i,i) = (n, n+1) = \mathbf{dim}P' + \mathbf{dim}I' + (j,j) + (0,1).$$

Now apply Theorem 3 we get

$$[S_2] \cdot r_n = \frac{1-q^{n+1}}{1-q} \sum_{0 \leq i \leq n} r_i \cdot [I_{n-i+1}] - \sum_{i,j} r([S_2] \cdot [P_i]) \cdot r_j \cdot [I_{n-i-j+1}] + Y$$

where

$$Y = X - \sum_{i,j} q^{i-1} [P_i] \cdot [S_2] \cdot r_j \cdot [I_{n-i-j+1}]$$

where i, j in the last sum run over $i \geq 1, j \geq 0, i+j \leq n$.

We claim that Y is zero. In order to see this, write Y as a combination of the natural bases $[M]$, note that each M has a preprojective direct summand. However, the expression of $[S_2] \cdot r_n$ has no such a term, since an extension of a regular module by a preinjective module can not have a preprojective direct summand. This proves $Y = 0$, i.e. the formula (9) for $m = 1$ holds. Then (9) follows from Lemma 4(iii), the fact that $\mathcal{T}$ is commutative and the induction, we leave the details to the reader. $\Box$

Note that $\mathcal{C}(K) = \bigoplus_{(s,t) \in \mathbb{N}_0^2} \mathcal{C}_{(s,t)}$ where $\mathcal{C}_{(s,t)}$ denotes the homogeneous component $\mathcal{C}(K)_{(s,t)}$.

Lemma 7. Let $n \geq 1$.
(i) If $\mathcal{C}_{(s,t)} \subseteq \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$ for $(s,t) \prec (n, n+1)$, then $\mathcal{C}_{(n,n+1)} \subseteq \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$.
(ii) If $\mathcal{C}_{(s,t)} \subseteq \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$ for $(s,t) \prec (n+1, n)$, then $\mathcal{C}_{(n+1,n)} \subseteq \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$.

Proof. For any monomial $X = [S_2]^{n_1} \cdot [S_1]^{m_1} \cdots [S_2]^{n_t} \cdot [S_1]^{m_t} \in \mathcal{C}_{(n,n+1)}$, we prove that X lie in $\mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$. By assumption we may assume that $n_1 \neq 0$. Let $X = [S_2] \cdot Y$, then by assumption $Y \in \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$. Without loss of generality, let $Y = [P] \cdot T \cdot [I]$ with $T \in \mathcal{T}$, and using the assumption we may assume that $I = 0$. In this case P must be zero since the degree of Y is (n,n). Thus, it suffices to prove $[S_2] \cdot \mathcal{T}_n \subseteq \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$.

Let $X = [S_2] \cdot r_{n_1} \cdots r_{n_t} = Z \cdot r_{n_t}$ with $t \geq 1, n_1 + \cdots n_t = n$. By the formula (9) it suffices to consider the case $t > 1$. By assumption this reduces to the case $Z = [I_{n-n_t+1}]$. Again from (9) the assertion follows. □

Proof of Theorem 4.

Use induction to prove that $\mathcal{C}_{(m,n)} \subseteq \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$ for $(m,n) \in \mathbb{N}_0^2$. If (m,n) is of the forms $(s, s+1)$ or $(s+1, s)$, then from the hypothesis of induction and Lemma 7 the assertion follows. Otherwise, consider the monomial $X = [S_2]^{n_1} \cdot [S_1]^{m_1} \cdots [S_2]^{n_t} \cdot [S_1]^{m_t} = [S_2] \cdot Y \in \mathcal{C}_{(m,n)}$, then by induction we may assume that $X = [S_2] \cdot ([P] \cdot T)$ with $[P]$ preprojective and $T \in \mathcal{T}$, moreover, P is non-zero (by comparing the degree). We discuss as follows:

(i) $T = 0$: i.e. $X = [S_2] \cdot [P]$. If P is indecomposable, then $X \in \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$ by Theorem 3(ii). Let P be decomposable, then from the degree of X we see that P must have more than two indecomposable direct summands, say, $X = [S_2] \cdot [P'] \cdot [P'']$ with P'' indecomposable and P' decomposable, then we may write $X = ([P'''] \cdot T' \cdot [I']) \cdot [P'']$. Note that $P''' \neq 0$ (again by comparing the degrees), by induction $X \in \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$.

(ii) $T \neq 0$: similarly we may assume that $X = ([S_2] \cdot [P]) \cdot T = ([P'] \cdot T' \cdot [I']) \cdot T$. Note that if $P' = 0$ then $I' = 0$ (this can be seen from the degree of $[S_2] \cdot [P]$), so $X \subseteq \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$. If $P' \neq 0$, then $X \in \mathcal{P} \cdot \mathcal{T} \cdot \mathcal{I}$ by induction.

This completes the proof. □

As a direct consequence of Theorem 5 we have

Corollary 3. The $\mathbb{N}_0$-graded commutative algebra $\mathcal{T} = \bigoplus_n \mathcal{T}_n$ is exactly the subalgebra of $\mathcal{C}(K)$ generated by all regular elements in $\mathcal{C}(K)$.

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