POSITIVE CHARACTERISTIC CLASSES OF VECTOR SHEAVES ON ARITHMETIC VARIETIES

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Abstract. Sufficient conditions for the existence of regular global sections of vector sheaves on arithmetic varieties are given. As an application, the intersection theoretic positivity of monomials in Chern classes of certain vector sheaves on arithmetic varieties is proved.

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Notation

The sign $\rhd$ separates hypotheses and assertions. The symbols $\overset{\circ}{\subset}$ and $\overset{\circ}{\supset}$ denote **open** inclusions, $\operatorname{pt} X$ is the set of closed points of a topological space X, $\operatorname{cl}_B A$ is the integral closure of a ring A in a ring $B \supset A$.

1. Regular global sections of vector sheaves

1.0 Skolem's Theorem. *Let K be a number field, T a finite set of places of K, R the ring of T-integers of K, $S := \operatorname{Spec} R$. Let $n \in \mathbb{N}$ and $X \overset{\circ}{\subset} \mathbb{A}^n_R$ R-dense*
$\rhd$ *There exists a finite field extension $L \supset K$ such that $X(\operatorname{cl}_L R) \neq \emptyset$*

Proof. [M, Th. 1.6] □

Notation. Let $f: Y \to X$ be a morphism of schemes, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}|_Y$ means $f^*\mathcal{F}$ and $\Gamma(Y, \mathcal{F})$ means $\Gamma(\mathcal{F}|_Y)$.

1.1 Definition. Let X be a locally Noetherian scheme, $\mathcal{E}$ a vector sheaf on X, $s \in \Gamma(\mathcal{E})$, and $V(s)$ the zero subscheme of X defined by s.
s is *regular* iff for any irreducible component X' of X the condition

$$\operatorname{codim}_{X'}(V(s) \cap X') \geq \min\{\operatorname{rg}(\mathcal{E}|_{X'}), \dim X' + 1\}$$

is satisfied.
It is called *strictly regular* iff it is regular and for every irreducible component X' of X with $\dim X' \geq \operatorname{rg}(\mathcal{E}|_{X'})$ we have $V(s) \cap X' \neq \emptyset$ (hence $\operatorname{codim}_{X'}(V(s) \cap X') = \operatorname{rg}(\mathcal{E}|_{X'})$).
If $X \to Y$ is a morphism of schemes, $s \in \Gamma(\mathcal{E})$ is called *fiberwise (strictly) regular* iff for all $y \in Y$ the restricted section $s_y := s|_{X_y} \in \Gamma(X_y, \mathcal{E})$ is (strictly) regular.

1.2 Remark. If X is not Cohen–Macaulay, this definition of regularity of sections differs from Fulton's [F, B 3.4].

1.3 Lemma. *Let S be as above, X an irreducible scheme, and $f: X \to S$ a flat morphism of finite type and pure relative dimension d. Let $\mathcal{E}$ be a vector sheaf on X, $s \in \Gamma(\mathcal{E})$.*

(1) *s is fiberwise regular* $\rhd$ *s is regular*
(2) *Suppose that $r := \operatorname{rg} \mathcal{E} \neq d + 1$, s is regular, and if $r \leq d$ there exists an $y \in S$ with s_y strictly regular.* $\rhd$ *s is strictly regular*

Proof. ad (1): If $V(s) = \emptyset$ there is nothing to prove. Otherwise let $Z \neq \emptyset$ be an irreducible component of $V(s)$, X' the irreducible component of X with $X' \supset Z$, $n := \operatorname{rg}(\mathcal{E}|_{X'})$. Then $\operatorname{codim} Z \leq n$. Let $x \in \operatorname{pt} Z$, $y := f(x)$. $Z_y \neq \emptyset$ implies $n \leq \dim X_y$, and

$$\dim_x Z_y = \dim_x X_y - \operatorname{rg}_x \mathcal{E}|_{X_y} = \dim X_y - n$$

In this situation [EGA IV.14.5.1] teaches us that $\operatorname{codim} Z = n$, hence s is regular.

ad (2): If $r > d+1$, the assertion is trivial. Otherwise

$$\dim X_y = d \geq r,$$

hence

$$V(s) \cap X_y = V(s_y) \neq \emptyset,$$

so $V(s) \neq \emptyset$. $\square$

1.4 Remark. If $r = d+1$, strict regularity and fiberwise regularity contradict each other.

1.5 Definition. Given a scheme X over a field k, a vector sheaf $\mathcal{E}$ on X, and a set $E \subset \Gamma(\mathcal{E})$ of global generators of $\mathcal{E}$, we will have occasion to consider the following condition $B(X, E)$:

$$\exists x \in \operatorname{pt}\operatorname{reg} X\colon\, M \cap \Gamma(\bar{X}, \bar{\mathcal{E}} \otimes \mathcal{I}) \twoheadrightarrow (\bar{\mathcal{E}} \otimes \mathcal{I})(x),$$

where $\bar{X} = X \underset{k}{\otimes} k(x)$, $\bar{\mathcal{E}} = \mathcal{E}|_{\bar{X}}$, $\bar{e}$ is the image of $e \in E$ in $\Gamma(\bar{\mathcal{E}})$,

$$M = \sum_{e \in E} k(x)\bar{e} \subset \Gamma(\bar{\mathcal{E}})$$

and $\mathcal{I}$ is the ideal sheaf of x in $\bar{X}$.

If $f\colon Y \to X$ is a morphism of k-schemes, $B(Y, E) := B(Y, \{f^*e \mid e \in E\})$.

1.6 Remark. Let R, S be as in 1.0, let X be a flat R-scheme of finite type, $\mathcal{E}$ a vector sheaf on X, and $\{e_1, \ldots, e_n\} \subset \Gamma(\mathcal{E})$ a set of global generators of $\mathcal{E}$.

Let the epimorphism $\rho\colon \mathcal{O}_X^n \twoheadrightarrow \mathcal{E}$ be defined by

$$\rho(a) = \textstyle\sum a_i e_i|_U \qquad (U \overset{\circ}{\subset} X,\ a \in \mathcal{O}_X^n(U))$$

$\mathcal{K} := \ker \rho$. The homomorphism $\mathcal{O}_X^n \twoheadrightarrow \check{\mathcal{K}}$ dual to $\mathcal{K} \hookrightarrow \mathcal{O}_X^n$ induces a closed immersion

$$W := \mathbb{V}(\check{\mathcal{K}}) = \operatorname{Spec}\operatorname{sym}\check{\mathcal{K}} \overset{j}{\hookrightarrow} \mathbb{A}_X^n.$$

$m := \dim(W|S)$.

We have the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[d, "j"] \arrow[dr, "f"] & & \\ X & \mathbb{A}_X^n \arrow[l] \arrow[r] & \mathbb{A}_R^n \arrow[r, "q"] & S \end{tikzcd}$$

W is also an irreducible flat S-scheme, f is a morphism of finite type.

Let A be an R-algebra.

$$\begin{aligned} \operatorname{im}(j)(A) &= \left\{ (a,x) \in \mathbb{A}^n(A) \times X(A) \;\middle|\; \text{the composition } \mathcal{O}_A \overset{a}{\to} \mathcal{O}_A^n \xrightarrow{x^*\rho} x^*\mathcal{E} \text{ is } 0 \right\} \\ &= \{ (a,x) \in A^n \times X(A) \mid \textstyle\sum a_i x^* e_i = 0 \} \end{aligned}$$

Hence, for $a \in A^n$, the subset $f^{-1}(a) \subset W(A)$ is mapped by p bijectively onto

$$\{ x \in X(A) \mid \textstyle\sum a_i x^* e_i = 0 \} = V(\textstyle\sum a_i e_i)(A)$$

(where $V(\sum a_i e_i) \subset X \underset{R}{\otimes} A$ is the zero subscheme of $\sum a_i e_i \in \Gamma(X \underset{R}{\otimes} A, \mathcal{E})$). In other words, the fibers of f are the zero schemes of the corresponding sections of $\mathcal{E}$.

1.7 Proposition. *Let K be a field, X a K-scheme of finite type. Let $\mathcal{E}$ be a vector sheaf on X, and $E = \{e_1, \dots, e_n\} \subset \Gamma(\mathcal{E})$ a finite set of global generators of $\mathcal{E}$.*

(1) $\triangleright$ *There exists a dense $U \overset{\circ}{\subset} \mathbb{A}^n_K$ such that for all $a \in U$ the section*

$$\sum a_i e_i \in \Gamma(X \underset{K}{\otimes} k(a), \mathcal{E})$$

is regular.

(2) *For any irreducible component X' of X, endowed with the reduced induced closed subscheme structure, we have*

$$B(X', E) \text{ or } \operatorname{rg}(\mathcal{E}|_{X'}) > \dim X'$$

$\triangleright$ *There exists a dense $U \overset{\circ}{\subset} \mathbb{A}^n_K$ such that for all $a \in U$ the section*

$$\sum a_i e_i \in \Gamma(X \underset{K}{\otimes} k(a), \mathcal{E})$$

is strictly regular.

Proof. If we replace X by X_{red}, $\mathcal{E}$ by $\mathcal{E}|_{X_{\text{red}}}$, and the e_i by $e_i|_{X_{\text{red}}}$, the hypotheses continue to be satisfied, and the scheme U parametrizing (strictly) regular sections of $\mathcal{E}|_{X_{\text{red}}}$ parametrizes (strictly) regular sections of $\mathcal{E}$ as well. Therefore we will assume X to be reduced.

Exactly as in the remark above we arrive at the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[d, "j"] \arrow[dr, "f"] & \\ X & \mathbb{A}^n_X \arrow[l] \arrow[r] & \mathbb{A}^n_K \end{tikzcd}$$

with the same interpretation of the fibers of f; $m := \dim W$.

Let us first assume X to be irreducible.

Case 1: f dominant

$$0 \le \dim(\text{generic fiber of } f) = \dim W - n = m - n$$

by [EGA IV.5.6.6]. $Z := \{ w \in W \mid \dim_w W_{f(w)} > m - n \}$ is closed by virtue of Chevalley's Theorem [EGA IV.13.1.5]. Hence $f(Z)$ is a constructible subset of $\mathbb{A}^n_K$ avoiding the generic point of $\mathbb{A}^n_K$. Therefore $\mathbb{A}^n_K \setminus f(Z)$ contains an open neighborhood U (of the generic point).

If $a \in U$, then W_a is of equidimension $m - n$, but

$$W_a \underset{p}{\xrightarrow{\sim}} V(\sum a_i e_i|_{X \otimes k(a)}),$$

so $\sum a_i e_i|_{X \otimes k(a)}$ is strictly regular.

Case 2: f not dominant

$f(W)$ is a constructible subset of $\mathbb{A}^n_K$ avoiding the generic point of $\mathbb{A}^n_K$. Let $U \overset{\circ}{\subset} \mathbb{A}^n_K \setminus f(W)$ be a neighborhood. If $a \in U$ then $W_a = \emptyset$, so $\sum a_i e_i$ is a regular global section. $d := \dim X$, $r := \operatorname{rg} \mathcal{E}$. If in addition $d < r$, the section $\sum a_i e_i$ is even strictly regular.

It remains to prove (2) in the case $d \geq r$. Here we will follow Gieseker [G, Lemma 1.1].

Let $x \in \operatorname{reg} X^{(d)}$, $\bar{X}$, $\bar{\mathcal{E}}$, $\mathcal{I}$ etc. as in the hypothesis $B(X, E)$ (Def. 1.5), $k := k(x)$, so $(A, \mathfrak{m}) := (\mathcal{O}_{\bar{X},x}, \mathfrak{m}_{\bar{X},x})$ is a d-dimensional regular Noetherian local ring, hence

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = d \geq r$$

Thus $\mathfrak{m}/\mathfrak{m}^2$ contains a k-linearly independent subset $\{\bar{g}_1, \dots, \bar{g}_r\}$. Let $(h_1, \dots, h_r)$ be an A-basis of $\bar{\mathcal{E}}_x$. We identify

$$\mathfrak{m}\bar{\mathcal{E}}_x/\mathfrak{m}^2\bar{\mathcal{E}}_x = \bar{\mathcal{E}}_x \underset{A}{\otimes} \mathfrak{m}/\mathfrak{m}^2$$

Let $s \in \Gamma(\bar{\mathcal{E}} \otimes \mathcal{I}) \cap M$ such that

$$s(x) = s_x \bmod \mathfrak{m}^2\bar{\mathcal{E}}_x = \sum_{i=1}^{r} h_i \otimes \bar{g}_i$$

Let $g_1, \dots, g_r \in \mathfrak{m}$ such that

$$s_x = \sum g_i h_i \in \mathfrak{m}\bar{\mathcal{E}}_x$$

Then

$$\bar{g}_i = g_i \bmod \mathfrak{m}^2 \quad (1 \leq i \leq r)$$

$Z := V(s)$, where now s is regarded as a section of $\bar{\mathcal{E}}$, i.e.

$$Z = V(\operatorname{im}(\check{s}\colon \check{\bar{\mathcal{E}}} \to \mathcal{O}_{\bar{X}})).$$

$$Z_x := V(s_x) = Z \underset{\bar{X}}{\times} \operatorname{Spec} A$$

$$\mathfrak{a} := \sum_{i=1}^{r} g_i A \subset \mathfrak{m}.$$

$$\{\bar{g}_1, \dots, \bar{g}_r\} \subset (\mathfrak{a} + \mathfrak{m}^2)/\mathfrak{m}^2$$

implies

$$\dim_k((\mathfrak{a} + \mathfrak{m}^2)/\mathfrak{m}^2) \geq r.$$

$$\begin{aligned}
\dim_x Z &= \dim Z_x \\
&= \dim(A/\mathfrak{a}) \\
&\leq \dim_k((\mathfrak{m}/\mathfrak{a})/(\mathfrak{m}^2 + \mathfrak{a}/\mathfrak{a})) \\
&= \dim_k(\mathfrak{m}/\mathfrak{m}^2 + \mathfrak{a}) \\
&= \dim_k((\mathfrak{m}/\mathfrak{m}^2)/(\mathfrak{a} + \mathfrak{m}^2/\mathfrak{m}^2)) \\
&= \dim_k(\mathfrak{m}/\mathfrak{m}^2) - \dim_k(\mathfrak{a} + \mathfrak{m}^2/\mathfrak{m}^2) \\
&\leq d - r
\end{aligned}$$

Therefore Z has an irreducible component of dimension $\leq d-r$.

Let $b_1,\ldots,b_r \in k$ such that $s = \sum_{i=1}^{r} b_i\bar{e}_i$. Then $Z \simeq W_b$, so the fiber W_b has an irreducible component of dimension $\leq d-r$. Applying [EGA IV.13.1.1] we infer that f has to be dominant, a contradiction.

Thus the case under consideration cannot occur if $B(X,E)$, and the proposition is proved for irreducible X.

For general X, let $X_1,\ldots,X_l$ be the irreducible components of X. By virtue of the special case of the theorem that is already valid, for each $i \in \{1,\ldots,l\}$ there exists a dense $U_i \overset{\circ}{\subset} \mathbb{A}^n_K$ whose points correspond to global sections s of $\mathcal{E}$ with $s|_{X_i}$ (strictly) regular. Now $U := \bigcap_{i=1}^{l} U_i$ will do the job. $\square$

1.8 Theorem. *Let K, R, S be as in 1.0, let X be a flat R-scheme of finite type, $\mathcal{E}$ a vector sheaf on X, and $E=\{e_1,\ldots,e_n\} \subset \Gamma(\mathcal{E})$ a set of global generators of $\mathcal{E}$*

(1) $\triangleright$ *There exists an S-dense $H \overset{\circ}{\subset} \mathbb{A}^n_R$ such that for every finite extension $\tilde{K} \supset K$ and every $a \in H(\tilde{R})$, where $\tilde{R} = \mathrm{cl}_{\tilde{K}} R$, the corresponding global section of $\mathcal{E}|_{X\otimes\tilde{R}}$ is fiberwise regular.*

(2) *For any irreducible component X' of X, denoting $d = \dim(X'|S)$ and $r = \operatorname{rg}\mathcal{E}|_{X'}$, let $r \neq d+1$. If $r \leq d$, suppose moreover $B(X'_K, E)$*

$\triangleright$ *There exists an S-dense $H \overset{\circ}{\subset} \mathbb{A}^n_R$ such that for every finite extension $\tilde{K} \supset K$ and every $a \in H(\tilde{R})$ the corresponding section is strictly regular.*

Proof. Reasoning as in the proof of Prop. 1.7 we reduce to the case that X is integral.

Consider the diagram of Remark 1.6 and assume first that f is dominant. Then

$$0 \leq \dim(\text{generic fiber of } f) = \dim(\text{generic fiber of } f \underset{R}{\otimes} K) = \dim W_K - n = m-n$$

by [EGA IV.5.6.6]. $Z := \{\, w \in W \mid \dim_w W_{f(w)} > m-n \,\}$ is again closed by Chevalley's Theorem [EGA IV.13.1.5].

If f is not dominant, $Z := W$. Note that in this case we may ignore the question of *strict* regularity, because if $d+1 < r$ then regular sections are automatically strictly regular (Lemma 1.3), and if $d \geq r$ the condition $B(X_K,E)$ is violated. (For assume $B(X_K,E)$. Then, as we have seen in Prop. 1.7, the morphism of the generic fibers $f_K\colon W_K \to \mathbb{A}^n_K$ would be dominant. Thus the closure of $f(W)$ would contain the closure of $\mathbb{A}^n_K$ in $\mathbb{A}^n_R$, which is all of $\mathbb{A}^n_R$. Hence it would follow that f was dominant.)

In either case $f(Z) \subset \mathbb{A}^n_R$ is constructible and avoids the generic point of $\mathbb{A}^n_R$. Thus there exists a neighborhood $U \overset{\circ}{\subset} \mathbb{A}^n_R \setminus f(Z)$. $S' := q(U)$. $\Sigma := S \setminus S'$ is a finite closed subset of S, the induced map $U \to S'$ is surjective. For $\mathfrak{p} \in \Sigma$, let $V_{\mathfrak{p}} \overset{\circ}{\subset} \mathbb{A}^n_{k(\mathfrak{p})}$ be the dense subset parametrizing (strictly) regular sections which is provided by Prop. 1.7. Let Y denote the closure of $\mathbb{A}^n_{S'} \setminus U$ in $\mathbb{A}^n_S$. $Y_{\mathfrak{p}} \neq \mathbb{A}^n_{k(\mathfrak{p})}$ for all $\mathfrak{p} \in \Sigma$. Now

$$H := \mathbb{A}^n_S \setminus \Big(Y \cup \coprod_{\mathfrak{p}\in\Sigma} (\mathbb{A}^n_{k(\mathfrak{p})} \setminus V_{\mathfrak{p}})\Big)$$

is an S-dense open subset of $\mathbb{A}^n_R$ with $H \cap \mathbb{A}^n_{S'} = U$ and $H_\mathfrak{p} \subset V_\mathfrak{p} \quad (\mathfrak{p} \in \Sigma)$.

By Skolem's Theorem there exist a finite extension $\tilde{K} \supset K$ and a section $a \in H(\tilde{R})$, where $\tilde{R} := \mathrm{cl}_{\tilde{K}} R$. Let $s \in \tilde{S} := \operatorname{Spec} \tilde{R}$.

If s lies over S' then $a(s) \in U_s$, hence $W_{a(s)} \simeq V(\sum a_i e_i)_s$ is empty if f is not dominant, and is of equidimension $m-n$ if f is dominant. Therefore $\sum a_i e_i|_{X_s}$ is regular, and is even strictly regular under the hypotheses of (2).

If not, then s is a maximal ideal $\tilde{\mathfrak{p}}$ of $\tilde{R}$ with $\mathfrak{p} := \tilde{\mathfrak{p}} \cap R \in \Sigma$. $a \in H(\tilde{R})$ means in particular that $a(\tilde{\mathfrak{p}}) \in \mathbb{A}^n_{k(\mathfrak{p})}$ lies in $H_\mathfrak{p}$, hence in $V_\mathfrak{p}$, therefore the restricted section

$$\sum a_i e_i|_{X_s} = \sum a_i e_i|_{X_\mathfrak{p} \underset{k(\mathfrak{p})}{\otimes} k(\tilde{\mathfrak{p}})}$$

is (strictly) regular by the very definition of $V_\mathfrak{p}$. □

1.9 Theorem. *Under the hypotheses of Thm. 1.8, let Y be an irreducible closed subset of X.*

(1) $\triangleright$ *There exist a finite field extension $\tilde{K} \supset K$ and a fiberwise regular section $e \in \Gamma(X \underset{R}{\otimes} \tilde{R}, \mathcal{E})$, where $\tilde{R} = \mathrm{cl}_{\tilde{K}} R$, such that $e|_{Y \otimes \tilde{R}}$ is fiberwise regular*

(2) $\mathrm{rg}(\mathcal{E}|_Y) \neq \dim Y$. *If $\mathrm{rg}(\mathcal{E}|_Y) \leq \dim Y$ suppose in addition $B(Y_\sigma, E)$, where $\sigma \in S$ is the generic point of the image of Y in S.*
$\triangleright$ *There exist a finite extension $\tilde{K} \supset K$ and a fiberwise regular $e \in \Gamma(X \underset{R}{\otimes} \tilde{R}, \mathcal{E})$ such that $e|_{Y \otimes \tilde{R}}$ is strictly regular*

Proof. Let $H \overset{\circ}{\subset} \mathbb{A}^n_R$ be the R-dense subscheme parametrizing fiberwise regular sections of $\mathcal{E}$ constructed in Thm. 1.8.

If Y is flat, then Thm. 1.8 is applicable to Y, $\mathcal{E}|_Y$, $\{e_1|_Y, \dots, e_n|_Y\}$ in place of X, $\mathcal{E}$, E, and provides an R-dense $H' \overset{\circ}{\subset} \mathbb{A}^n_R$, whose integral points correspond to sections of $\mathcal{E}$ which when restricted to Y become (strictly) regular.

If Y is not flat, there exists a $\mathfrak{p} \in \operatorname{Max} R$ with $Y \subset X_\mathfrak{p}$ (and then $\sigma = \mathfrak{p}$ in (2)), and Prop. 1.7 can be applied to obtain a dense $U' \overset{\circ}{\subset} \mathbb{A}^n_{k(\mathfrak{p})}$ parametrizing (strictly) regular sections of $\mathcal{E}|_Y$. In this situation we define H' to be the S-dense open subset $\mathbb{A}^n_R \setminus (\mathbb{A}^n_{k(\mathfrak{p})} \setminus U')$ of $\mathbb{A}^n_R$.

Now an integral point of $H \cap H'$, which is provided by Skolem's Theorem, corresponds to a global section of $\mathcal{E}$ enjoying the asserted properties. □

Remark. The reason that this theorem is not used in §2 is that in §2 we will always be able to reduce the considerations to the case $Y = X$. Nevertheless, Thm. 1.9 might be useful in more complicated intersection theoretic contexts.

2. Positive characteristic classes

2.0 Definition. Let X be a noetherian scheme.

$$Z^+(X)_\mathbb{Q} := \left\{ \sum_{i=1}^n \alpha_i x_i \;\middle|\; x_1, \dots, x_n \in X,\ \alpha_1, \dots, \alpha_n \in \mathbb{Q}, \right.$$
$$\left. \alpha_i > 0 \quad (1 \leq i \leq n),\ n \in \mathbb{N}^+ \right\}$$

$$
\begin{aligned}
A_. X &:= CH_.(X) \otimes \mathbb{Q} \\
A^+ X &:= \text{image of } Z^+(X)_{\mathbb{Q}} \text{ in } A(X) \\
A^{\geq} X &:= A^+(X) \cup \{0\}
\end{aligned}
$$

Chern classes $c_p(\mathcal{E})\colon A_. X \to A_{.-p} X$ of a vector sheaf $\mathcal{E}$ on X will be understood following [F; Ch. 3, 20.1].

2.1 Remark. In general, it may well happen that $0 \in A^+X$.

If X is proper over a Dedekind domain, such that $\deg\colon A_0 X \to \mathbb{Q}$ is defined, we have $\deg z > 0$ for every $z \in A^+X$.

2.2 Lemma. *Let $f\colon \tilde{X} \to X$ be a dominant finite morphism.*

$$\triangleright \qquad f_* A^+ \tilde{X} = A^+ X$$

Proof. Being closed, f is surjective. Let Y be an integral closed subscheme of X. We have to show $[Y] \in f_* A^+ \tilde{X}$. Replacing f by $f \underset{X}{\times} Y$ we may assume $Y = X$. There exists an irreducible component X' of $\tilde{X}$ such that $f|_{X'}$ is dominant. Let $n = \deg(f|_{X'}) = [R(X') : R(X)]$. Then $f_*[X'] = n[X]$, hence $[X] = f_*(\frac{1}{n}[X']) \in f_* A^+ \tilde{X}$ □

2.3 Definition (P A Griffiths). A module sheaf $\mathcal{E}$ on a scheme X is called *strongly ample* iff it is generated by finitely many global sections and for each $x \in \operatorname{pt} X$ the fiber $(\mathcal{E} \otimes \mathcal{I}_{x|X})(x)$ is generated by global sections of the sheaf $\mathcal{E} \otimes \mathcal{I}_{x|X}$, where $\mathcal{I}_{x|X}$ denotes the ideal sheaf of the reduced closed subscheme $\{x\}$ of X.

2.4 Remark. The strong ampleness property is obviously inherited by quotient sheaves.

2.5 Lemma. *Let X be a locally Noetherian scheme with a strongly ample coherent module $\mathcal{F}$, and let $f\colon Y \to X$ be an étale finite morphism of schemes*

$\triangleright \quad$ *$f^*\mathcal{F}$ is strongly ample*

Proof. The canonical homomorphisms

$$f^* \mathcal{I}_{x|X} \longrightarrow \mathcal{I}_{x|X} \cdot \mathcal{O}_Y \subset \mathcal{I}_{y|Y}$$

induce a "pull-back" homomorphism

$$\Gamma(\mathcal{E} \otimes \mathcal{I}_{x|X}) \longrightarrow \Gamma(f^*\mathcal{E} \otimes f^*\mathcal{I}_{x|X}) \longrightarrow \Gamma(f^*\mathcal{E} \otimes \mathcal{I}_{y|Y})$$

with a commutative diagram

$$
\begin{array}{ccc}
\Gamma(f^*\mathcal{E} \otimes \mathcal{I}_{y|Y}) & \longrightarrow & (f^*\mathcal{E} \otimes \mathcal{I}_{y|Y})(y) \\
\big\uparrow & & \big\uparrow \tau \\
\Gamma(\mathcal{E} \otimes \mathcal{I}_{x|X}) & \longrightarrow & (\mathcal{E} \otimes \mathcal{I}_{x|X})(x)
\end{array}
$$

Let $(A,\mathfrak{m}) := (\mathcal{O}_{X,x},\mathfrak{m}_x)$ and $(B,\mathfrak{n}) := (\mathcal{O}_{Y,y},\mathfrak{m}_y)$. Tensoring the exact sequence $\mathfrak{m} \hookrightarrow A \twoheadrightarrow k(x)$ of A-modules with B yields an exact sequence

$$0 \longrightarrow \mathfrak{m} \underset{A}{\otimes} B \xrightarrow{\upsilon} B \longrightarrow k(x) \underset{A}{\otimes} B \longrightarrow 0$$

since B is assumed to be flat over A. On the other hand B is unramified over A, so $\operatorname{im}\upsilon = \mathfrak{n}$. Thus we know $\mathfrak{m} \underset{A}{\otimes} B \xrightarrow{\sim} \mathfrak{n}$, which in turn gives a sequence of canonical isomorphisms

$$(\mathfrak{m} \underset{A}{\otimes} k(x)) \underset{k(x)}{\otimes} k(y) \simeq \mathfrak{m} \underset{A}{\otimes} k(y) \simeq \mathfrak{m} \underset{A}{\otimes} B \underset{B}{\otimes} k(y) \xrightarrow{\sim} \mathfrak{n} \underset{B}{\otimes} k(y),$$

which means

$$\mathcal{I}_{x|X}(x) \underset{k(x)}{\otimes} k(y) \xrightarrow{\sim} \mathcal{I}_{y|Y}(y)$$

Consequently τ induces an isomorphism

$$(\mathcal{E} \otimes \mathcal{I}_{x|X})(x) \underset{k(x)}{\otimes} k(y) \xrightarrow{\sim} (f^*\mathcal{E} \otimes \mathcal{I}_{y|Y})(y)$$

proving the assertion $\square$

2.6 Lemma. *Let X be a locally Noetherian scheme, $\mathcal{E}$ a coherent X-module, $E \subset \Gamma(\mathcal{E})$ finite, Y a closed subscheme, $y \in Y$, $E_Y := E|_Y \subset \Gamma(Y,\mathcal{E})$. Let $E \cap \Gamma(\mathcal{E} \otimes \mathcal{I}_{y|X})$ generate $(\mathcal{E} \otimes \mathcal{I}_{y|X})(y)$*

$\triangleright$ *$E_Y \cap \Gamma(\mathcal{E}|_Y \otimes \mathcal{I}_{y|Y})$ generates $(\mathcal{E}|_Y \otimes \mathcal{I}_{y|Y})(y)$*

Proof. $\bar{X} := X \underset{\mathbb{Z}}{\otimes} k(y)$, $\bar{Y} := Y \underset{\mathbb{Z}}{\otimes} k(y)$, $\bar{\mathcal{E}} := \mathcal{E}|_{\bar{X}}$. Then $\mathcal{I}_{y|\bar{Y}} = \mathcal{I}_{y|\bar{X}} \restriction_{\bar{Y}}$ and

$$(\bar{\mathcal{E}} \otimes \mathcal{I}_{y|\bar{X}})(y) \xrightarrow{\sim} \left((\bar{\mathcal{E}}|_{\bar{Y}}) \otimes \mathcal{I}_{y|\bar{Y}}\right)(y);$$

the corresponding vector space

$$M_Y := \sum_{e \in E} k(y) e|_{\bar{Y}}$$

for Y is the image of

$$M := \sum_{e \in E} k(y) e|_{\bar{X}} \subset \Gamma(\bar{\mathcal{E}})$$

in $\Gamma(\bar{Y},\bar{\mathcal{E}})$, and the surjectivity of α in the commutative diagram

$$\begin{array}{ccc} M \cap \Gamma(\bar{X}, \bar{\mathcal{E}} \otimes \mathcal{I}_{y|\bar{X}}) & \xrightarrow{\alpha} & (\bar{\mathcal{E}} \otimes \mathcal{I}_{y|\bar{X}})(y) \\ \downarrow & & \downarrow \simeq \\ M_Y \cap \Gamma((\bar{\mathcal{E}}|_{\bar{Y}}) \otimes \mathcal{I}_{y|\bar{Y}}) & \xrightarrow{\beta} & \left((\bar{\mathcal{E}}|_{\bar{Y}}) \otimes \mathcal{I}_{y|\bar{Y}}\right)(y) \end{array}$$

shows that β is surjective, too. $\square$

2.7 Corollary. *Let X and Y as above, and let $\mathcal{E}$ be a strongly ample coherent X-module.*

$\triangleright$ *$\mathcal{E}|_Y$ is strongly ample* $\square$

The following theorem is a generalization of [G, Thm. 1.2]:

2.8 Theorem. *Let K, R, X, S be as in Thm. 1.8, let $\mathcal{E}$ be a vector sheaf on X generated by finitely many global sections. Let $j \in \mathbb{N}$.*

(1) $\quad\triangleright\qquad c_j(\mathcal{E}) \cdot A^{\geq} X \subset A^{\geq} X$

(2) *Let X be of equidimension n, $\mathcal{E}$ of constant rank r, suppose that for all $s \in S$ the vector sheaf $\mathcal{E}_s := \mathcal{E}|_{X_s}$ is strongly ample. Let $j \leq r$ and $l \in \mathbb{N}$ with $j \leq l \leq n$.*

$\quad\triangleright\qquad c_j(\mathcal{E}) \cdot A_l^{+} X \subset A_{l-j}^{+} X$

Proof. Thm. 1.8 provides a finite extension L of K such that there exists a regular element of $\Gamma(\tilde{X}, \mathcal{E})$, where $\tilde{X} = X \underset{R}{\otimes} \mathrm{cl}_L R$.

Let $\xi \in A^{+} X$. By Lemma 2.2 there is a $\tilde{\xi} \in A^{+} \tilde{X}$ with $\xi = f_* \tilde{\xi}$, where $f\colon \tilde{X} \to X$ is the canonical morphism, which is finite and surjective, and the projection formula [F; Thm. 3.2(c), 20.1] yields

$$c_j(\mathcal{E}) \cdot \xi = f_*\big(c_j(f^*\mathcal{E}) \cdot \tilde{\xi}\big),$$

hence it is sufficient to show $c_j(f^*\mathcal{E}) \cdot A^{+} \tilde{X} \subset A^{+} \tilde{X}$ (resp. $c_j(f^*\mathcal{E}) \cdot A^{\geq} \tilde{X} \subset A^{\geq} \tilde{X}$).

Let $\tilde{s} \in \operatorname{Spec} \mathrm{cl}_L R$, and let s denote its image in S. Since $k(s)$ is a perfect field, $X_{\tilde{s}}$ is a finite étale cover of X_s, so if $\mathcal{E}_s$ is strongly ample, by virtue of Lemma 2.5 the pull-back $\mathcal{E}_{\tilde{s}}$ is also strongly ample. Thus we may replace X by $\tilde{X}$, i.e. we may assume that $\mathcal{E}$ possesses a regular global section e.

Now if $r > n$, then e has no zero, hence it determines an exact sequence

$$\mathcal{O}_X \overset{e}{\hookrightarrow} \mathcal{E} \twoheadrightarrow \mathcal{E}'$$

where $\mathcal{E}'$ is a vector sheaf with $\operatorname{rg} \mathcal{E}' = r - 1$ and $c(\mathcal{E}') = c(\mathcal{E})$. In this way the proof is reduced to the case $r \leq n$. If, however, $r = n$, the assertion is trivial, since $A_0 X = 0$ [KS, §6]. The remaining case $r < n$ will be treated by induction on r.

Let $i\colon Y \hookrightarrow X$ be an l-dimensional integral closed subscheme. Certainly $i_* A^{+} Y \subset A^{+} X$ and [F, Thm. 3.2(c), 20.1]

$$c_j(\mathcal{E}) \cdot [Y] = i_* \left(c_j(i^*\mathcal{E}) \cdot [Y]\right),$$

hence it suffices to prove $c_j(i^*\mathcal{E}) \cdot [Y] \in A^{+} Y$ (resp. $A^{\geq} Y$).

For $s \in S$ the strong ampleness of $\mathcal{E}|_{X_s}$ implies the strong ampleness of $\mathcal{E}|_{Y_s}$ by Cor. 2.7.

Case 1: Y flat

In this case we may assume $i = \mathrm{id}_X$.

ad (2): $X_K \neq \emptyset$. Let $x \in \operatorname{pt} \operatorname{reg} X_K$, $X' := \overline{\{x\}}$. Let E'_K be a finite set of global generators of the sheaf $\mathcal{E}_K \otimes \mathcal{I}_{x|X_K}$ generating its fiber at x. Now $\mathcal{I}_{x|X_K} = (\mathcal{I}_{X'|X})_K$ by the definition of X', hence $\mathcal{E}_K \otimes \mathcal{I}_{x|X_K} = (\mathcal{E} \otimes \mathcal{I}_{X'|X})_K$, so by flat base change [H, III.9.3]

$$H^0(X_K, \mathcal{E}_K \otimes \mathcal{I}_{x|X_K}) = H^0(\mathcal{E} \otimes \mathcal{I}_{X'|X}) \underset{R}{\otimes} K$$

Thus on multiplying the elements of E'_K with suitable elements of R we may assume that the elements of E'_K extend to X, i.e. there is a set $E' \subset \Gamma(X, \mathcal{E} \otimes \mathcal{I}_{X'|X})$ having E'_K as its image on X_K. Let E_0 denote the image of E' in $\Gamma(\mathcal{E})$. Since $\mathcal{E}$ is finitely globally generated, there exists a finite set $E \supset E_0$ of global generators of $\mathcal{E}$. Now $(\mathcal{E}_K \otimes \mathcal{I}_{x|X_K})(x)$ is clearly generated by $E|_{X_K} \cap \Gamma(\mathcal{E}_K \otimes \mathcal{I}_{x|X_K})$, and thus $B(X_K, E)$.

Therefore we can apply Thm. 1.8 to provide a dominating finite X-scheme $\tilde{X}$ and a strictly regular $s \in \Gamma(\tilde{X}, \mathcal{E})$. Again we may and do assume $\tilde{X} = X$. The zero subset $Z := V(s)$ is of pure dimension $n - r$.

Case 1.1: $j = r$

[F; Prop. 14.1(a),(b), 20.1] yields $c_r(\mathcal{E}) \cdot [X] \in A^+X$. Having settled this case we have also established the starting case $r = 1$ for the induction (c_1 is treated and $c_0 = 1$ does not pose serious problems).

Case 1.2: $j < r$

$X' := X \setminus Z \overset{i}{\hookrightarrow} X$. The section $i^*s \in \Gamma(X', \mathcal{E})$ does not vanish anywhere and induces an exact sequence $\mathcal{O}_{X'} \hookrightarrow i^*\mathcal{E} \twoheadrightarrow \mathcal{E}'$ with some vector sheaf $\mathcal{E}'$ of rank $r-1$ on X' satisfying $c_j(\mathcal{E}') = c_j(i^*\mathcal{E})$. As $\dim Z = n - r < n - j$, hence $A_{n-j}Z = 0$, we have

$$i^*\colon A_{n-j}X \overset{\sim}{\to} A_{n-j}X'$$

(cf. [F, Prop. 1.8]), and $i^*A^+_{n-j}X = A^+_{n-j}X'$. Our contention now follows from the induction hypothesis on $\mathcal{E}'$, since

$$i^*(c_j(\mathcal{E}) \cdot [X]) = c_j(i^*\mathcal{E}) \cdot [X'] = c_j(\mathcal{E}') \cdot [X'] \in A^+_{n-j}X'$$

ad (1): If s is strictly regular, the above proof applies to show $c_j(\mathcal{E}) \cdot [X] \in A^{\geq}_{n-j}X$ in the case $j \leq r$, $j \leq n$. But if $j > r$ or $j > n$ the assertion is trivial, for then $c_j(\mathcal{E}) = 0$.

If finally s is not strictly regular, then $Z = \emptyset$, and $c_r(\mathcal{E}) \cdot [X] = 0 \in A^{\geq}_{n-r}X$, while the inductive argument for $j < r$ still works.

Case 2: Y **not flat.**

As Y is irreducible, there exists a maximal ideal $\mathfrak{p}$ of R with $Y \subset X_{\mathfrak{p}}$. Now the proof proceeds analogous to Case 1 (only simpler), with R replaced by $k(\mathfrak{p})$, X by $X_{\mathfrak{p}}$, and Thm. 1.8 by Prop. 1.7.

In view of the remarks stated before Case 1 we can forget about X and R and consider only $\mathcal{E}|_Y$ (which in part (2) is strongly ample by Cor. 2.7) on Y over $k(\mathfrak{p})$.

ad (2): Let $y \in \operatorname{pt}\operatorname{reg} Y$, then the strong ampleness hypothesis guarantees the existence of a set E' of global generators of $\mathcal{E} \otimes \mathcal{I}_{y|Y}$ generating $(\mathcal{E} \otimes \mathcal{I}_{y|Y})(y)$. Let $E \subset \Gamma(Y, \mathcal{E})$ be a finite set of generators of $\mathcal{E}$ containing the image of E' in $\Gamma(Y, \mathcal{E})$. Then $B(Y, E)$, so Prop. 1.7 yields a strictly regular $s \in \Gamma(Y, \mathcal{E})$. Using this s, the continuation of the proof of (2) is completely similar to the corresponding part of the proof in the flat case.

ad (1): Analogous to Case 1. $\square$

2.9 Corollary. *Let $X, \mathcal{E}$ be subject to the same hypotheses as in Thm. 2.8, and let* $i_1, \dots, i_s \in \mathbb{N}$

(1) $\quad \triangleright \quad c_{i_1}(\mathcal{E}) \cdot \ldots \cdot c_{i_s}(\mathcal{E}) \cdot A^{\geq}X \subset A^{\geq}X$

(2) *Under the additional hypotheses of Thm. 2.8(2), let* $l \in \mathbb{N}$, $d := \sum_{\nu=1}^{s} \nu i_\nu \leq l \leq n$, *and* $d \leq r$

$$\triangleright \qquad c_{i_1}(\mathcal{E}) \cdot \ldots \cdot c_{i_s}(\mathcal{E}) \cdot A_l^{+}X \subset A_{l-d}^{+}X$$

□

2.10 Remark. The result (1) of Cor. 2.9, for general Schur polynomials in place of monomials, is contained in [F; Example 12.1.7(a)], where its proof, which is much simpler than the one given above, uses the effectiveness of the intersection of an effective cycle with an effective divisor rather than regular sections.

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