LOCALISATION OF THE DONALDSON'S INVARIANTS ALONG SEIBERG-WITTEN CLASSES.

Victor Pidstrigach and Andrei Tyurin

Abstract. This article is a first step in establishing a link between the Donaldson polynomials and Seiberg-Witten invariants of a smooth 4-manifold.

§0. Introduction.

As the result of analysis of the structure of the Donaldson's invariant of a smooth simply connected 4 manifold X Kronheimer and Mrowka [K-M1] (see also [F-S]) proved the existence of k_X classes

$$K_1, \dots, K_{k_X} \in H_2(X, \mathbb{Z})$$

and k_X polynomials

$$f_1, \dots, f_{k_X} \in \mathbb{Q}[z]$$

such that the Donaldson's invariant of X is defined uniquely by these pairs.

These classes are called the basic classes of X and the set $\{K_i\}$, for $i = 1, \dots, k_X$, in $H^2(X, \mathbb{Z})$ is a diffeomorphism invariant. So the diffeomorphism group $Diff X$ admits a representation to the symmetric group on k_X letters. A little bit later Seiberg and Witten introduced another set of classes $\{SW\}$ and polynomials and Witten predicted the shape of the Donaldson's invariant in terms of these data. S-W classes are labelling the moduli spaces $\{\mathcal{M}_{SW}\}$ of the solutions of the Seiberg-Witten system of equations (see §1) and the dimensions of these moduli spaces are the degrees of the polynomials. Using other type of system of equations we will prove that the classes of Kronheimer and Mrowka and Seiberg-Witten coincide. This paper is the first step of this proof and contains those facts from gauge theory which we need for the proof.

At various stages of this project useful were conversations with Ed Witten, S.Donaldson, S.Bauer. We are grateful to S.Donaldson who invited us to the conference at the Newton Institute in December 1994 where we described these results and to I. Hambleton for his kind invitation to the MPI, Bonn. The first author acknowledges with gratitude financial support of the AMS (fSU grant) and the VW-Stiftung during the work over this project as well as the kind invitation of S.Bauer and the hospitality of Bielefeld University.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§1. Configuration spaces and its cohomology.

Let $c \in H^2(X)$ be a $Spin^c$ structure on X, i.e. a pair of complex hermitian rank 2 bundles $W_c^{\pm}, c_1(W_c^{\pm}) = c$ (we shall omit the subscript c if there is no confusion). Then the tangent bundle

$$TX = Hom_h(W^-, W^+) \tag{1.1}$$

is the bundle of homomorphisms preserving the Hermitian structures.

Fix a pair of connections $\nabla^{\pm}$ on $W^{\pm}$ such that their tensor product is the Levi Civita connection on TX (which is the same as fixing the determinant $\nabla_{det} = det\nabla^{\pm}$). This defines a Dirac operator:

$$\mathcal{D}^{g,\nabla_{det}} : \Gamma(W^+) \to \Gamma(W^-) \tag{1.2}$$

as a composition of the connection $\nabla^+ : \Gamma(W^+) \to \Gamma(W^+ \otimes T^*X)$ and the convolution along (1.1). We shall omit the superscripts if this will not cause confusion.

U(1) - monopoles.

Let us recall that the configuration space $\mathcal{C}_{detW^+}$ for $U(1)$-Seiberg-Witten moduli space is

$$\mathcal{A}_{detW^+} \times \Gamma(W^+)/\mathcal{G}_{detW^+} = \mathcal{N} \times \Gamma(W^+)/S^1 \tag{1.3}$$

where $\mathcal{N} = ker(d^* : \Omega^1 \to \Omega^0)$ is a slice of an action of $\mathcal{G}_{detW^+}$. The Seiberg-Witten moduli spaces $\mathcal{M}_{SW}(0, c)$ are defined as the space of solutions of the system

$$\mathcal{D}^{\nabla_{det}}\phi = 0$$

$$F^+_{\nabla_{det}} = -(\phi \otimes \overline{\phi})_0, \tag{1.4}$$

modulo action of the gauge group, where $(\nabla_{det}, \phi) \in \mathcal{A}_{detW^+} \times \Gamma(W^+)$. Over the subspace $\mathcal{C}^*_{detW^+} = \mathcal{N} \times \mathbb{P}(\Gamma(W^+))$ of non-reducible points of $\mathcal{C}_E$ there is a complex linear bundle $\mathcal{O}_{\mathcal{C}^*}(1) = \mathcal{O}_{\mathbb{P}(\Gamma(W^+))}(1)$ with the first Chern class $t = c_1(\mathcal{O}_{\mathcal{C}^*}(1))$ generating the cohomology ring $H^*(\mathcal{C}^*, \mathbb{Q})$ in the case X is simply connected.

The system of equations (1.4) can be gauge invariantly perturbed as

$$\mathcal{D}^{\nabla_{det}}\phi = 0$$

$$(\omega + F_{\nabla_{det}})^+ = -(\phi \otimes \overline{\phi})_0, \tag{1.4$_\omega$}$$

where ω is any 2-form.

The Seiberg-Witten moduli spaces $\mathcal{M}_{SW}(\omega, c)$ are defined as the space of solutions of the perturbed system $(1{,}4_\omega)$ modulo the action of the gauge group. For a generic form ω the moduli space is cut out transversely and therefore smooth.

We shall also consider another perturbation - perturbation of the metric g. The similar "generic position" statement for this space of parameters is provided by the following

Lemma 1.1. *Fix some 2-form ω and consider the map*

$$\mathcal{M}etr \times \Omega^1 \times \mathcal{A}_{detW^+} \times \Gamma(W^+) \xrightarrow{f} \Gamma(W^-) \times \Omega^2_+ \tag{1.5}$$

given by sending

$$(g, \xi, \nabla_{det}, \phi) \to (\mathcal{D}^{g,\nabla_{det}}\phi, (\omega + F_{\nabla_{det}})^+ + (\phi \otimes \overline{\phi})_0. \tag{1.6}$$

This map is a submersion.

Proof. We shall remark first that since the principal symbol of the Dirac operator, as well as of the Levi-Civita connection, doesn't depend on the metric, the dependence is given by the formula:

$$\mathcal{D}^{g+r,\nabla_{det}} = \mathcal{D}^{g,\nabla_{det}} + \delta_r \tag{1.7}$$

where $\delta_r \in \Omega^1$. Thus

$$\delta f(r, \alpha, \delta\nabla_{det}, \delta r) = (\delta_1 f, \delta_2 f)$$

where

$$\delta_1 f = (\alpha + \delta\nabla_{det} + \delta_r) \otimes \phi + \mathcal{D}^{g,\nabla_{det}}(\delta\phi),$$
$$\delta_2 f = r^*(F^+_{\nabla_{det}} + \omega) + d^+_{\nabla_{det}}(\delta\nabla_{det}) + (\delta\phi \otimes \overline{\phi})_0 + (\phi \otimes \overline{\delta\phi})_0. \tag{1.8}$$

It is a standard observation that the restriction

$$\delta_1 f_{|r=\delta\nabla_{det}=0} = (\alpha + \delta_r) \otimes \phi + \mathcal{D}^{g,\nabla_{det}}(\delta\phi)$$

is an epimorphism. Therefore it is enough to prove that the restriction

$$\delta_2 f|_{\alpha=\delta\phi=0}$$

is onto. This follows from the fact that for any non zero 2-form $(F^+_{\nabla_{det}} + \omega)$ and for any harmonic self dual 2-form χ one can find an infinitesimal deformation of the metric r such that the pairing

$$\langle r^*(F^+_{\nabla_{det}} + \omega), \chi\rangle \neq 0.$$

Indeed, by Cor.4.2.23 of [D-K] such a χ vanishes on an open set if only $\chi = 0$ everywhere.

There is an another gauge invariant perturbation of the equation. Let

$$\mathcal{T}_{\frac{1}{2}} \subset \Gamma(End(\Lambda^0 \oplus \Lambda^2_+)) \tag{1.9}$$

be a subspace of endomorphisms t of the bundle $\Lambda^0 \oplus \Lambda^2_+$ subject to the condition $|t| \leq \frac{1}{2}$. Any such t defines an equation as follows

$$\mathcal{D}^{\nabla_{det}}\phi = 0$$
$$(\omega + F_{\nabla_{det}})^+ = ((t-1)(\phi \otimes \overline{\phi}))_0, \tag{1.4$_{\omega,t}$}$$

At points where $|\phi|$ has its maximal value, the Weitzenböck formula implies

$$0 \leq \Delta|\phi| \leq \frac{-s}{2}|\phi|^2 + \langle((t-1)(\phi\otimes\overline{\phi}))_0(\phi),\phi\rangle =$$

$$= \frac{-s}{2}|\phi|^2 - |\phi|^4 + \langle(t(\phi\otimes\overline{\phi}))_0(\phi),\phi\rangle \leq \frac{|\phi|^2}{2}(-s-|\phi|^2+|t||\phi|^2) \tag{1.10}$$

and as a consequence

$$|\phi|^2 \leq \frac{-s}{(1-|t|)} \leq -2s$$

provided our condition $|t| \leq \frac{1}{2}$ is true (compare [K-M2]).

Thus, following well known properties of $\mathcal{M}_{SW}(\omega,c)$ for a generic metric g (see for example [K-M2], [W]) are true also for the moduli $\mathcal{M}_{SW}(\omega,t,c)$ of solutions of $1.4_{\omega,t}$:

1) there exists only a finite set of classes $c \in H^2(X)$ such that $\mathcal{M}_{SW}(\omega,t,c) \neq \emptyset$;

2) every moduli space $\mathcal{M}_{SW}(\omega,t,c)$ is compact;

3) if X admits a metric with positive scalar curvature then $\mathcal{M}_{SW}(0,t,c)=\emptyset$ (in the relevant chamber in case $b_2^+(X)=1$).

So we can see that the Seiberg-Witten system of equations (1.4) and ($1.4_{\omega,t}$) is very strong and that the space of solutions is very rigid.

The strategy to get a non trivial solution is to consider a degenerated solution $(\nabla_{det},0) \in \mathcal{M}_{SW}(\omega,c)$ for some ω and to use the description of Kuranishi map for this singular point.

Non abelian monopoles.

Now take a $PU(r)$-bundle ξ on X and fix its lift to a $U(r)$-bundle E that is fix $c_1(E) \in H^2(X)$. As a configuration space consider the set of triples

$$(a_0, \nabla_{det}, \phi) \in \mathcal{A}(P(E)) \times \mathcal{A}(det(E\otimes W^+)) \times \Gamma(E\otimes W^+) \tag{1.11}$$

where a_0 is a traceless part of $U(r)$-connection, ∇_{det} is an abelian connection on the line bundle $det(E\otimes W^+)$ and ϕ is a section of the vector bundle $E\otimes W^+$. The first two components of a triple (1.11) define a coupled Dirac operator

$$\mathcal{D}_{a_0}^{\nabla_{det}} : \Gamma(E\otimes W^+) \to \Gamma(E\otimes W^-) \tag{1.12}$$

by the Leibnitz rule for the action of the connection along E and the ordinary Dirac operator (1.2) along W^+. Now we can consider the non abelian analog of the system (1.4) (see [V-W]) :

$$\mathcal{D}_{a_0}^{\nabla_{det}}(\phi) = 0$$

$$(F_{a_0} + \frac{1}{2}F_{\nabla_{det}} \otimes id_E)^+ = -(\phi\otimes\overline{\phi})_0, \tag{1.13}$$

and the symmetry group of this system is given as the central extension

$$1 \to \mathcal{G}_{det(E\otimes W^+)} \to \mathcal{G} \to \mathcal{G}_\xi \to 1. \tag{1.14}$$

The moduli spaces $\mathcal{M}_{SW}(0, 2c_1 + rc, p_1)$ of Seiberg-Witten monopoles are defined as a space of solutions of this system modulo the action of the gauge group (1.14), where p_1 is the Pontrjagin number of E.

In the same vein we can define the ω, t perturbed system considering

$$(F_{a_0} + (\frac{1}{2}F_{\nabla_{det}} + \omega) \otimes id_E)^+ = ((t-1)(\phi \otimes \overline{\phi}))_0 \qquad (1.13_\omega)$$

as the second equation of the system (1.13) and get the moduli space

$$\mathcal{M}_{SW}(\omega, t, 2c_1 + rc, p_1).$$

In particular

$$\mathcal{M}_{asd} \subset \mathcal{M}_{SW}(-\frac{1}{2}F_{\nabla_{det}}, t, 2c_1 + rc, p_1) \qquad (1.15)$$

as the subspace of degenerated solutions of type $(a_0, \nabla_{det}, 0)$.

Non abelian Seiberg-Witten equations are very strong too (in spite of fact that we lost a compactness). In particular it is easy to see that

1) in the symplectic (or Kähler) case we have nontrivial solution of (1.13) only if

$$(2c_1 + rc) \cdot [\omega] \leq 0 \qquad (1.16)$$

where $[\omega]$ is the class of the symplectic (Kähler) form;

2) if p_1 is negative enough then a generic degenerated solution $(a_0, \nabla_{det}, 0)$ can be deformed to a non trivial solution of the system (1.13_ω) for any ω.

3) if X admits a metric with positive scalar curvature then $\mathcal{M}_{SW}(0, 2c_1 + rc, p_1) = \emptyset$ always.

To get a more plastic moduli space we construct a system of equations which does not have any analog in the abelian case.

For simplicity we will consider the case of $rk2$ bundles only and use the following trick: if we fix some connection b_0 on $detW^+$ then the first pair of components (1.11) (a_0, ∇_{det}) is given uniquely by a $U(2)$-connection a on E. So instead of the space of triples (1.11) we can consider the space of pairs

$$(a, \phi) \in \mathcal{A}(E) \times \Gamma(E \otimes W^+) \qquad (1.17)$$

as a configuration space and $\mathcal{G}_E$ as a gauge group.

Let $\omega \in \Omega^2(X)$ be a two form in the cohomology class of $-\pi i c_1(E)$. Denote by $\mathcal{A}_\omega \subset \mathcal{A}_E$ a subspace of the space of all $U(2)$ connections a on the bundle E subject to the condition

$$trF_a = 2\omega, \qquad (1.18)$$

$tr : EndE \to \mathbb{C}$ is a trace. The traceless part of the curvature $F_a - \frac{1}{2}tr(F_a)$ will be denoted as $(F_a)_0 = F_{a_0}$.

A gauge group $\mathcal{G}_E$ of E acts on both spaces $\mathcal{A}_E$ and $\mathcal{A}_\omega$ with orbit spaces, resp., $\mathcal{B}_E$ and $\mathcal{B}_\xi$.

Let us consider the natural action of this group on the space $\mathcal{A}_\omega \times \Gamma(W^+ \otimes E)$ with the orbit space denoted as $\mathcal{C}_E$. So $\mathcal{C}_E$ is the fibration

$$\mathcal{C}_E \to \mathcal{B}_\xi \qquad (1.19)$$

with fibre $\Gamma(W^+ \otimes E)$.

Let $(a, \phi) \in \mathcal{A}_\omega \times \Gamma(W^+ \otimes E)$. Denote by $N_{(a,\phi)}$ a slice of the action at (a, ϕ):

$$N_{(a,\phi)} = Ker(d_a^* \oplus m_\phi^*) \tag{1.20}$$

where

$$d_a \oplus m_\phi : \Omega^0(ad(E)) \to \Omega^1(ad(E)) \times \Gamma(W^+ \otimes E), \tag{1.21}$$

is a tangent map of the action, m_ϕ is the multiplication of a vector spinor ϕ by the endomorphism.

Stabiliser of the gauge action.

The stabiliser $St = St_{(a,\phi)}$ of a point (a, ϕ) is at most $U(2)$ - the largest possible stabiliser of the connection itself. There are three possibilities for $St_a \subset U(2)$ (provided by the fact that St_a is a centraliser of some subgroup of $U(2)$) except the trivial subgroup: $S^1, S^1 \times S^1, U(2)$. The last one corresponds to the pair consisting of the trivial connection and the zero spinor. Its neighbourhood is modelled as $N_{(0,0)}/U(2)$.

The centre $S^1 \subset U(2)$ of the structure group is the stabiliser of a point of type $(a, 0)$, where a is non-reducible. Orbits of such points form the subspace of the zero-section of the fibration (1.19) $\mathcal{B}_\xi \subset \mathcal{C}_E$ with neighbourhood $U = \tilde{U}/S^1$, where $\tilde{U}$ is modelled on the total space of the bundle (1.19):

$$\tilde{U} = \Gamma(W^+ \otimes E) \times_{\mathcal{G}_E/S^1} \mathcal{A}_\omega^* \to \mathcal{B}_\xi^*, \tag{1.22}$$

S^1 acts with a weight one representation on the fibres. The determinant of the universal bundle $det\mathbb{E}$ can be described as the S^1 -equivariant bundle over $X \times \tilde{U}$ associated to the tautological representation of S^1 in $detE$.

If the stabiliser S^1 of (a, ϕ) is a different subgroup of $U(2)$, namely the one which maps onto a maximal torus of $PU(2)$, it means that the connection $a = \lambda_1 \oplus \lambda_2$ is reducible due to some reduction of the bundle $E = L_1 \oplus L_2$, and with spinor of the form $(\phi_1 \oplus \phi_2), \phi_1 = 0, \phi_2 \neq 0$ with respect to the same reduction of the bundle. The subspace of all orbits of such a singularities is a subspace of

$$\mathcal{A}_{L_1} \times \mathcal{A}_{L_2} \times \Gamma(W^+ \otimes L_2)/(\mathcal{G}_{L_1} \times \mathcal{G}_{L_2}) \tag{1.23}$$

consisting of points subject to the condition $F_{\lambda_1} + F_{\lambda_2} = 2\omega$ (which, given λ_2, determines λ_1 up to gauge equivalence) i.e. isomorphic to $\mathcal{C}_{L_2}$. The neighbourhood U of this locus is modelled as follows:

$$U = \mathcal{N} \times (\Omega^1(L_1 \otimes L_2^{-1}) \times \Gamma(W^+ \otimes L_1) \times (\Gamma(W^+ \otimes L_2) - \{0\}))/(S^1 \times S^1). \tag{1.24}$$

This is the total space of the real cone fibration

$$(\Omega^1(L_1 \otimes L_2^{-1}) \otimes \mathcal{O}_{\mathcal{C}^*_{det(W^+ \otimes L_2)}}(-1) \oplus \Gamma(W^+ \otimes L_1) \otimes \mathcal{O}^*_{\mathcal{C}^*_{det(W^+ \otimes L_2)}})/S^1 \tag{1.25}$$

with weight one action of S^1 on the fibres of the vector bundle, over $\mathcal{C}^*_{det(W^+\otimes L_2)}$. The restriction of the universal bundle to U is given as an $S^1\times S^1$ -equivariant fibration over the $S^1\times S^1$-space

$$X\times\mathcal{N}\times(\Omega^1(L_1\otimes L_2^{-1})\times\Gamma(W^+\otimes L_1)\times(\Gamma(W^+\otimes L_2)-\{0\})) \tag{1.26}$$

given by the natural action of $S^1\times S^1$ on $L_1\oplus L_2$. This is equivalent to an S^1-equivariant bundle $L_1\oplus L_2\otimes\mathcal{O}_{\mathcal{C}^*_{det(W^+\otimes L_2)}}(1)$ over the total space of the bundle

$$(\Omega^1(L_1\otimes L_2^{-1})\otimes\mathcal{O}_{\mathcal{C}^*_{det(W^+\otimes L_2)}}(-1)\oplus\Gamma(W^+\otimes L_1)\otimes\mathcal{O}_{\mathcal{C}^*_{det(W^+\otimes L_2)}}). \tag{1.27}$$

The cohomology ring of $\mathcal{C}^*_E$.

Let $(\mathcal{A}_\omega\times\Gamma(W^+\otimes E))^*$ denote the subspace of all points in $\mathcal{A}_\omega\times\Gamma(W^+\otimes E)$ with trivial stabiliser. Denote by $\mathcal{C}^*_E$ the corresponding orbit space. There is a universal bundle $\mathbb{E}$ over the space $X\times\mathcal{C}^*_E$

$$\mathbb{E}=E\times_{\mathcal{G}_E}(\mathcal{A}_\omega\times\Gamma(W^+\otimes E))^*. \tag{1.28}$$

The cohomology ring $H^*(\mathcal{C}^*_E,\mathbb{Q})$ is generated by the 2-dimensional classes

$$\mu(\Sigma)=\frac{1}{4}p_1(\mathbb{E})/[\Sigma],$$

$$t=c_1(\mathbb{E})/[pt] \tag{1.29}$$

and the 4-dimensional class $\nu=\mu(pt)=\frac{1}{4}p_1(\mathbb{E})/[pt]$ as it follows from the fibration

$$\mathbb{R}_+\times\mathbb{CP}^\infty\to(\mathcal{A}_\omega\times\Gamma(W^+\otimes E))^*/\mathcal{G}_E\to\mathcal{A}^*_\xi/\mathcal{G}_\xi. \tag{1.30}$$

The space

$$\mathcal{P}_E=(\mathcal{A}_\omega\times\mathbb{S}(\Gamma(W^+\otimes E))^*/\mathcal{G}_E), \tag{1.31}$$

where $\mathbb{S}()$ is the unit sphere bundle, is a deformation retract of $\mathcal{C}^*_E$ and of the blow-up $\hat{\mathcal{C}}_E$ (cf. §3 below). Therefore the inclusion $\mathcal{P}_E\xrightarrow{i}\hat{\mathcal{C}}_E$ together with the projection $\hat{\mathcal{C}}_E\xrightarrow{i}\mathcal{P}_E$ induces an isomorphism in cohomology. Thus the restriction of the μ,ν classes to the locus of reducibles of the first type (as well as on it's link in the modified definition) coincides with those for $\mathcal{P}_E$. The restriction of the t class is the generator of the fibre $\mathbb{CP}^\infty$ of the trivial fibration (1.30).

For the locus of reductions of the second type description (1.26-1.27) of the restriction of the universal bundle to the neighbourhood of this locus gives the restriction of our generators. The link of this locus is a projective bundle

$$\mathbb{P}_{L_1,L_2}=\mathbb{P}(\Omega^1(L_1\otimes L_2^{-1})\otimes\mathcal{O}_{\mathcal{C}^*_{L_2}}(-1)\oplus\Gamma(W^+\otimes L_1)\otimes\mathcal{O}) \tag{1.32}$$

with the restriction of the universal bundle given as

$$L_1\otimes\mathcal{O}_{\mathbb{P}(\Omega^1(L_1\otimes L_2^{-1})\otimes\mathcal{O}_{\mathcal{C}^*_{L_2}}(-1)\oplus\Gamma(W^+\otimes L_1)\otimes\mathcal{O})}(1)\oplus L_2\otimes\mathcal{O}_{\mathcal{C}^*_{L_2}}(1). \tag{1.33}$$

A small computation shows

$$\mu(\Sigma)=\frac{1}{2}\langle l_1-l_2,\Sigma\rangle(t+c_1(\mathcal{O}_{\mathbb{P}(\Omega^1(L_1\otimes L_2^{-1})\otimes\mathcal{O}_{\mathcal{C}^*_{L_2}}(-1)\oplus\Gamma(W^+\otimes L_1)\otimes\mathcal{O})}(1))) \tag{1.34}$$

$$\nu=(t+c_1(\mathcal{O}_{\mathbb{P}(\Omega^1(L_1\otimes L_2^{-1})\otimes\mathcal{O}_{\mathcal{C}^*_{L_2}}(-1)\oplus\Gamma(W^+\otimes L_1)\otimes\mathcal{O})}(1)))^2. \tag{1.35}$$

§2. The equation.

Let $a \in \mathcal{A}_E, \phi \in \Gamma(E \otimes W^+)$ and $\nabla^\pm \in \mathcal{A}_{W^\pm}$. On the tensor product of the bundles $Hom(W^-, W^+) \otimes adE$ the connections a and $\nabla^\pm$ define a connection which is the tensor product of the Levi - Civita connection and the traceless part a_0 of a. The dirac operator (1.12) coupled with a connection a is defined as a map:

$$\mathcal{D}_{a_0}^{det(a\otimes\nabla^\pm)} : \Gamma(W^+ \otimes E) \to \Gamma(W^- \otimes E). \tag{2.1}$$

Here the dependence on the traceless part a_0 and the $U(1)$-connection

$$\nabla_{det} = det(a \otimes \nabla)$$

on $det(E \otimes W^+)$ will be exploited in different ways and this is reflected in the notations.

Fix any 2-form ω in the cohomology class of $-4\pi i f = -4\pi i(c_1 + c)$ and $t \in \mathcal{T}_{\frac{1}{4}}$. Then our system of equations is:

$$\begin{aligned} F_{\nabla_{det}} &= \omega, \\ \mathcal{D}_{a_0}^{\nabla_{det}} \phi &= 0, \\ F_{a_0}^+ &= ((t-1)(\phi \otimes \overline{\phi}))_{00} \end{aligned} \tag{2.2}$$

where $((1-t)(\phi \otimes \overline{\phi}))_{00}$ is the "double" traceless component of

$$((1-t)(\phi \otimes \overline{\phi})) \in sl(E) \otimes \Lambda_+^2 = su(E) \otimes su(W^+) \subset u(E) \otimes u(W^+).$$

So the symmetry group of this system is given as the central extension (1.14) which we can identify as $\mathcal{G}_E$ fixing any connection b_0 on $detW^\pm$. Under this identification the coupled Dirac operator (2.1) depends on a and the metric g on X. So

$$\mathcal{D}_{a_0}^{\nabla_{det}} = \mathcal{D}_a^{g, c+c_1(E)}. \tag{2.3}$$

Definition 2.1. We shall denote moduli space of orbits of solutions of (2.2), modulo the action of the gauge group as $\mathcal{M}_B^{g,\omega,t}(c + c_1(E), p_1(E))$.

The infinitesimal properties of this system such as the deformation complex, normal cones of singularities and so on are related closely to the properties of the following system of equations

$$\begin{aligned} F_{\nabla_{det}} &= \omega, \\ \mathcal{D}_{a_0}^{\nabla_{det}} \phi &= 0, \\ F_{a_0}^+ &= 0 \end{aligned} \tag{2.2_0}$$

which was investigated in the series of papers [P-T], [T1-T5] and [P].

The moduli space of orbits of the solutions of (2.2_0), modulo the action of the gauge group was denoted as $\mathcal{MP}(X, p_1(E), c+c_1(E))$ (the moduli space of jumping pairs, see Def. of §1 of [P] or (1.1.26) of [P-T]). Sending a pair (a, ϕ) to a_0 we get a map

$$\pi : \mathcal{MP}(X, p_1(E), c + c_1(E)) \to \mathcal{M}_{asd} \tag{2.4}$$

to the moduli space of $SO(3)$-instantons and the image of this map

$$\mathcal{M}_1^{g,\omega}(c+c_1(E),p_1(E)) = \pi(\mathcal{MP}(X,p_1(E),c+c_1(E))) \subset \mathcal{M}_{asd} \tag{2.5}$$

is called the moduli space of jumping instantons.

On the set of solutions of the system (2.2) the gauge group action is not free. Possible stabiliser groups are $S^1, S^1\times S^1, U(2)$. The last one occurs as the stabilizers group of the solutions with trivial connection (on the trivial bundle) and zero spinor only.

The group S^1 may be a stabiliser of two types of solutions:

Singularities of the first type. Solutions with $\phi = 0$. This implies $(F_a^+)_0 = F_{a_0}^+ = 0$ i.e. the (uniquely) corresponding $PU(2)$ connection $(a)_0$ is antiselfdual. The stabiliser is the centre of $U(2)$. We shall call these solutions reducibles of the first type. The subspace of reducible solutions of the first type is diffeomorphic to the moduli space of asd connection for the adjoint bundle ξ.

Formally the set of singularities of this type is all $\mathcal{M}_{asd}$ but let

$$\mathcal{M}_B^{g,\omega,t}(c+c_1(E),p_1(E))_0 \subset \mathcal{M}_B^{g,\omega}(c+c_1(E),p_1(E)) \tag{2.6}$$

be the subset of solutions with $\phi \neq 0$. Then in $\mathcal{M}_{asd}$ the subset

$$Sing_1 = lim \mathcal{M}_B^{g,\omega,t}(c+c_1(E),p_1(E))_0 \tag{2.7}$$

is defined correctly as the subset of limits of solutions with non trivial twisted spinor field ϕ.

Proposition 2.1. *For a generic metric g and a generic form ω*

$$Sing_1 = \mathcal{M}_1^{g,\omega}(c+c_1(E),p_1(E)) \tag{2.8}$$

This Proposition follows immediately from the Transversality Theorem (see the next section) and the description of the Kuranishi family at a pair $(a,0)$.

Remark. There exists an elementary proof of this statement using the iterative procedure provided by the third equation of (2.2).

Singularities of the second type. Let $Sing_2$ be the set of solutions with reducible connections, which we lift to a $U(2)$-connection a, then the Levi-Civita $PU(2)$-connection on W^+ is lifted to $U(2)$-connection ∇^+ and

$$F_{\nabla^+} + F_{deta} = \frac{1}{2}\omega. \tag{2.9}$$

Moreover, $a = \lambda_1 \oplus \lambda_2$, due to some reduction of the bundle $E = L_1 \oplus L_2$, and with a spinor of the form $\phi = 0 \oplus \phi_2$ with respect to the same reduction of the bundle. For a reducible connection the harmonic spinor always splits in this way and we require the vanishing of the first component. Here the stabiliser is of the form $S^1 \times 1 \subset S^1 \times S^1 \subset U(2)$. Let $l_i = c_1(L_i)$, $\alpha = l_1 - l_2$ and $f = c + c_1(E)$. The spinor $\phi_2 \in \Gamma(W_c^+ \otimes L_2) = \Gamma(W_{c+2l_2}^+)$ is a harmonic one:

$$\mathcal{D}^{(2\lambda_2 \otimes det \nabla^* = \nabla_{det})}\phi_2 = 0.$$

Now

$$F_a = \begin{pmatrix} F_{\lambda_1} & 0 \\ 0 & F_{\lambda_2} \end{pmatrix} = \begin{pmatrix} \omega - F_{det\nabla^+} - F_{\lambda_2} & 0 \\ 0 & F_{\lambda_2} \end{pmatrix},$$

$$(F_a)_0 = \frac{1}{2} \begin{pmatrix} F_{\lambda_1} - F_{\lambda_2} & 0 \\ 0 & F_{\lambda_2} - F_{\lambda_1} \end{pmatrix}$$

and

$$-(\phi \otimes \overline{\phi}) = \begin{pmatrix} 0 & 0 \\ 0 & -(\phi_2 \otimes \overline{\phi_2}) \end{pmatrix}$$

$$-(\phi \otimes \overline{\phi})_0 = \frac{1}{2} \begin{pmatrix} (\phi_2 \otimes \overline{\phi_2}) & 0 \\ 0 & -(\phi_2 \otimes \overline{\phi_2}) \end{pmatrix}.$$

Therefore our second equation $(F_a^+)_0 = ((t-1)(\phi \otimes \overline{\phi}))_{00}$ provides

$$F^+_{2\lambda_2 + det\nabla^+} = ((t-1)(\phi_2 \otimes \overline{\phi_2}))_0 - \omega^+. \tag{2.10}$$

So a reducible solution of the second type produces a solution to the $U(1)$ - Seiberg Witten equation:

$$(2\lambda_2 + det\nabla^+, \phi_2) \in \mathcal{M}_{SW}(\omega^+, t, f - \alpha). \tag{2.11}$$

It is easy to see that this is a one to one correspondence and we proved

Proposition 2.2.

$$Sing_2 = \bigcup_{\beta | (f-\beta)^2 = p_1} \mathcal{M}_{SW}(\omega^+, t, \beta). \tag{2.12}$$

Remark. The very important and very interesting case is when $f = 0$. Here we can choose $\omega = 0$ and we have the standard Seiberg - Witten equation (1.4). In this case the diffeomorphisms group $Diff_X$ acts on the moduli space $\mathcal{M}_B^{g,0}(0, p_1(E))$ and all its singularities.

And finally $S^1 \times S^1$ occurs as a stabiliser of a solution consisting of an *asd* reducible connection and a vanishing spinor.

Linearization. The Linearization of the system (2.2) is elliptic and, therefore, Fredholm:

$$\mathcal{D}_a \delta\phi + \delta a * \phi = 0$$

$$d_a^+(\delta a) = ((t-1)(\delta\phi \otimes \overline{\phi} + \phi \otimes \overline{\delta\phi}))_{00} \tag{2.13}$$

where $\delta a \in \Omega^1(u(E)), \delta\phi \in \Gamma(E \otimes W^+)$ are components of a tangent vector at the point (a, ϕ).

This linear map together with the linearization of the action of gauge group gives the following deformation complex, which is homotopic to a direct sum of the deformation complex for the moduli space of *asd*-connections and the Dirac operator:

$$\Omega^0(u(E)) \xrightarrow{\delta_1} \Omega^1(pu(E)) \times \Gamma(W^+ \otimes E) \xrightarrow{\delta_2} \Omega^2_+(pu(E)) \times \Gamma(W^- \otimes E), \tag{2.14}$$

with the first map given by $\delta_1 = d_a \oplus m_\phi$ and the second by a matrix

$$\delta_2(\delta a, \delta\phi) = (d_a^+(\delta a) + ((t-1)(\phi \otimes \overline{\delta\phi} + \delta\phi \otimes \overline{\phi}))_{00}, \mathcal{D}_a(\delta\phi) + (\delta a)\cdot\phi) \qquad (2.15)$$

The index of the complex is given by the following formula:

$$ind = -\frac{3}{2}p_1 - 3(1 + b_2^+) + \frac{1}{2}(f^2 - \sigma_X) - 1 \qquad (2.16)$$

where $p_1 = p_1(E)$, $f = c + c_1(E)$, σ_X - signature of X.

The Standard technique of Kuranishi models gives a model of a neighbourhood of the point (a,ϕ) as the space of $St_{(a,\phi)}$ orbits in a neighbourhood of zero in the preimage of zero of a certain real-analytic map $H^1_{(a,\phi)} \xrightarrow{\Psi} H^2_{(a,\phi)}$, where $H^i_{(a,\phi)}$ are homologies of the deformation complex.

For a reduction of the first type one has all the cross-terms in the deformation complex vanishing and it turns to be a direct sum of the deformation complex for the moduli of *asd*-connections and Dirac operator with stabiliser S^1 acting only on the spaces of sections of twisted spinors $\Gamma(W^\pm \otimes E)$ with weight one.

Let $(a,\phi) = (\lambda_1 \oplus \lambda_2, 0 \oplus \phi_2)$ be a reduction of the second type. Its deformation complex is a sum of the deformation complexes of the $U(1)$-monopole equations and the following complex:

$$\Omega^0(L_1 \otimes L_2^{-1}) \xrightarrow{\delta_1} \Omega^1(L_1 \otimes L_2^{-1}) \times \Gamma(W^+ \otimes L_1) \xrightarrow{\delta_2} \Omega^2_+(L_1 \otimes L_2^{-1}) \times \Gamma(W^- \otimes L_1), \qquad (2.17)$$

with first map given by $\delta_1 = d_\lambda \oplus m_{\phi_2}$ and second by a matrix

$$\delta_2(\alpha, \delta\phi_1) = (d_\lambda^+(\alpha) + ((t-1)(\phi_2 \otimes \overline{\delta\phi_1} + \delta\phi_1 \otimes \overline{\phi_2})_0, \mathcal{D}_{\lambda_1}(\delta\phi_1) + (\alpha)\cdot\phi_2). \quad (2.18)$$

§3. Transversality.

Let us return to the beginning of the previous section. If we fix the connections $\nabla^\pm$ on the spinor bundles $W^\pm$ compatible with the Levi - Civita connection, then the Dirac operator (2.1) will depend on the $U(2)$-connection a and as the configuration space we can consider the space $\mathcal{A}_\omega \times \Gamma(W^+ \otimes E)$ with the condition (1.18) instead of the first equation of (2.2). We will do it in this section.

From the Kuranishi description it follows that the point (a,ϕ) is smooth if the obstruction space vanishes: $H^2_{(a,\phi)} = 0$. Our moduli space depends also on different continuous parameters i.e. metric, connection on $detW^+$ etc. It is the task of this section to prove that for generic parameters one has $H^2_{(a,\phi)} = 0$ for all solutions (a,ϕ).

First we shall discuss transversality at reducible points. In order to get it we shall change the configuration space of our system. The following illustrates the reasons in the case of reducible solutions of the first type.

Reductions of the first type, i.e. $(a,0)$ with *asd*-connection a occur for a generic metric only if $v.dim(\mathcal{M}_{asd}) = -2p_1(E) - 3(1+b_2^+) \geq 0$ (cf [FU], Thm 3.1). It also follows that for a generic metric the asd-component of the deformation complex has vanishing second cohomology $H^2_a = 0$. Therefore the only possibility for $H^2_{(a,\phi)} \neq 0$

is when $coker\mathcal{D}_a \neq 0$. Generally it is the case since the Chern class of the minus index bundle $c_{ind\mathcal{D}_a+1}(-Ind\mathcal{D}_a)$ does not vanish. This is a topological obstruction for the vanishing of the obstruction space $H^2_{(a,0)}$ for all asd connections a.

The Kuranishi description is given by a map

$$\Psi : H^1_a \otimes ker\mathcal{D}_a \to coker\mathcal{D}_a \tag{3.1}$$

with $\Psi(\alpha,\sigma) = \alpha * \sigma + O((\alpha,\sigma)^3)$ has vanishing differential.

Thus we shall change equation, i.e. configuration space to get rid of this topological obstruction for $H^2_{(a,0)}$ to vanish everywhere. Take instead of the space $\mathcal{C}_E\mathcal{A}_\omega \times \Gamma(W^+ \otimes E)$ its blow up in the locus $\mathcal{A}_{\omega,red} \times \{0\}$ of pairs consisting of the reducible connection and the zero spinor:

$$\widehat{\mathcal{C}_E} = \mathcal{A}_\omega \times \widehat{\Gamma(W^+} \otimes E). \tag{3.2}$$

The Gauge group acts in a natural way on the blow up. Lift the equation to the blow up in a natural way. For example in a neighbourhood of the reducibles of the first type for

$$(a,\phi,\psi) \in \mathcal{A}_\omega \times \Gamma(W^+ \otimes E) \times \mathbb{P}(\Gamma(W^+ \otimes E))$$

take the restriction of the solutions of the system

$$\mathcal{D}_a\psi = 0$$

$$(F^+_a)_0 = -(\phi \otimes \overline{\phi})_{00}$$

to $Y \subset \Gamma(W^+ \otimes E) \times \mathbb{P}(\Gamma(W^+ \otimes E))$ where Y is a blow up of $\Gamma(W^+ \otimes E)$ in zero or equivalently the total space of the linear bundle $\mathcal{O}_{\mathbb{P}(\Gamma(W^+\otimes E))}(-1)$.

Definition 3.1. Denote by $\widehat{\mathcal{M}^{g,\omega}_B}(c + c_1(E), p_1(E))$ the moduli of solutions in the space (3.2) modulo the action of the gauge group.

There is an obvious projection

$$\pi : \widehat{\mathcal{M}^{g,\omega}_B}(c + c_1(E), p_1(E)) \to \mathcal{M}^{g,\omega}_B(c + c_1(E), p_1(E)) \tag{3.3}$$

(compare 2.4).

Now the space of reductions of the first type $\pi^{-1}(\mathcal{M}_{asd})$ can be identified with the moduli space of pairs $\mathcal{MP}(X, p_1(E), c + c_1(E))$ (2.4). In [P-T] it is proved that this moduli space is smooth in the sense that the second homology of the corresponding deformation complex vanishes.

Consider reducible solutions of the second type $\pi^{-1}(\mathcal{M}_{SW}(\beta))$. Let

$$(a = \lambda_1 \oplus \lambda_2, \phi = 0 \oplus \phi_2) \in \mathcal{M}_{SW}(\beta)$$

and

$$(a, \phi_2, \langle\theta, \delta\phi_1\rangle) \in \pi^{-1}(\mathcal{M}_{SW}(\beta)), \tag{3.4}$$

where

$$\langle\theta, \delta\phi_1\rangle \in \mathbb{P}(ker\delta^*_1) \subset \mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^+ \otimes L_1)).$$

The curvature of $\lambda = \lambda_1 - \lambda_2$ has the form $F_\lambda = \sigma \otimes u$ for a two-form σ and $u \in ker d_{a_0}$.

Deformations of the solution of the modified system are described by a sum of the deformation complex of the Seiberg-Witten moduli space at the point (λ_2, ϕ_2) and the following picture for the deformations in the normal direction (compare 2.17-2.18): take the bundle

$$\begin{array}{c}(\Omega^2_+(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^- \otimes L_1)) \otimes \mathcal{O}_{\mathbb{P}}(1) \\ \downarrow \\ \mathbb{P}(ker \delta_1^*),\end{array} \tag{3.5}$$

and take the section of this defined by the map δ_2, with δ_i given by the formulas

$$\delta_1 = d_\lambda \oplus m_{\phi_2}$$

$$\delta_2(\theta, \delta\phi_1) = (d^+_\lambda(\theta) + ((t-1)(\phi_2 \otimes \overline{\delta\phi_1} + \delta\phi_1 \otimes \overline{\phi_2}))_0, \mathcal{D}_{\lambda_1}(\delta\phi_1) + (\theta) \cdot \phi_2).$$

Now directions where the deformation is not obstructed are given as the zero set of this section denoted by s_{δ_2}.

The dependence on metric is given through that of δ_2 and dependence on the 1-form η is given by replacing ω by $\omega + d\eta$ presents δ_2 as a section of

$$\begin{array}{c}(\Omega^2_+(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^- \otimes L_1)) \otimes \mathcal{O}_{\mathbb{P}}(1) \\ \downarrow \\ \mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^+ \otimes L_1)/\delta_1(\Omega^0(L_1 \otimes L_2^{-1}))) \times \mathcal{M}etr \times \Omega^1\end{array}$$

with linearization at the point (3.4) given by the differential

$$((\Omega^1(L_1 \otimes L_2^{-1}) \times \Gamma(W^+ \otimes L_1))) \times T\mathcal{M}etr \times \Omega^1 \to$$

$$\to \Omega^0(L_1 \otimes L_2^{-1}) \times \Omega^2_+(L_1 \otimes L_2^{-1}) \times \Gamma(W^- \otimes L_1), \tag{3.6}$$

$$(\theta, \delta\phi_1, r, \eta) \mapsto (\delta_1^*(\theta, \delta\phi_1), (d^+_\lambda + r^* d^-_\lambda)\theta + \overline{\phi_2} \otimes \delta\phi_1, \mathcal{D}_{\lambda_1}\delta\phi_1 + \eta * \delta\phi_1 + \theta * \phi_2)$$

Assume that

$$c_{dim \mathcal{M}_{SW}}(Ind(d_\lambda \oplus \mathcal{D}_{\lambda_1})) \neq 0. \tag{3.7}$$

The assumption provides us with the nonzero point $(\theta, \delta\phi_1) \in ker \delta_2$ for some solution $a = \lambda_1 \oplus \lambda_2, \phi = 0 \oplus \phi_2$ or in other words with a point in $\widehat{\mathcal{M}^{g,\omega}_B}(c + c_1(E), p_1(E))$ lying over the corresponding Seiberg - Witten moduli space.

Lemma 3.1. *With the assumptions as above the differential (3.6) is epimorphic.*

Proof. We shall remark first that if $s_{\delta_2}(a, \phi_2, \langle\theta, \delta\phi_1\rangle) = 0$ for $(a, \phi_2, \langle\theta, \delta\phi_1\rangle) \in \mathbb{P}(ker \delta_1^*)$ then

$$(\theta, \delta\phi_1) \neq 0 \Rightarrow \theta \neq d_\lambda \xi \ \ \& \ \ \delta\phi_1 \neq 0.$$

Indeed, assume $\theta = d_\lambda \xi$. Vanishing of the section implies in particular

$$d_\lambda^* \theta + m_{\phi_2}^*(\delta\phi_1) = 0, \; d_\lambda^+ \theta + (\delta\phi_1 \otimes \phi_2)_0 = 0.$$

This is equivalent to

$$d_\lambda^* d_\lambda \xi + (\delta\phi_1 \otimes \phi_2)_{tr} = 0 \tag{3.8}$$

$$\sigma \otimes [u, \xi] + (\delta\phi_1 \otimes \phi_2)_0 = 0. \tag{3.9}$$

Taking scalar product of (3.8) with ξ one has

$$(\sigma \otimes [u, \xi], \xi) + ((\delta\phi_1 \otimes \phi_2)_0, \xi) = 0 + ((\delta\phi_1 \otimes \phi_2)_0, \xi) = 0$$

and a small calculation shows that

$$((\delta\phi_1 \otimes \phi_2)_0, \xi) = 0 \Rightarrow ((\delta\phi_1 \otimes \phi_2)_{tr}, \xi) = 0$$

Now taking scalar product of (3.9) with ξ one has

$$(d_\lambda^* d_\lambda \xi, \xi) + ((\delta\phi_1 \otimes \phi_2)_{tr}, \xi) = (d_\lambda^* d_\lambda \xi, \xi) = 0 \Rightarrow |d_\lambda \xi| = 0. \Rightarrow \xi = 0.$$

Substituting this into (3.8) and (3.9) we get $(\delta\phi_1 \otimes \phi_2) = 0$ which, since ϕ_2, being a solution to an elliptic equations with Laplace - type symbol, vanishes totally if it vanishes on an open subset, means that $\delta\phi_1 = 0$. This is a contradiction to the assumption $(\theta, \delta\phi_1) \neq 0$.

In a similar way, let us assume that $\delta\phi_1 = 0$. From the vanishing of the section it follows in particular that

$$\mathcal{D}_{\lambda_1}(\delta\phi_1) + (\theta) \cdot \phi_2 = 0$$

which means that $(\theta) \cdot \phi_2 = 0$ and therefore that $\theta = 0$ for the same reason as above.

Take a nonzero point $(\theta, \delta\phi_1) \in ker d_\lambda \oplus \mathcal{D}_{\lambda_1}$. Then, as we showed before, $\theta \neq d_\lambda \xi$, and by Thm 4.19 [FU] the restriction of the differential of our map to the infinitesimal variations of the metric $g = g + r$ covers $\Omega^2_+(L_1 \otimes L_2^{-1}) : r \to r^* d_\lambda^- \theta$ is onto $coker d_\lambda^+$. As well $\delta\phi_1 \neq 0$ and the infinitesimal variation of 1-forms covers $\Gamma(W^- \otimes L_1)$: $\eta \to \eta * \delta\phi_1$ is onto $coker \mathcal{D}_{\lambda_1}$. Together with Lemma 1.1 this proves the statement.

Therefore one can assume that for a generic metric and 1-form the second homology group vanishes and the reducibles of the second type are in general position. Thus for a generic parameter, the zero locus of the section s_{δ_2} is a smooth finite-dimensional submanifold of the before mentioned projective fibration with fibre $\mathbb{P}(\Omega^1(L_1 \otimes L_2^{-1}) \oplus \Gamma(W^+ \otimes L_1)/\delta_1(\Omega^0(L_1 \otimes L_2^{-1})))$ over the point of $\mathcal{M}_{SW}(\beta)$. This submanifold will be referred to as the link of $\mathcal{M}_{SW}(\beta)$ in $\widehat{\mathcal{M}_B}$.

The following theorem shows that when some extra continuous parameters are added this is true also for non reducible solutions. As above, the dependence on the choice of metric and the form ω gives a gauge invariant smooth map:

$$\mathcal{T} \times \mathcal{M}etr \times \Omega^1 \times \mathcal{A}_\omega \times \Gamma(W^+ \otimes E) \xrightarrow{w \oplus v} \Omega^2_+(pu(E)) \times \Gamma(W^- \otimes E) \tag{3.10}$$

defined as

$$(t, g, \omega, a, \phi) \to (\mathcal{D}_a \phi, F_a^{+_g}(t-1)((-\phi \otimes \overline{\phi}))_{00} \tag{3.11}$$

If we prove that it is a submersion it will follow from Sard-Smale theorem that for generic g and ω the moduli space is smooth in the sense that the second homology of the corresponding deformation complex vanishes.

Theorem 3.1. *The map w is a submersion.*

Proof. Let Dw, Dv denote the differentials of maps w, v resp. It follows from [P-T, Prop.1.3.5] that if $D_{(a,\phi)}v_{|\delta a=\delta g=\delta t=0}$ isn't onto then the connection a is reducible $a = \lambda_1 \oplus \lambda_2$ and the spinor has the form $\phi = 0 \oplus \phi_2$, that is the case of the previous lemma. Now we shall use variations of the metric and connection to prove that $D_{(a,\phi)}w_{|\delta\phi=\xi=0} = \Omega^2_+(pu(E))$.

Condition

$$\langle d_a^+(\delta a) + r^* F_a^- + (\delta t(\phi \otimes \overline{\phi}))_{00}, \Phi\rangle = 0 \tag{3.12}$$

for a self dual non zero $\Phi \in \Omega^2_+(u(E))$ is equivalent to

$$d_a(\Phi) = 0 = d_a^*(\Phi) \tag{3.13}$$

and images of F_a^- and Φ considered as a map from Λ^2_-, Λ^2_+ resp. to $pu(E)$ are orthogonal as well as those of $(\phi \otimes \overline{\phi})$ and Φ (cf. [F-U], Lemma 3.7). That is, Φ does not vanish on an open set. Thus the maximal possible rank of the image of F_a^- is 2. In that case Φ has rank one in a generic point and therefore may be written as $\Phi = \chi \otimes u$, where $\chi \in \Omega^2_+, u \in \Omega^0(u(E))$ and $|u| = 1$. A standard computation as in [F-U, Thm 3.4] shows that $d\chi = d_a u = 0$. Thus the image of F_a^- is perpendicular to u. The same reference gives $(F_a^-, u) = 0 \Rightarrow [F_a^-, u] \neq 0$. So $[F_a, u] = [F_a^+, u] \oplus [F_a^-, u] \neq 0$ and

$$d_a u = [F_a, u] = [F_a^-, u] \oplus [-\phi \otimes \overline{\phi}, u] \neq 0 \tag{3.14}$$

which gives a contradiction. This means that F_a^- and $(-\phi \otimes \overline{\phi})$ have rank at most one, and it is exactly one at a generic point and their images are parallel. This is possible only if ϕ, considered as a map $W^+ \to E$, has rank one. In this case by [PT, Prop. 1.3.5] a is reducible as well and we are done.

The results of this section can be formulated as follows:

Theorem 3.2. *For a generic choice of parameter in $\mathcal{T} \times \mathcal{M}etr \times \Omega^1 \times$ the moduli space $\widehat{\mathcal{M}_B^{t,g,\omega}}(c + c_1(E), p_1(E))$ is a smooth manifold of dimension given by (2.16) with boundary*

$$\mathcal{MP}^{g,\omega}(c + c_1(E), p_1(E)) \cup \bigcup_{\beta|(f-\beta)^2=p_1} \textit{Link of } \mathcal{M}_{SW}(\omega^+, t, \beta)$$

Remark. *In case $b_2^+(X) = 1$ one has a chamber structure for both invariants. Specifying chambers for Seiberg-Witten invariants related to some fixed chamber for Donaldson polynomial one has to take in account shift of the period by ω^+ which may a priory move it out of the positive cone (i.e. period space of the Donaldson theory).*

§4. Compactification.

The Weitzenböck formula

$$\mathcal{D}_a^*\mathcal{D}_a\phi = \nabla_a^*\nabla_a\phi - (F_{a_0}^+ + F_{\nabla_{det}})\phi + \frac{s}{4}\phi$$

provides universal estimates:

$$|\phi| \le const_1$$
$$|F_a^+|^2 \le const_2 \tag{4.1}$$

in the same way it does for $U(1)$ case (compare (1.10)) although in this case it depends not only on the scale curvature of the manifold but also on the maximum of the curvature $F_{\nabla_{det}}$:

$$0 \le \Delta|\phi| \le \frac{-s}{4}|\phi|^2 + \langle((t-1)(\phi\otimes\overline{\phi}))_{00}(\phi),\phi\rangle + \langle(F_{\nabla_{det}}(\phi),\phi\rangle =$$

$$= \frac{-s}{4}|\phi|^2 - \frac{1}{2}|\phi|^4 + +\langle(t(\phi\otimes\overline{\phi}))_{00}(\phi),\phi\rangle + \langle(F_{\nabla_{det}}(\phi),\phi\rangle \le$$

$$\le (\frac{-s}{4} + |F_{\nabla_{det}}|)|\phi|^2 - (\frac{1}{2} - |t|)|\phi|^4 \le (\frac{-s}{4} + |F_{\nabla_{det}}|)|\phi|^2 - \frac{1}{4}|\phi|^4$$

provided $|t| \le \frac{1}{4}$ (i.e. $t \in \mathcal{T}_{\frac{1}{4}}$). Therefore

$$|\phi| \le -s + 4|F_{\nabla_{det}}|. \tag{4.1'}$$

The Weil formula for $c_2(E)$ gives

$$8\pi^2 c_2(E) \le \|F_a^-\|_{L_2} \le 8\pi^2 c_2(E) + const_2 \cdot vol_X. \tag{4.2}$$

This inequality provides the following

Lemma 4.1. *Let (a_i,ϕ_i) be a sequence of solutions of our system and ϵ - a real positive number. Then there is a subsequence i_j and a finite set $x_1,...,x_p \in X$ such that*

$$\forall y \in X - \{x_i\}\ \exists D_y \ni y \text{ such that } \forall j\ \|F_{a_{i_j}}\|_{L_2}^2 \le \epsilon.$$

By this lemma and Theorem 2.3.7 of [DK], for each such D_y there exist a gauge in which one has an estimate

$$\|a_{|D_y}\|_{L_1^2} \le M\|F_{a_{|D_y}}\|_{L^2}$$

for some constant M.

To get C^∞ - convergence on the ball D_y (or in fact on a smaller ball $\hat{D}_y \Subset D_y$) for any $y \in X - \{x_i\}$, and therefore on a punctured manifold $X - \{x_i\}$, one uses the approach of [DK], Ch 2, for the proof of the Uhlenbeck theorem.

The only place one needs antiselfduality for the proof of the Theorem 2.3.8 of [DK] is getting from an L_1^2 bound on the connection to an L_2^2 bound on a smaller domain. This one has to be replaced by an estimate:

$$\|a_{|\hat{D}_y}\|_{L_2^2} \le const\cdot(\|a_{|D_y}\|_{L_1^2} + \|\phi\|_{L_1^2}) \le const\cdot\|a_{|D_y}\|_{L_1^2} + const_2\cdot vol_{D_y} \tag{4.3}$$

where $\hat{D}_y \Subset D_y$ which gives at the end an estimate

$$\|a_{|\hat{D}_y}\|_{L^2_l} \le M_{l,D_y}\|F_a\|_{L^2(D_Y)} + const \cdot vol(D_y) \tag{4.3'}$$

while the rest of the scheme applied is unchanged. The estimate (4.3') provides C^∞ -convergence on any compact subset of $X - \{x_i\}$ together with an estimate on the norm of the curvature of the limit (a_∞, ϕ_∞):

$$\|F_{a_\infty}\|_{L^2} \le 8\pi^2 c_2(E) + const_2 \cdot vol_X$$

The removable singularity theorem is also a slight modification of that for asd-connections. Since an harmonic spinor is smooth (provided we coupled the Dirac operator with a smooth connection a) one has regularity of L^2_1 solutions of our system. Cutting off the singularity $(\psi^2 a_0, \psi\phi)$ in a small ball D_x centred in a singularity with a smooth cut-off function ψ vanishing in a r-ball and equal to one outside of a $2r$-ball gives a small error to the first equation of the system:

$$F_{\psi^2 a_0} + (\psi\phi \otimes \psi\phi)_{00} = \psi^2(F_{a_0} + (\phi \otimes \phi)_{00}) + d(\psi^2) \cdot a_0 + (\psi^4 - \psi^2)[a_0, a_0]^+$$

$$\|F_{\psi^2 a_0} + (\psi\phi \otimes \psi\phi)_{00}\|_{L^2(D_x)} \le \|d(\psi^2)\|_{L^4(D_x)}\|a_0\|_{L^4(D_x)} + \|a_0\|_{L^4(D_x)} \le$$
$$\le const\|F_{a_0}\|_{L^2(D_x)},$$

with a similar estimate for $\|\mathcal{D}_{\psi a}(\psi\phi)\|_{L^2}$. Taking weak L^2_1 limits in a Coulomb gauge when r tends to zero gives a limit pair (a', ϕ') with a smooth connection and spinor provided by the above mentioned regularity of L^2_1 solutions.

Let us add ideal points to our moduli space. These are triples:

$$((a,\phi),(x_1,...,x_l)), \text{ where } (a,\phi) \in \mathcal{M}_B(p_1 + 4l), x_i \in X. \tag{4.4}$$

We shall assign a (curvature) density

$$|F_a|^2 + 8\pi \sum_{i=1}^{l} \delta_{x_i} \tag{4.5}$$

to each of these and say that a sequence of points (a_j, ϕ_j) converges to an ideal point if the curvature densities converge as measures and there is a C^∞ convergence on compact subsets in $X \backslash \{x_j\}$. Let us omit for simplicity some indexes: $\mathcal{M}_B^{g,\omega}(c + c_1(E), p_1(E)) = \mathcal{M}_B(p_1)$.

Definition 4.1. *Let $\overline{\mathcal{M}_B(p_1)}$ be the closure of the moduli space in the union*

$$\mathcal{M}_B(p_1) \cup \mathcal{M}_B(p_1 + 4) \times X \cup ... \cup \mathcal{M}_B(p_1 + 4k) \times s^k X \cup ..., \tag{4.6}$$

endowed with the convergence topology.

As a result one has the following

Theorem 4.1. *The space $\overline{\mathcal{M}_B}$ is compact*

A dimension counting shows that for generic parameters, only a subvariety of codimension $2k$ of the stratum

$$\mathcal{M}_B(p_1 + 4k) \times s^k X \tag{4.7}$$

is in the closure of $\mathcal{M}_B$.

Corollary. *If $m_i \in \mathcal{M}_B(p_1) \cap V_\Sigma$ is a sequence converging to a point $(m_\infty, \{x_1, ..., x_k\})$ of the compactification, then either $x_i \in \Sigma$ or $m_\infty \in \mathcal{M}_B(p_1 + 4k) \cap V_\Sigma$.*

§5. Localisation of the Donaldson polynomial.

Now one can look at our moduli spaces as a bordism connecting the links of reducibles points. Reducibles of the first type, moduli space $\mathcal{MP}$, is a boundary of $\mathcal{M}_B$ and its link is a copy of $\mathcal{MP}$ itself (or a relevant compactification if considering $\overline{\mathcal{M}_B}$). The set of reducibles of the second type is the union of all Seiberg-Witten moduli spaces $\mathcal{M}_{SW}(\beta)$ such, that $(-\beta+f)^2 = p_1$ or $(-\beta+f)^2 \geq p_1$ if one takes in account the compactification $\overline{\mathcal{M}_B}$.

We shall take the Poincare dual of certain cohomology classes on our moduli space $\mathcal{M}_B$ and on its compactification $\overline{\mathcal{M}_B}$. If one takes Poincare dual on $\overline{\mathcal{M}_B}$ to the cohomology class of degree $dim\mathcal{M}_B - 1$ one has a 1-dimensional singular manifold $\mathcal{I}$ with singularities diffeomorphic to the real cone over a finite number of points. We shall assume that this manifold intersects lower strata of the compactification only at reducible points of the second type. Indeed the dimension of the moduli space drops by 6:

$$dim\mathcal{M}_B(p_1) - dim\mathcal{M}_B(p_1+4) = 6. \tag{5.1}$$

Therefore if one bubbling point "kills" not more then two degree 2 classes (like μ_Σ or t) or one degree 4 class (ν -class) then $\mathcal{I}$ intersects lower strata of the compactification in reducible points only. This works since one takes representatives "localised" over Riemannian surfaces (for μ_Σ) or a point (for ν or t) unless there is a trivial connection in the compactification. Since all μ-classes are lifted from the moduli of asd-connections one can use standard dimension counting to show that $\mathcal{I}$ avoids lower strata in the compactification of reducibles of the first type $\overline{\mathcal{MP}}$ and the number $\sharp\partial\mathcal{I} \cap \mathcal{MP}$ is the value of the relevant Donaldson polynomial (or Spin polynomial more generally). Thus the manifold with boundary $\mathcal{I}$ is relating a certain intersection number on $\mathcal{MP}$ with certain integrals over links of singularities of the second type in $\overline{\mathcal{M}_B}$. If one takes, say, the class $\prod_1^d \mu_\Sigma \times t^n$, $2d = v.dim\mathcal{M}_{asd}(p_1) = -2p_1 - 3(1+b_2^+)$ one has following equality:

$$\gamma^d_{p_1, f\,mod 2 - w_2(X)}(\Sigma) = \sum_{(-\beta+f)^2 \geq p_1} \int_{\text{Link of} \mathcal{M}_{SW}(\beta)} \prod_1^d \mu_\Sigma u^n. \tag{5.2}$$

Of course the summand in the r.h.s is nontrivial only if the moduli space $\mathcal{M}_{SW}(\beta)$ is nonempty.

Now to describe the value of the Donaldson polynomial we have to describe for any Seiberg-Witten class β the link of

$$\mathcal{M}_{SW}(\beta) \subset Sing_2 \subset \mathcal{M}_B(f, p_1) \tag{5.3}$$

(We omit the upper indices of the moduli space because it doesn't depend on the continuous parameters) and the integral of the type (5.2) over this link. The point is that this integral is a standard polynomial of the linear forms

$$\langle\beta, \rangle, \langle f, \rangle$$

and the quadratic intersection form q_X. Let us call it the local polynomial and denote it as $loc\gamma$.

Now in the localisation formula (5.2) the left hand side doesn't depend on f and we can consider one as a relation between the local polynomials. Using this relations we will compute the local polynomials precisely in a subsequent paper.

For this computation we will construct a purely geometric finite space $MH^1(\beta, r)$, where $4r = (-\beta + f)^2 - p_1$ is the number of points on X, as the space of "gluing parameters", (which doesn't depend on 2-cohomology class f) and the virtual vector bundle on $MH^1(\beta, r)$ such that the intersections of μ-classes with the "top" Chern class of this bundle gives the local polynomials. These constructions are purely geometrical and here we can say "goodbye" to gauge theory, connections , moduli spaces and all the standard stuff of Donaldson's Theory.

References

[D–K] S. Donaldson and P. Kronheimer, *The Geometry of Four-Manifolds*, Clarendon Press, Oxford, 1990.

[F–S] R. Fintushel, R. Stern, *Donaldson invariants of 4-manifolds of simple type*, Preprint (1994), 1–16.

[F–U] D. Freed and K. Uhlenbeck, *Instantons and four-manifolds*, M.S.R.I. Publ. Springer, New York, 1988.

[K–M1] P. B. Kronheimer and T. Mrowka, *Recurrence relations and asymptotics for four-manifold invariant*, Bull. Amer. Math. Soc **30** (1994), 215–221.

[K–M2] P.B. Kronheimer and T. Mrowka, *The genus of embedded surface in the projective plane*, Preprint. Oxford. (1994).

[P] V. Pidstrigach, *Some glueing formulas for spin polynomials and Van de Van conjecture*, Izv. RAN **54** (1994).

[P–T] V. Pidstrigach and A. Tyurin, *Invariants of the smooth structures of an algebraic surfaces a*, Russian Acad. Sci. Izv. Math. **40:2** (1993), 267–351.

[T1] A. Tyurin, *Spin polynomial invariants of the smooth structures of algebraic surfaces*, Russian Acad. Sci. Izv. Math. **42:2** (1993), 333–369.

[T2] A. Tyurin, *The canonical spin polynomials of algebraic surfaces I*, Iz. AN SSSR **58:6** (1994).

[T3] A. Tyurin, *Spin canonical invariants of 4-manifolds and algebraic surfaces*, Warwick Preprint **38** (1994).

[T4] A. Tyurin, *The simple method of distinguishing the underlying differentiable structures of algebraic surfaces*, Mathematica Gottingenis, Sonderforschungsbereichs Geometry and Analysis **Heft 25** (1992), 1–24.

[T5] A. Tyurin, *Canonical and almost canonical spin polynomials of algebraic surface*, Proceedings of the Durham Conference "Vector bundles", 1993.

[V-W] E. Witten, *A strong coupling test of S-duality.*, Preprint. hep-th / 9408074 (1994).

[W] E. Witten, *Monopoles and four-manifolds.*, Preprint IASSNS–HEP–94 (1994).

Steklov Mathematical Institute, ul. Vavilova 42, Moscow, 117966, GSP-1, Russia and

Mathematical Institute, 24-29 St.Giles, Oxford OX1 3LB, UK

e-mail: pidstrig@class.mian.su, pidstrig@maths.ox.ac.uk , tyurin@tyurin.mian.su