Finite dimensional approximation of diffusion processes on infinite dimensional spaces

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Abstract

We prove that the laws of diffusion processes $\mathbf{M}$ on E associated with Dirichlet forms of type $\mathcal{E}(u,v) = \int_E \langle A(z)\nabla u(z), \nabla v(z)\rangle_H \mu(dz)$, where H, E are separable Hilbert spaces, are the weak limits of laws of finite dimensional diffusions. These are associated with the image Dirichlet forms obtained from $\mathcal{E}$ under projections from E onto finite dimensional subspaces in H. As a by-product we obtain Hoelder continuity of the sample paths as well as a new existence proof for the infinite dimensional diffusion $\mathbf{M}$.

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1 Introduction, framework and main results

The purpose of this paper is to construct natural finite dimensional approximations to infinite dimensional stochastic processes associated with Dirichlet forms of diffusion type (cf. e.g. [AR91]). Though the existence of such infinite dimensional diffusions is well-known (see [Sch90], [RS92] or [MR92, Chap.IV, Subsection 4b]), our approximation procedure also gives a new existence proof (at least if the state space is a Hilbert space and condition (1.1) below holds). Apart from general interest these approximations are motivated since they can be used to obtain *new* information about the limiting process. For example, we can now prove Hoelder continuity of its sample paths. To state and illustrate our results and their applications more precisely, as well as to give an overview of the organization of this paper we need some preparations. We start with describing our framework.

Let H, E be separable Hilbert spaces (with inner product $\langle\ ,\ \rangle_H$, $\langle\ ,\ \rangle_E$ respectively), $H \subset E$ densely and continuously, and such that, in addition,

$$\text{the embedding } H \subset E \text{ is Hilbert-Schmidt.} \tag{1.1}$$

Let μ be a probability measure on the Borel σ-algebra $\mathcal{B}(E)$ of E such that $E' \subset L^2(E;\mu)$. Here E' denotes the (topological) dual of E. Define the linear space of smooth bounded cylinder functions, as usual, by

$$\mathcal{F}C_b^\infty := \{f(l_1,\ldots,l_m) \mid m \in \mathbb{N},\ f \in C_b^\infty(\mathbb{R}^m),\ l_1,\ldots l_m \in E'\}\ .$$

Here $C_b^\infty(\mathbb{R}^m)$ is the space of all bounded infinitely differentiable real-valued functions on $\mathbb{R}^m$ with all partial derivatives bounded.

Let $\mathcal{L}^\infty(H)$ denote the space of all bounded linear operators on H with usual operator norm $\|\ \|_{\mathcal{L}^\infty(H)}$. Let $A : E \to \mathcal{L}^\infty(H)$ be a strongly measurable map such that $A(z)$ is self-adjoint and positive definite for all $z \in E$ and

$$\text{there exists } q \in]1,\infty[\text{ such that } \int \|A(z)\|^q_{\mathcal{L}(H)}\ \mu(dz) < \infty\ . \tag{1.2}$$

Consider the following symmetric positive definite bilinear form on $L^2(E;\mu)$ $(:= L^2(E \to \mathbb{R};\mu))$

$$\mathcal{E}(u,v) := \int_E \langle A(z)\nabla u(z), \nabla v(z)\rangle_H\ \mu(dz) \quad ;\ u,v \in \mathcal{F}C_b^\infty\ . \tag{1.3}$$

Here, for simplicity, we assume that $\mu(U) > 0$ for every weakly open set $U \subset E$, hence each $u \in \mathcal{F}C_b^\infty$ can be identified with its corresponding class in $L^2(E;\mu)$. Furthermore, $\nabla u(z)$ is the unique element in H representing the bounded linear functional

$$h \mapsto \frac{\partial u}{\partial h}(z) := \frac{d}{ds} u(z+sh)|_{s=0} \quad , h \in H \ .$$

$(\mathcal{E}, \mathcal{F}C_b^\infty)$ is densely defined (cf. [MR92, Chap.II, Subsect. 3a]). We additionally assume that

$$(\mathcal{E}, \mathcal{F}C_b^\infty) \text{ is closable on } L^2(E;\mu). \tag{1.4}$$

For sufficient (and easily checkable) conditions ensuring (1.4) we refer to [AR91, Theorem 3.1] (see also [AR90a]). Let $(\mathcal{E}, D(\mathcal{E}))$ denote the *closure* (i.e., the smallest closed extension) of $(\mathcal{E}, \mathcal{F}C_b^\infty)$ on $L^2(E;\mu)$. It is well-known and easy to see that $(\mathcal{E}, D(\mathcal{E}))$ is a (symmetric) *Dirichlet form* (cf. e.g. [MR92, Chap.II, Exercises 3.6 and 3.9] or [AR91, Theorem 3.1]). The general definition of a Dirichlet form can e.g. be found in [MR92, Chap.I, Definition 4.5] (see also [F80], [FOT94]).

For our finite dimensional approximations we now fix an orthonormal basis $\{e_n | n \in \mathbb{N}\}$ of H such that $\{e_n/\lambda_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of E, where $\lambda_n \in]0,\infty[$ such that $\sum_{n=1}^\infty \lambda_n^2 < \infty$. e_n , λ_n , $n \in \mathbb{N}$, exist since $H \subset E$ is Hilbert-Schmidt. Clearly,

$$\langle e_n, h\rangle_E = \lambda_n^2 \langle e_n, h\rangle_H \quad \text{for all } h \in H \ (\subset E),\ n \in \mathbb{N}. \tag{1.5}$$

Fix $n \in \mathbb{N}$. Define $k_n \in E'$ by

$$k_n(z) := \lambda_n^{-2} \langle e_n, z\rangle_E \quad , z \in E \ . \tag{1.6}$$

Furthermore, define $\Pi_n : E \longrightarrow \mathbb{R}^n$ by

$$\Pi_n(z) := (k_1(z), \cdots, k_n(z)) \ .$$

For $i,j \in \mathbb{N}$ set

$$a_{ij}(z) := \langle A(z) e_i, e_j\rangle_H \quad , z \in E \ ,$$

and let $a_{ij}^{(n)} : \mathbb{R}^n \longrightarrow \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^n)$–measurable such that

$$a_{ij}^{(n)} \circ \Pi_n = E_\mu[a_{ij} | \sigma(\Pi_n)] \quad \mu\text{-a.e.}, \tag{1.7}$$

where $E_\mu[\,\cdot\,|\sigma(\Pi_n)]$ denotes the conditional expectation of μ given the σ-algebra generated by Π_n. Define $\mu_n := \mu \circ \Pi_n^{-1}$ and

$$\mathcal{E}^{(n)}(u,v) := \sum_{i,j=1}^{n} \int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}^{(n)}\ d\mu_n\ ; u,v \in C_0^\infty(\mathbb{R}^n)\ , \tag{1.8}$$

where $C_0^\infty(\mathbb{R}^n)$ denotes the set of all functions in $C_b^\infty(\mathbb{R}^n)$ with compact support. Then $(\mathcal{E}^{(n)}, C_0^\infty(\mathbb{R}^n))$ is a (well-defined) densely defined symmetric positive definite bilinear form on $L^2(\mathbb{R}^n;\mu_n)$.

Proposition 1.1 *$(\mathcal{E}^{(n)}, C_0^\infty(\mathbb{R}^n))$ is closable on $L^2(\mathbb{R}^n;\mu_n)$.*

Proof. see Sect. 2. □

Hence it is well-known that the closure $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ of $(\mathcal{E}^{(n)}, C_0^\infty(\mathbb{R}^n))$ is a Dirichlet form on $L^2(\mathbb{R}^n;\mu_n)$ (cf. [F80, Example 1.2.1] or [MR92, Chap.II, Subsect. 2c]) which is *regular* in the sense of [F80], i.e., $C_0^\infty(\mathbb{R}^n) \cap D(\mathcal{E}^{(n)})$ is dense in $D(\mathcal{E}^{(n)})$ w.r.t. $(\mathcal{E}_1^{(n)})^{1/2} := (\mathcal{E}^{(n)} + (\ ,\ )_{L^2(\mathbb{R}^n;\mu_n)})^{1/2}$ and in $C_0(\mathbb{R}^n)$(:= the set of all continuous functions on $\mathbb{R}^n$ with compact support) w.r.t. the uniform norm. Consequently, by [F80, Chap.6] there exists a conservative diffusion process $\mathbf{M}^{(n)} := (\Omega^{(n)}, \mathcal{F}^{(n)}, (X_t^{(n)})_{t\geq 0}, (P_z^{(n)})_{z\in\mathbb{R}^n})$ *associated with* $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$, i.e., for every bounded $\mathcal{B}(\mathbb{R}^n)$-measurable function u and all $t > 0$

$$z \mapsto \int_\Omega u(X_t^{(n)}) dP_z^{(n)} \quad \text{is a } \mu\text{-version of } e^{tL^{(n)}}u\ . \tag{1.9}$$

Here $L^{(n)}$ with domain $D(L^{(n)})$ is the unique negative definite self-adjoint operator on $L^2(\mathbb{R}^n;\mu_n)$ such that

$$D(\mathcal{E}^{(n)}) = D(\sqrt{-L^{(n)}}) \quad \text{and} \quad \mathcal{E}^{(n)}(u,v) = (\sqrt{-L^{(n)}}u, \sqrt{-L^{(n)}}v)_{L^2(\mathbb{R}^n;\mu_n)}$$
$$\text{for all } u,v \in D(\mathcal{E}^{(n)}).$$

Furthermore, $\Omega_n := C([0,\infty[\ ,\mathbb{R}^n)$, i.e., the space of all continuous paths from $[0,\infty[$ to $\mathbb{R}^n$, and $X_t^{(n)}(\omega) := \omega(t), t \geq 0, \omega \in \Omega_n$. In particular, μ_n is an invariant measure for $\mathbf{M}^{(n)}$.

Let $\Omega := C_0([0,\infty[, E)$ and let $I_n : \Omega_n \to \Omega$ be defined by

$$I_n(X)_t := i_n(X_t^{(n)}) \quad ,\ t \in [0,\infty[\ ,\ X \in \Omega_n\ ,$$

where

$$i_n(x) := \sum_{i=1}^{n} x_i\, e_i \quad \text{if } x = (x_1, \cdots, x_n) \in \mathbb{R}^n \ .$$

Now we consider the image measure under I_n of the distribution of $(X_t^{(n)})_{t\geq 0}$ when started with μ_n, that is

$$P_n := P_{\mu_n}^{(n)} \circ I_n^{-1} \ , \tag{1.10}$$

which is a probability measure on $(\Omega, \mathcal{F})$. Here $P_{\mu_n}^{(n)} := \int_{\mathbb{R}^n} P_x^{(n)}\, \mu_n(dx)$, $\mathcal{F} := \sigma(X_t \mid t \in [0,\infty[)(= \mathcal{B}(\Omega))$, $X_t(\omega) := \omega(t), t \geq 0$, and Ω is equipped with the topology of locally uniform convergence. For $\alpha \in]0, \frac{1}{2}[$ let Ω_α denote all paths in Ω which are locally Hoelder continuous of order α.

Now we can formulate the main result of this paper:

Theorem 1.2 *There exists a probability measure P on $(\Omega, \mathcal{F})$ such that $P_n \underset{n\to\infty}{\longrightarrow} P$ weakly. Furthermore, $P(\Omega_\alpha) = 1$ for all $\alpha \in]0, \frac{1}{2} - \frac{1}{2q}[$.*

Corollary 1.3 *(i) There exists a diffusion process $\mathbf{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_z)_{z\in E})$ on E associated with $(\mathcal{E}, D(\mathcal{E}))$ (cf. above) such that*

$$P = \int P_z\ \mu(dz)\ (=: P_\mu)\ , \tag{1.11}$$

where P is as in Theorem 1.2. In particular, the finite dimensional distributions of $(X_t)_{t\geq 0}$ under P are given by the operator semigroup associated with $(\mathcal{E}, D(\mathcal{E}))$.
(ii) Let $\alpha \in]0, \frac{1}{2} - \frac{1}{2q}[$. For μ-a.e. $z \in E$, $t \mapsto X_t(\omega)$ is Hoelder continuous of order α for P_z-a.e. $\omega \in \Omega$.

We also have a result of type 1.3 (ii) for quasi-every $z \in E$ (cf. Remark 4.1). Under these general assumptions 1.3 (ii) even seems to be new if $H = E = \mathbb{R}^n$.

The proofs of both results will be given in detail in Sections 2–4 below. We only want to mention here that the proof of Theorem 1.2 goes in two steps: First we prove tightness of $\{P_n \mid n \in \mathbb{N}\}$ in Sect. 2 using the Lyons-Zheng decomposition (cf. [LZhe88], [LZ94]); in the second step we identify the limit (cf. Sect. 3). We would also like to draw the reader's attention to Remark 3.3 and to the appendix.

As mentioned before the existence of a diffusion process $\mathbf{M}$ associated to $(\mathcal{E}, D(\mathcal{E}))$ is well-known, even if E is only a Banach space without condition (1.1) and with merely $q = 1$ in (1.2) (cf. e.g. [MR92, Chap.IV, Theorem 3.5 and Subsect. 4b, in particular Exercise 4.3]) So, the new results in Corollary 1.3 are the identification of P_μ as the weak limit of $(P_n)_{n\in\mathbb{N}}$ and the Hoelder continuity of the sample paths (cf. Sect. 4 for details).

Finally, in Sect. 5 we present some meanwhile well-studied examples of infinite-dimensional diffusions (like e.g. those connected with Euclidean field theory, first investigated by P. Jona-Lasinio and P. K. Mitter in [J-LM85]) which the above results apply to. For the convenience of the reader we recall the necessary definitions.

2 Tightness

We start with the

Proof of Proposition 1.1.
Consider the image Dirichlet form $(\mathcal{E}^\#, D(\mathcal{E}^\#))$ of $(\mathcal{E}, D(\mathcal{E}))$ under Π_n on $L^2(\mathbb{R}^n; \mu_n)$, i.e.,

$$D(\mathcal{E}^\#) := \{u \in L^2(\mathbb{R}^n; \mu_n) |\ u \circ \Pi_n \in D(\mathcal{E})\}$$

and

$$\mathcal{E}^\#(u, v) := \mathcal{E}(u \circ \Pi_n, v \circ \Pi_n)\ , \quad u, v \in D(\mathcal{E}^\#)\ .$$

By [MR92, Chap.VI, Exercise 1.1] $(\mathcal{E}^\#, D(\mathcal{E}^\#))$ is closed on $L^2(\mathbb{R}^n; \mu_n)$. Furthermore, $C_0^\infty(\mathbb{R}^n) \subset D(\mathcal{E}^\#)$ and for $u, v \in C_0^\infty(\mathbb{R}^n)$ by the chain rule

$$\begin{aligned} \mathcal{E}^\#(u, v) &= \sum_{i,j=1}^n \int_E a_{ij} \frac{\partial u}{\partial x_i} \circ \Pi_n \frac{\partial v}{\partial x_j} \circ \Pi_n d\mu \\ &= \sum_{i,j=1}^n \int_{\mathbb{R}^n} a_{ij}^{(n)} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d\mu_n \\ &= \mathcal{E}^{(n)}(u, v)\ . \end{aligned} \tag{2.1}$$

Consequently, $(\mathcal{E}^{(n)}, C_0^\infty(\mathbb{R}^n))$ is closable on $L^2(\mathbb{R}^n; \mu_n)$. □

We also need

Lemma 2.1 *Let $z \in E$. Then $i_n \circ \Pi_n(z) \longrightarrow z \quad$ in E as $n \to \infty$.*

Proof. Since $\{e_n/\lambda_n \mid n \in \mathbb{N}\}$ is an orthonormal base of E, the assertion is an immediate consequence of (1.5), (1.6) $\square$

The first step to prove Theorem 1.2 is the following

Proposition 2.2 *$\{P_n | n \in \mathbb{N}\}$ is tight and if P is one of its accumulation points, then $P[\Omega_\alpha] = 1$ for all $\alpha \in]0, \frac{1}{2} - \frac{1}{2q}[$.*

Proof. Let $n \in \mathbb{N}$. Since $E' \subset L^2(E;\mu)$ by assumption, the coordinate functions x^i on $\mathbb{R}^n$ belong to $L^2(\mathbb{R}^n;\mu_n)$. Then it is easy to see that even $x^i \in D(\mathcal{E}^{(n)})$, $1 \leq i \leq n$. Hence it follows by [LZhe88], [LZ94] that if $\left(X_i^{(n)}(t)\right)_{t\geq 0}$, $1 \leq i \leq n$, denote the coordinate processes of $(X_t^{(n)})_{t\geq 0}$, then the following decomposition holds for $1 \leq i \leq n$:

$$X_i^{(n)}(t) - X_i^{(n)}(0) = \frac{1}{2}M_t^i - \frac{1}{2}(\bar{M}_1^i - \bar{M}_{1-t}^i)\,,\ 0 \leq t \leq 1\,,\ P_{\mu_n}^{(n)} - \text{a.s.} \tag{2.2}$$

Here (M_t^i) is an $(\mathcal{F}_t^{(n)})$-martingale and $(\bar{M}_t^i)$ is an $(\bar{\mathcal{F}}_t^{(n)})$-martingale with $\mathcal{F}_t^{(n)} := \sigma(X_s^{(n)} | s \leq t)$, $\bar{\mathcal{F}}_t := \sigma(X_{1-s}^{(n)} | s \leq t)$. Moreover, their brackets are given by

$$\begin{aligned} \langle M^i \rangle &= \int_0^\cdot a_{ii}^{(n)}(X_s^{(n)})ds\,, \\ \langle \bar{M}^i \rangle &= \int_0^\cdot a_{ii}^{(n)}(X_{1-s}^{(n)})ds\,, \end{aligned}$$

$1 \leq i \leq n$. Hence if $p := 2q$ by the Burkholder-Gundy inequalities we can find $C_p \in]0,\infty[$ such that for all $1 \leq i \leq n$, $0 \leq s,t \leq 1$,

$$\begin{aligned} & \left(E^{(n)}[|X_i^{(n)}(t) - X_i^{(n)}(s)|^p]\right)^{1/p} \\ \leq\ & E^{(n)}[|M_t^i - M_s^i|^p]^{1/p} + E^{(n)}[|\bar{M}_{1-s}^i - \bar{M}_{1-t}^i|^p]^{1/p} \\ \leq\ & C_p \left[E^{(n)}\left[\left(\int_s^t a_{ii}^{(n)}(X_{s'}^{(n)})ds'\right)^{p/2}\right]^{1/p} + E^{(n)}\left[\left(\int_{1-t}^{1-s} a_{ii}^{(n)}(X_{s'}^{(n)})ds'\right)^{p/2}\right]^{1/p}\right] \\ \leq\ & C_p(t-s)^{1/2-1/p}\left[\left(\int_s^t E^{(n)}\left[a_{ii}^{(n)}(X_{s'}^{(n)})^{p/2}\right]ds'\right)^{1/p} + \left(\int_{1-t}^{1-s} E^{(n)}\left[a_{ii}^{(n)}(X_{s'}^{(n)})^{p/2}\right]ds'\right)^{1/p}\right] \\ \leq\ & 2C_p(t-s)^{1/2}\left(\int_{\mathbb{R}^n} a_{ii}^{(n)}(x)^{p/2}\mu_n(dx)\right)^{1/p} \\ \leq\ & C_p^{(1)}(t-s)^{1/2}\,, \end{aligned} \tag{2.3}$$

where $C_p^{(1)} := 2C_p \left(\int \|A(z)\|^{p/2}_{\mathcal{L}^\infty(H)}\mu(dz)\right)^{1/p}$ and $E^{(n)}$ denotes the expectation w.r.t. $P^{(n)}_{\mu_n}$. Consequently , since $\{e_n/\lambda_n | n \in \mathbb{N}\}$ is an orthonormal basis of E

$$\begin{aligned}
\left(\int \|X_t - X_s\|^p_E dP_n\right)^{2/p} &= E^{(n)}\left[\|\sum_{i=1}^n \left(X_i^{(n)}(t) - X_i^{(n)}(s)\right) e_i\|^p_E\right]^{2/p} \\
&= E^{(n)}\left[\left\{\sum_{i=1}^n \lambda_i^2 \left(X_i^{(n)}(t) - X_i^{(n)}(s)\right)^2\right\}^{p/2}\right]^{2/p} \\
&\leq \sum_{i=1}^n \lambda_i^2 E^{(n)}\left[|X_i^{(n)}(t) - X_t^{(n)}(s)|^p\right]^{2/p} \\
&\leq C_p^{(2)}(t-s) \qquad (2.4)
\end{aligned}$$

by (2.3), with $C_p^{(2)} := \left(C_p^{(1)}\right)^2 \sum_{i=1}^\infty \lambda_i^2 \quad (< \infty)$. On the other hand for all $t \geq 0$, $n \in \mathbb{N}$,

$$P_n \circ X_t^{-1} = \mu_n \circ i_n^{-1} = \mu \circ (i_n \circ \Pi_n)^{-1} \ .$$

Therefore, by Lemma 2.1 $(P_n \circ X_t^{-1})_{n\in\mathbb{N}}$ converges weakly, hence $\{P_n \circ X_t^{-1} | n \in \mathbb{N}\}$ is a tight set of measures on $(E, \mathcal{B}(E))$ for all $t \geq 0$. By [EK86, Chap.3, Theorem 7.2] it thus follows that $\{P_n | n \in \mathbb{N}\}$ is tight on $C([0,1], E)$. Similarly, it follows that $\{P_n | n \in \mathbb{N}\}$ is tight on $C([0,T], E)$ for all $T > 0$, hence tight on $C([0,\infty[, E)$.

To prove the second part of the assertion let P be an accumulation point of $\{P_n | n \in \mathbb{N}\}$. Replacing $\|X_t - X_s\|^p_E$ in the left hand side of (2.4) by $\|X_t - X_s\|^p_E \wedge N$, $N \in \mathbb{N}$, and P_n by P_{n_k} where $P_{n_k} \underset{k\to\infty}{\longrightarrow} P$ weakly, and first letting $k \to \infty$ and then $N \to \infty$, we see that

$$\int \|X_t - X_s\|^p_E \, dP \leq C_p^{(2)}(t-s)^{p/2}.$$

Hence by standard arguments (see [StV79, Corollary 2.1.4], whose proof directly extends from $\mathbb{R}^d$– to Banach-space –valued processes), it follows that

$$P[\Omega_\beta] = 1$$

for all $\beta \in]0, \frac{1}{2} - \frac{1}{p}[$. □

3 Convergence and identification of the limit

Let $(L, D(L))$ be the generator of $(\mathcal{E}, D(\mathcal{E}))$, i.e., the unique negative definite self-adjoint operator on $L^2(E;\mu)$ such that

$$D(\mathcal{E}) = D(\sqrt{-L}) \quad \text{and} \quad \mathcal{E}(u,v) = (\sqrt{-L}u, \sqrt{-L}v)_{L^2(E;\mu)} \\ \text{for all } u, v \in D(\mathcal{E}) \tag{3.1}$$

Let $T_t := e^{tL}$, $t \geq 0$, and for $\alpha \in]0, \infty[$

$$G_\alpha u := \int_0^\infty e^{-\alpha t} T_t u \, dt \ , \ u \in L^2(E;\mu), \tag{3.2}$$

i.e., $(T_t)_{t>0}$ resp. $(G_\alpha)_{\alpha>0}$ is the (operator) semigroup resp. (operator) resolvent associated with $(\mathcal{E}, D(\mathcal{E}))$. For $n \in \mathbb{N}$ let $(T_t^{(n)})_{t>0}$, $(G_\alpha^{(n)})_{\alpha>0}$ be the corresponding objects relative to $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)})$ on $L^2(\mathbb{R}^n;\mu_n)$.

Proposition 3.1 *Let $g_n \in L^2(\mathbb{R}^n;\mu_n)$, $n \in \mathbb{N}$, such that $g_n \circ \Pi_n \to g$ in $L^2(E;\mu)$ as $n \to \infty$. Then for all $\alpha > 0$*

$$\left(G_\alpha^{(n)} g_n\right) \circ \Pi_n \underset{n\to\infty}{\longrightarrow} G_\alpha g \quad \text{in } L^2(E;\mu).$$

Proof. Fix $\alpha > 0$ and $n \in \mathbb{N}$. Define as usual $\mathcal{E}_\alpha := \mathcal{E} + \alpha(\ ,\ )_{L^2(E;\mu)}$ and set

$$F_n := \{v \circ \Pi_n | v \in D(\mathcal{E}^{(n)})\} \ .$$

Since (2.1) extends to all $u, v \in D(\mathcal{E}^{(n)})$, it follows that F_n is a closed linear subspace of $D(\mathcal{E})$ (the latter being equipped with the norm $\mathcal{E}_\alpha^{1/2}$). Furthermore, by the same reason for all $\alpha > 0$ and all $u \in L^2(\mathbb{R}^n;\mu_n)$

$$\begin{aligned}
\mathcal{E}_\alpha\left((G_\alpha^{(n)}u)\circ\Pi_n, v\circ\Pi_n\right) &= \mathcal{E}_\alpha^{(n)}\left(G_\alpha^{(n)}u, v\right) = (u,v)_{L^2(\mathbb{R}^n;\mu_n)} \\
&= (u\circ\Pi_n, v\circ\Pi_n)_{L^2(E;\mu)} \\
&= \mathcal{E}_\alpha\left(G_\alpha(u\circ\Pi_n), v\circ\Pi_n\right) \ .
\end{aligned}$$

Hence, if $P_n^{(\alpha)}$ denotes the orthogonal projection onto F_n in the Hilbert space $(D(\mathcal{E}), \mathcal{E}_\alpha)$, then

$$(G_\alpha^{(n)}u)\circ\Pi_n = P_n^{(\alpha)} G_\alpha(u\circ\Pi_n) \ . \tag{3.3}$$

Since the linear span of $\{k_n | n \in \mathbb{N}\}$ is dense in E', it is easy to see that every element in $\mathcal{F}C_b^\infty$ can be approximated by a sequence in $\cup_{n\in\mathbb{N}} F_n$ w.r.t. $\mathcal{E}_\alpha^{1/2}$ (cf. [AR89b, Proposition 2.10]). Since $\mathcal{F}C_b^\infty$ is dense in $D(\mathcal{E})$ w.r.t. $\mathcal{E}_1^{1/2}$, hence w.r.t. $\mathcal{E}_\alpha^{1/2}$, it follows that $P_n^{(\alpha)} \underset{n\to\infty}{\longrightarrow} Id$ strongly on the Hilbert space $(D(\mathcal{E}), \mathcal{E}_\alpha)$. Consequently, (3.3) with g_n replacing u implies the assertion. □

As a consequence we obtain the following corollary which in connection with Proposition 2.2 completes the proof of Theorem 1.2, since it identifies the finite dimensional distributions of any accumulation point of $\{P_n | n \in \mathbb{N}\}$.

Corollary 3.2 *Let P be an accumulation point of $\{P_n | n \in \mathbb{N}\}$. Then for any $f_0, f_1, \ldots, f_m \in C_b(E)$ (= all bounded continuous functions on E) and any $t_1, \ldots, t_m \in]0, \infty[$*

$$\begin{aligned} & \int f_0(X_0) f_1(X_{t_1}) f_2(X_{t_1+t_2}) \cdots f_m(X_{t_1+\cdots+t_m})\, dP \\ = & \int f_0 T_{t_1} \left(f_1 T_{t_2} \left(f_2 \cdots T_{t_m}(f_m)\right) \cdots\right) \, d\mu \ . \end{aligned}$$

In particular, all accumulation points of $\{P_n | n \in \mathbb{N}\}$ coincide.

Proof. For $j \in \{0, 1, 2, \ldots, m\}$ and $n \in \mathbb{N}$ define $f_j^{(n)} : \mathbb{R}^n \to \mathbb{R}$ by

$$f_j^{(n)}(x) := f_j(i_n(x)) \ , x \in \mathbb{R}^n \ .$$

Note that by Lemma 2.1, since $f_j \in C_b(E)$, we have that

$$f_j^{(n)} \circ \Pi_n \underset{n\to\infty}{\longrightarrow} f_j \qquad \text{in } L^2(E; \mu) \ . \tag{3.4}$$

Let $\alpha_1, \ldots, \alpha_m \in]0, \infty[$.
Claim

$$\begin{aligned} & G_{\alpha_1}^{(n)} \left(f_1^{(n)} G_{\alpha_2}^{(n)} \left(f_2^{(n)} \cdots G_{\alpha_m}^{(n)}(f_m^{(n)})\right) \cdots\right) \circ \Pi_n \\ \underset{n\to\infty}{\longrightarrow} & G_{\alpha_1} \left(f_1 G_{\alpha_2} \left(f_2 \cdots G_{\alpha_m}(f_m)\right) \cdots\right) \quad \text{in } L^2(E; \mu). \end{aligned}$$

We prove the claim by induction on m.

$m = 1$: This case follows by (3.4) from Proposition 3.1.

$m - 1 \mapsto m$: By assumption we know that

$$\begin{aligned} & G_{\alpha_2}^{(n)} \left(f_2^{(n)} \cdots G_{\alpha_m}^{(n)}(f_m^{(n)})\right) \cdots\right) \circ \Pi_n \\ \underset{n\to\infty}{\longrightarrow} & G_{\alpha_2} \left(f_2 \cdots G_{\alpha_m}(f_m)\right) \ldots) \quad \text{in } L^2(E; \mu), \end{aligned}$$

hence by (3.4)

$$\begin{aligned} & \left(f_1^{(n)} \circ \Pi_n\right) G_{\alpha_2}^{(n)} \left(f_2^{(n)} \cdots G_{\alpha_m}^{(n)}(f_m^{(n)}) \cdots\right) \circ \Pi_n \\ \underset{n\to\infty}{\longrightarrow} & f_1 G_{\alpha_2} \left(f_2 \cdots G_{\alpha_m}(f_m)\right) \ldots) \quad \text{in } L^2(E; \mu). \end{aligned}$$

Applying Proposition 3.1 again we obtain that

$$\begin{aligned} & G_{\alpha_1}^{(n)}\left(f_1^{(n)}G_{\alpha_2}^{(n)}\left(f_2^{(n)}\cdots G_{\alpha_m}^{(n)}(f_m^{(n)})\right)\cdots\right)\circ\Pi_n \\ \underset{n\to\infty}{\longrightarrow} \; & G_{\alpha_1}\left(f_1G_{\alpha_2}\left(f_2\cdots G_{\alpha_m}(f_m)\right)\cdots\right) \quad \text{in } L^2(E;\mu). \end{aligned}$$

But by (1.9), (1.10) for all $n\in\mathbb{N}$

$$\begin{aligned} & \int f_0(X_0)f_1(X_{t_1})f_2(X_{t_1+t_2})\cdots f_m(X_{t_1+\cdots+t_m})dP_n \\ = \; & \int \left(f_0^{(n)}\circ\Pi_n\right)T_{t_1}^{(n)}\left(f_1^{(n)}T_{t_2}\left(f_2^{(n)}\cdots T_{t_m}^{(n)}(f_m^{(n)})\right)\cdots\right)\circ\Pi_n d\mu \; . \end{aligned}$$

Integrating both sides w.r.t. $e^{-(\alpha_1t_1+\alpha_2t_2+\cdots+\alpha_mt_m)}dt_1\dots dt_m$ over $[0,\infty[^m$ we obtain that

$$\begin{aligned} & \int_0^\infty\cdots\int_0^\infty\int f_0(X_0)f_1(X_{t_1})f_2(X_{t_1+t_2})\cdots f_m(X_{t_1+\cdots+t_m})dP_ne^{-(\alpha_1t_1+\cdots+\alpha_mt_m)}dt_1\dots dt_m \\ = \; & \int\left(f_0^{(n)}\circ\Pi_n\right)G_{\alpha_1}^{(n)}\left(f_1^{(n)}G_{\alpha_2}^{(n)}\left(f_2^{(n)}\cdots G_{\alpha_m}^{(n)}(f_m^{(n)})\right)\cdots\right)\circ\Pi_n d\mu \; . \end{aligned}$$

By Lebesgue's dominated convergence theorem the left hand side with $(n)_{n\in\mathbb{N}}$ replaced by some subsequence $(n_k)_{k\in\mathbb{N}}$ converges to the same expression with P replacing P_n. But by the claim the right hand side converges to

$$\begin{aligned} & \int f_0G_{\alpha_1}\left(f_1G_{\alpha_2}\left(f_2\cdots G_{\alpha_m}(f_m)\right)\cdots\right)d\mu \\ \underset{(3.2)}{=} \; & \int_0^\infty\cdots\int_0^\infty\int f_0T_{t_1}\left(f_1T_{t_2}(f_2\cdots T_{t_m}(f_m))\dots\right)d\mu e^{-(\alpha_1t_1+\cdots+\alpha_mt_m)}dt_1\dots dt_m\; . \end{aligned}$$

By the uniqueness of the Laplace transform the assertion follows. □

Remark 3.3 It is also possible to prove the analogue of Proposition 3.1 with $T_t^{(n)}$ resp. T_t replacing $G_\alpha^{(n)}$ resp. G_α (cf. Proposition A.1 below). In fact, this is a consequence of Proposition 3.1 and the proof of the Hille-Yoshida theorem and technically quite difficult. Though, as shown above, for our purpose, Proposition 3.1 as it stands, is sufficient, we include a proof of this analogue in the appendix, since this result is of its own interest. Note that the standard theory about the equivalence of strong resolvent convergence and strong convergence of the associated semigroup does not apply here, since $(G_\alpha^{(n)})_{\alpha>0}$, $(T_t^{(n)})_{t>0}$ are defined on L^2-spaces depending on n.

4 Proof of Corollary 1.3

Since part (i) of Corollary 1.3 is well-known and we only have a new variant of its proof we only briefly indicate the main arguments. These are based on Theorem 1.2 obove and [AR89a, Theorem 2.7] (the latter in turn relies on Fukushima's process construction for regular Dirichlet forms with locally compact separable metric spaces as state spaces (cf. [F80, Chap.6]) already used obove). Note that we cannot apply [AR89a, Theorem 2.7] directly since condition (2.8) is not verified yet.

Proof of Corollary 1.3 (i) : We adopt the terminology in [AR89a] used in the proof of Theorem 2.7 therein. So, let $\bar{E}$ be the compactification of E constructed in [AR89a, Theorem 2.7], let $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$ be the image Dirichlet form on $L^2(\bar{E};\bar{\mu})$ of $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(E;\mu)$ under the continuous embedding $E \subset \bar{E}$, and $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{X}_t)_{t\geq 0}, (\bar{P}_z)_{z\in\bar{E}})$ the diffusion process associated with $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$. Here $\bar{\Omega} := C([0,\infty[, \bar{E})$, $\bar{X}_t, t \geq 0$, is the coordinate process and $\bar{\mathcal{F}} := \sigma(\bar{X}_t | t \geq 0)$. Let $\overline{\mathrm{Cap}}$ be the capacity associated to $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$. As in the proof of [AR89a, Theorem 2.7] the assertion (including (1.11)) is proved if we can find compact sets $K_n \subset E$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \overline{\mathrm{Cap}}(E \setminus K_n) = 0.$$

But by Theorem 1.2 $\bar{P}_\mu(:= \int_{\bar{E}} \bar{P}_z\, \bar{\mu}(dz)) = P$, so by [MR92, Chap.IV, Theorem 1.15] there exist compact sets $K_n \subset E$, $n \in \mathbb{N}$, such that $\bar{\sigma}_{\bar{E}\setminus K_n} := \inf\{t > 0 | \bar{X}_t \in E \setminus K_n\} \underset{n\to\infty}{\longrightarrow} \infty$ $\bar{P}_\mu$-a.s. Hence (cf. e.g. [MR92, Chap.IV, Theorem 5.2])

$$\begin{aligned} \overline{\mathrm{Cap}}(E \setminus K_n) \leq \overline{\mathrm{Cap}}(\bar{E} \setminus K_n) &= \int e^{-\bar{\sigma}_{\bar{E}\setminus K_n}} d\bar{P}_\mu \\ &\underset{n\to\infty}{\longrightarrow} 0 \, . \end{aligned}$$

(ii) : The assertion is obvious from (1.11) and Theorem 1.2. □

Remark 4.1 For $\alpha \in]0, \frac{1}{2}[$ let Ω'_α denote all paths in Ω which are merely locally Hoelder continuous on $]0,\infty[$. Let θ_t, $t > 0$, denote the usual shift operators on Ω. Then by the Markov property for every $z \in E$

$$E_z\left[P_{X_t}[\Omega'_\alpha]\right] = E_z\left[\theta_t^{-1}(\Omega'_\alpha)\right] \underset{t\downarrow 0}{\longrightarrow} P_z[\Omega'_\alpha] \, .$$

Since the left hand side is a version of $T_t(P_\cdot[\Omega'_\alpha])$, it follows by [F80, Lemma 1.3.3 (iii)] and [MR92, Chap.III, Proposition 3.5] that $z \mapsto E_z[\Omega'_\alpha]$ is $\mathcal{E}$-*quasi-continuous*, hence by Corollary 1.3(ii) $P_z[\Omega'_\alpha] = 1$ for $\mathcal{E}$-*quasi-every* $z \in E$. We refer to [MR92, Chap.III, Definitions 2.1 and 3.2] for the notions of $\mathcal{E}$-*quasi-continuity* and $\mathcal{E}$-*quasi-everywhere*. So far we do not know whether $P_z[\Omega_\alpha] = 1$ for $\mathcal{E}$-q.e. $z \in E$ for the smaller sets Ω_α.

5 Examples

In this section we give an as concise as possible survey of a number of meanwhile well-studied infinite dimensional diffusions associated with Dirichlet forms of type $(\mathcal{E}, D(\mathcal{E}))$ as in Sect. 1 which our results apply to. For simplicity we confine ourselves to cases where $A(z) = \frac{1}{2} Id_H$ for all $z \in E$.

5.1 Some non-diagonalizable Gaussian cases

Let $\mathcal{S}' := \mathcal{S}'(\mathbb{R}^d)$ be the space of tempered Schwartz distributions on $\mathbb{R}^d$ and $\mathcal{S} := \mathcal{S}(\mathbb{R}^d)$ the associated test function space equipped with the usual topology. We have $\mathcal{S}'' = \mathcal{S}$. Let μ be the mean zero Gaussian measure on $(\mathcal{S}'(\mathbb{R}^d), \mathcal{B}(\mathcal{S}'(\mathbb{R}^d)))$ with covariance

$$\begin{aligned} &\int {}_{\mathcal{S}}\langle k_1, z\rangle_{\mathcal{S}'}\ {}_{\mathcal{S}}\langle k_2, z\rangle_{\mathcal{S}'}\mu(dz) \\ &= \int\int (-\Delta + m^2)^{-\alpha}(x-y) k_1(x) k_2(y) dx dy;\ k_1, k_2 \in \mathcal{S} = E'\ . \end{aligned}$$

Here $\alpha, m \in]0, \infty[$, $(-\Delta + m^2)^{-\alpha}(x)$ denotes the Green function of the operator $(-\Delta + m^2)^\alpha$ on $\mathbb{R}^d$ and dx denotes Lebesgue measure. Now for $n \in \mathbb{N}$, let $\mathcal{S}_{-n} := \mathcal{S}_{-n}(\mathbb{R}^d)$ denote the Hilbert subspace of $\mathcal{S}'(\mathbb{R}^d)$ which is the dual of $\mathcal{S}_n(\mathbb{R}^d)$ defined as the completition of $\mathcal{S}$, w.r.t. the norm

$$\|k\|_n := \left[\sum_{|m| \le n} \int_{\mathbb{R}^d} (1 + |x|^2)^n \left| \left(\frac{\partial^{m_1}}{\partial x_1^{m_1}} \cdots \frac{\partial^{m_d}}{\partial x_d^{m_d}} \right) k(x) \right|^2 dx \right]^{1/2}$$

where $m := (m_1, \ldots, m_d) \in (\mathbb{Z}_+)^d$ and $|m| = m_1 + \ldots + m_d$.

Then by [AR91, Example 7.9] there exists $n_0 \in \mathbb{N}$, such that

$$\mathcal{E}(u, v) := \frac{1}{2} \int_{\mathcal{S}_{-n}} \langle \nabla u, \nabla v \rangle_{L^2(\mathbb{R}^2; dx)} d\mu\ ;\ u, v \in \mathcal{F}C_b^\infty,$$

(i.e., $E := \mathcal{S}_{-n_0}$, $H := L^2(\mathbb{R}^d; dx)$) is closable on $L^2(\mathcal{S}_{-n_0}; \mu)$. Also, clearly, conditions (1.1), (1.2) are satisfied. Hence Theorem 1.2 and Corollary 1.3 apply. The diffusion process $\mathbf{M}$ in Corollary 1.3 associated with the closure $(\mathcal{E}, D(\mathcal{E}))$, for which we have thus constructed finite dimensional approximations, is known to solve the stochastic differential equation

$$dX_t = dW_t + (-\Delta + m^2)^{\alpha}(X_t)dt \tag{5.1}$$

in the weak sense. Here $(W_t)_{t\geq 0}$ is an $\mathcal{S}_{-n_0}$-valued Brownian motion with covariance given by $L^2(\mathbb{R}^d; dx)$ (cf. [AR91, Example 7.9] for details).

5.2 Infinite volume space-time quantum fields

Let μ_0^* be the *space-time free field* of mass m on $\mathbb{R}^2$, i.e., μ_0^* is defined on $(\mathcal{S}'(\mathbb{R}^2), \mathcal{B}(\mathcal{S}'(\mathbb{R}^2)))$ as in Subsection 5.1 with $\alpha = 1$. We have the well-known (Wiener-Itô) *chaos decomposition*

$$L^2(\mathcal{S}'; \mu_0^*) = \bigoplus_{n\geq 0} \mathcal{H}_n$$

where $\mathcal{H}_0 := \mathbb{R}$, $\mathcal{H}_n := P^{(\leq n)} \ominus P^{(\leq (n-1))}$, $n \in \mathbb{N}$, with $P^{(\leq n)}$ being the closed linear span of the monomials $\prod_{j=1}^m \ {}_{\mathcal{S}}\langle k_j, .\rangle_{\mathcal{S}'}$, $m \leq n$, $k_1, \ldots, k_m \in \mathcal{S}$, in $L^2(\mathcal{S}'; \mu_0^*)$. Let $: \ :_n$ denote the orthogonal projection onto $\mathcal{H}_n$ in $L^2(\mathcal{S}'; \mu_0^*)$. If $\epsilon > 0$, $h \in L^{1+\epsilon}(\mathbb{R}^2; dx)$ and $n \in \mathbb{N}$, define $: z^n : (h)$ to be the unique element in $\mathcal{H}_n$ such that

$$\begin{aligned}
&\int : z^n : (h) \ : \prod_{j=1}^n \ {}_{\mathcal{S}}\langle k_j, .\rangle_{\mathcal{S}'} :_n d\mu_0^* \\
= \ & n! \int_{\mathbb{R}^2} \prod_{j=1}^n \left(\int_{\mathbb{R}^2} (-\Delta + m^2)^{-1}(x - y_j) k_j(y_j) dy_j \right) h(x) dx
\end{aligned}$$

for all $k_1, \ldots, k_n \in \mathcal{S}(\mathbb{R}^2)$ (see [S74, §V.1] for existence of $: z^n : (h)$). Clearly, for $h_1, h_2 \in L^{1+\epsilon}(\mathbb{R}^2; dx)$, $\alpha, \beta \in \mathbb{R}$,

$$: z^n : (\alpha h_1 + \beta h_2) = \alpha : z^n : (h_1) + \beta : z^n : (h_2) \quad \mu_0^*\text{-a.e.}$$

From now on we fix $N \in \mathbb{N}$, $a_n \in \mathbb{R}$, $0 \leq n \leq 2N$, and define for $h \in L^{1+\epsilon}(\mathbb{R}^2; dx)$

$$:P(z):(h) := \sum_{n=0}^{2N} a_n : z^n : (h) \quad \text{with } a_{2N} > 0. \tag{5.2}$$

We have that $\exp(- : P(z) : (h)) \in L^p(\mathcal{S}'(\mathbb{R}^2); \mu_0^*)$ for all $p \in [1, \infty[$, if $h \geq 0$ (cf. e.g. [S74, §V.2]). Hence the following probability measures (called *space-time cut-off quantum fields*) are well-defined for $\Lambda \in \mathcal{B}(\mathbb{R}^2)$, Λ bounded,

$$\mu_\Lambda^* := \frac{\exp(- : P(z) : (1_\Lambda))}{\int \exp(- : P(z) : (1_\Lambda)) d\mu_0^*} \cdot \mu_0^* \ ,$$

where 1_Λ denotes the indicator function of Λ. It has been proven that the weak limit

$$\lim_{\Lambda \nearrow \mathbb{R}^2} \mu_\Lambda^* =: \mu^*$$

exits as a probability measure on $(\mathcal{S}'(\mathbb{R}^2), \mathcal{B}(\mathcal{S}'(\mathbb{R}^2)))$ (see [GJ86] and also [R86]) having moments of all orders. In particular, it follows by [AR89a, Proposition 3.7] that $\mu^*(\mathcal{S}_{-n}(\mathbb{R}^2)) = 1$ for some $n \in \mathbb{N}$. Since $\mathcal{B}(\mathcal{S}_{-n}) \subset \mathcal{B}(\mathcal{S}')$ we restrict μ^* to $\mathcal{B}(\mathcal{S}_{-n})$ and by [AR90b, Proposition 2.7] we have that $\operatorname{supp} \mu^* = \mathcal{S}_{-n}$ (i.e., $\mu^*(U) > 0$ for each open subset U of $\mathcal{S}_{-n}$). Then by [AR91, Subsection 7.II.a]

$$\mathcal{E}(u, v) := \frac{1}{2} \int_{\mathcal{S}_{-n}} \langle \nabla u, \nabla v \rangle_{L^2(\mathbb{R}^2; dx)} d\mu^* \ ; \ u, v \in \mathcal{F}C_b^\infty$$

(i.e., $E := \mathcal{S}_{-n}$, $H := L^2(\mathbb{R}^2; dx)$) is closable on $L^2(\mathcal{S}_{-n}; \mu^*)$. Also conditions (1.1), (1.2) are obviously satisfied. Hence Theorem 1.2 and Corollary 1.3 apply. This time the corresponding approximated diffusion in Corollary 1.3 weakly solves

$$dX_t = dW_t - \left[: P'(X_t) : + (-\Delta + m^2)(X_t) \right] dt \qquad (5.3)$$

where P' is the derivative of $P(s) = \sum_{n=0}^{2N} a_n s^n, \ s \in \mathbb{R}$ and $(W_t)_{t \geq 0}$ is an $\mathcal{S}_{-n}$-valued Brownian motion with covariance given by $L^2(\mathbb{R}^2; dx)$. We refer to [AR91, Subsection 7.II.a] for details and the precise meaning of (5.3).

5.3 Infinite volume time-zero quantum fields

Let μ_0 be the *time-zero free field* of mass m on $\mathbb{R}$, i.e., μ_0 is defined on $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})))$ as in Subsection 5.1 with $\alpha = \frac{1}{2}$. Let $(L_0, D(L_0))$ denote the generator of the corresponding Dirichlet form on $L^2(\mathcal{S}_{-n_0}; \mu_0)$ in Subsection 5.1. For $h \in L^{1+\epsilon}(\mathbb{R}; dx)$, $\epsilon \in]0, 1]$ and $n \in \mathbb{N}$, define

$: z^n : (h)$ in exactly the same way as in Subsection 5.2 above but with $\left(-\frac{d^2}{dx^2}+m^2\right)^{-1/2}(x),\ x \in \mathbb{R}$, replacing $(-\Delta+m^2)^{-1}(x),\ x \in \mathbb{R}^2$, and μ_0 replacing μ_0^*. Now define $: P(z) : (h)$ for $h \in L^{1+\epsilon}(\mathbb{R}; dx)$ as in (5.2) and for $\Lambda \in \mathcal{B}(\mathbb{R})$, Λ bounded, let $V_\Lambda := \,: P(z) : (1_\Lambda)$ considered as a multiplication operator on $L^2(\mathcal{S}_{-n_0}; \mu_0)$ with (maximal) domain $D(V_\Lambda)$. Let L_Λ be the operator on $L^2(\mathcal{S}_{-n_0}; \mu_0)$ with domain $D(L_\Lambda) = D(L_0) \cap D(V_\Lambda)$ defined by

$$L_\Lambda := L_0 + V_\Lambda \ .$$

It is known that L_Λ is essentially self-adjoint on $D(L_\Lambda)$, lower bounded and that the infimum of its spectrum is a simple isolated eigenvalue E_Λ with a strictly positive corresponding eigenvector Ω_Λ in $L^2(\mathcal{S}_{-n_0}; \mu_0)$ (cf. e.g. [ReS75, Sect.XIII.12] for details). We may assume that Ω_Λ has norm equal to one. The probability measure

$$\mu_\Lambda := \Omega_\Lambda^2 \cdot \mu_0,$$

on $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})))$ is called the *space cut-off $P(\Phi)_2$-quantum field.* Let us assume from now on that the coefficients $a_n,\ 0 \le n \le 2N$ in (5.2) are sufficiently small. It has been proven that the weak limit

$$\lim_{\Lambda \nearrow \mathbb{R}} \mu_\Lambda =: \mu$$

exists as a probability measure on $(\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R})))$ for $\Lambda = [-a, a]$ as $a \to \infty$ (cf. [GJS73], [GJS74]) having finite moments. μ is called the *weakly coupled $P(\Phi)_2$-time-zero quantum field.* By [AR91, Subsection 7.II.b] we know that $\mu(\mathcal{S}_{-n}(\mathbb{R})) = 1$ for some $n \in \mathbb{N}$, that $\operatorname{supp} \mu = \mathcal{S}_{-n}(\mathbb{R})$ and that

$$\mathcal{E}(u, v) := \frac{1}{2} \int_{\mathcal{S}_{-n}(\mathbb{R})} \langle \nabla u, \nabla v \rangle_{L^2(\mathbb{R}; dx)} d\mu \ ; \ u, v \in \mathcal{F}C_b^\infty$$

(i.e., $E := \mathcal{S}_{-n}(\mathbb{R}),\ H := L^2(\mathbb{R}; dx)$) is closable on $L^2(\mathcal{S}_{-n}; \mu)$. Again (1.1), (1.2) are satisfied and thus Theorem 1.2 and Corollary 1.3 apply.

5.4 Space-time quantum fields in finite volume

Let Λ be an open rectangle in $\mathbb{R}^2$. Let $(-\Delta + 1)_N$ be the generator of the following quadratic form on $L^2(\Lambda; dx) : (u, v) \mapsto \int_\Lambda \langle \nabla u, \nabla v \rangle_{\mathbb{R}^2} dx + \int_\Lambda uv\, dx$

with $u, v \in \{g \in L^2(\Lambda; dx) | \nabla g \in L^2(\Lambda; dx)\}$ (where ∇ is in the sense of distributions). Let $\{e_n | n \in \mathbb{N}\} \subset C^\infty(\bar{\Lambda})$ be an (orthonormal) eigenbasis of $(-\Delta+1)_N$ and $\{\lambda_n | n \in \mathbb{N}\}$ $(\subset]0, \infty[\ )$ the corresponding eigenvalues (cf. [ReS78, p. 266]). Define for $\alpha \in \mathbb{R}$

$$H_\alpha := \left\{ u \in L^2(\Lambda; dx) | \sum_{n=1}^{\infty} \lambda_n^\alpha \langle u, e_n \rangle_{L^2(\Lambda; dx)}^2 < \infty \right\}$$

equipped with the inner product

$$\langle u, v \rangle_{H_\alpha} := \sum_{n=1}^{\infty} \lambda_n^\alpha \langle u, e_n \rangle_{L^2(\Lambda; dx)} \langle v, e_n \rangle_{L^2(\Lambda; dx)}.$$

Clearly, we have that

$$H_\alpha = \begin{cases} \text{completition of } C^\infty(\bar{\Lambda}) \text{ w.r.t. } \| \ \|_{H_\alpha} \text{ if } \alpha \geq 0 \\ \text{completition of } C_0^\infty(\Lambda) \text{ w.r.t. } \| \ \|_{H_\alpha} \text{ if } \alpha \leq 0 \end{cases}$$

(cf. [LiM72, p. 79] for the latter).

Fix $\delta > 0$. Since $\sum_{n=1}^\infty \lambda_n^{-1-\delta} < \infty$, we have, applying [Y89, Theorem 3.2] (i.e., the Gross-Minlos-Sazonov theorem) with $H := L^2(\Lambda; dx)$, $\| \,.\, \| := \| \,.\, \|_{H_{-\delta}}$, $A_1 := (-\Delta+1)_N^{-\delta/2}$, and $A_2 := (-\Delta+1)_N^{-1/2}$, that there exists a unique mean zero Gaussian probability measure μ_0 on $E := H_{-\delta}$ (called *free field* on Λ; see [N73]) such that

$$\int_E {}_{E'}\langle l, z \rangle_E^2 \ \mu_0(dz) = \|l\|_{H_{-1}}^2 \text{for all } l \in E' = H_\delta. \tag{5.4}$$

In (5.4) we have realized the dual of $H_{-\delta}$ as H_δ using as usual the chain

$$H_\alpha \subset H_0 = L^2(\Lambda; dx) \subset H_{-\alpha}, \quad \alpha \geq 0$$

Let $h \in L^2(\Lambda; dx)$, $n \in \mathbb{N}$, and define $: z^n : (h)$ analogously as in Subsection 5.2 but with $(-\Delta + m^2)$ on $\mathbb{R}^2$ replaced by $(-\Delta+1)_N$ on Λ. Now fix $N \in \mathbb{N}$, $a_n \in \mathbb{R}$, $0 \leq n \leq 2N$ with $a_{2N} > 0$ and define

$$V := \sum_{n=0}^{2N} a_n : z^n : (1_\Lambda).$$

Let

$$\varphi := \exp(-\frac{1}{2} V).$$

Then $\varphi > 0$ μ-a.e. and $\varphi \in L^p(E;\mu)$ for all $p \in [1,\infty[$ (cf., e.g., [S74, Sect. 5.2] or [GJ86, Sect. 8.6]). Set

$$\mu = \varphi^2 \cdot \mu_0 \ .$$

Then by [RZ92, Sect. 7]

$$\mathcal{E}(u,v) := \int_{H_{-\delta}} \langle \nabla u, \nabla v \rangle_{H_\alpha} \, d\mu \ ; \ u,v \in \mathcal{F}C_b^\infty,$$

(i.e., $E := H_{-\delta},\ H := H_\alpha$) is closable on $L^2(H_{-\delta};\mu)$. Condition (1.2) and, provided $\alpha > 1-\delta$, also (1.1) is satisfied and, therefore, Theorem 1.2 and Corollary 1.3 apply. In this case, if $\alpha > 1-\delta/2$, the corresponding approximated diffusion in Corollary 1.3 is the weak solution of

$$dX_t = dW_t^\alpha - \Big[(-\Delta+1)_N^{1-\alpha}(X_t) + (-\Delta+1)_N^{-\alpha} \ : P'(X_t) : \Big] \, dt \qquad (5.5)$$

(cf. (5.3)) where $(W_t^\alpha)_{t\geq 0}$ is an $H_{-\delta}$-valued Brownian motion with covariance given by $(H_\alpha, \langle\ ,\ \rangle_{H_\alpha})$ (cf. [RZ92, Remark 7.4(ii)] for details). For $\delta = 1$ a weak solution to (5.5) has also been constructed by P. Jona-Lasinio and P.K. Mitter in [J-LM85] using a different technique. The two solutions, however, have been proved to have the same laws in [RZ92, Corollary 7.6].

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Appendix

Proposition A.1 Let $g_n \in L^2(\mathbb{R}^n;\mu_n)$, $n \in \mathbb{N}$, such that $g_n \circ \Pi_n \to g$ in $L^2(E;\mu)$ as $n \to \infty$. Then for all $t > 0$

$$(T_t^{(n)} g_n) \circ \Pi_n \underset{n\to\infty}{\longrightarrow} T_t g \quad \text{in } L^2(E;\mu).$$

Proof. We first prove the proposition for some special choices of g_n and g, and then extend it to the general case. So, assume that $g := (I-L)^{-2}h = (G_1)^2 h$ for some $h \in L^2(E;\mu)$. Let $n \in \mathbb{N}$ and $h_n \in L^2(\mathbb{R}^n;\mu_n)$ be such that $h_n \circ \Pi_n = E_\mu[h|\sigma(\Pi_n)]$. It is clear that $h_n \circ \Pi_n \underset{n\to\infty}{\longrightarrow} h$ in $L^2(E;\mu)$. Define

$$g_n := \left(I - L^{(n)}\right)^{-2} h_n = (G_1^{(n)})^2 h_n.$$

We have that by (3.3) and Proposition 3.1

$$\begin{aligned} g_n \circ \Pi_n &= \left((G_1^{(n)})^2 h_n\right) \circ \Pi_n \\ &= P_n^{(1)} G_1 \left((G_1^{(n)} h_n) \circ \Pi_n\right) \\ &\underset{n\to\infty}{\longrightarrow} (G_1)^2 h = g \quad \text{in } L^2(E;\mu). \end{aligned}$$

As in the proof of the Hille-Yoshida theorem we introduce the operator

$$T_t^{(n,\beta)} := e^{-t\beta} \sum_{m=0}^{+\infty} \frac{(t\beta)^m}{m!} \left(\beta G_\beta^{(n)}\right)^m; \quad t, \beta > 0,\ n \in \mathbb{N}.$$

Claim : For fixed $t > 0$, as $\beta \to +\infty$, $T_t^{(n,\beta)} g_n$ converges in $L^2(\mathbb{R}^n;\mu_n)$ uniformly in n.
Proof of the claim: Note that for all $u \in D(L^{(n)})$

$$\begin{aligned} \|T_t^{(n,\alpha)}u - T_t^{(n,\beta)}u\|_{L^2(\mathbb{R}^n;\mu_n)} &= \int_0^t \|e^{sL^{(n,\alpha)}} e^{(t-s)L^{(n,\beta)}}\| \|L^{(n,\alpha)}u - L^{(n,\beta)}u\|_{L^2(\mathbb{R}^n;\mu_n)} ds \\ &\le t\|L^{(n,\alpha)}u - L^{(n,\beta)}u\|_{L^2(\mathbb{R}^n;\mu_n)}, \end{aligned}$$

where $L^{(n,\alpha)} := \alpha(\alpha G_\alpha^{(n)} - 1)$. Thus it sufficies to show that as $\alpha \to +\infty$, $L^{(n,\alpha)} g_n$ converges in $L^2(\mathbb{R}^n;\mu_n)$ uniformly in n. In fact,

$$\begin{aligned} L^{(n,\alpha)} g_n &= \alpha^2 \int_0^\infty e^{-\alpha t} \left(T_t^{(n)} g_n - g_n\right) dt \\ &= \alpha^2 \int_0^\infty e^{-\alpha t} dt \int_0^t T_s^{(n)} L^{(n)} g_n ds \\ &= \alpha \int_0^\infty T_s^{(n)} L^{(n)} g_n ds\, \alpha \int_s^\infty e^{-\alpha t} dt \\ &= \alpha \int_0^\infty T_s^{(n)} L^{(n)} g_n) e^{-\alpha s} ds \\ &= \int_0^\infty T_{\frac{t}{\alpha}}^{(n)} L^{(n)} g_n e^{-t} dt, \end{aligned}$$

hence

$$
\begin{aligned}
L^{(n,\alpha)}g_n - L^{(n)}g_n &= \int_0^\infty e^{-t}dt \left(T^{(n)}_{\frac{t}{\alpha}} L^{(n)}g_n - L^{(n)}g_n\right) \\
&= \int_0^\infty e^{-t}dt \int_0^{\frac{t}{\alpha}} L^{(n)}T^{(n)}_s L^{(n)}g_n ds \\
&= \int_0^\infty e^{-t}dt \int_0^{\frac{t}{\alpha}} T^{(n)}_s (L^{(n)})^2 g_n ds \quad .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|L^{(n,\alpha)}g_n - L^{(n)}g_n\|_{L^2(\mathbb{R}^n;\mu_n)} &\leq \int_0^\infty e^{-t}dt \int_0^{\frac{t}{\alpha}} \left\|T^{(n)}_s (L^{(n)})^2 g_n\right\|_{L^2(\mathbb{R}^n;\mu_n)} ds \\
&\leq \int_0^\infty e^{-t}\frac{t}{\alpha}dt \, \|(L^{(n)})^2 g_n\|_{L^2(\mathbb{R}^n;\mu_n)} \, .
\end{aligned}
$$

By the choice of g_n one can prove that $\sup_n \|(L^{(n)})^2 g_n\|_{L^2(\mathbb{R}^n;\mu_n)} < +\infty$. Indeed, since $h_n = (I-L^{(n)})^2 g_n$, it follows that $(L^{(n)})^2 g_n = h_n - g_n + 2L^{(n)}g_n = h_n + g_n - 2G^{(n)}_1 h_n$. This shows that $\sup_n \|(L^{(n)})^2 g_n\|_{L^2(\mathbb{R}^n;\mu_n)} < +\infty$. Thus we obtain that

$$
\|L^{(n,\alpha)}g_n - L^{(n)}g_n\|_{L^2(\mathbb{R}^n;\mu_n)} \longrightarrow 0 \quad \text{as } \alpha \to +\infty
$$

uniformly in n and the claim is proved.

Using the claim and interchanging the order of limits we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \left(T^{(n)}_t g_n\right) \circ \Pi_n &= \lim_{n\to\infty} \lim_{\beta\to+\infty} \left(T^{(n,\beta)}_t g_n\right) \circ \Pi_n \\
&= \lim_{\beta\to+\infty} \lim_{n\to\infty} e^{-t\beta} \sum_{m=0}^{+\infty} \frac{(t\beta)^m}{m!} \left[\left(\beta\, G^{(n)}_\beta\right)^m g_n\right] \circ \Pi_n \\
&= \lim_{\beta\to+\infty} e^{-t\beta} \sum_{m=0}^{+\infty} \frac{(t\beta)^m}{m!} \lim_{n\to\infty} \left[\left(\beta G^{(n)}_\beta\right)^m g_n\right] \circ \Pi_n \\
&= \lim_{\beta\to+\infty} e^{-t\beta} \sum_{m=0}^{+\infty} \frac{(t\beta)^m}{m!} (\beta G_\beta)^m g = T_t g \quad \text{in } L^2(E;\mu).
\end{aligned}
$$

where we used Lebesgue's dominated convergence theorem and the fact that $\beta G^{(n)}_\beta$, $\beta > 0$, $n \in \mathbb{N}$, are contractions on $L^2(\mathbb{R}^n;\mu_n)$ in the third step. Furthermore, we used the claim in the proof of Corollary 3.2 in the fourth step, and the Hille-Yoshida approximation of T_t.

Now, if $f_n \in L^2(\mathbb{R}^n;\mu_n)$ such that $f_n \circ \Pi_n \underset{n\to\infty}{\longrightarrow} g$ in $L^2(\mathbb{R}^n;\mu_n)$, then we have

$$
\begin{aligned}
\|(T_t^{(n)} f_n)\circ \Pi_n - T_t g\|_{L^2(E;\mu)} &\le \|(T_t^{(n)} f_n)\circ\Pi_n - (T_t^{(n)} g_n)\circ \Pi_n\|_{L^2(E;\mu)} \\
&\quad + \|(T_t^{(n)} g_n)\circ\Pi_n - T_t g\|_{L^2(E;\mu)} \\
&\le \|f_n - g_n\|_{L^2(\mathbb{R}^n;\mu_n)} + \|(T_t^{(n)} g_n)\circ \Pi_n - T_t g\|_{L^2(E;\mu)} \\
&\longrightarrow 0 \quad \text{as } n \to +\infty,
\end{aligned}
$$

where g_n is the special choice above. Thus we have proved the assertion for $g \in D((I-L)^2)$. Now we prove the assertion for arbitrary $g \in L^2(E;\mu)$. Following the same arguments as before, we only need to prove the lemma for a special choice of g_n. For example, we choose $g_n \in L^2(\mathbb{R}^n;\mu_n)$ such that

$$
g_n \circ \Pi_n = E_\mu\left[g|\sigma(\Pi_n)\right].
$$

Since $D((I-L)^2)$ is dense in $L^2(E;\mu)$, there exist $h_k \in D((I-L)^2),\ k \in \mathbb{N}$, such that $h_k \underset{k\to\infty}{\longrightarrow} g$ in $L^2(E;\mu)$. Let $k, n \in \mathbb{N}$ and let $h_{k,n}$ denote a function in $L^2(\mathbb{R}^n;\mu_n)$ satisfying $h_{k,n}\circ \Pi_n = E_\mu\left[h_k|\sigma(\Pi_n)\right]$. Then we have

$$
\begin{aligned}
\|h_{k,n}\circ\Pi_n - g_n \circ \Pi_n\|_{L^2(E;\mu)} &\le \left(\int E_\mu[|h_k - g|\,|\sigma(\Pi_n)]^2 d\mu\right)^{1/2} \\
&\le \left(\int (h_k - g)^2 d\mu\right)^{1/2} = \|h_k - g\|_{L^2(E;\mu)}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\|(T_t^{(n)} g_n)\circ\Pi_n - T_t g\|_{L^2(E;\mu)} &\le \|(T_t^{(n)} g_n)\circ \Pi_n - (T_t^{(n)} h_{k,n})\circ\Pi_n\|_{L^2(E;\mu)} \\
&+ \|(T_t^{(n)} h_{k,n})\circ\Pi_n - T_t h_k\|_{L^2(E;\mu)} + \|T_t h_k - T_t g\|_{L^2(E;\mu)} \\
&\le 2\|h_k - g\|_{L^2(E;\mu)} + \|(T_t^{(n)} h_{k,n})\circ\Pi_n - T_t h_k\|_{L^2(E;\mu)}\,.
\end{aligned}
$$

Now, by letting first $n \to \infty$, we get

$$
\lim_{n\to\infty} \|(T_t^{(n)} g_n)\circ\Pi_n - T_t g\|_{L^2(E;\mu)} \le 2\|h_k - g\|_{L^2(E;\mu)} \text{ for all } k \in \mathbb{N}.
$$

Then letting $k \to \infty$, we finish the proof. $\square$

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