ON THE NORM RESIDUE HOMOMORPHISM FOR FIELDS

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Abstract. It is proved that the triviality of the Bockstein map of degree n for all fields of characteristic different from 2 implies the bijectivity of the norm residue homomorphism of degree n modulo 2. In particular, the bijectivity of the norm residue homomorphism follows from its surjectivity.

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Let F be a field and m be a natural number prime to the characteristic of F. The interaction between Milnor K-theory and Galois cohomology of the field F is presented by the *norm residue homomorphism* of degree n modulo m:

$$h_{n,m,F} : K_nF/mK_nF \to H^n(F, \mu_m^{\otimes n}).$$

There is a well-known conjecture saying that $h_{n,m,F}$ is always an isomorphism. This is clear if $n = 0$, simply follows from Hilbert theorem 90 for $n = 1$, and is known to be true if $n = 2$ for arbitrary m [8]. It is also known to be true if m is a power of 2 and $n = 3$ [9],[14] or $n = 4$ (Rost, unpublished) for an arbitrary field F.

In the present paper we consider the case of arbitrary n, $m = 2$ and a field F of characteristic different from 2. It is easy to see (Lemma 2.1) that the surjectivity of the norm residue homomorphism $h_n = h_{n,2,F}$ implies the triviality of the Bockstein connecting map of degree n

$$B_n = B_{n,F} : H^n(F, \mu_2^{\otimes n}) \to H^{n+1}(F, \mu_2^{\otimes n})$$

given by the short exact sequence

$$1 \to \mu_2^{\otimes n} \to \mu_4^{\otimes n} \to \mu_2^{\otimes n} \to 1.$$

We prove that, conversely, the purely cohomological property of the Bockstein map to be trivial implies the bijectivity of the norm residue homomorphism h_n. More precisely, our main result is the following

Theorem. *Suppose that for some $n \geqslant 0$ the Bockstein map B_n is trivial for all fields. Then the norm residue homomorphism h_n is an isomorphism for all fields.*

Corollary. *The surjectivity of h_n for all fields implies the bijectivity of h_n also for all fields.*

1. Notation and preliminary results

Milnor K-theory. Let F be a field. The Milnor graded ring K_*F of the field F is the factor ring of the tensor $\mathbb{Z}$-algebra $T(F^*)$ modulo the ideal generated by the tensors $a\otimes(1-a)$ for all $a\neq 0,1$ in F [13]. Hence, $K_*F=\coprod_{n\geqslant 0} K_nF$, $K_0F=\mathbb{Z}$, $K_1F=F^*$.

For any field extension E/F there is a natural ring homomorphism $i_{E/F}\colon K_*F\to K_*E$ induced by the inclusion $F\hookrightarrow E$.

If E/F is a finite extension, then there is defined the norm homomorphism $n_{E/F}: K_*E\to K_*F$ [5]. The *projection formula* $n_{L/F}(u\cdot i_{L/F}(v))=n_{L/F}(u)\cdot v$ holds for any $u\in K_*E$, $v\in K_*F$. In particular the composition $n_{E/F}\circ i_{E/F}$ is the multiplication by the degree $[E:F]$ of the extension E/F.

Throughout the paper we use the notation k_*F for $K_*F/2K_*F$. The image of the tensor $a_1\otimes a_2\otimes\ldots\otimes a_n$ in k_*F will be denoted by $\{a_1,a_2,\ldots,a_n\}$ and called a *symbol.*

If E/F is a quadratic extension, then k_nE is generated by symbols of the type $u=\{x,a_1,a_2,\ldots a_{n-1}\}$ where $x\in E^*$, $a_i\in F^*$ [2] and

$$n_{E/F}(u)=\{n_{E/F}(x),a_1,a_2,\ldots a_{n-1}\}$$

by the projection formula.

Let X be an algebraic variety defined over F. For any $i\geqslant 0$ denote by $X^{(i)}$ the subset of points in X of codimension i. The group

$$C_n^i(X)=\coprod_{x\in X^{(i)}} k_nF(x)$$

is called the group of k_n-*cycles of codimension i on X.* There is the complex [6]:

$$\cdots\to C_{n+1}^{i-1}(X)\to C_n^i(X)\to C_{n-1}^{i+1}(X)\to\ldots.$$

We denote by $H^i(X,k_{n+i})$ the homology group of this complex in $C_n^i(X)$.

If $X=\mathbb{A}_F^1$ is the affine line, then $H^0(X,k_n)=k_nF$ and $H^i(X,k_n)=0$ if $i>0$. In other words, there is an exact sequence [13]:

$$0\to k_nF\to k_nF(t)\to\coprod_{x\in\mathbb{A}^1} k_{n-1}F(x)\to 0.$$

Cohomology. Denote by F_s a separable closure of the field F, and let $\Gamma=\mathrm{Gal}(F_s/F)$. For any discrete Γ-module M by $H^*(F,M)$ we denote the cohomology groups $H^*(\Gamma,M)$ [16].

For any field extension E/F there is a natural *restriction* homomorphism:

$$i_{E/F}\colon H^*(F,M)\to H^*(E,M).$$

If E/F is a finite extension, then there exists the *norm homomorphism* (*corestriction*):

$$n_{E/F}\colon H^*(E,M)\to H^*(F,M).$$

The composition $n_{E/F} \circ i_{E/F}$ is the multiplication by $[E : F]$.

The group $H^*(F, \mathbb{Z}/2\mathbb{Z})$ will be simply denoted by H^*F. For a quadratic extension $E = F(\sqrt{a})/F$, $\operatorname{char} F \neq 2$, there is an infinite exact sequence [1]:

$$\cdots \to H^{n-1}F \xrightarrow{?\cup(a)} H^nF \xrightarrow{i_{L/F}} H^nL \xrightarrow{n_{L/F}} H^nF \to \dots$$

Let X be an algebraic variety defined over F.

The group

$$Z_n^i(X) = \coprod_{x \in X^{(i)}} H^nF(x)$$

is called the group of *H^n-cycles of codimension i on X*. There is the complex [3], [12] ($\operatorname{char} F \neq 2$) :

$$\cdots \to Z_{n+1}^{i-1}(X) \to Z_n^i(X) \to Z_{n-1}^{i+1}(X) \to \dots$$

The cohomology groups of this complex in $Z_n^i(X)$ we denote by $H^i(X, H^{n+i})$.

If $X = \mathbb{A}_F^1$, then $H^0(X, H^n) = H^nF$ and $H^i(X, H^n) = 0$ for $i \geq 1$. In other words, there exists an exact sequence [4]:

$$0 \to H^nF \to H^nF(t) \to \coprod_{x \in \mathbb{A}^1} H^{n-1}F(x) \to 0.$$

If $(m, \operatorname{char} F) = 1$, then the group $\mu_m \subset F_s^*$ of m-th roots of unity is a Γ-module. By $\mu_m^{\otimes n}$ we denote the n-th tensor power $\underbrace{\mu_m \otimes \ldots \otimes \mu_m}_{n}$ of μ_m. The exact sequence of Γ-modules $1 \to \mu_m \to F_s^* \xrightarrow{m} F_s^* \to 1$ induces the connecting homomorphism

$$F^*/F^{*m} \to H^1(F, \mu_m), a \cdot F^{*m} \mapsto (a),$$

which is an isomorphism in view of Hilbert theorem 90 [16].

The cup-product in the cohomology theory induces the homomorphism

$$\underbrace{F^* \otimes \ldots \otimes F^*}_{n} \to H^n(F, \mu_m^{\otimes n}); a_1 \otimes \ldots \otimes a_n \mapsto (a_1) \cup \cdots \cup (a_n) \stackrel{\text{def}}{=} (a_1, a_2, \ldots, a_n),$$

which factors through the *norm residue homomorphism of degree n modulo m* [18]:

$$h_{n,m,F} \colon K_nF/mK_nF \to H^n(F, \mu_m^{\otimes n})$$

If $m = 2$ the norm residue homomorphism is simply denoted by

$$h_n = h_{n,F} \colon k_nF \to H^nF$$

Comparing the exact sequences of Milnor groups and Galois cohomology for an affine line $\mathbb{A}^1$ one deduces that if h_{n-1} is an isomorphism for any field, then $\ker h_{n,F} \simeq \ker h_{n,F(t)}$, $\operatorname{coker} h_{n,F} \simeq \operatorname{coker} h_{n,F(t)}$, i.e., kernel and cokernel of the norm residue homomorphism do not change under purely transcendental extensions.

The norm residue homomorphism commutes with norms: for a finite field extension E/F the diagram

$$\begin{array}{ccc} k_nE & \xrightarrow{h_{n,E}} & H^nE \\ {\scriptstyle n_{E/F}}\downarrow & & \downarrow{\scriptstyle n_{E/F}} \\ k_nF & \xrightarrow{h_{n,F}} & H^nF \end{array}$$

commutes [17].

If F is a finite, local or global field, then $h_{n,F}$ is an isomorphism [13],[18].

If X is an algebraic variety over F, then for an étale sheaf M on $X_{\text{ét}}$ by $H^*(X,M)$ we denote étale cohomology groups [12]. The group $H^*(X,\mathbb{Z}/2\mathbb{Z})$ we shall denote by H^*X.

The degree $\deg x$ of a closed point $x\in X$ is the degree of the extension $F(x)/F$.

By $[1,n]$ we denote the set $\{1,2,\dots,n\}$.

In the section 2 we prove certain properties of the Bockstein map. The next section is devoted to the study of étale cohomology of conic curves (projective and affine). In section 4 we express the fact that a given element in k_nF is trivial in terms of algebraic equations. This statement generalizes Lemma 4 in [7]. In section 5, we develop a specialization technique, part of which was already used in [7]. The next section is devoted to the computation of K-cohomology groups of affine conic curve. In section 7 we prove that under certain conditions the surjectivity of $h_{n,F}$ implies the surjectivity of $h_{n,F(X)}$ for a conic curve X over F. The proof of the Theorem is presented in section 8. We follow the approach of [7] in the case $n=2$.

2. Bockstein map

Let F be a field of characteristic different from 2. The map $B_n=B_{n,F}$ in the exact sequence

$$H^n(F,\mu_4^{\otimes n})\to H^nF\xrightarrow{B_n}H^{n+1}F\to H^{n+1}(F,\mu_4^{\otimes n})$$

associated to the short exact sequence

$$1\to\mu_2^{\otimes n}\to\mu_4^{\otimes n}\to\mu_2^{\otimes n}\to 1$$

will be called the *Bockstein map.*

Lemma 2.1. *The surjectivity of the norm residue homomorphism $h_{n,F}$ implies $B_{n,F}=0$.*

Proof. Follows from the commutative diagram

$$\begin{array}{ccccc} K_nF & = & K_nF & & \\ {\scriptstyle h_{4,F}}\downarrow & & {\scriptstyle h_{2,F}}\downarrow & & \\ H^n(F,\mu_4^{\otimes n}) & \longrightarrow & H^nF & \xrightarrow{B_n} & H^{n+1}F. \end{array}\quad\square$$

Later we shall need the following two statements:

Lemma 2.2. *If $B_{n,F}=0$, then the restriction of the natural map $H^{n+1}(F,\mu_4^{\otimes n})\to H^{n+1}F$ to the subgroup of elements of exponent 2 is trivial.*

Proof. Follows from the fact that the composition

$$H^{n+1}(F,\mu_4^{\otimes n})\to H^{n+1}F\to H^{n+1}(F,\mu_4^{\otimes n})$$

is the multiplication by 2 and the second map is injective since $B_{n,F}=0$. □

Lemma 2.3. *If B_n is trivial for all fields, then $B_{n-1}=0$ also for all fields.*

Proof. Let $E=F((t))$ be the field of formal power series over F. The statement follows from the diagram with the surjective rows [13]:

$$\begin{array}{ccc} H^nE & \xrightarrow{\partial} & H^{n-1}F \\ \Big\downarrow{\scriptstyle B_{n,E}} & & \Big\downarrow{\scriptstyle B_{n-1,F}} \\ H^{n+1}E & \xrightarrow{\partial} & H^nF. \end{array}$$ □

3. Étale cohomology of conic curves

Let C be a smooth curve over a field F, S be a finite set of closet points in C, $U=C-S$. Let m be a natural number prime to $\operatorname{char}F$.

There is Gysin exact sequence [12]:

$$\cdots\to\coprod_{c\in S}H^{n-2}(F(c),\mu_m^{\otimes(j-1)})\to H^n(C,\mu_m^{\otimes j})\to H^n(U,\mu_m^{\otimes j})\to\dots$$

Passing to the inductive limit with respect to S we get the localization sequence :

$$\cdots\to\coprod_{c\in C}H^{n-2}(F(c),\mu_m^{\otimes(j-1)})\to H^n(C,\mu_m^{\otimes j})\to H^n(F(C),\mu_m^{\otimes j})\to\dots$$

The arrows in the sequence are compatible with the multiplication on $H^*(F,\mu_m^{\otimes *})$ up to sign.

In the special case $n=2$, $j=1$ we get a monomorphism

$$\operatorname{Pic}C/m\operatorname{Pic}C\to H^2(C,\mu_m)$$

There is the Hochschild–Serre spectral sequence [12]:

$$E_2^{p,q}=H^p(F,H^q(C_s,\mu_m^{\otimes j}))\Rightarrow H^{p+q}(C_s,\mu^{\otimes j})$$

where $C_s=C\otimes_F F_s$. To compute E_2-term one must calculate the groups

$$H^q(C_s,\mu_m^{\otimes j})=H^q(C_s,\mu_m)\otimes\mu_m^{\otimes(j-1)}.$$

It is known that $H^0(C_s,\mu_m^{\otimes j})=\mu_m^{\otimes j}$ and $H^q(C_s,\mu_m^{\otimes j})=0$ if $q\geqslant 3$ [12]. The exact sequence

$$1\to\mu_m\to\mathbb{G}_m\xrightarrow{m}\mathbb{G}_m\to 1$$

implies that $H^2(C_s, \mu_m) \simeq \operatorname{Pic} C_s/m \operatorname{Pic} C_s$ since $H^1(C_s, \mathbb{G}_m) \simeq \operatorname{Pic} C_S$ and m-torsion in $H^2(C_s, G_m)$ is trivial [12], and we also obtain the exact sequence

$$1 \to F_s[C_s]^*/F_s[C_s]^{*m} \to H^1(C_s, \mu_m) \to \operatorname{Pic} C_s \xrightarrow{m} \operatorname{Pic} C_s.$$

Let Y be a projective conic curve given in $\mathbb{P}^2_F$ by the equation $aU^2 + bV^2 = W^2$, $a,\ b \in F^*$ (we assume that char $F \neq 2$). The following conditions are clearly equivalent:

1. Y has a F-rational point.
2. Y is isomorphic to $\mathbb{P}^1_F$.
3. Quaternion algebra $\left(\frac{a,b}{F}\right)$ splits, i.e., $(a, b) = 0 \in H^2 F$.

If Y satisfies these conditions, then it is called *split* conic curve.

If Y is not split, then the degree of each closed point is even. The exact sequence [15]

$$0 \to \operatorname{Pic} Y \to H^0(F, \operatorname{Pic} Y_s) \to \operatorname{Br} F \to \operatorname{Br} F(Y)$$

shows that the kernel of $\operatorname{Br} F \to \operatorname{Br} F(Y)$ is generated by $(a, b) \in H^2 F$, $\operatorname{Pic} Y = \mathbb{Z}$ is generated by the class of any F-rational point if Y is split and $\operatorname{Pic} Y = 2\mathbb{Z}$ is generated by the class of arbitrary closed point of degree 2.

Since $Y_s \simeq \mathbb{P}^1_s$, we have $F_s[Y_s] = F_s$ and $\operatorname{Pic} Y_s = \mathbb{Z}$. Applying the above computation one gets :

$$H^q(Y_s, \mu_m^{\otimes j}) = \begin{cases} \mu_m^{\otimes j}, & \text{if } q = 0 \\ \mu_m^{\otimes (j-1)}, & \text{if } q = 2 \\ 0, & \text{if } q = 1 \text{ or } q \geqslant 3 \end{cases}$$

Therefore, we have a commutative diagram

$$\begin{tikzcd}
& & H^n(F, \mu_m^{\otimes j}) \arrow[d] \arrow[dr, "i"] & & \\
\cdots \arrow[r, "\partial"] & \coprod_{y \in Y} H^{n-2}(F(y), \mu_m^{\otimes (j-1)}) \arrow[r] \arrow[dr, "j"'] & H^n(Y, \mu_m^{\otimes j}) \arrow[r] \arrow[d, "\pi"] & H^n(F(Y), \mu_m^{\otimes j}) \arrow[r] & \cdots \\
& & H^{n-2}(F, \mu_m^{\otimes (j-1)}) & &
\end{tikzcd}$$

where i is the natural map.

Lemma 3.1. *The homomorphism j is the sum of the norm homomorphisms.*

Proof. Consider first the case where $n = 2$, $j = 1$ and $y \in Y$ is F-rational point. The statement follows from the commutative diagram

$$\begin{tikzcd}
& H^0(F(y), \mathbb{Z}/m\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^2(Y, \mu_m) \arrow[d] \\
\operatorname{Pic} Y_s/m \operatorname{Pic} Y_s \arrow[r, "\sim"] & \mathbb{Z}/m\mathbb{Z} \arrow[r, "\sim"] & H^2(Y_s, \mu_m).
\end{tikzcd}$$

The case of a rational point $y \in Y$ (with arbitrary n and j) follows from the above and the fact that the homomorphisms in the diagram are compatible up to sign with the multiplication on $H^*(F, \mu_m^{\otimes *})$.

For the general case let $E = F(y)$ and y be an E-rational point of Y_E over y, $E(y) = F(y)$. The statement follows from the commutative diagram

$$\begin{array}{ccccc} H^{n-2}(E(y'), \mu_m^{\otimes(j-1)}) & \longrightarrow & H^n(Y_E, \mu_m^{\otimes j}) & \xrightarrow{\pi_E} & H^{n-2}(E, \mu_m^{\otimes(j-1)}) \\ \| & & \downarrow{\scriptstyle n_{E/F}} & & \downarrow{\scriptstyle n_{E/F}} \\ H^{n-2}(F(y), \mu_m^{\otimes(j-1)}) & \longrightarrow & H^n(Y, \mu_m^{\otimes j}) & \xrightarrow{\pi_F} & H^{n-2}(F, \mu_m^{\otimes(j-1)}). \end{array} \quad \square$$

It follows from Lemma 3.1 that the norm maps induce a well-defined norm map

$$N\colon H^1(Y, H^{n-1}) \to H^{n-2}F.$$

In the special case $n = m = 2$ we have a monomorphism $\operatorname{Pic} Y/2 \operatorname{Pic} Y \to H^2Y$.

Lemma 3.2. *If Y is not split, then the image of the canonical generator of* $\operatorname{Pic} Y$ *in H^2Y equals $(a, b)_Y$.*

Proof. The group $\operatorname{Pic} Y$ is generated by the class $[y]$ of arbitrary point $y \in Y$ of degree 2. Since $[y] \notin \operatorname{im} \partial$ and $j([y]) = 0$, we see that the image of $[y]$ in H^2Y comes from a nontrivial element of the kernel of $H^2F \to H^2F(Y)$. But the only nontrivial element in this kernel is (a, b). $\square$

Let X be an open subvariety in Y defined by $w \neq 0$, so that X is an affine conic curve : $u^2 - av^2 = b$, where $u = \frac{U}{W}$, $v = \frac{V}{W}$. The complement $Y - X$ is isomorphic to $\operatorname{Spec} L$ where $L = F(\sqrt{a})$. It is only one "infinite" point $\infty \in Y$ of degree 2 if $a \notin F^{*2}$ and a union of two points if $a \in F^{*2}$. If X is not split, then $\operatorname{Pic} X = \operatorname{Pic} Y / <\infty>= 0$ since $\operatorname{Pic} Y$ is generated by the class of the point ∞ of degree 2.

If X is split but $a \notin F^{*2}$, then the group $\operatorname{Pic} X \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by the class of any rational point. If $a \in F^{*2}$, then $\operatorname{Pic} X = 0$ since $X \simeq \mathbb{A}^1 - pt$.

Assume that $(m, \operatorname{char} F) = 1$. It follows from the computation above that the group $H^1(X_s, \mu_m)$ is isomorphic to $F_s[X]^*/F_s^* \simeq \mathbb{Z}/m\mathbb{Z}$ and is generated by the class of invertible regular function $u + v\sqrt{a}$ on X. The Galois group Γ acts through the action of $\operatorname{Gal}(L/F)$ whose generator takes $k \in \mathbb{Z}/m\mathbb{Z}$ to $-k$. Denote the group $\mathbb{Z}/m\mathbb{Z}$ with this Γ-action by $\widetilde{\mathbb{Z}/m\mathbb{Z}}$ and put $\tilde{\mu}_m^{\otimes j} = \widetilde{\mathbb{Z}/m\mathbb{Z}} \otimes \mu_m^{\otimes j}$. Hence we have :

$$H^q(X_s, \mu_m^{\otimes j}) = \begin{cases} \mu_m^{\otimes j} & \text{if } q = 0 \\ \tilde{\mu}_m^{\otimes(j-1)} & \text{if } q = 1 \\ 0 & \text{if } q \geqslant 2 \end{cases}$$

The Hochschield–Serre spectral sequence induces the exact sequence

$$\cdots \to H^n(F, \mu_m^{\otimes j}) \to H^n(X, \mu_m^{\otimes j}) \to H^{n-1}(F, \tilde{\mu}_m^{\otimes(j-1)}) \xrightarrow{\Delta} H^{n+1}(F, \mu_m^{\otimes j}).$$

Lemma 3.3. *If $m = 2$, then the differential map $\Delta\colon H^{n-1}F \to H^{n+1}F$ is the multiplication by $(a,b) \in H^2F$.*

Proof. Since Δ commutes with the multiplication on H^*F, it follows that Δ is the multiplication on $u = \Delta(1+2\mathbb{Z}) \in H^2F$.

The commutative diagram

$$\begin{tikzcd} & & \operatorname{Pic} X \arrow[d] \arrow[dr, "j"] & \\ H^0F \arrow[r, "\Delta"] & H^2F \arrow[r] \arrow[dr, "i"] & H^2X \arrow[r] \arrow[d] & H^1F \\ & & H^2F(X) & \end{tikzcd}$$

shows that $u \in \ker i$. If X is split, then $u = 0 = (a,b)$. If X is not split, then $\operatorname{Pic} X = 0$ and therefore u generates $\ker i$, i.e., $u = (a,b)$. $\square$

Consider a commutative diagram

$$\begin{tikzcd} & H^nF \arrow[d] \arrow[dr, "i"] & \\ \coprod_{x\in X} H^{n-2}F(x) \arrow[r] \arrow[dr, "j"] & H^nX \arrow[r] \arrow[d] & H^nF(X) \\ & H^{n-1}F & \end{tikzcd}$$

Lemma 3.4. *The map j coincides with the composition of the norm homomorphism and the multiplication by $(a) \in H^1F$ on each summand.*

Proof. As in the proof of Lemma 3.1 it is sufficient to consider the case of F-rational point $x \in X$ (X is therefore split) and $n = 2$, so we must show that $j(1+2\mathbb{Z}) = (a) \in H^1F$. If $(a) = 0$, i.e., $a \in F^{*2}$, then $\operatorname{Pic} X = 0$, hence $1+2\mathbb{Z}$ belongs to the image of $H^1F(X) \to \coprod_{x\in X} H^0F(x)$ and $j(1+2\mathbb{Z}) = 0 = (a)$.

If $(a) \neq 0$, then the group $\operatorname{Pic} X = \mathbb{Z}/2\mathbb{Z}$ is generated by the class of x. Since the map i is injective the diagram in the proof of Lemma 3.3 shows that $j(1+2\mathbb{Z}) \neq 0$.

Since $(a)_L = 0 \in H^1L$, by considering X over L and the case described above one shows that $j(1+2\mathbb{Z})_L = 0$, hence $j(1+2\mathbb{Z}) \in \ker(H^1F \to H^1L)$ and $j(1+2\mathbb{Z}) = (a)$. $\square$

Corollary 3.5. *The sum of the norm homomorphisms induces a well-defined norm map*

$$N\colon H^1(X, H^{n-1}) \to H^{n-2}F/n_{L/F}(H^{n-2}L).$$

Proof. The image of the map j belongs to

$$\ker(H^{n-1}F \to H^{n-1}L) \simeq H^{n-2}F/n_{L/F}(H^{n-2}L)\,. \quad \square$$

Consider the following sequence:

$$(*) \qquad 0 \to H^1(X, H^n) \xrightarrow{N} H^{n-1}F/n_{L/F}(H^{n-1}L) \xrightarrow{\cup(a,b)} H^{n+1}F \to H^{n+1}F(X).$$

The map in the middle is the multiplication by (a,b); it is well defined since

$$n_{L/F}(H^{n-1}L) \cup (a) = 0.$$

Lemma 3.6. *The sequence* $(*)$ *is a complex.*

Proof. Since $(a,b)_{F(X)} = 0$, the composition of the last two maps is zero. The equality $(a,b)\circ N = 0$ follows from the fact that N commutes with the multiplication on (a,b) and $(a,b)_{F(x)} = 0$ for any $x \in X$. $\square$

Remark 3.7. The homology of the complex $(*)$ at the second term is naturally isomorphic to the homology of the complex

$$H^1(Y, H^n) \xrightarrow{N} H^{n-1}F \xrightarrow{(a,b)} H^{n+1}F.$$

Lemma 3.8. *For any* $x \in X$ *the composition*

$$H^{n-1}F(x) \to H^{n+1}X \to H^{n+1}(X, \mu_4^{\otimes n})$$

is trivial.

Proof. As in the proof of Lemma 3.1 we may assume that $n = 1$ and x is F-rational point of X. But in this case the statement follows from the fact that $2 \cdot \operatorname{Pic} X = 0$ and the commutativity of the diagram

$$\begin{array}{ccccc} H^0F(x) & \longrightarrow & \operatorname{Pic} X & \longrightarrow & H^2X \\ & & {\scriptstyle 2}\downarrow & & \downarrow \\ & & \operatorname{Pic} X & \longrightarrow & H^2(X, \mu_4). \end{array} \quad \square$$

Lemma 3.9. *Assume that the Bockstein map* $B_{n-1,F}$ *is trivial. Then the image of the composition*

$$H^n(X, \mu_4^{\otimes n}) \to H^nX \to H^{n-1}F$$

is contained in $n_{L/F}(H^{n-1}L)$.

Proof. It suffices to show that the map ∂ in the diagram

$$\begin{array}{ccccc} H^n(X, \mu_4^{\otimes n}) & \longrightarrow & H^nX & & \\ \downarrow & & \downarrow & & \\ H^{n-1}(F, \tilde{\mu}_4^{\otimes(n-1)}) & \longrightarrow & H^{n-1}F & \xrightarrow{\partial} & H^nF \end{array}$$

coincides with the multiplication by $(a) \in H^1F$. The map δ is the connected map relative to the short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} = \mu_2^{\otimes(n-1)} \to \tilde{\mu}_4^{\otimes(n-1)} \to \mu_2^{\otimes(n-1)} = \mathbb{Z}/2\mathbb{Z} \to 0.$$

The class of this sequence in $\operatorname{Ext}^1_\Gamma(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is the sum of classes of the following sequences: $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}[G] \to \mathbb{Z}/2\mathbb{Z} \to 0$ where $G = \operatorname{Gal}(L/F)$ and $0 \to \mathbb{Z}/2\mathbb{Z} \to \mu_4^{\otimes(n-1)} \to \mathbb{Z}/2\mathbb{Z} \to 0$. The connecting homomorphism for the first sequence in the multiplication by (a) [1] and for the second sequence is equal to $B_{n-1,F} = 0$. $\square$

Consider the following diagram:

$$
\begin{tikzcd}
 & & & \coprod\limits_{x\in X} H^{n-1}F(x) \arrow[r] \arrow[d] & H^{n+1}X \arrow[d] \\
 & H_4^n(X) \arrow[r] \arrow[d] & H_4^n(F(X)) \arrow[r] \arrow[d, "p"] & \coprod\limits_{x\in X} H_4^{n-1}(F(x)) \arrow[r] \arrow[d] & H^{n+1}(X,\mu_4^{\otimes n}) \\
\coprod\limits_{x\in X} H^{n-2}F(x) \arrow[r] & H^nX \arrow[r] \arrow[d] & H^nF(X) \arrow[r] & \coprod\limits_{x\in X} H^{n-1}F(x) & \\
 & H^{n-1}F \arrow[d, "{(a,b)}"] & & & \\
 & H^{n+1}F & & &
\end{tikzcd}
$$

where we write for short $H_4^n(-)$ for $H^n(-,\mu_4^{\otimes n})$.

The diagram chase and Lemmas 3.8 and 3.9 show that there is a well-defined map

$$\theta\colon \coprod_{x\in X} H^{n-1}F(x) \to H^{n-1}F/n_{L/F}(H^{n-1}L).$$

Lemma 3.10. *The homomorphism θ is induced by the norm map*

$$N\colon H^1(X,H^n) \to H^{n-1}F/n_{L/F}(H^{n-1}L).$$

Proof. We may assume that $(a) \neq 0 \in H^1F$. As in Lemma 3.1 it suffices to consider the case $n = 1$ and X is split. Let x be F-rational point in X. We must show that the nonzero element in $H^0F(x)$ corresponds to nonzero element in H^0F. Choose a function $f \in F(X)^*$ such that $\operatorname{div} f = 2x$. Since $\operatorname{div} f \,\vdots\, 2$, we have $(f) = g_{F(X)}$ for some $g \in H^1X$. The image of g in H^0F is nontrivial. For if it is trivial, then $g \in \operatorname{im}(H^1F \to H^1X)$ and therefore $f \in F^* \cdot F(X)^{*2}$, but this is impossible since $0 \neq [x] \in \operatorname{Pic} X \simeq \mathbb{Z}/2\mathbb{Z}$. On the other hand the image of f in $H^0(F(x),\mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$ equals $2 \mod 4\mathbb{Z}$ and hence nontrivial. □

Lemma 3.11. *Let $B_{n-1,F}$ and $B_{n,F(x)}$ be trivial. Then the sequence $(*)$ is exact at the second term.*

Proof. Diagram chase using the surjectivity of p and Lemma 3.10. □

Lemma 3.12. *Assume that $B_{n,F} = 0$ and h_{n-1} is surjective for all fields. Let $u = (u_y) \in \coprod\limits_{y\in Y} H^{n-1}F(y)$ such that $\sum n_y(u_y) = 0 \in H^{n-1}F$. Then the image of u in $H^{n+1}Y$ is equal to $(a,b)\cdot v_Y$ for some $v \in H^{n-1}F$.*

Proof. Choose $w_y \in H^{n-1}(F(y),\mu_4^{\otimes(n-1)}) \mapsto u_y \in H^{n-1}F(y)$ and assume first that $\sum n_y(w_y) = 0 \in H^{n-1}(F,\mu_4^{\otimes(n-1)})$. Then the image of w in $H^{n+1}(Y,\mu_4^{\otimes n})$ equals t_Y for some $t \in H^{n+1}(F,\mu_4^{\otimes n})$. In particular, $t_{F(Y)} = 0$.

Since L splits Y, the map

$$H^{n+1}(L,\mu_4^{\otimes n}) \to H^{n+1}(L(Y),\mu_4^{\otimes n})$$

is injective, and hence $t_L = 0$. Applying the norm map we have : $2t = n_{L/F}(t_L) = 0 \in H^{n+1}(F, \mu_4^{\otimes n})$. By Lemma 2.2 the image of t in $H^{n+1}F$ is trivial, therefore, the image of u in $H^{n+1}Y$ is trivial.

In general $\sum n_y(w_y) = k(s)$ where $s \in H^{n-1}F$ and

$$k : H^{n-1}F \to H^{n-1}(F, \mu_4^{\otimes(n-1)}).$$

Choose $s' \in H^{n-1}(F, \mu_4^{\otimes(n-1)}) \mapsto s \in H^{n-1}F$. Then

$$k(s) = 2s' \in H^{n-1}(F, \mu_4^{\otimes(n-1)})$$

and by the first part of the proof we can replace w by w' with the support at one point ∞ of degree 2:

$$w'_y = \begin{cases} s'_L \in H^{n-1}(L, \mu_4^{\otimes(n-1)}) & \text{if } y = \infty \\ 0 & \text{if } y \neq \infty \end{cases}$$

Therefore, we can replace u by u' :

$$u'_y = \begin{cases} s_L \in H^{n-1}L & \text{if } y = \infty \\ 0 & \text{if } y \neq \infty \end{cases}$$

Finally, using the multiplicative property we may assume that $n = 1$ and apply Lemma 3.2. $\square$

Corollary 3.13. *In the conditions of the Lemma the sequence $(*)$ is exact at the first term.*

Proof. Let $(u_x) \in \coprod H^{n-1}F(x)$ and $\sum n_x(u_x) = n_{L/F}(w)$ for some $w \in H^{n-1}L$. Consider $u \in \coprod_{y \in Y} H^{n-1}F(y)$ such that

$$u_y = \begin{cases} u_x & \text{if } y = x \in X \\ w & \text{if } y = \infty. \end{cases}$$

Then $\sum n_y(u_y) = 0$ and by Lemma 3.12 the image of u in $H^{n+1}Y$ equals $(a,b) \cdot v_Y$ for some $v \in H^{n-1}F$. Hence the image of (u_x) in $H^{n+1}X$ equals zero aud we apply the exact sequence

$$H^nF(X) \to \coprod_{x \in X} H^{n-1}F(x) \to H^{n+1}X. \quad \square$$

Lemma 3.14. *The homology of the complex $(*)$ at the first and the third terms are isomorphic.*

Proof. Follows from the commutative diagram:

$$\begin{tikzcd}
& & H^{n-1}F \arrow[d, "{(a,b)}"] & \\
& & H^{n+1}F \arrow[d] \arrow[dr, "i"] & \\
H^nF(X) \arrow[r] & \coprod_{x \in X} H^{n-1}F(x) \arrow[r] \arrow[dr, "j"'] & H^{n+1}X \arrow[r] \arrow[d] & H^{n+1}F(X) \\
& & H^nF &
\end{tikzcd}$$

Lemmas 3.3, 3.4 and Lemma of the 700th [11]. $\square$

Combining Lemmas 3.11, 3.14 and Corollary 3.13 we get the following.

Proposition 3.15. *Let B_n be trivial and h_{n-1} be surjective for all fields. Then the sequence $(*)$ is exact.* $\square$

4. When some element in k_n is zero

In this section we generalize Lemma 4 in [7].

Let $n, m \in \mathbb{N}$, $a_1, a_2, \dots, a_m \in F^*$ and $b_\alpha \in F^*$ for any map $\alpha\colon [1, n-1] \to [1, m]$. Put $\underline{a}_\alpha = \{a_{\alpha(1)}, a_{\alpha(2)}, \dots, a_{\alpha(n-1)}\} \in k_{n-1}F$.

Proposition 4.1. *Let the images of $a_1, \dots, a_m$ be linearly independent in F^*/F^{*2} (considered as a vector space over $\mathbb{Z}/2\mathbb{Z}$). The following conditions are equivalent :*

1. *$\sum_\alpha b_\alpha \cdot \underline{a}_\alpha = 0 \in k_nF$ where the sum is taken over all maps $\alpha\colon [1, n-1] \to [1, m]$.*
2. *There exist an integer $r \geqslant m$ and elements $a_{m+1}, \dots, a_r \in F^*$, $c_{s,j}, d_{s,j} \in F$ for $j = 1, 2, \dots, n-1$ and all maps*

$$s\colon [1, n-1] \to \{\textit{nonempty subsets in } [1, r]\}$$

such that

$$b_\alpha = \prod_s \Big(\prod_{j=1}^{n-1} (c_{s,j}^2 - a_{s(j)} \cdot d_{s,j}^2)\Big)$$

for any map $\alpha\colon [1, n-1] \to [1, r]$, where the product is taken over all s such that $s(k) \ni \alpha(k)$ for any $k = 1, 2, \dots, n-1$ (also $a_T = \prod_{i \in T} a_i$ for any subset $T \subset [1, r]$; $b_\alpha = 1$, if $\operatorname{im} \alpha \not\subset [1, m]$)

Proof. (2) $\Rightarrow$ (1);

$$\begin{aligned}\sum_\alpha b_\alpha \cdot \underline{a}_\alpha &= \sum_{j=1}^{n-1} \Big(\sum_s (c_{s,j}^2 - a_{s(j)} \cdot d_{s,j}^2) \cdot \sum_{\alpha(k) \in s(k)} \underline{a}_\alpha\Big) \\ &= \sum_{j=1}^{n} \Big(\sum_s \{c_{s,j}^2 - a_{s(j)} \cdot d_{s,j}^2, a_{s(1)}, \dots, a_{s(j)}, \dots, a_{s(n-1)}\}\Big) = 0\end{aligned}$$

since $\{c^2 - ad^2, a\} = 0 \in k_2F$ for $a, c, d \in F$. Note that here we do not need a_i to be linearly independent in F^*/F^{*2}.

(1) $\Rightarrow$ (2); The group k_nF is isomorphic to the factor group of

$$\underbrace{F^*/F^{*2} \otimes \dots \otimes F^*/F^{*2}}_{n}$$

by the subgroup generated by $u \otimes v_1 \otimes \dots \otimes v_{n-1}$ where u is a norm in the quadratic extension $F(\sqrt{v_j})/F$ for some $j = 1, 2, \dots, n-1$. Therefore, the element $\theta = \sum_\alpha b_\alpha \otimes a_{\alpha(1)} \otimes \dots \otimes a_{\alpha(n-1)}$ is the sum of elements of the type described above. Consider elements $a_{m+1}, \dots, a_r$ such that the images of $a_1, \dots, a_r$ in F^*/F^{*2} form a base of a vector space $V \subset F^*/F^{*2}$ such that the above relation holds in $F^* \otimes V \otimes \dots \otimes V$.

Hence

$$\begin{aligned}\theta &= \sum_{j=1}^{n-1}\Big(\sum_{s}(c_{s,j}^2 - a_{s(j)}d_{s,j}^2)\otimes a_{s(1)}\otimes\ldots\otimes a_{s(j)}\otimes\ldots\otimes a_{s(n-1)}\Big)\\ &= \sum_{j=1}^{n-1}\sum_{\alpha}\prod_{s}(c_{s,j}^2 - a_{s(j)}d_{s,j}^2)\otimes a_{\alpha(1)}\otimes a_{\alpha(2)}\otimes\ldots\otimes a_{\alpha(n-1)}\Big)\end{aligned}$$

where the product is taken over all s such that $s(k)\ni\alpha(k)$ for any $k=1,2,\ldots,n-1$. Hence the desired relations hold modulo F^{*2}. But squares can be included to $c_{s,1}$ and $d_{s,1}$ for s with $s(k)=\{\alpha(k)\}$, $k=1,2,\ldots,n-1$. $\square$

5. Specialization

We develop the specialization technique which will be used later many times.

Let X be an irreducible variety over a field F, $x\in X$ be a closed point. Assume that for any $n\geqslant 0$ and any point $y\in X$ such that $x\in\overline{\{y\}}$ there is an abelian group $V_n(y)$ such that:

(1) For any points $y,z\in X$ such that $x\in\overline{\{y\}}$ and y is a regular point of codimension 1 in $\overline{\{z\}}$ there is a homomorphism

$$\partial_{y,z}^n\colon V_n(z)\to V_{n-1}(y)$$

called residue map, for any $n\geqslant 1$.

(2) For any $z\in X$ such that $x\in\overline{\{z\}}$ there is a multiplication

$$V_n(z)\otimes F(z)^*\to V_{n+1}(z)$$

for any $n\geqslant 0$.

Let $y,z\in X$ be the two points as in (1), $\pi\in F(z)^*$ be a prime element of the discrete valuation on the field $F(z)$ defined by the point y. Consider the composition

$$s_{y,z}^n\colon V_n(z)\to V_{n+1}(z)\xrightarrow{\partial_{y,z}^n} V_n(y)$$

where the first map is the multiplication by π.

Assume that x is a regular point of X. Then there exists a sequence of points in X: $x=x_0,x_1,\ldots,x_m$ such that $x_m=\xi$ is the generic point of X and for any $i=0,1,\ldots,m-1$ x_i is the regular point in $\overline{\{x_{i+1}\}}$ of codimension 1. Choose prime elements $\pi_{i+1}\in F(x_{i+1})$ with respect to the discrete valuation defined by x_i on $F(x_{i+1})$. The composition

$$s^n = s_{x_0,x_1}^n\circ s_{x_1,x_2}^n\circ\cdots\circ s_{x_{m-1},x_m}^n : V_n(\xi)\to V_n(x)$$

is called the specialization map. It depends on the choice of the sequence of points x_i and prime elements π_i.

Examples. 1) $V_n(y) = k_nF(y)$. The residue map $\partial^n_{y,z}$ in (1) is the usual residue map [2],[13] and the multiplication is obvious. We have therefore a specialization map

$$s^n \colon k_n(F(X)) \to k_nF(x).$$

If the functions $f_1, f_2, \dots, f_n \in F(X)^*$ are defined and nonzero in x, then

$$s^n(\{f_1, f_2, \dots, f_n\}) = \{f_1(x), f_2(x), \dots, f_n(x)\}.$$

2) Let $\pi \colon V \to X$ be a flat dominating morphism. Fix some integer $i \geqslant 0$. For any $y \in X$ define

$$V_n(y) = H^i(V_y, k_n), \quad \text{where } V_y = \pi^{-1}(y).$$

Let $y, z \in X$ be as in (1) and $\mathcal{O}$ be the discrete valuation ring $\mathcal{O}_{y,z}$ with fraction field $F(z)$ and residue field $F(y)$. Consider $V_{\mathcal{O}} = V \times_X \operatorname{Spec} \mathcal{O}$. Since $V_{\mathcal{O}}^{(i)} = V_z^{(i)} \cup V_y^{(i-1)}$, we have the exact sequence of complexes:

$$0 \to \coprod_{s \in V_y^{(*-1)}} K_*F(s) \to \coprod_{s \in V_{\mathcal{O}}^{(*)}} K_*F(s) \to \coprod_{s \in V_z^{(*)}} K_*F(s) \to 0.$$

Taking the exact sequence of homology groups we get the residue map $\partial^n_{y,z} \colon V_n(z) \to V_{n-1}(y)$.

The multiplication map clearly exists. We have therefore the specialization map

$$s^n \colon H^i(V_\xi, k_n) \to H^i(V_x, k_n).$$

6. K-cohomology of a conic curve

We keep the notation of Section 3. Since the conic curve Y is projective, the norm maps $k_nF(y) \to k_nF$ for each $y \in Y$ determine the norm map [6]:

$$\overline{N} \colon H^1(Y, k_n) \to k_{n-1}F.$$

Therefore, the norm map $N \colon H^1(X, k_n) \to k_{n-1}F/n_{L/F}(k_{n-1}L)$ for an affine conic curve X is determined by the commutative diagram

$$\begin{tikzcd}
k_{n-1}F(\infty) \arrow[r] \arrow[d, equal] & H^1(Y, k_n) \arrow[r, two heads] \arrow[d, "\overline{N}"] & H^1(X, k_n) \arrow[d, "N"] \\
k_{n-1}L \arrow[r, "n_{L/F}"] & k_{n-1}F \arrow[r] & k_{n-1}F/n_{L/F}k_{n-1}L.
\end{tikzcd}$$

We shall show in this section that under certain conditions N is the injective map.

Assume that X is not split. For any $m \geqslant 0$ we consider the subspace $L_m \subset F[X]$ consisting of the functions $f(u, v)$ where f is a polynomial in two variables of degree $\leqslant m$ and u, v are regular functions on X such that $u^2 - av^2 = b$.

It is clear that $\dim L_m = 2m + 1$. Any function $g \in L_m$ has a pole at infinity of order at most m, therefore, the degree of the divisors of poles and zeros of g is at most $2m$. Hence for any $x \in X$ such that $g(x) = 0$ we have $\deg(x) \leqslant 2m$. Therefore for any $x \in X$ of degree $2m$ the map $L_{m-1} \xrightarrow{\rho} F(x)$, $g \mapsto g(x)$, is injective.

Lemma 6.1. *For any $x \in X$ of degree $2m$ and any $w \in F(x)^*$ there exist functions $g \in L_{m-1}$, $h \in L_1$ such that $w = (gh^{-1})(x)$.*

Proof. Let A be the image of the map ρ above, $\dim F(x)/A = 1$. Consider the map

$$L_1 \to F(x)/A, \quad h \mapsto h(x) \cdot w \mod A.$$

Since $\dim L_1 = 3$, there exists a nonzero function $h \in L_1$ such that $0 \neq h(x) \cdot w \in A$, i.e., $h(x) \cdot w = g(x)$ for some function $g \in L_{m-1}$. $\square$

For any $m \geqslant 1$ by A_m we denote the image of the natural map

$$\coprod_{\deg x \leqslant 2m} k_{n-1}F(x) \to H^1(X, k_n).$$

It is clear that $A_1 \subset A_2 \subset \cdots \subset H^1(X, k_n) = \bigcup_m A_m$.

Lemma 6.2. *$H^1(X, k_n) = A_1$, i.e., any element in $H^1(X, k_n)$ can be represented by some k_{n-1}-cycle with support in the set of points of degree 2.*

Proof. We shall show that $A_1 = A_m$ by induction on m. Let $x \in X$ be a point of degree $2m > 2$ and $w = \{w_1, w_2, \ldots, w_{n-1}\} \in k_{n-1}F(x)$. By lemma 6.1 we can find functions $g_i \in L_{m-1}$, $h_i \in L_1$ such that $w_i = (g_i h_i^{-1})(x)$ for any $i = 1, 2, \ldots, n-1$. Since $\operatorname{Pic} X = 0$, there exists a function $f \in F(X)^*$ such that $\operatorname{div} f = x$. Consider the symbol

$$t = \{g_1 h_1^{-1}, g_2 h_2^{-1}, \ldots, g_{n-1} h_{n-1}^{-1}, f\} \in k_n F(X).$$

It is clear that $\partial(t) = u \cdot x + (k_{n-1}$-cycle with support in points of degree

$$\leqslant (2m-2) \in \coprod_{z \in X} k_{n-1}F(z).$$

Therefore, $u \cdot x \in A_{m-1} = A_1$ by the inductive hypothesis. $\square$

Lemma 6.3. *If $x \in X$ such that $F(x) \simeq L$, then the image of $k_{n-1}F(x)$ in $H^1(X, k_n)$ is trivial.*

Proof. The image of $k_{n-1}F(x)$ in $H^1(X, k_n)$ belongs to the image of the norm map $n_{L/F} \colon H^1(X_L, k_n) \to H^1(X, k_n)$. But $H^1(X_L, k_n) = 0$ since $X_L = \mathbb{A}^1_L - pt$. $\square$

Corollary 6.4. *Any element in the kernel of the norm map*

$$N \colon H^1(X, k_n) \to k_{n-1}F / n_{L/F}(k_{n-1}L)$$

can be represented by k_{n-1}-cycle $\sum w_x \cdot x$, $w_x \in K_{n-1}F(x)$, with support in the set of points of degree 2 such that $\sum n_x(w_x) = 0 \in k_{n-1}F$.

Proof. By Lemma 6.2 we can find the representation we need but $\sum n_x(w_x) = n_{L/F}(s)$ for some $s \in k_{n-1}L$. Choose a point $x_0 \in X$ such that $F(x_0) \simeq L$. By Lemma 6.3 we can replace w_x by

$$w'_x = \begin{cases} w_x, & \text{if } x \neq x_0 \\ w_{x_0} + s, & \text{if } x = x_0. \end{cases} \quad \square$$

Let $D = \binom{a,b}{F}$ be the quaternion algebra. There is a canonical map [10]:

$$p\colon D^* - F^* \to \{\text{points of degree 2 in } Y\}$$

such that for any $w \in D^* - F^*$ there is a natural isomorphism of the (maximal) quadratic subfield $F[w] \subset D$ and the residue field of the point $p(w) \in Y$. Therefore we have a map

$$D^* - F^* \to \{K_1\text{-cycles on } Y\}, \quad w \mapsto w \cdot p(w).$$

We extend this map to D^* taking $c \in F^*$ to $c \cdot \infty$. For any $w \in D^*$ the corresponding K_1-cycle on Y we simply denote by $[w]$. It is proved in [10] that the composition $D^* \to \{K_1\text{-cycles on } Y\} \to H^1(Y, K_2)$ is a group homomorphism. Using this property, Corollary 6.4 and also the fact that for any quadratic extension E/F the group $K_m E$ is generated by the symbols $\{x, a_1, \dots, a_{m-1}\}$ where $x \in E^*$, $a_i \in F^*$, one proves the following

Lemma 6.5. *Any element in the kernel of the map*

$$H^1(X, k_n) \to k_{n-1}F/n_{L/F}(k_{n-1}L)$$

can be represented by k_{n-1}-cycle of the type $\sum_\alpha [u_\alpha] \cdot \underline{a}_\alpha$, where the sum is taken over all the maps $\alpha\colon [1, n-2] \to [1, m]$ for some $m > 0$, $u_\alpha \in D^*$, $\underline{a}_\alpha = \{a_{\alpha(1)}, \dots, a_{\alpha(n-2)}\} \in k_{n-2}F$, where the images of $a_1, a_2, \dots, a_m \in F^*$ are linearly independent in F^*/F^{*2} and

$$\sum_\alpha (\operatorname{Nrd} u_\alpha) \cdot \underline{a}_\alpha = 0 \in k_{n-1}F. \quad \square$$

Let θ be any element in the kernel of the map $H^1(X, k_n) \to k_{n-1}F/n_{L/F}(k_{n-1}L)$. We write $\theta = \sum_\alpha [u_\alpha] \cdot \underline{a}_\alpha$ as in Lemma 6.5. By Proposition 4.1. there exist $a_{m+1}, \dots, a_r \in F^*$, $c_{s,j}, d_{s,j} \in F$ such that

$$\operatorname{Nrd}(u_\alpha) = \prod_s \Big(\prod_{j=1}^{n-2} (c_{s,j}^2 - a_s(j) \cdot d_{s,j}^2) \Big).$$

Let $u_\alpha = s_\alpha + t_\alpha \cdot i + v_\alpha \cdot j + w_\alpha \cdot ij$ where $s_\alpha, t_\alpha, v_\alpha, w_\alpha \in F$ and i, j are generators of the quaternion algebra D such that $i^2 = a$, $j^2 = b$, $ij = -ji$. We have: $\operatorname{Nrd} u_\alpha = s_\alpha^2 - at_\alpha^2 - bv_\alpha^2 + abw_\alpha^2$.

Consider indeterminates $A, B, P, Q, S_\alpha, T_\alpha, V_\alpha, W_\alpha$ for all $\alpha\colon [1, n-2] \to [1, r]$ and the affine variety V over F defined by the equations:

$$\begin{cases} S_\alpha^2 - A \cdot T_\alpha^2 - B \cdot V_\alpha^2 + AB \cdot W_\alpha^2 = \operatorname{Nrd}(u_\alpha) \\ P^2 - A \cdot Q^2 = B \end{cases}$$

and also the variety W defined in coordinates $A, B, S_\alpha, T_\alpha, V_\alpha, W_\alpha$ by the equations of the first type.

There is a natural dominating flat morphism $\pi\colon V \to W$. The point $x \in X$ with the coordinates $A = a, B = b, S_\alpha = s_\alpha, T_\alpha = t_\alpha, V_\alpha = v_\alpha, W_\alpha = w_\alpha$ is nonsingular. The fiber Y_x is isomorphic to the affine conic X.

Let ξ be the generic point of W. Choose any specialization map

$$s\colon H^1(V_\xi, k_n) \to H^1(V_x, k_n) = H^1(X, k_n).$$

Let $\left(\frac{A,B}{F(W)}\right)$ be the quaternion algebra over $F(W)$ with the anticommutative generators I and J such that $I^2 = A$, $J^2 = B$. Consider the element

$$\Theta = \sum_\alpha [S_\alpha + T_\alpha \cdot I + V_\alpha \cdot J + W_\alpha \cdot IJ] \cdot \underline{a}_\alpha \in H^1(V_\xi, k_n).$$

Since $N(\Theta) = \sum_\alpha (\operatorname{Nrd} u_\alpha) \cdot \underline{a}_\alpha = 0$, we have

$$\Theta \in \ker[H^1(V_\xi, k_n) \to k_{n-1}F(W)/n(k_{n-1}L(W))].$$

It is clear that $s(\Theta) = \theta$.

Consider the natural dominating projection $V \to \mathbb{A}^3$ with respect to coordinates A, P and Q. The generic fiber of this morphism is the product of several affine quadrics corresponding to the isotropic quadratic form $\langle 1, -A, -B, AB\rangle$ over $F(\mathbb{A}^3)$, and, therefore is a rational variety over $F(\mathbb{A}^3)$. Hence V is the rational variety over F.

Lemma 6.6. *Assume that h_{n-1} is an isomorphism for all fields. Then for any purely transcendental extension E/F the homomorphism $k_nE/k_nF \to H^nE/H^nF$ induced by the map $h_{n,E}$ is an isomorphism.*

Proof. We may assume that $E = F(\mathbb{A}^1)$. In this case the homomorphism in question is isomorphic to

$$\coprod h_{n-1,F(x)}\colon \coprod_{x\in\mathbb{A}^1} k_{n-1}F(x) \to \coprod_{x\in\mathbb{A}^1} H^{n-1}F(x). \quad \square$$

Corollary 6.7. *The natural homomorphism $H^1(V_\xi, k_n) \to H^1(V_\xi, H^n)$ is an isomorphism.*

Proof. The field $F(\xi)(V_\xi) = F(V)$ is a purely transcendental over F. $\square$

Proposition 6.8. *If B_n is trivial and h_{n-1} is an isomorphism for all fields, then the norm map $N\colon H^1(X, k_n) \to k_{n-1}F/n_{L/F}(k_{n-1}L)$ is surjective.*

Proof. We may clearly assume that X is not split. Let θ be in the kernel. It follows from Proposition 3.15, Corollary 6.7 and the commutative diagram:

$$\begin{array}{ccc}
H^1(V_\xi, k_n) & \longrightarrow & H^1(V_\xi, H^n) \\
\Big\downarrow N & & \Big\downarrow N \\
k_{n-1}F(W)/n(k_{n-1}L(W)) & \underset{\sim}{\longrightarrow} & H^{n-1}F(W)/n(H^{n-1}L(W))
\end{array}$$

that the left vertical map is injective, hence $\Theta = 0 \in H^1(V_\xi, k_n)$. Applying the specialization map s we get: $\theta = s(\Theta) = 0$. $\square$

7. Lifting property for a conic curve

Let X be an affine conic curve as in section 3. Denote the groups $H^0(X,k_n)/k_nF$ and $H^0(X,H^n)/H^nF$ by $\overline{H}^0(X,k_n)$ and $\overline{H}^0(X,H^n)$ respectively.

Lemma 7.1. *Let B_n be trivial and h_{n-1} be an isomorphism for all fields. Then the map*

$$\coprod_{E/F \text{ splits } X} \overline{H}^0(X_E,H^n) \xrightarrow{\sum n_{E/F}} \overline{H}^0(X,H^n)$$

is surjective.

Proof. It follows from the commutative diagram:

$$\begin{array}{ccccc} H^nF & \to & H^nX & \to H^{n-1}F \xrightarrow{(a,b)} H^{n+1}F \\ & \searrow & \downarrow & \\ & & H^nF(X) & \\ & & \downarrow & \\ & & \coprod_{x\in X} H^{n-1}F(x) & \end{array}$$

and Proposition 3.15 that there is a canonical surjection (see Remark 3.7)

$$H^1(Y,H^n) \twoheadrightarrow (\ker(H^{n-1}F \xrightarrow{(a,b)} H^{n+1}F)) \twoheadrightarrow \overline{H}^0(X,H^n).$$

The statement of lemma follows from the commutative diagram:

$$\begin{array}{ccc} \coprod_{E \text{ splits } X} H^1(Y_E,H^n) & \longrightarrow & \coprod_{E \text{ splits } X} \overline{H}^0(X_E,H^n) \\ \downarrow\!\!\!\downarrow & & \downarrow \\ H^1(Y,H^n) & \twoheadrightarrow & \overline{H}^0(X,H^n) \end{array}$$

and the surjectivity of the left vertical map. $\square$

Proposition 7.2. *Let B_n be trivial and h_{n-1} be an isomorphism for all fields. Then:*

(1) $\overline{H}^0(X,k_n) \to \overline{H}^0(X,H^n)$ is surjective map,
(2) $H^1(X,k_n) \to H^1(X,H^n)$ is injective map.

Proof. (1) If X is split, then by Lemma 6.6 $k_nF(X)/k_nF \simeq H^nF(X)/H^nF$, hence the first map is surjective. In general statement (1) follows from Lemma 7.1 and the commutative diagram:

$$\begin{array}{ccc} \coprod_{E \text{ splits } X} \overline{H}^0(X_E,k_n) & \xrightarrow{\sim} & \coprod_{E \text{ splits } X} \overline{H}^0(X_E,H^n) \\ \downarrow & & \downarrow \\ \overline{H}^0(X,k_n) & \longrightarrow & \overline{H}^0(X,H^n) \end{array}$$

(2) Follows from the commutative diagram:

$$\begin{array}{ccc}
H^1(X,k_n) & \longrightarrow & H^1(X,H^n) \\
\Big\downarrow\cap & & \Big\downarrow \\
k_{n-1}F/n_{L/F}(k_{n-1}L) & \xrightarrow{\sim} & H^{n-1}F/n_{L/F}(H^{n-1}L)
\end{array}$$

and Proposition 6.8. □

Corollary 7.3. *In the conditions of the Proposition assume that $h_{n,F}$ is surjective. Then $h_{n,F(X)}$ is also surjective.*

Proof. Follows from the commutative diagram:

$$\begin{array}{ccccccccc}
k_nF & \longrightarrow & k_nF(X) & \longrightarrow & \coprod_{x\in X} k_{n-1}F(x) & \longrightarrow & H^1(X,k_n) & \longrightarrow & 0 \\
\Big\downarrow h_{n,F} & & \Big\downarrow h_{n,F(X)} & & \Big\downarrow\wr & & \Big\downarrow & & \\
H^nH & \longrightarrow & H^nF(X) & \longrightarrow & \coprod_{x\in X} H^{n-1}F(x) & \longrightarrow & H^1(X,H^n) & \longrightarrow & 0
\end{array}$$

□

8. Proof of the Theorem

We follow the approach given in [7] in the case $n=2$. Consider any quadratic extension $L=F(\sqrt{a})/F$.

Proposition 8.1. *If B_n is trivial and $h_{n-1,F}$ is an isomorphism for all fields, then the sequence*

$$k_nF \to k_nL \xrightarrow{n_{L/F}} k_nF$$

is exact.

Proof. Let $u\in k_nL$ and $n_{L/F}(u)=0\in k_nF$. We can write u in the form $u=\sum_\alpha x_\alpha\cdot \underline{a}_\alpha$ where the sum is taken over all maps $\alpha\colon [1,n-1]\to[1,m]$, $x_\alpha\in L^*$, $\underline{a}_\alpha=\{a_{\alpha(1)},\dots,a_{\alpha(n-1)}\}\in k_{n-1}F$ and $a_1,\dots,a_m\in F^*$ are independent in F^*/F^{*2}.

Since $0=n_{L/F}(u)=\sum_\alpha n_{L/F}(x_\alpha)\cdot\underline{a}_\alpha\in k_nF$, it follows from Proposition 4.1 that there exist $a_{m+1},\dots,a_r\in F^*$, $c_{s,j},d_{s,j}\in F^*$ such that

$$n_{L/F}(x_\alpha)=\prod_s\Big(\prod_{j=1}^{n-1}(c_{s,j}^2-a_{s(j)}d_{s,j}^2)\Big)$$

for all $\alpha\colon[1,n-1]\to[1,r]$.

Consider the indeterminates A_i $(i=1,\dots,r)$, X'_α, X''_α $(\alpha\colon[1,n-1]\to[1,r])$, $C_{s,j}$, $D_{s,j}$. Denote $\prod_{i\in T}A_i$ by A_T for $T\subset[1,r]$. We consider the affine variety

X defined over the field $F_1 = F_0(a)$, where F_0 is the prime subfield in F, by the equations:

$$X'^{\,2}_\alpha - aX'^{\,2}_\alpha = \prod_s (\prod_{j=1}^{n-1} (C^2_{s,j} - A_{s(j)} D^2_{s,j}))$$

for all $\alpha\colon [1, n-1] \to [1, r]$.

Let $L_1 = F_1(\sqrt{a})$; consider the element

$$U = \sum_\alpha (X'_\alpha + X''_\alpha \sqrt{a}) \cdot \underline{A}_\alpha \in k_n L_1(X),$$

where $\underline{A}_\alpha = \{A_{\alpha(1)}, \dots, A_{\alpha(n-1)}\} \in k_{n-1} F_1(X)$. By Proposition 4.1

$$n_{L_1(X)/F_1(X)}(U) = 0.$$

Let E be a purely transcendental extension $F_1(A_i, C_{s,j}, D_{s,j})$ of F_1. The field F_1 is either finite, or global or isomorphic to $\mathbb{Q}(t)$. In all the cases h_{n,F_1} is an isomorphism, hence $h_{n,E}$ is an isomorphism.

The function field $F_1(X)$ is isomorphic over E to the function field of the product of several conics, therefore, by Corollary 7.3 $h_{n,F_1(X)}$ is surjective.

The conics mentioned above are split by the field $E(\sqrt{a})$. Hence $L_1(X)$ is a purely transcendental extension of L_1 and since h_{n,L_1} is an isomorphism, $h_{n,L_1(X)}$ is also an isomorphism. It follows from the commutative diagram:

$$\begin{tikzcd}
k_n F_1(X) \arrow[r, "i"] \arrow[d, two heads] & k_n L_1(X) \arrow[r, "n_{L/F}"] \arrow[d, "\wr"] & k_n F_1(X) \arrow[d] \\
H^n F_1(X) \arrow[r] & H^n L_1(X) \arrow[r] & H^n F_1(X)
\end{tikzcd}$$

that $U \in \operatorname{im}(i)$.

There is a nonsingular point $x \in X$ and the specialization maps s and s' fitting to the commutative diagram

$$\begin{tikzcd}
k_n F_1(X) \arrow[r, "i"] \arrow[d, "s'"'] & k_n K_1(X) \arrow[d, "s"] \\
k_n F \arrow[r, "j"] & k_n L
\end{tikzcd}$$

such that $s(U) = u$, hence $u = s(U) \in \operatorname{im} j$. $\square$

Corollary 8.2. *If α_F is injective, then α_L is also injective.*

Proof. Follows from the commutative diagram:

$$\begin{tikzcd}
k_{n-1} F \arrow[r, "\{a\}"] \arrow[d, "\wr"] & k_n F \arrow[r] \arrow[d, hook] & k_n L \arrow[r, "n_{L/F}"] \arrow[d] & k_n F \arrow[d, hook] \\
H^{n-1} F \arrow[r, "(a)"] & H^n F \arrow[r] & H^n L \arrow[r, "n_{L/F}"] & H^n F.
\end{tikzcd}$$ $\square$

Proposition 8.3. *Assume that B_n is trivial and h_{n-1} is an isomorphism for all fields. Then the sequence*

$$k_{n-1}F \xrightarrow{\{a\}} k_nF \to k_nL$$

is exact.

Proof. Let $u \in k_nF$ and $u_L = 0$. We can write u in the form $u = \sum_\alpha b_\alpha \cdot \underline{a}_\alpha$ where the sum is taken over all maps $\alpha\colon [1, n-1] \to [1, m]$, $b_\alpha \in F^*$, $\underline{a}_\alpha = \{a_{\alpha(1)}, \dots, a_{\alpha(n-1)}\} \in k_{n-1}F$, $a_1, a_2, \dots, a_m \in F^*$ and we may assume that a_i are linearly independent in L^*/L^{*2}. By Proposition 4.1 there exist $a_{m+1}, \dots, a_r \in L^*$, $c_{s,j}, d_{s,j} \in L$ such that

$$b_\alpha = \prod_s \Big(\prod_{j=1}^{n-1} (c_{s,j}^2 - a_{s(j)} d_{s,j}^2) \Big)$$

for all maps $\alpha\colon [1, n-1] \to [1, r]$.

Consider the indeterminates

$$A_i (i = 1, \dots, m); A_i', A_i'' (i = m+1, \dots, r);$$

$$B_\alpha(\alpha\colon [1, n-1] \to [1, r]), C_{s,j}', C_{s,j}'', D_{s,j}', D_{s,j}''.$$

Put $A_i = A_i' + A_i'' \cdot \sqrt{a}$ for $i = m+1, \dots, r$ and $A_T = \prod_{i \in T} A_i$ for any $T \subset [1, r]$.

We consider the affine variety Y over a field $F_1 = F_0(a)$ given by the equations:

$$B_\alpha = \prod_s \Big[\prod_{j=1}^{n-1} ((C_{s,j}' + C_{s,j}''\sqrt{a})^2 - A_{s(j)} \cdot (D_{s,j}' + D_{s,j}''\sqrt{a})^2 \Big].$$

Each equation is considered as a system of two equations over F_1.

Consider the element $U = \sum_\alpha B_\alpha \cdot \underline{A}_\alpha \in k_nF_1(Y)$, where

$$\underline{A}_\alpha = \{A_{\alpha(1)}, \dots, A_{\alpha(n-1)}\} \in k_{n-1}F_1(X).$$

By Proposition 4.1

$$U \in \ker(k_nF_1(X) \to k_nL_1(X)),$$

where $L_1 = F_1(\sqrt{a})$.

There is a nonsingular F-point $x \in X$ and a specialization map $s\colon k_nF_1(Y) \to k_nF$ such that $s(U) = u$.

Consider a purely transcendental extension E of F obtaining by joining all indeterminates except $C_{s,1}'$, $C_{s,1}''$, $D_{s,1}'$, $D_{s,1}''$ where $s(k)$ consists of one element for all $k = 1, \dots, n-1$. The function field $F_1(Y)$ is isomorphic over E to the function field of the product of varieties of the type $Z = R_{E(\sqrt{a})/E}(Q)$, where Q is an affine conic curve over $E(\sqrt{a})$ and R is the Weil transfer [16].

Lemma 8.4. *There are quadratic extensions $E(Z) \supset E_1 \supset E_2$ such that E_2 is a purely transcendental extension of E.*

Proof. Let $Q \to \mathbb{A}^1_{E(\sqrt{a})}$ be any double cover. Then it is easy to see that the extension $E(Z)$ of degree 4 over $R_{E(\sqrt{a})/E}(\mathbb{A}^1_{E(\sqrt{a})})$ contains an intermediate subfield. $\square$

As in the proof of the Proposition 8.1 $h_{n,E}$ is an isomorphism. By Corollary 8.2 and Lemma 8.4 $\alpha_{F_1(Y)}$ is injective. It follows from the commutative diagram:

$$\begin{array}{ccccc} k_{n-1}F_1(Y) & \xrightarrow{\{a\}} & k_nF_1(Y) & \longrightarrow & k_nL_1(Y) \\ \downarrow\wr & & \cap\downarrow & & \downarrow \\ H^{n-1}F_1(Y) & \xrightarrow{(a)} & H^nF_1(Y) & \longrightarrow & H^nL_1(Y) \end{array}$$

that the top row is exact, hence U is divisible by $\{a\}$. Applying the specialization map in the commutative diagram

$$\begin{array}{ccc} k_{n-1}F_1(Y) & \xrightarrow{\{a\}} & k_nF_1(Y) \\ \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s} \\ k_{n-1}F & \xrightarrow{\{a\}} & k_nF \end{array}$$

we deduce that u is in the image of the bottom map. $\square$

Proof of the Theorem. Using Lemma 2.3 by induction on n we may assume that h_{n-1} is an isomorphism for all fields.

Injectivity of $h_{n,F}$. Let $u \in k_nF$ and $h_n(u) = 0 \in H^nF$. Assume that u is the sum of k symbols in k_nF. We prove by induction on k that $u = 0$. One can find a quadratic extension $L = F(\sqrt{a})/F$ such that u_L is the sum of $k-1$ symbols, hence by inductive hypothesis $u_L = 0$. It follows from the commutative diagram:

$$\begin{array}{ccccccc} k_{n-1}L & \xrightarrow{n_{L/F}} & k_{n-1}F & \xrightarrow{\{a\}} & k_nF & \longrightarrow & k_nL \\ \downarrow\wr & & \downarrow\wr & & \downarrow{\scriptstyle h_n} & & \downarrow \\ H^{n-1} & \xrightarrow{n_{L/F}} & H^{n-1}F & \xrightarrow{(a)} & H^nF & \longrightarrow & H^nL \end{array}$$

and Proposition 8.3 that $u = 0$.

Surjectivity of $h_{n,F}$. Assume first that F has no odd degree extensions. Let $u \in H^nF$ and E be a finite extension of F such that $u_E = 0$. We prove by induction on the degree $[E : F]$ that $u \in \operatorname{im} h_{n,F}$. By the assumption one can find an intermediate field L, $F \subset L \subset E$ such that $[L : F] = 2$. By the inductive hypothesis $u_L \in \operatorname{im} h_{n,L}$. The inclusion $u \in \operatorname{im} h_{n,F}$ follows from the injectivity of h_n, which is already proved, Proposition 8.1 and the diagram in the proof of Corollary 8.2.

In general, using the first case one can find an odd degree extension E/F such that $u_E \in \operatorname{im} h_{n,E}$. Then $u = n_{E/F}(u_E) \in \operatorname{im} h_{n,F}$ since h_n commutes with norms. $\square$

Corollary. *The surjectivity of h_n for all fields implies the bijectivity of h_n also for all fields.*

Proof. Follows from Theorem and Lemma 2.1. $\square$

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