ON THE DERIVATIVE OF THE STABLE HOMOTOPY OF MAPPING SPACES

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Introduction

In this paper we shall provide a homotopy-theoretic computation of the *derivative*, in the sense of Goodwillie [G], of the functor

$$X \quad \longmapsto \quad Q_+F(K,X) \ ,$$

where K denotes a simplicial complex, $F(K,X)$ denotes the space of unpointed maps $K \to X$, and Q_+ denotes unreduced stable homotopy.

The derivative of this functor was first determined using framed bordism theory in [loc.cit.]; the answer was expressed as the section space of a certain fibration over K. The first homotopy theoretic computation of it appeared in [H] and independently in [A], using configuration spaces. However, these proofs also used manifolds at various places in the argument.

We shall make the computation by a different route which avoids the use of configuration spaces and manifolds. The main idea is to apply a special case of the *chain rule* of [KR] together with certain elementary results about homotopy limits. One advantage of our approach is that it generalizes to give a determination of the derivative of

$$X \quad \longmapsto \quad \Omega^\infty \mathbf{E} \wedge_{hH} F(K,X) \ ,$$

where H is a topological group, K is a space with pointed H-action, and $\mathbf{E}$ is a spectrum with H-action (see 6.2 below).

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1. Preliminaries

1.1. In this paper **Top** denotes category of topological spaces, $\mathbf{Top}_*$ is the category of pointed spaces, and **Sp** denotes the category of (omega) spectra. We will implicity assume that all spaces appearing below are the homotopy type of a CW complex.

If Z is a pointed space, and $\mathbf{E}$ is a spectrum, then $F(Z, \mathbf{E})$ denotes the function spectrum whose n-th space is the space of based maps from Z to the n-th space of $\mathbf{E}$. If $\mathbf{E}$ is a spectrum equipped with action by a topological group G, we let $\mathbf{E}_{hG}$ denote the associated homotopy orbit spectrum defined by $(\mathbf{E} \wedge EG_+)/G$.

We cite some results about homotopy limits in closed model categories. Let $\mathbf{C}$ be a closed model category. We will assume that $\mathbf{C}$ admits a functorial cylinders. One therefore has the notion of *homotopy equivalence* in $\mathbf{C}$.

If $f\colon D \to \mathbf{C}$ denotes a finite diagram (i.e. a covariant or contravariant functor from a very small category D into $\mathbf{C}$), it makes sense to speak of the *homotopy limit* and *homotopy colimit of f* (see e.g., [DS,10.13]).

We denote

$$H^\bullet(D; f) \quad := \quad \operatorname*{holim}_D f \ .$$

and

$$H_\bullet(D; f) \quad := \quad \operatorname*{hocolim}_D f \ ,$$

to emphasize the analogy with (co)-homology with coefficients in a sheaf.

We shall say that a natural transformation of $f \to g$ functors is a *natural weak equivalence* if $f(d) \to g(d)$ is a (weak) equivalence of $\mathbf{C}$. More generally, we say that f and g are *naturally weak equivalent*, if there exists a finite chain of natural weak equivalences,

$$f = f_0 \overset{\simeq}{\leftarrow} f_1 \overset{\simeq}{\rightarrow} \dots \overset{\simeq}{\leftarrow} f_n = g \ .$$

We have,

Lemma 1.2. *(1). Suppose that $f(d)$ and $g(d)$ are fibrant for all $d \in D$. If $f \to g$ is a natural weak equivalence of functors $D \to \mathbf{C}$ then the induced map*

$$H^\bullet(D; f) \to H^\bullet(D; g)$$

is a homotopy equivalence of $\mathbf{C}$.

(2). If f and g are naturally weak equivalent, then $H^\bullet(D; f)$ and $H^\bullet(D; g)$ are connected by a chain of homotopy equivalences of $\mathbf{C}$. In particular, they are weak equivalent.

Proof. This is shown in [BK,XI§6] when $\mathbf{C}$ is the category of simplicial sets. The proof in the present context is given by direct generalization. $\square$

In this paper, we will take $\mathbf{C}$ to be the category of spectra. The category of spectra can be equipped with a closed model structure (see e.g., [BF]). The cautious reader may alternatively work in $\mathbf{Top}_*$ (with closed model structure determined by Serre fibrations and weak homotopy equivalences), if she/he so wishes, to obtain the theorem on the level of zero-th spaces.

1.3. Definition of the derivative. We briefly recall here the definition of the derivative of a homotopy functor $f\colon \mathbf{Top} \to \mathbf{Top}_*$. We shall assume that f is *stably excisive* (see [G,1.8]). Given a pointed space X, we may consider the map

$$f(X \vee S^j) \to f(X)\ .$$

Call $\partial_{X,j} f$ the homotopy fiber of this map. Varying j, we get a prespectrum. The associated spectrum is then the definition of the derivative of f at X. We shall denote it by $\partial_X f$.

2. Statement of the main result

2.1. We recall a certain construction of [G,2.1]. Let $E \to K$ be a Serre fibration over a finite complex K. For each $i \geq 0$, let E_i denote the pushout of

$$K \leftarrow S^{i-1} \times E \overset{\subset}{\to} D^i \times E.$$

The induced map $E_i \to K$ is then a fibration over K with preferred section. The fiber over $k \in K$ is the space $\Sigma^i(E(k)_+)$, where Σ denotes reduced suspension, and $E(k)_+$ denotes the fiber $E(k)$ at k with a disjoint basepoint added. Let $\Gamma_{i,K}(E)$ denote the (pointed) space of all sections of $E_i \to K$. These spaces form a prespectrum after a minor rectification (see [G,0.1]). Let $\Gamma_K(E; \mathbf{S}^0_+)$ denote the corresponding spectrum.

We apply the preceding in the following special case: Let $x \in X$ be a choice of basepoint. Define a fibration

$$E_x(K, X) \to K$$

with total space

$$E_x(K, X) := \{(k, f)|\ k \in K, f\colon K \to X\}$$

and map to K given by the first coordinate projection. Note that the fiber over $k \in K$ is the function space of maps $K \to X$ sending k to x. Using the above construction, we obtain a spectrum

$$\Gamma_K(E_x(K, X); \mathbf{S}^0_+)\ .$$

We are now in a position to state Goodwillie's result [G,2.4].

2.2. Main Theorem. *Let X be space, K a finite complex. Then the derivative of the functor $X \mapsto Q_+F(K, X)$ at the pointed space (X, x) is given by the spectrum $\Gamma_K(E_x(K, X); \mathbf{S}^0_+)$.*

3. Fibrations and section spaces

Let K be a finite geometric simplicial complex. We shall sometimes abuse notation and implicitly identify the simplices σ of K with the standard geometric simplices $|\sigma| = \Delta^{\dim \sigma}$. Set Δ_K equal to the category whose objects are simplices of K and whose morphisms are the inclusions of such simplices.

$E \to K$ is a fibration and $L \subset K$ is a subcomplex, we let $\Gamma_L(E)$ denote the space of sections of the restriction $E \to K$ along L. By the construction in §2, we have a spectrum $\Gamma_L(E; \mathbf{S}^0_+)$.

For $\sigma \in \Delta_K$, define a space

$$E_x(K, X; \sigma) \quad := \quad \{f \colon K \to X \mid f(\sigma) = x\} \ . \tag{3.1}$$

The assignment

$$\sigma \quad \longmapsto \quad \Sigma^\infty E_x(K, X; \sigma)_+ \tag{3.2}$$

is then a contravariant functor which we denote by $\Sigma^\infty_+ E_x(K, X; -)$.

3.3. Proposition. *The homotopy limit of 3.2 is canonically homotopy equivalent to $\Gamma_K(E_x(K, X); \mathbf{S}^0_+)$, where $E_x(K, X) \to K$ is the fibration defined above.*

Proof. Associated to $E_x(K, X) \to K$, we may define a contravariant functor $\Delta_K \to \mathbf{Sp}$ given by

$$\sigma \longmapsto \Gamma_\sigma(E_x(K, X); \mathbf{S}^0_+) \ . \tag{3.4}$$

It is fairly straightforward to check that the ordinary limit of 3.4 is homeomorphic to $\Gamma_K(E_x(K, X); \mathbf{S}^0_+)$.

Let $E_x(K, X; \sigma) \times \sigma \to \sigma$ denote the trivial fibration, and let $E_x(K, X)(\sigma) \to \sigma$ denote the restriction of $E_x(K, X) \to K$ to σ.

By definition, there is a natural inclusion

$$E_x(K, X; \sigma) \times \sigma \xrightarrow{\subset} E_x(K, X)(\sigma)$$

which is a fiber homotopy equivalence of fibrations. This induces a homotopy equivalence

$$\Gamma_\sigma(E_x(K, X; \sigma) \times \sigma; \mathbf{S}^0_+) \xrightarrow{\simeq} \Gamma_\sigma(E_x(K, X); \mathbf{S}^0_+) \ .$$

Consequently, taking limits, we get a map

$$\lim_{\sigma \in \Delta_K} \Gamma_\sigma(E_x(K, X; \sigma) \times \sigma); \mathbf{S}^0_+) \to \lim_{\sigma \in \Delta_K} \Gamma_\sigma(E_x(K, X); \mathbf{S}^0_+) \ . \tag{3.5}$$

Since the source of 3.5 is homeomorphic to $H^\bullet(\Delta_K; E_x(K, X; -))$ and the target is homeomorphic to $\Gamma_K(E_x(K, X); \mathbf{S}^0_+)$ we may rewrite 3.5 as

$$H^\bullet(\Delta_K; E_x(K, X; -)) \to \Gamma_K(E_x(K, X); \mathbf{S}^0_+) \ , \tag{3.6}$$

It will be sufficient to show that the latter is a homotopy equivalence. We will prove this by induction on the simplices of K.

The induction makes use of the following observations: Firstly, for any $\sigma \in \Delta_K$, the restriction maps

$$\Gamma_\sigma(E_x(K, X; \sigma) \times \sigma; \mathbf{S}^0_+) \to \Gamma_{\partial\sigma}(E_x(K, X; \partial\sigma) \times \partial\sigma; \mathbf{S}^0_+)$$

and

$$\Gamma_\sigma(E_x(K,X);\mathbf{S}^0_+) \to \Gamma_{\partial\sigma}(E_x(K,X);\mathbf{S}^0_+)$$

are fibrations. Secondly, suppose that $L \subset K$ is a subcomplex The functor $L \mapsto G(L)$ given by

$$L \quad \mapsto \quad \lim_{\tau\in\Delta_L} \Gamma_\tau(E_x(K,X;\tau)\times\tau);\mathbf{S}^0_+)$$

and the functor $L \mapsto H(L)$ given by

$$L \quad \mapsto \quad \lim_{\tau\in\Delta_L} \Gamma_\tau(E_x(K,X);\mathbf{S}^0_+)$$

are defined on the category (i.e., poset) consisting of the subcomplexes of K ordered with respect to inclusion. Note that $G(K)$ is the source of 3.6 and $H(K)$ is the target of 3.6.

Suppose that the map $G(L_0) \to H(L_0)$ is a homotopy equivalence for some $L_0 \subset K$. Let $L = L_0 \cup \sigma$ where $\partial\sigma \subset L_0$. Let $i\colon \partial\sigma \to \sigma$ be the inclusion Then we have a commutative diagram

$$\begin{array}{ccccc}
G(L_0) & \longrightarrow & G(\partial\sigma) & \xleftarrow{G(i)} & G(\sigma) \\
\simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq \\
H(L_0) & \longrightarrow & H(\partial\sigma) & \xleftarrow{H(i)} & H(\sigma)
\end{array}$$

Moreover, the pullback of the top row is $G(L)$ and the pullback of bottom row is $H(L)$. Since $G(i)$ and $H(i)$ are fibrations, it follows that the induced map of pullbacks, i.e., the map $G(L) \to H(L)$, is also a homotopy equivalence. This completes the inductive step. $\square$

4. Prelude to the main result

Given (continuous, stably excisive homotopy) functors $f, g\colon \mathbf{Top}_* \to \mathbf{Top}_*$, the chain rule of [KR,4.8] says that the derivative of $g\circ f$ at a pointed space X is given by the formula

$$\partial_X(g\circ f) \quad = \quad \partial_{f(X)}g \wedge_{h\Omega f(X)} \partial_X f \tag{4.1}$$

(Here and elsewhere, $\Omega f(X)$ denotes the topological group model for the loop space $f(X)$ which is given by taking the realization of the Kan loop group of the total singular complex of $f(X)$.) Thus, the derivative of $g\circ f$ is determined by the derivative of g, the derivative of f, a certain left action of $\Omega f(X)$ on $\partial_X f$, and a certain right action of $\Omega f(X)$ on $\partial_{f(X)}g$.

Set $f(X) = F(K,X)$ (with basepoint given by the constant function to x), and $g(Z) = Q_+Z$. The composite functor $g\circ f$ is then

$$X \quad \longmapsto \quad Q_+F(K,X)\ .$$

As a means toward computing the derivative of the latter using the chain rule, we first identify the derivatives of g and f. The derivative of g is well-known: it is given by the sphere spectrum $\mathbf{S}^0$ (see [G,1.19(1)]). The derivative of f can be computed using the definition of derivative as follows: recall that it is the spectrification of the pre-spectrum whose j-th space is given by

$$h\mathrm{fib}(f(X \vee S^j) \to f(X)) ,$$

where $h$fib denotes homotopy fiber. Let $G = \Omega X$. Since the homotopy fiber of $X \vee S^j \to X$ is just $S^j \wedge G_+$ (up to homotopy), it follows that there is a fibration up to homotopy

$$F(K, S^j \wedge G_+) \to F(K, X \vee S^j) \to F(K, X) .$$

Spectrifying, the derivative of f at X is given by

$$\partial_X f \quad = \quad F(K, \mathbf{S}^0 \wedge G_+) .$$

The next step is to identify the actions. Using the definition of the actions as given in [KR], it is fairly straightforward to verify that the right action of $\Omega f(X) = \Omega F(K, X) = F(K, G)$ on $\mathbf{S}^0$ is trivial, and the left action of $\Omega F(K, G)$ on $F(K, \mathbf{S}^0 \wedge G_+)$ is given pointwise, i.e., by the formula

$$(\phi, \lambda) \quad \longmapsto \quad (k \mapsto \phi(k) \cdot \lambda(k)) ,$$

for $\phi \in F(K, G)$ and $\lambda \in F(K, \mathbf{S}^0 \wedge G_+)$, where we are letting G action on $\mathbf{S}^0 \wedge G_+$ by left translation.

In summary, we see that

4.2. Lemma. *(cf.* [KR,A.7]*). The derivative of $X \mapsto Q_+ F(K, X)$ is given by*

$$\mathbf{S}^0 \wedge_{hF(K,G)} F(K, \mathbf{S}^0 \wedge G_+) \quad = \quad F(K, \mathbf{S}^0 \wedge G_+)_{hF(K,G)}.$$

5. Proof of the main result

We will make use of the following fact:

Lemma 5.1. *If G is a topological group, then the functor $\mathbf{Y} \mapsto \mathbf{Y}_{hG}$ (from spectra with G-action to spectra) commutes with finite homotopy limits (up to weak equivalence).*

Proof. The homotopy orbit construction commutes with cofibration sequences. Since every fibration is a cofibration up to homotopy in category of spectra, it follows that the homotopy orbit construction commutes with fibrations up to homotopy. As every finite homotopy limit can be expressed as an iterated homotopy fiber, the result follows. □

5.2. Part one of the proof. By 4.2 and 3.3, it will be sufficient to show that the homotopy limit of the contravariant functor $\Sigma^\infty_+ E_x(K, X; -)$ of 3.2, given by

$$\sigma \quad \longmapsto \quad \Sigma^\infty E_x(K, X; \sigma)_+ \tag{5.3}$$

is equivalent to the spectrum

$$F(K, \mathbf{S}^0 \wedge G_+)_{hF(K,G)} \tag{5.4}$$

where $G = \Omega X$. We may rewrite 5.4 as

$$(\lim_{\Delta_K}(\sigma \mapsto F(\sigma, \mathbf{S}^0 \wedge G_+)))_{hF(K,G)} \tag{5.5}$$

The latter is in turn equivalent to the corresponding expression involving a homotopy limit

$$(H^\bullet(\Delta_K; \sigma \mapsto F(\sigma, \mathbf{S}^0 \wedge G_+)))_{hF(K,G)} \ . \tag{5.6}$$

By 5.1, and the finiteness of K, 5.6 is equivalent to the homotopy limit of the functor

$$\sigma \quad \longmapsto \quad F(\sigma, \mathbf{S}^0 \wedge G_+))_{hF(K,G)} \ . \tag{5.7}$$

with $F(K, G)$ acting on $F(\sigma, \mathbf{S}^0 \wedge G_+)$ by means of the restriction map

$$F(K, G_+) \xrightarrow{\text{rest.}} F(\sigma, G_+) \ .$$

We summarize this statement in

Lemma 5.8. *The derivative of $X \mapsto Q_+F(K, X)$ is equivalent to the homotopy limit of the functor 5.7.*

5.9. Completion of the proof. For $\sigma \in \Delta_K$, consider the fibration of topological groups

$$E_x(K, G; \sigma) \to F(K, G) \xrightarrow{\text{rest.}} F(\sigma, G) \ . \tag{5.10}$$

There is therefore an isomorphism of functors

$$(\sigma \mapsto F(\sigma, \mathbf{S}^0 \wedge G_+)_{hF(K,G)} \quad \cong \quad (\sigma \mapsto F(\sigma, \mathbf{S}^0 \wedge G_+)_{hE_x(K,G;\sigma)})_{hF(\sigma,G)} \ . \tag{5.11}$$

By definition of action of $F(K, G)$ on $F(\sigma, \mathbf{S}^0 \wedge G_+)$, we find that the restriction of the action to $E_x(K, G; \sigma)$ is trivial. Consequently, the action factors through the quotient group $F(\sigma, G)$. We therefore have a natural weak equivalence

$$\begin{aligned} \sigma \quad \longmapsto \quad & (F(\sigma, \mathbf{S}^0 \wedge G_+)_{hE_x(K,G;\sigma)})_{hF(\sigma,G)} \\ & \simeq \quad \sigma \quad \longmapsto \quad (F(\sigma, \mathbf{S}^0 \wedge G_+) \wedge E_x(K, X; \sigma)_+)_{hF(\sigma,G)} \ , \end{aligned} \tag{5.12}$$

where we identify $BE_x(K,G;\sigma)$ with $E_x(K,X;\sigma)$.

On the one hand, the continuous homomorphism $G \to F(\sigma, G)$, given by sending a group element g to the constant function with value g is a G-equivariant homotopy equivalence. Moreover, the corresponding induced action of G on $F(\sigma, \mathbf{S}^0 \wedge G_+)$ is seen to be free. This implies that there is a natural weak equivalence

$$
\begin{aligned}
\sigma \quad &\longmapsto \quad (F(\sigma, \mathbf{S}^0 \wedge G_+) \wedge E_x(K,X;\sigma)_+)_{hF(\sigma,G)} \\
&\qquad\qquad \simeq \quad \sigma \quad \longmapsto \quad F(\sigma, \mathbf{S}^0 \wedge G_+) \wedge E_x(K,X;\sigma)_+)_{hG} \; . \qquad (5.13)
\end{aligned}
$$

On the other hand, the map of spectra

$$\mathbf{S}^0 \wedge G_+ \to F(\sigma, \mathbf{S}^0 \wedge G_+)$$

which sends a point to the constant function at that point is a G-equivariant homotopy equivalence. So, taking homotopy G-orbits, we have a natural weak equivalence

$$
\begin{aligned}
\sigma \quad &\longmapsto \quad (F(\sigma, \mathbf{S}^0 \wedge G_+) \wedge E_x(K,X;\sigma)_+)_{hG} \\
&\qquad\qquad \simeq \quad \sigma \quad \longmapsto \quad (\mathbf{S}^0 \wedge G_+ \wedge E_x(K,X;\sigma)_+)_{hG} \\
&\qquad\qquad \simeq \quad \sigma \quad \longmapsto \quad \mathbf{S}^0 \wedge E_x(K,X;\sigma)_+ \; . \qquad (5.14)
\end{aligned}
$$

Consequently, 5.11–5.14 imply that there is a weak natural weak equivalence of functors

$$\sigma \quad \longmapsto \quad F(\sigma, \mathbf{S}^0 \wedge G_+)_{hF(K,G)} \quad \simeq \quad \sigma \quad \longmapsto \quad \mathbf{S}^0 \wedge E_x(K,X;\sigma)_+ \qquad (5.15)$$

Applying 5.8 and 1.2, the theorem follows. □

6. Other homotopy functors

Let $\mathbf{E}$ be a spectrum with an action of H, the latter the realization of a simplicial group. Suppose that K is a finite simplicial complex equipped with H-action. Then $F(K,X)$ is equipped with an H-action given by precomposition. The goal of this section is to compute the derivative of the functor

$$X \quad \longmapsto \quad \Omega^\infty \mathbf{E} \wedge_{hH} F(K,X)_+ \; . \qquad (6.1)$$

6.2. Theorem. *The derivative of 6.1 is given by*

$$\mathbf{E} \wedge_{hH} \Gamma_K(E_x(K,X); \mathbf{S}^0_+) \; .$$

6.3. Remarks. (1). The answer for the latter was known previously in the following two cases: (i) when $\mathbf{E} = \mathbf{S}^0$ for a finite group H acting on a discrete set K (see [G,1.19(iii)]), and (ii) when $\mathbf{E} = \mathbf{S}^0$, with $H = S^1$ acting on $K = S^1$ by rotation (see e.g., [B...]).

(2). Let $\Gamma_K(E_x(K,X); \mathbf{E}_+)$ be the spectrum with H-action defined as follows: Add a disjoint basepoint to $E_x(K,X) \to K$ fiberwise, and then apply $\Omega^\infty \mathbf{E}$ fiberwise. Convert the resulting space over K into a fibration. Then $\Gamma_K(E_x(K,X);\mathbf{E}_+)$ is the spectrum with H-action whose zero-th space is the space of sections of this fibration.

It is straightforward to check that there is a weak equivalence

$$\Gamma_K(E_x(K,X);\mathbf{E}_+)_{hH} \quad \simeq \quad \mathbf{E}\wedge_{hH}\Gamma_K(E_x(K,X);\mathbf{S}^0_+)\ .$$

This gives an alternative way of expresssing the derivative of 6.1.

Proof of 6.2. Let R^H denote the category of spaces equipped with pointed H-action. Let $G = \Omega X$. We may then write the functor in question as a composite

$$\begin{array}{ccccc} \mathbf{Top}_* & \xrightarrow{f} & R^H & \xrightarrow{g} & \mathbf{Top}_* \\ X & \longmapsto & F(K,X) & & \\ & & Z & \longmapsto & \Omega^\infty\mathbf{E}\wedge_{hH} Z\ . \end{array}$$

The derivative of g is $\mathbf{E}$. The derivative of f, without regard to the action of H, was computed above; it is the spectrum

$$F(K,\mathbf{S}^0\wedge G_+)$$

with $F(K,G)$-action defined by pointwise multiplication. It also fairly straightforward to check that the action of H on the latter is the evident one, induced by the action on K. Thus we have identified $\partial_X f$ as a spectrum with $H\times F(K,G)$-action.

Applying the chain rule [KR,4.6], the derivative of 6.1 is given by

$$\mathbf{E}\wedge_{hH\times F(K,G)} F(K,\mathbf{S}^0\wedge G_+)\ , \tag{6.4}$$

where $F(K,G)$ acts trivially on $\mathbf{E}$. We showed above (§5) that

$$F(K,\mathbf{S}^0\wedge G_+)_{hF(K,G)} \quad \simeq \quad \Gamma_K(E_x(K,X);\mathbf{S}^0_+)\ . \tag{6.5}$$

There is an H-action on $\Gamma_K(E_x(K,X);\mathbf{S}^0_+)$ which is induced by one on $E_x(K,X)$, given by $(k,f)\mapsto(hk,fh^{-1})$ for $h\in H$. With a bit of effort, one sees that the equivalence 6.5 is an equivariant map with respect to this choice.

Hence 6.4 can be simplified to

$$\mathbf{E}\wedge_{hH}\Gamma_K(E_x(K,X);\mathbf{S}^0_+)\ .$$

This completes the proof. □

References

[A] G. Arone, *The Taylor tower of the stable homotopy of mapping spaces*, submitted to Trans. AMS.

[BF] A.K. Bousfield and E. M. Friedlander, *Homotopy theory of* Γ*-spaces, spectra, and bisimplicial sets*, Applications of homotopy theory II, Springer LNM 658, 1978, pp. 80–130.

[B...] M. Boekstedt, G. Carlsson, R. Cohen, T. Goodwillie, W.C. Hsiang and I. Madsen, *On the algebraic* K*-theory of simply connected spaces*, Aarhus University Prepint (1991).

[BK] A.K. Bousfield and D.M. Kan, *Homotopy limits, completions and localizations*, LNM 304, Springer, 1972.

[DS] W.G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, Handbook of algebraic topology (I.M. James, ed.), Elsevier Science, 1995.

[G] T. Goodwillie,, *Calculus I, the derivative of a homotopy functor*, K-theory **4** (1990), 1–27.

[H] L. Hesseholt, *A homotopy theoretical derivation of* $QMAP(K,-)_+$, Math. Scan **70** (1992), 193-203.

[KR] J. Klein and J. Rognes, *A chain rule in the calculus of homotopy functors*, preprint.

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