DIRICHLET FORMS AND MARKOV PROCESSES: A GENERALIZED FRAMEWORK INCLUDING BOTH ELLIPTIC AND PARABOLIC CASES

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Abstract. We extend the framework of classical Dirichlet forms to a class of bilinear forms, called generalized Dirichlet forms, which are the sum of a coercive part and a linear unbounded operator as a perturbation. The class of generalized Dirichlet forms, in particular, includes symmetric and coercive Dirichlet forms (cf. [Fu2], [M/R]) as well as time dependent Dirichlet forms (cf. [O1]) as special cases and also many new examples. Among these are, e.g. transformations of time dependent Dirichlet forms by α–excessive functions h (h–transformations), Dirichlet forms with time dependent linear drift and fractional diffusion operators. One of the main results is that we identify an analytic property of these forms which ensures the existence of associated strong Markov processes with nice sample path properties, and give an explicit construction for such processes. This construction extends previous constructions of the processes in the elliptic and the parabolic cases, is, in particular, carried out on general topological state spaces (as in [M/R]), and is applied to the above examples.

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I. Introduction

This paper is concerned with the development of an extended framework for Dirichlet forms including both elliptic cases (i.e.,symmetric and coercive Dirichlet forms as treated in [Fu2] resp. [M/R]) and parabolic ones (i.e.,time dependent Dirichlet forms as considered in [O1]) .

To this end we introduce a new class of bilinear forms, defined on some L^2-space $L^2(E,m)$, called *generalized Dirichlet forms*, which are in general neither symmetric nor coercive nor of the type considered in [O1], but still have corresponding generators, strongly continuous contraction semigroups and resolvents. These forms are the sum of a coercive closed form $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ (in the sense of [M/R]) and a linear (unbounded) operator (Λ, D) on $L^2(E,m)$, where the latter is a small perturbation of the first in the following sense: (Λ, D) generates a C_0–semigroup $(U_t)_{t\geq 0}$ of contractions such that $(U_t)_{t\geq 0}$ and the adjoint semigroup $(\hat{U}_t)_{t\geq 0}$ can be extended to C_0–semigroups on the dual space $D(\mathcal{A})'$ of $D(\mathcal{A})$. Then the generalized bilinear form is defined as follows:

$$
\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{if } u\in D(\Lambda)\cap D(\mathcal{A}), v\in D(\mathcal{A}) \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle & \text{if } u\in D(\mathcal{A}), v\in D(\hat{\Lambda})\cap D(\mathcal{A})\ . \end{cases} \tag{1.1}
$$

Here $\langle\cdot,\cdot\rangle$ denotes the dualization between $D(\mathcal{A})$ and $D(\mathcal{A})'$, and $(\Lambda, D(\Lambda))$, $(\hat{\Lambda}, D(\hat{\Lambda}))$ the generators of the extension of $(U_t)_{t\geq 0}$, $(\hat{U}_t)_{t\geq 0}$ resp. Forms of this type have been studied in [L/Ma] and also in [L].

We are interested in a special subclass of these forms, called *generalized Dirichlet forms*, and their probabilistic counterparts. We call a form of type (1.1) a generalized (semi–) Dirichlet form if the associated strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ is *sub–Markovian* and if there is a dense subspace $\mathcal{Y}\subset D(\Lambda)\cap D(\mathcal{A})$ such that $u\wedge\alpha\in\mathcal{Y}$ for $u\in\mathcal{Y}$ and $\alpha\geq 0$. It is clear that in the case $\Lambda = 0$ we end up in the class of coercive Dirichlet forms (cf. [M/R],[M/Ov/R]). But the class of generalized Dirichlet forms also includes the time dependent Dirichlet forms, as considered in [O1], which are defined as follows: Let $(\mathcal{E}^{(t)}, V)_{t\in\mathbb{R}}$ be a family of coercive Dirichlet forms with uniform sector constant K and common domain V in some L^2–space such that $t\mapsto \mathcal{E}^{(t)}(u,v)$ is Borel–measurable, $\forall u,v\in V$, and $c\mathcal{E}_1^{(0)}(u,u)\leq \mathcal{E}_1^{(t)}(u,u)\leq c^{-1}\mathcal{E}_1^{(0)}(u,u)$, $\forall u\in V$, $t\in\mathbb{R}$, for some positive constant c. Then the corresponding *time dependent Dirichlet form*

$$
\mathcal{E}(u,v) := \begin{cases} \int_{\mathbb{R}} \mathcal{E}^{(t)}(u(t),v(t))\,dt - \int_{\mathbb{R}}\langle \frac{du}{dt}(t), v(t)\rangle\,dt & \text{if } u\in L^2(\mathbb{R},V)\cap \\ & H^{1,2}(\mathbb{R},V'), v\in L^2(\mathbb{R},V) \\ \int_{\mathbb{R}} \mathcal{E}^{(t)}(u(t),v(t))\,dt + \int_{\mathbb{R}}\langle \frac{dv}{dt}(t), u(t)\rangle\,dt & \text{if } u\in L^2(\mathbb{R},V), v\in \\ & L^2(\mathbb{R},V)\cap H^{1,2}(\mathbb{R},V'), \end{cases}
$$

is a generalized Dirichlet form in the above sense. Here $\langle\cdot,\cdot\rangle$ denotes the dualization between V and the dual space V'. We emphasize, however, that if we replace the time–derivative in these examples by the time derivative $(\frac{d}{dt})^\alpha$ of power α, $\alpha\in(0,1)$, we still have a generalized Dirichlet form (cf. Example II.11(iv)). Moreover, also the *h–transformation* (cf. Example II.11(v) for the precise definition) of a time dependent Dirichlet form is again a *generalized* (semi–) Dirichlet form, while it is, of course, not a time dependent Dirichlet form.

Dirichlet forms with time dependent linear drift can also be treated, as shown by the following example: Let $(A_t)_{t\in\mathbb{R}}$ be a family of (non–symmetric) $d\times d$–matrices with trace$A_t \geq 0$, $t\in\mathbb{R}$, such that $t\mapsto \langle A_t x, y\rangle_{\mathbb{R}^d}$ is measurable, $\forall x,y\in\mathbb{R}^d$, and $\int_{\mathbb{R}} \|A_t\|^2_{L(\mathbb{R}^d)}\,dt<\infty$. Then there exists an extension (Λ, D) in $L^2(\mathbb{R}\times\mathbb{R}^d, dt\otimes dx)$ of the operator $\Lambda f(t,x)=\frac{\partial f}{\partial t}(t,x)+A_t x\cdot\nabla f(t,x)$, $f\in C_0^\infty(\mathbb{R}^{d+1})$ which is a small perturbation in the sense described above of the gradient form $\int_{\mathbb{R}}\int_{\mathbb{R}^d}\nabla u(t,x)\cdot\nabla v(t,x)\,dx\,dt$ on $L^2(\mathbb{R},H^{1,2}(\mathbb{R}^d))$ (see Example II.11(vi) for details).

Let us now give a (systematic) summary of our results. We start in Section II with proving the necessary functional analytic results complementing the corresponding results in [L/Ma] and [L] (cf. Proposition II.7). Then we characterize the sub–Markov property of the resolvent in terms of the form $\mathcal{E}$ (cf. Proposition II.8). At the end of Section II we discuss the above mentioned examples in detail (cf. Examples II.11).

The main result of the paper, however, is to identify an analytic property, called *quasi–regularity* in analogy with [A/M/R] and [M/R], which allows the construction of associated strong Markov processes with additional regularity properties. Let us recall that the main result in [A/M/R] (see also [M/R] and [Ov/M/R]) is a complete characterization of all Dirichlet forms associated with m–tight special standard processes. This extends fundamental work by M. Fukushima [Fu] and also S. C. Menendez [Me]. Our construction hence generalizes one half of this result (namely the "sufficiency part"), but it also extends the construction carried out by Y. Oshima [O1].

In order to prove our main result we have to develop the analytic potential theory of generalized Dirichlet forms (cf. Section III). We introduce the notion of nests and exceptional sets which are defined directly in terms of the given generalized Dirichlet form (cf. Definition III.10). Moreover, we construct a Choquet capacity whose zero sets are exactly the exceptional sets (cf. Proposition III.17). This will be an essential tool in the construction of the process. This part of the paper can be considered as a continuation and extension of the classical work on parabolic potential theory as developed by M. Pierre in [Pi1], [Pi2] and [Pi3] and the recent extensions thereof by Y. Oshima in [O1].

In particular, in Section IV we are then prepared to define the notion of quasi–regularity for *generalized* Dirichlet forms analogously to [A/M/R] and [M/R], and we prove that also in this case it is sufficient to ensure the existence of an m–tight special standard process (cf. Theorem IV.8). To motivate the process construction we prove in detail (cf. Corollary IV.5 and Examples IV.6) that almost all of the examples above are quasi–regular generalized (semi–) Dirichlet forms and hence have associated processes.

All the results in this paper have been announced in [St1].

We first introduce our framework. Let $\mathcal{H}$ be a separable real Hilbert space, $(\mathcal{A}, \mathcal{V})$ a real–valued coercive closed form on $\mathcal{H}$ with sector constant $K \geq 1$. Then $\mathcal{V}$ equipped with inner product $\widetilde{\mathcal{A}}_1(\cdot, \cdot)$ (where $\widetilde{\mathcal{A}}_\alpha(u, v) := 1/2(\mathcal{A}(u, v) + \mathcal{A}(v, u)) + \alpha(u, v)_\mathcal{H}$ for $u, v \in \mathcal{V}, \alpha > 0$) is a separable real Hilbert space. Let $\| \cdot \|_\mathcal{V}$ be the corresponding norm. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$, we have that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densely and continuously. Denote by $\langle \cdot, \cdot \rangle$ the dualization between $\mathcal{V}$ and $\mathcal{V}'$.

Let Λ with domain $D(\Lambda, \mathcal{V}')$ be a linear operator on $\mathcal{V}'$ satisfying the following conditions:

D1 (i) $(\Lambda, D(\Lambda, \mathcal{V}'))$ generates a strongly continuous semigroup $(U_t)_{t \geq 0}$ of bounded linear operators on $\mathcal{V}'$.

(ii) $(U_t)_{t \geq 0}$ can be restricted both to a strongly continuous *contraction* semigroup on $\mathcal{H}$ and to a strongly continuous semigroup of bounded linear operators on $\mathcal{V}$.

Since $(U_t)_{t \geq 0}$ is strongly continuous on $\mathcal{V}'$ as well as $\mathcal{V}$ there are constants $C, D \geq 1$ with $\|U_t\|_{L(\mathcal{V})} \leq Ce^{Dt}$ and $\|U_t\|_{L(\mathcal{V}')} \leq Ce^{Dt}$. Denote by $(\Lambda, D(\Lambda, \mathcal{H}))$ and $(\Lambda, D(\Lambda, \mathcal{V}))$ the generators corresponding to the restricted semigroups. It is easy to check that the dual operator $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{V}'))$ of $(\Lambda, D(\Lambda, \mathcal{V}))$ also satisfies condition D1.

We can now define the Hilbert spaces $\mathcal{F} := D(\Lambda, \mathcal{V}') \cap \mathcal{V}$ with norm $\|u\|_\mathcal{F}^2 := \|u\|_\mathcal{V}^2 + \|\Lambda u\|_{\mathcal{V}'}^2$, and $\widehat{\mathcal{F}} := D(\hat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$ with norm $\|u\|_{\widehat{\mathcal{F}}}^2 := \|u\|_\mathcal{V}^2 + \|\hat{\Lambda} u\|_{\mathcal{V}'}^2$.

Remark 1. (i) By the theorem of Hille–Yosida–Phillips (cf. [Re/Si II, p. 247, Theorem X47.b]) the linear operator $V_\alpha := (\alpha - \Lambda)^{-1}$ exists for $\alpha > D$ both on $\mathcal{V}$ and $\mathcal{V}'$ with norm $\|V_\alpha\|_{L(\mathcal{V})} \leq \frac{C}{\alpha - D}$ (resp. $\|V_\alpha\|_{L(\mathcal{V}')} \leq \frac{C}{\alpha - D}$). Therefore $\lim_{\alpha \to \infty} \alpha V_\alpha v = v$ in $\mathcal{V}$ for $v \in \mathcal{V}$ and in $\mathcal{V}'$ for v in $\mathcal{V}'$. Since $\Lambda(\alpha V_\alpha v) = \alpha V_\alpha(\Lambda v) \to \Lambda v$ in $\mathcal{V}'$ for $v \in \mathcal{F}$ we have that $D(\Lambda, \mathcal{V}) \subset \mathcal{F}$ dense. Similarly we can show $D(\hat{\Lambda}, \mathcal{V}) \subset \widehat{\mathcal{F}}$ dense.

(ii) Since $U_t : \mathcal{H} \to \mathcal{H}$ is a contraction for $t \geq 0$ it follows $(\Lambda u, u)_\mathcal{H} \leq 0$ for $u \in D(\Lambda, \mathcal{H})$. By the density of $D(\Lambda, \mathcal{V})$ in $\mathcal{F}$ we can find for each $u \in \mathcal{F}$ a sequence $(u_n)_{n \geq 1} \subset D(\Lambda, \mathcal{V})$ converging to u in $\mathcal{F}$. Consequently $\langle \Lambda u, u \rangle = \lim_{n \to \infty} \langle \Lambda u_n, u_n \rangle = \lim_{n \to \infty} (\Lambda u_n, u_n)_\mathcal{H} \leq 0$. Similarly we can show $\langle \hat{\Lambda} u, u \rangle \leq 0$ for $u \in \widehat{\mathcal{F}}$.

(iii) Suppose that (Λ, D) is a linear operator on $\mathcal{H}$ that generates a stongly continuous contraction semigroup $(U_t)_{t \geq 0}$ such that $(U_t)_{t \geq 0}$ and the adjoint semigroup $(\hat{U}_t)_{t \geq 0}$ can be extended to strongly continuous semigroups of bounded operators on $\mathcal{V}'$. Denote the corresponding generators with $(\Lambda, D(\Lambda, \mathcal{V}'))$ and $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{V}'))$. Then $(\Lambda, D(\Lambda, \mathcal{V}'))$ and $(\hat{\Lambda}, D(\hat{\Lambda}, \mathcal{V}'))$ both satisfy condition D1.

Definition 2. Let the form $\mathcal{E}$ be given by

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{if } u \in \mathcal{F}, v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle & \text{if } u \in \mathcal{V}, v \in \widehat{\mathcal{F}}\ , \end{cases} \tag{2.1}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{\mathcal{H}}$ for $\alpha > 0$.

Remark 3. Interchanging Λ and $\hat{\Lambda}$ and the entries of the bilinear form $\mathcal{A}$ we can also define the *co–form* $\widehat{\mathcal{E}}$ by

$$\widehat{\mathcal{E}}(u,v) := \begin{cases} \mathcal{A}(v,u) - \langle \hat{\Lambda} u, v\rangle & \text{for } u \in \widehat{\mathcal{F}}, v \in \mathcal{V} \\ \mathcal{A}(v,u) - \langle \Lambda v, u\rangle & \text{for } u \in \mathcal{V}, v \in \mathcal{F}\ . \end{cases}$$

The form $\mathcal{E}$ and the co–form $\widehat{\mathcal{E}}$ are well–defined since $\langle \Lambda u, v\rangle = \langle \hat{\Lambda} v, u\rangle$ for $u \in \mathcal{F}$, $v \in \hat{\mathcal{F}}$. For the proof of the next proposition we refer to [L/Ma, p. 248, Théorème 1.1]:

Proposition 4. *For all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \to \mathcal{F}$ and $\widehat{W}_\alpha : \mathcal{V}' \to \widehat{\mathcal{F}}$ such that*

$$\mathcal{E}_\alpha(W_\alpha f, u) = \langle f, u\rangle = \mathcal{E}_\alpha(u, \widehat{W}_\alpha f)\ \forall f \in \mathcal{V}', u \in \mathcal{V}\ .$$

$(W_\alpha)_{\alpha>0}$ and $(\widehat{W}_\alpha)_{\alpha>0}$ satisfy the resolvent equation, i.e.

$$W_\alpha - W_\beta = (\beta-\alpha)W_\alpha W_\beta \text{ and } \widehat{W}_\alpha - \widehat{W}_\beta = (\beta-\alpha)\widehat{W}_\alpha \widehat{W}_\beta \text{ for } \alpha,\beta > 0\ . \tag{2.2}$$

Denote by G_α and $\widehat{G}_\alpha$ for $\alpha > 0$ the restrictions of W_α and $\widehat{W}_\alpha$ on $\mathcal{H}$.

Remark 5. Since $W_1(\mathcal{V}') = \mathcal{F}$ it follows from the continuity of W_1 that $G_1(\mathcal{H}) = W_1(\mathcal{H})$ is dense in $\mathcal{F}$. Similarly, $\widehat{G}_1(\mathcal{H})$ is dense in $\widehat{\mathcal{F}}$.

Proposition 6. *(i) $(G_\alpha)_{\alpha>0}$ is a strongly continuous contraction resolvent on $\mathcal{H}$.*

(ii) $(\widehat{G}_\alpha)_{\alpha>0}$ is the adjoint strongly continuous contraction resolvent of $(G_\alpha)_{\alpha>0}$.

Proof. Each G_α is a contraction since

$$\alpha \|G_\alpha f\|^2_{\mathcal{H}} \le \mathcal{E}_\alpha\left(G_\alpha f, G_\alpha f\right) = (f, G_\alpha f)_{\mathcal{H}} \le \|f\|_{\mathcal{H}} \|G_\alpha f\|_{\mathcal{H}}\ .$$

The resolvent equation for $(G_\alpha)_{\alpha>0}$ follows easily from (2.2). For $f = G_1 h$ we have that

$$\begin{aligned} \alpha\|\alpha G_{\alpha+1} f - f\|^2_{\mathcal{H}} &\le \mathcal{E}_{\alpha+1}\left(\alpha G_{\alpha+1} f - f, \alpha G_{\alpha+1} f - f\right) = -\mathcal{E}_1\left(f, \alpha G_{\alpha+1} f - f\right) \\ &= -\left(h, \alpha G_{\alpha+1} f - f\right)_{\mathcal{H}} \\ &\le \|h\|_{\mathcal{H}} \|\alpha G_{\alpha+1} f - f\|_{\mathcal{H}}\ . \end{aligned}$$

Consequently $\lim_{\alpha\to\infty} \|\alpha G_\alpha f - f\|_{\mathcal{H}} = 0$. Since $G_1(\mathcal{H})$ is dense in $\mathcal{H}$ by the last remark, the strong continuity of $(G_\alpha)_{\alpha>0}$ now follows by a 3-ε-argument.

For the proof of (ii) it suffices now to show that $\widehat{G}_\alpha$ is the adjoint operator of G_α on $\mathcal{H}$. This is true since $(G_\alpha f, g)_{\mathcal{H}} = \mathcal{E}_\alpha\left(G_\alpha f, \widehat{G}_\alpha g\right) = \left(f, \widehat{G}_\alpha g\right)_{\mathcal{H}}$. $\square$

As in the case of resolvents associated with coercive closed forms, $(G_\alpha)_{\alpha>0}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ have additional continuity properties.

Proposition 7. *(i) If $f \in \mathcal{V}'$ then $\lim_{\alpha\to\infty} \alpha W_\alpha f = f$ in $\mathcal{V}'$.*

(ii) If $f \in \mathcal{V}$ then $\lim_{\alpha\to\infty} \alpha G_\alpha f = f$ in $\mathcal{V}$.

(iii) If $f \in \mathcal{F}$ then $\lim_{\alpha\to\infty} \alpha G_\alpha f = f$ in $\mathcal{F}$.

Corresponding statements hold for $(\widehat{G}_\alpha)_{\alpha>0}$.

Proof. Let us show (ii) first. For $f \in \mathcal{V}, \alpha > 2D$, and $w_\alpha := \alpha G_\alpha f - \alpha V_\alpha f$ it follows

$$\begin{aligned}\|w_\alpha\|_V^2 &\le \mathcal{E}_\alpha(w_\alpha, w_\alpha) = \alpha\,(f, w_\alpha)_{\mathcal{H}} - \alpha\,(\alpha V_\alpha f - \Lambda V_\alpha f, w_\alpha)_{\mathcal{H}} - \mathcal{A}(\alpha V_\alpha f, w_\alpha)\\ &= -\mathcal{A}\,(\alpha V_\alpha f, w_\alpha)\\ &\le 4KC\|f\|_{\mathcal{V}}\|w_\alpha\|_{\mathcal{V}}\ , \text{ hence}\end{aligned}$$

$$\|\alpha G_\alpha f\|_V \le \|\alpha V_\alpha f\|_V + \|w_\alpha\|_V \le 6KC\|f\|_{\mathcal{V}}\ . \tag{2.3}$$

For $f = G_1 h$ we have that

$$\begin{aligned}\|\alpha G_\alpha f - f\|_{\mathcal{V}}^2 &\le \mathcal{E}_1\,(\alpha G_\alpha f - f, \alpha G_\alpha f - f)\\ &= (\alpha G_\alpha h - h, \alpha G_\alpha f - f)_{\mathcal{H}} \to 0 \text{ for } \alpha \to \infty\ .\end{aligned}$$

Since $G_1(\mathcal{H})$ is dense in $\mathcal{V}$ by the last remark the strong continuity now follows by a 3-ε-argument.

(i) Similarly to (2.3) we can show that $\|\alpha \widehat{G}_\alpha v\|_{\mathcal{V}} \le 6KC\|v\|_{\mathcal{V}}$ for $v \in \mathcal{V}$ and $\alpha > 2D$. Consequently for $f \in \mathcal{V}'$ and $\alpha > 2D$

$$\begin{aligned}\langle \alpha W_\alpha f, v\rangle &= \mathcal{E}_\alpha\left(\alpha W_\alpha f, \widehat{G}_\alpha v\right) = \langle f, \alpha \widehat{G}_\alpha v\rangle\\ &\le 6KC\|f\|_{\mathcal{V}'}\|v\|_{\mathcal{V}},\ \forall v \in \mathcal{V}\end{aligned}$$

and therefore

$$\|\alpha W_\alpha f\|_{\mathcal{V}'} \le 6KC\|f\|_{\mathcal{V}'}\ . \tag{2.4}$$

Since $\overline{\lim}_{\alpha\to\infty} \|\alpha W_\alpha f - f\|_{\mathcal{V}'} \le \overline{\lim}_{\alpha\to\infty} \|\alpha G_\alpha f - f\|_{\mathcal{H}} = 0$ for $f \in \mathcal{H}$ and $\mathcal{H} \subset \mathcal{V}'$ dense, the strong continuity follows now by a 3-ε-argument.

(iii) Finally let $f \in \mathcal{F}$. Then $f = W_1 h$ for some $h \in \mathcal{V}'$ and

$$\begin{aligned}\langle \Lambda(\alpha G_\alpha f - f), v\rangle &= -\mathcal{E}_1(\alpha G_\alpha W_1 h - W_1 h, v) + \mathcal{A}_1(\alpha G_\alpha f - f, v)\\ &= -\mathcal{E}_1(W_1(\alpha W_\alpha h - h), v) + \mathcal{A}_1(\alpha G_\alpha f - f, v)\\ &\le \|\alpha W_\alpha h - h\|_{\mathcal{V}'}\|v\|_{\mathcal{V}} + K\|\alpha G_\alpha f - f\|_{\mathcal{V}}\|v\|_{\mathcal{V}}\ .\end{aligned}$$

Thus $\|\Lambda(\alpha G_\alpha f - f)\|_{\mathcal{V}'} \le \|\alpha W_\alpha h - h\|_{\mathcal{V}'} + K\|\alpha G_\alpha f - f\|_{\mathcal{V}}$ and $\overline{\lim\limits_{\alpha\to\infty}}\|\alpha G_\alpha f - f\|_{\mathcal{F}} \le \overline{\lim\limits_{\alpha\to\infty}}(K+1)\|\alpha G_\alpha f - f\|_{\mathcal{V}} + \|\alpha W_\alpha h - h\|_{\mathcal{V}'} = 0$ by what we have already proved. The corresponding statements for $(\widehat{G_\alpha})_{\alpha>0}$ can be proved analogously. $\square$

In the next step we generalize the Dirichlet property to this new setting. Let $(E, \mathcal{B}, m)$ be a σ-finite measure space and $\mathcal{H} = L^2(E, \mathcal{B}, m)$. We assume the following condition:

D2 $u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V}$ and $\mathcal{E}(u, u - u^+ \wedge 1) \ge 0$.

For the terminology in the following proposition we refer to [M/R], [M/Ov/R].

Proposition 8. *(i) D2 holds if and only if $(G_\alpha)_{\alpha>0}$ is sub–Markovian.*

(ii) If $(\mathcal{A}, \mathcal{V})$ is a semi–Dirichlet form and $(\Lambda, D(\Lambda, \mathcal{H}))$ a Dirichlet operator, then D2 is satisfied.

Proof. (i) First suppose that D2 is satisfied. Let $f \in \mathcal{H}$ with $0 \le f \le 1$ and $u := \alpha G_\alpha f$. Then

$$\begin{aligned}0 \ge -\mathcal{E}(u, u - u^+ \wedge 1) &= \alpha(u - f, u - u^+ \wedge 1)_{\mathcal{H}}\\ &= \alpha\|u - u^+ \wedge 1\|^2_{\mathcal{H}} + \alpha(u^+ \wedge 1 - f, u - u^+ \wedge 1)_{\mathcal{H}}\\ &\ge \alpha\|u - u^+ \wedge 1\|^2_{\mathcal{H}}\end{aligned}$$

since $(u^+ \wedge 1 - f, u - u^+ \wedge 1)_{\mathcal{H}} = \int\limits_{\{u>1\}}(1-f)(u-1)dm - \int\limits_{\{u<0\}} f u\, dm \ge 0$. Therefore $u = u^+ \wedge 1$, which implies $0 \le u \le 1$.

On the other hand let $(G_\alpha)_{\alpha>0}$ be sub–Markovian. For $g \in \mathcal{H}$ we have that

$$\begin{aligned}&(g^+ \wedge 1 - \alpha G_\alpha(g^+ \wedge 1), g - g^+ \wedge 1)_{\mathcal{H}}\\ &= \int_{\{g>1\}}\left(1 - \alpha G_\alpha(g^+ \wedge 1)\right)(g-1)dm - \int_{\{g<0\}} \alpha G_\alpha(g^+ \wedge 1) g\, dm \ge 0\ ,\end{aligned} \tag{2.5}$$

since $0 \le \alpha G_\alpha(g^+ \wedge 1) \le 1$. It follows

$$\begin{aligned}&\mathcal{A}(\alpha G_\alpha(g - g^+ \wedge 1), \alpha G_\alpha(g - g^+ \wedge 1))\\ &\qquad \le \mathcal{E}\left(\alpha G_\alpha\left(g - g^+ \wedge 1\right), \alpha G_\alpha\left(g - g^+ \wedge 1\right)\right)\\ &\qquad \le \alpha\left(g - g^+ \wedge 1 - \alpha G_\alpha\left(g - g^+ \wedge 1\right), g - g^+ \wedge 1\right)_{\mathcal{H}}\\ &\qquad \le \alpha\left(g - \alpha G_\alpha g, g - g^+ \wedge 1\right)_{\mathcal{H}}\ .\end{aligned} \tag{2.6}$$

Consequently for $g = G_1 h$ by the resolvent equation

$$\sup_{\alpha>0} \mathcal{A}\left(\alpha G_\alpha \left(g - g^+ \wedge 1\right), \alpha G_\alpha \left(g - g^+ \wedge 1\right)\right) \\ \le \sup_{\alpha>0} \left(\alpha G_\alpha h - \alpha G_\alpha g, g - g^+ \wedge 1\right)_{\mathcal{H}} < +\infty ,$$

which implies $g^+ \wedge 1 \in \mathcal{V}$ (cf. [M/R, p. 21, Lemma 2.12]) and

$$\begin{aligned} \mathcal{A}\left(g - g^+ \wedge 1, g - g^+ \wedge 1\right) &= \lim_{\alpha\to\infty} \mathcal{A}\left(\alpha G_\alpha (g - g^+ \wedge 1), \alpha G_\alpha (g - g^+ \wedge 1)\right) \\ &\le \overline{\lim_{\alpha\to\infty}}\, \alpha \left(g - \alpha G_\alpha g, g - g^+ \wedge 1\right)_{\mathcal{H}} \\ &\le \overline{\lim_{\alpha\to\infty}}\, \mathcal{E}\left(\alpha G_\alpha g, g - g^+ \wedge 1\right) \\ &= \mathcal{E}(g, g - g^+ \wedge 1) \\ &\le 3K \|g\|_{\mathcal{F}} \|g - g^+ \wedge 1\|_{\mathcal{V}} . \end{aligned}$$

Thus

$$\mathcal{A}_1\left(g - g^+ \wedge 1, g - g^+ \wedge 1\right) \le 4K \|g\|_{\mathcal{F}} \|g - g^+ \wedge 1\|_{\mathcal{V}} . \tag{2.7}$$

For general $g \in \mathcal{F}$ take a sequence $(g_n)_{n\ge1} \subset G_1(\mathcal{H})$ such that $g_n \to g$ in $\mathcal{F}$. Since $g_n^+ \wedge 1 \to g^+ \wedge 1$ in $\mathcal{H}$ and $\sup_{n\ge1} \|g_n^+ \wedge 1\|_{\mathcal{V}} < +\infty$ by (2.7) it follows $g^+ \wedge 1 \in \mathcal{V}$ by [M/R, p. 21, Lemma 2.12]. Now the inequality follows easily from

$$\begin{aligned} \mathcal{E}(u, u - u^+ \wedge 1) &= \lim_{\alpha\to\infty} \mathcal{E}\left(\alpha G_\alpha u, u - u^+ \wedge 1\right) \\ &= \lim_{\alpha\to\infty} \alpha \left(u - \alpha G_\alpha u, u - u^+ \wedge 1\right)_{\mathcal{H}} \\ &= \lim_{\alpha\to\infty} \alpha \left(u - u^+ \wedge 1 - \alpha G_\alpha \left(u - u^+ \wedge 1\right), u - u^+ \wedge 1\right)_{\mathcal{H}} \\ &\qquad + \alpha \left(u^+ \wedge 1 - \alpha G_\alpha (u^+ \wedge 1), u - u^+ \wedge 1\right)_{\mathcal{H}} \\ &\ge 0 , \end{aligned}$$

where we have used (2.5) and the fact that αG_α is a contraction.

(ii) If $(\mathcal{A}, \mathcal{V})$ is a semi–Dirichlet form then $u \in \mathcal{V}$ implies $u^+ \wedge 1 \in \mathcal{V}$ and

$$\mathcal{A}\left(u^+ \wedge 1, u - u^+ \wedge 1\right) \ge 0 \tag{2.8}$$

(cf. [M/Ov/R]). Since $(\Lambda, D(\Lambda, \mathcal{H}))$ is a Dirichlet operator (i.e. $(\Lambda u, (u-1)^+)_{\mathcal{H}} \le 0\ \forall u \in D(\Lambda, \mathcal{H})$) it follows that $(U_t)_{t\ge0}$ is a sub–Markovian contraction semigroup and therefore positivity preserving (i.e. $U_t h \ge 0$ whenever $h \ge 0$). Therefore

$$\begin{aligned} \left(\Lambda u, u^-\right)_{\mathcal{H}} &= \lim_{t\downarrow0} \frac{1}{t} \left(U_t u - u, u^-\right)_{\mathcal{H}} \\ &= \lim_{t\downarrow0} \frac{1}{t} \left(U_t u^+, u^-\right) - \frac{1}{t} \left(U_t u^-, u^-\right) + \frac{1}{t} \left(u^-, u^-\right)_{\mathcal{H}} \\ &\ge \underline{\lim}_{t\downarrow0} - \frac{1}{t} \left(u^-, u^-\right)_{\mathcal{H}} + \frac{1}{t} \left(u^-, u^-\right)_{\mathcal{H}} = 0 . \end{aligned}$$

for $u \in D(\Lambda, \mathcal{H})$, and thus

$$-\left(\Lambda u, u - u^+ \wedge 1\right)_{\mathcal{H}} = \left(\Lambda u, u^-\right)_{\mathcal{H}} - \left(\Lambda u, (u-1)^+\right)_{\mathcal{H}} \geq 0\,.$$

For general $u \in \mathcal{F}$ take a sequence $(u_n)_{n\geq 1}$ in $D(\Lambda, \mathcal{V})$ such that $u_n \to u$ in $\mathcal{F}$ (cf. Remark 1). It follows $u_n^+ \wedge 1 \to u^+ \wedge 1$ weakly in $\mathcal{V}$ since $\sup_{n\geq 1} \mathcal{A}\left(u_n^+ \wedge 1, u_n^+ \wedge 1\right) < +\infty$ by (2.8) and $u_n^+ \wedge 1 \to u^+ \wedge 1$ in $\mathcal{H}$. Hence

$$-\langle \Lambda u, u - u^+ \wedge 1\rangle = \lim_{n\to\infty} -\langle \Lambda u_n, u_n - u_n^+ \wedge 1\rangle \geq 0$$

and therefore $\mathcal{E}(u, u - u^+ \wedge 1) \geq 0$. $\square$

Remark 9. Clearly $\left(\hat{G}_\alpha\right)_{\alpha>0}$ is sub–Markovian if and only if D2 is satisfied for the co–form $\widehat{\mathcal{E}}$, i.e.:

$$u \in \widehat{\mathcal{F}} \Rightarrow u^+ \wedge 1 \in \mathcal{V} \text{ and } \mathcal{E}\left(u - u^+ \wedge 1, u\right) \geq 0\,. \tag{2.9}$$

In general $\mathcal{F}$ is neither inf–stable (i.e. $u, v \in \mathcal{F} \not\Rightarrow u \wedge v \in \mathcal{F}$) nor contains an algebra of functions. Hence we impose *one* of the following regularity conditions on the domain $\mathcal{F}$:

D3 There exists a dense linear subspace $\mathcal{Y} \subset \mathcal{F}$ such that $u \wedge \alpha \in \mathcal{Y}$ for $u \in \mathcal{Y}$ and $\alpha \geq 0$.

D3' There exists a dense linear subspace $\mathcal{Y} \subset \mathcal{F}$ of bounded functions such that $u \cdot v \in \mathcal{Y}$ for $u, v \in \mathcal{Y}$.

Definition 10. The form $\mathcal{E}$ with domain $\mathcal{F} \times \mathcal{V} \cup \mathcal{V} \times \widehat{\mathcal{F}}$ is called a *generalized semi–Dirichlet form* if D1, D2 and D3 (or D1, D2 and D3') hold and a *generalized Dirichlet form* if both $\mathcal{E}$ and its co–form $\widehat{\mathcal{E}}$ are generalized semi–Dirichlet forms.

Examples 11.

(i) Let $(\mathcal{A}, \mathcal{V})$ be a semi–Dirichlet form and $\Lambda = 0$. Clearly $\mathcal{F} = \mathcal{V} = \widehat{\mathcal{F}}$ and $\mathcal{E} = \mathcal{A}$ is a generalized semi–Dirichlet form.

(ii) Let $\mathcal{A} = 0$ on $\mathcal{V} := \mathcal{H}$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ be the generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ which is sub–Markovian. Then $\mathcal{F} = D(\Lambda, \mathcal{H}), \widehat{\mathcal{F}} = D(\widehat{\Lambda}, \mathcal{H})$ and the corresponding form $\mathcal{E}$ is a generalized semi–Dirichlet form if the domain $D(\Lambda, \mathcal{H})$ satisfies D3 or D3'.

(iii) Let $\mathcal{E}$ be a time dependent semi–Dirichlet form, i.e. $\left(\mathcal{E}^{(t)}, V\right)_{t\in\mathbb{R}}$ a family of coercive closed forms with sector constant K independent of t and common domain V in some L^2–space $H := L^2(X, \mu)$– where V is a Hilbert space with corresponding norm $\|\cdot\|_V$– satisfying the following conditions:

(a) $\mathcal{E}^{(t)}(u, v)$ is a Borel measurable function of $t \in \mathbb{R}$, $\forall u, v \in V$.

(b) There exists a positive constant c such that

$$c\|u\|_V^2 \le \mathcal{E}_1^{(t)}(u,u) \le c^{-1}\|u\|_V^2, u \in V, t \in \mathbb{R} \ .$$

(c) Every form $\left(\mathcal{E}^{(t)}, V\right)$ is a semi–Dirichlet form.

For $u, v \in \mathcal{V} := L^2(\mathbb{R}, V)$ define $\mathcal{A}(u,v) := \int_{\mathbb{R}} \mathcal{E}^{(t)}(u(t), v(t))\, dt$. Then $(\mathcal{A}, \mathcal{V})$ is a semi–Dirichlet form on $\mathcal{H} := L^2(\mathbb{R}, H) \cong L^2(\mathbb{R} \times X, dt \otimes \mu)$. Note that the time derivative $\frac{d}{dt}$ on $L^2(\mathbb{R}, V')$ with domain $H^{1,2}(\mathbb{R}, V')$ satisfies condition D1. D2 holds because of Proposition 8 and D3 with $\mathcal{Y} := H^{1,2}(\mathbb{R}, \mathcal{H}) \cap \mathcal{V}$ since $u \wedge \alpha \in H^{1,2}(\mathbb{R}, \mathcal{H})$ for $u \in H^{1,2}(\mathbb{R}, \mathcal{H})$ and $\alpha \ge 0$. Hence $\mathcal{E}$ is a generalized semi–Dirichlet form.

(iv) In the situation of Example (iii) with $X = \mathbb{R}^d$ and $\mu = dx$ denote the one–sided stable semigroup of order $\alpha \in (0,1)$ by $(\eta_t^\alpha)_{t \ge 0}$ and the semigroup corresponding to $(\frac{d}{dt}, H^{1,2}(\mathbb{R}, \mathcal{V}'))$ by $(U_t)_{t\ge 0}$. Let $(U_t^\alpha)_{t\ge0}$ be the semigroup subordinated to $(U_t)_{t\ge0}$ via $(\eta_t^\alpha)_{t\ge0}$. Then the corresponding generator $\left(\left(\frac{d}{dt}\right)^\alpha, D\left(\left(\frac{d}{dt}\right)^\alpha, \mathcal{V}'\right)\right)$ which is the time derivative of power α satisfies D1. The corresponding form $\mathcal{E}$ satisfies D2 since $\left(\left(\frac{d}{dt}\right)^\alpha, D\left(\left(\frac{d}{dt}\right)^\alpha, \mathcal{H}\right)\right)$ is a Dirichlet operator as the generator of a sub–Markovian semigroup. It can be shown that

$$(2.10)\quad D\left(\left(\tfrac{d}{dt}\right)^\alpha, \mathcal{H}\right) = \{f \in L^2\left(\mathbb{R}^{d+1}, dt \otimes dx\right) \mid \int |\hat{f}|^2(t,x)|t|^{2\alpha} d(t,x) < \infty\}\ ,$$

where $\hat{f}$ denotes the Fourier transform of f (cf. [St2, p. 58 f]). Since the right side of (2.10) is a Dirichlet space, for $\mathcal{Y} := D\left(\left(\frac{d}{dt}\right)^\alpha, \mathcal{H}\right) \cap \mathcal{V}$ condition D3 is satisfied, and therefore $\mathcal{E}$ is a generalized semi–Dirichlet form.

In the special case where $V = H^{1,2}\left(\mathbb{R}^d\right)$ and $\mathcal{E}^{(t)}(u,v) = \int \nabla u \cdot \nabla v\, dx$ for all t, the generator $(L^\alpha, D(L^\alpha))$ of the corresponding contraction resolvent is an extension of the fractional diffusion operator $(\frac{\partial}{\partial t})^\alpha u + \Delta u$, $u \in C_0^\infty\left(\mathbb{R}^{d+1}\right)$ (cf. [S/W]).

(v) Consider Example (iii) again. Let $h \in D\left(\frac{d}{dt}, \mathcal{H}\right), h > 0$, be an α–excessive function for some $\alpha > 0$ (i.e. $\beta G_{\beta+\alpha} h \le h\ \forall \beta > 0$). Then we can define the h–transform $\mathcal{E}^h$ of $\mathcal{E}$ on $\mathcal{H}^h := L^2(\mathbb{R} \times X, h^2 \cdot dt \otimes \mu)$ by $D(\mathcal{A}^h) := \{u \in \mathcal{H}^h \mid u \cdot h \in \mathcal{V}\}$, $\mathcal{A}^h(u,v) := \mathcal{A}_\alpha(uh, vh)$ and $D\left(\frac{d^h}{dt}, \mathcal{H}^h\right) := \{u \in \mathcal{H}^h \mid u \cdot h \in D\left(\frac{d}{dt}, \mathcal{H}\right)\}$, $\frac{d^h}{dt} u := \frac{1}{h}\frac{d}{dt}(u \cdot h)$. It is easy to see that $\left(\frac{d^h}{dt}, D\left(\frac{d^h}{dt}, \mathcal{H}^h\right)\right)$ generates a strongy continuous contraction semigroup $(U_t^h)_{t\ge0}$ on $\mathcal{H}$ that can be extended to a strongly continuous semigroup on $D(\mathcal{A}^h)'$. The generator $\left(\frac{d^h}{dt}, D\left(\frac{d^h}{dt}, D\left(\mathcal{A}^h\right)'\right)\right)$ of the semigroup on $D(\mathcal{A}^h)'$ satisfies condition D1. The strongly continuous contraction resolvent $\left(G_\beta^h\right)_{\beta>0}$ corresponding to $\mathcal{E}^h$ is sub–Markovian, hence D2 holds. Note that $u \wedge \alpha \in D\left(\frac{d^h}{dt}, \mathcal{H}^h\right)$ for $u \in D\left(\frac{d^h}{dt}, \mathcal{H}^h\right)$ and $\alpha \ge 0$ since $(u \wedge \alpha) \cdot h = (u \cdot h) \wedge (\alpha h)$ and $D\left(\frac{d}{dt}, \mathcal{H}\right)$ is inf–stable. Thus $\mathcal{Y} := D\left(\frac{d^h}{dt}, \mathcal{H}^h\right) \cap D\left(\mathcal{A}^h\right)$ satisfies condition D3. This implies that $\mathcal{E}^h$ is a generalized semi–Dirichlet form.

(vi) In the situation of Example (iii) with $X = \mathbb{R}^d$, $\mu = dx$, $V = H^{1,2}(\mathbb{R}^d)$ and $\mathcal{E}^{(t)}(u,v) = \int \nabla u \cdot \nabla v \, dx$ let $(A_t)_{t\in\mathbb{R}}$ be a familiy of (non–symmetric) $d \times d$–matrices $A_t = (a_{ij}(t))_{1\le i,j,\le d}$ such that trace $(A_t) \ge 0$ for all $t \in \mathbb{R}$. We suppose that $t \mapsto a_{ij}(t)$ is measurable, $1 \le i,j \le d$, and $M := (\int \|A_t\|^2_{L(\mathbb{R}^d)} \, dt)^{1/2} < \infty$. Define the linear operator $(\Lambda, C_0^\infty(\mathbb{R}^{d+1}))$ on $\mathcal{H}$ by $\Lambda f(s,x) = (\partial f/\partial t)(t,x) + A_t x \cdot \nabla f(t,x)$. We can now prove the following

Proposition. *There exists a closed extension $(\Lambda, D(\Lambda, \mathcal{H}))$ of $(\Lambda, C_0^\infty(\mathbb{R}^{d+1}))$ that generates a strongly continuous contraction semigroup $(U_t)_{t\ge 0}$ on $\mathcal{H}$, such that $(U_t)_{t\ge0}$ and the adjoint semigroup $(\hat{U}_t)_{t\ge0}$ both can be restricted to strongly continuous semigroups on $\mathcal{V}$. In particular the dual operator $(\Lambda, D(\Lambda, \mathcal{V}'))$ of the generator $(\hat{\Lambda}, D(\hat{\Lambda}))$ of the restriction of $(\hat{U}_t)_{t\ge0}$ on $\mathcal{V}$ satisfies condition D1.*

Proof. Let $\delta \in C_0^\infty(\mathbb{R}), \delta \ge 0$, such that $\int \delta \, dt = 1$, and $\delta_\varepsilon(t) := (1/\varepsilon)\delta(t/\varepsilon)\, , \varepsilon > 0$. For $\varepsilon > 0$ let $a_{ij}^\varepsilon(t) := \int a_{ij}(t-s)\delta_\varepsilon(s)\, ds$ and $A_t^\varepsilon = (a_{ij}^\varepsilon(t))_{1\le i,j\le d}$. It is well–known that for $(s,x) \in \mathbb{R} \times \mathbb{R}^d$ there exists a unique continuously differentiable solution $\xi_{s,x}^\varepsilon : [0,\infty) \to \mathbb{R}^d$ of the ordinary differential equation $(d/dt)\xi_{s,x}^\varepsilon(t) = A_{t+s}^\varepsilon \xi_{s,x}^\varepsilon(t)$, $\xi_{s,x}^\varepsilon(0) = x$ (cf. [Co/Le, Section 1.5]). Clearly $\|\xi_{s,x}^\varepsilon(t)\| \le \|x\|$ $\exp(\int_s^{s+t} \|A_u^\varepsilon\|_{L(\mathbb{R}^d)}\, du)$. Note that $B_{s,s+t}^\varepsilon : x \mapsto \xi_{s,x}^\varepsilon(t)$ is linear and (cf. [Co/Le, Section 1.7])

$$\det B_{s,s+t}^\varepsilon = \exp(\int_s^{s+t} \text{trace } (A_u^\varepsilon)\, du)\, .$$

Thus $\det B_{s,s+t}^\varepsilon \ge 1$, since trace $A_t^\varepsilon = \int$ trace $A_{s-t}\, \delta_\varepsilon(s)\, ds \ge 0$. For $t \ge 0$ define the linear operator U_t^ε on $\mathcal{H}$ by $U_t^\varepsilon f(s,x) = f(s+t, B_{s,s+t}^\varepsilon x)$ and observe that U_t^ε is a contraction since

$$\int_\mathbb{R}\int_{\mathbb{R}^d} (U_t^\varepsilon f)^2(s,x)\, dx\, ds = \int_\mathbb{R} (\det B_{s,s+t}^\varepsilon)^{-1} \int_{\mathbb{R}^d} f^2(s+t,x)\, dx\, ds.$$

Note that $B_{s+t,s+t+u}^\varepsilon B_{s,s+t}^\varepsilon = B_{s,s+t+u}^\varepsilon$, hence it is easy to see that $(U_t^\varepsilon)_{t\ge0}$ is a sub–Markovian strongly continuous semigroup whose corresponding generator $(\Lambda^\varepsilon, D(\Lambda^\varepsilon, \mathcal{H}))$ is an extension of $\Lambda^\varepsilon f(t,x) = (\partial f/\partial t)(t,x) + A_t^\varepsilon x \cdot \nabla f(t,x)$, $f \in C_0^\infty(\mathbb{R}^{d+1})$. The strong continuity can be proved as in [M/R, p. 64, Proposition 4.3]. Since $\nabla U_t^\varepsilon f(s,x) = (B_{s,s+t}^\varepsilon)^T \nabla f(s+t, B_{s,s+t}^\varepsilon x)$ for $f \in C_0^\infty(\mathbb{R}^{d+1})$ it follows that

$$\mathcal{A}(U_t^\varepsilon f, U_t^\varepsilon f) \le \int_\mathbb{R}\int_{\mathbb{R}^d} \|(B_{s,s+t}^\varepsilon)^T\|^2_{L(\mathbb{R}^d)} \|\nabla f(s+t, B_{s,s+t}^\varepsilon x)\|^2\, dx\, ds\, .$$

Note that

$$\begin{aligned}\|(B_{s,s+t}^\varepsilon)^T\|_{L(\mathbb{R}^d)} &= \|B_{s,s+t}^\varepsilon\|_{L(\mathbb{R}^d)} \\ &\le \exp(\int_s^{s+t} \|A_u^\varepsilon\|_{L(\mathbb{R}^d)}\, du) \le \exp(t^{1/2}M)\end{aligned}$$

and therefore

$$\begin{aligned}\mathcal{A}(U_t^\varepsilon f, U_t^\varepsilon f) &\le \exp(2t^{1/2}M) \int_\mathbb{R}\int_{\mathbb{R}^d} \|\nabla f(s+t, B_{s,s+t}^\varepsilon x)\|^2\, dx\, ds \\ &\le \exp(2t^{1/2}M)\mathcal{A}(f,f)\end{aligned}$$

which implies that $(U_t^\varepsilon)_{t\geq 0}$ induces a strongly continuous semigroup of bounded operators on $\mathcal{V}$. On the other hand, the adjoint semigroup $(\hat{U}_t^\varepsilon)_{t\geq 0}$ is given by

$$\hat{U}_t^\varepsilon f(s,x) = (\det B_{s-t,s}^\varepsilon)^{-1} f(s-t, (B_{s-t,s}^\varepsilon)^{-1}x) .$$

Hence

$$\begin{aligned}\mathcal{A}(\hat{U}_t^\varepsilon f, \hat{U}_t^\varepsilon f) \leq &\int_{\mathbb{R}} (\det B_{s-t,s}^\varepsilon)^{-2} \|(B_{s-t,s}^\varepsilon)^{-1}\|_{L(\mathbb{R}^d)}^2 \\ &\cdot \int_{\mathbb{R}^d} \|\nabla f(s-t, (B_{s-t,s}^\varepsilon)^{-1}x)\|^2 \, dx \, ds .\end{aligned}$$

Clearly

$$\begin{aligned} x = B_{s-t,s}^\varepsilon (B_{s-t,s}^\varepsilon)^{-1} x &= (B_{s-t,s}^\varepsilon)^{-1} x + \int_{s-t}^s A_u^\varepsilon B_{s-t,u}^\varepsilon (B_{s-t,s}^\varepsilon)^{-1} x \, du \\ &= (B_{s-t,s}^\varepsilon)^{-1} x + \int_{s-t}^s A_u^\varepsilon B_{s-t,u}^\varepsilon (B_{u,s}^\varepsilon B_{s-t,u}^\varepsilon)^{-1} x \, du \\ &= (B_{s-t,s}^\varepsilon)^{-1} x + \int_{s-t}^s A_u^\varepsilon (B_{u,s}^\varepsilon)^{-1} x \, du \end{aligned}$$

which implies

$$\|(B_{s-t,s}^\varepsilon)^{-1} x\| \leq \|x\| + \int_{s-t}^s \|A_u^\varepsilon\|_{L(\mathbb{R}^d)} \|(B_{u,s}^\varepsilon)^{-1} x\| \, du$$

and therefore

$$\|(B_{s-t,s}^\varepsilon)^{-1} x\| \leq \|x\| \exp(\int_{s-t}^s \|A_u^\varepsilon\|_{L(\mathbb{R}^d)} \, du) .$$

Consequently $\|(B_{s-t,s}^\varepsilon)^{-1}\|_{L(\mathbb{R}^d)} \leq \exp(\int_{s-t}^s \|A_u^\varepsilon\|_{L(\mathbb{R}^d)} \, du) \leq \exp(t^{1/2}M)$ and

$$\mathcal{A}(\hat{U}_t^\varepsilon f, \hat{U}_t^\varepsilon f) \leq \exp(2t^{1/2}M) \mathcal{A}(f,f) ,$$

which implies that also $(\hat{U}_t^\varepsilon)_{t\geq 0}$ can be restricted to a strongly continuous semigroup of bounded operators on $\mathcal{V}$. Since for ε, $\varepsilon' > 0$

$$\begin{aligned} \|B_{s,s+t}^\varepsilon x - B_{s,s+t}^{\varepsilon'} x\| \leq &\int_s^{s+t} \|A_u^\varepsilon - A_u^{\varepsilon'}\|_{L(\mathbb{R}^d)} \|B_{s,u}^\varepsilon x\| \, du \\ &+ \int_s^{s+t} \|A_u^{\varepsilon'}\|_{L(\mathbb{R}^d)} \|B_{s,u}^\varepsilon x - B_{s,u}^{\varepsilon'} x\| \, du \\ \leq &\exp(t^{1/2}M)\|x\| t^{1/2} (\int \|A_u^\varepsilon - A_u^{\varepsilon'}\|_{L(\mathbb{R}^d)}^2 \, du)^{1/2} \\ &+ \int_s^{s+t} \|A_u^{\varepsilon'}\|_{L(\mathbb{R}^d)} \|B_{s,u}^\varepsilon x - B_{s,u}^{\varepsilon'} x\| \, du \end{aligned}$$

it follows that

$$\begin{aligned}\|B^{\varepsilon}_{s,s+t}x - B^{\varepsilon'}_{s,s+t}x\| &\le \|x\| t^{1/2}\exp(t^{1/2}M)\Big(\int \|A^{\varepsilon}_u - A^{\varepsilon'}_u\|^2_{L(\mathbb{R}^d)}\,du\Big)^{1/2}\\ &\qquad\cdot \exp\Big(\int_s^{s+t} \|A^{\varepsilon'}_u\|_{L(\mathbb{R}^d)}\,du\Big)\\ &\le \|x\| t^{1/2}\exp(2t^{1/2}M)\Big(\int \|A^{\varepsilon}_u - A^{\varepsilon'}_u\|^2_{L(\mathbb{R}^d)}\,du\Big)^{1/2},\end{aligned}$$

and therefore $B^{\varepsilon}_{s,s+t}x$ converges for $\varepsilon \to 0$ to some limit in $\mathbb{R}^d$ which we denote with $B_{s,s+t}x$. Since $\{(s,x) \in \mathbb{R}\times\mathbb{R}^d | U^{\varepsilon}_t f(s,x) \neq 0 \text{ for some } \varepsilon > 0\}$ is contained in a compact set for $f \in C_0^{\infty}(\mathbb{R}^{d+1})$, due to the estimate $\|(B^{\varepsilon}_{s,s+t})^{-1}\|_{L(\mathbb{R}^d)} \le \exp(t^{1/2}M)$, we obtain from Lebesgues Theorem that $U^{\varepsilon}_t f$ is convergent in $\mathcal{H}$. Denote the limit with $U_t f$. Clearly $\|U_t f\|_{\mathcal{H}} \le \|f\|_{\mathcal{H}}$ and therefore $f \mapsto U_t f$ can be extended to a contraction on $\mathcal{H}$. It is now easy to see that $(U_t)_{t\ge 0}$ is a sub–Markovian strongly continuous semigroup on $\mathcal{H}$ whose generator $(\Lambda, D(\Lambda,\mathcal{H}))$ extends $(\Lambda, C_0^{\infty}(\mathbb{R}^{d+1}))$. Since $\sup_{\varepsilon>0}\mathcal{A}(U^{\varepsilon}_t f, U^{\varepsilon}_t f) \le \exp(2t^{1/2}M)\mathcal{A}(f,f)$ it follows that U_t can be restricted to a bounded operator on $D(\mathcal{A})$ and $U_t f \to f$ for $t \to 0$ weakly in $D(\mathcal{A})$ for $f \in D(\mathcal{A})$. Thus $(U_t)_{t\ge0}$ can be restricted to a strongly continuous semigroup on $\mathcal{V}$ (cf [Y, p.233, Theorem]). Denote by $(\hat{U}_t)_{t\ge0}$ the adjoint semigroup of $(U_t)_{t\ge0}$ on $\mathcal{H}$. Clearly $\hat{U}^{\varepsilon}_t f \to \hat{U}_t f$ weakly in $\mathcal{H}$ for $f \in \mathcal{H}$. Since $\sup_{\epsilon>0}\|\hat{U}^{\varepsilon}_t f\|_{\mathcal{H}} \le \|f\|_{\mathcal{H}}$ and $\sup_{\epsilon>0}\mathcal{A}(\hat{U}^{\varepsilon}_t f, \hat{U}^{\varepsilon}_t f) \le \exp(2t^{1/2}M)\mathcal{A}(f,f)$ we obtain in the same way as above that $(\hat{U}_t)_{t\ge0}$ can be restricted to a strongly continuous semigroup on $\mathcal{V}$. □

The corresponding form $\mathcal{E}$ satisfies D2 because of Proposition 8. Note that both $D(\Lambda,\mathcal{H})$ and $\mathcal{V}$ are inf–stable. Hence D3 is satisfied for $\mathcal{Y} := D(\Lambda,\mathcal{H}) \cap \mathcal{V}$, and $\mathcal{E}$ is therefore a generalized semi–Dirichlet form.

III. Analytic Potential Theory of generalized semi–Dirichlet forms

We study excessive and 1–reduced functions and introduce the notions of $\mathcal{E}$–nests and $\mathcal{E}$–exceptional sets which are sufficient in order to formulate the quasi–regularity property of generalized semi–Dirichlet forms in the same way as in [M/R] (cf. the next section). Moreover, we will show, that we can always construct a Choquet capacity whose zero sets are exactly the $\mathcal{E}$–exceptional sets. This capacity is an essential tool in the construction of associated processes which will be carried out in the next section.

Throughout this section let $(E,\mathcal{B},m)$ be some σ-finite measure space and $\mathcal{H} = L^2(E,\mathcal{B},m)$. Let $\mathcal{E}$ with domain $\mathcal{F}\times\mathcal{V}\cup\mathcal{V}\times\widehat{\mathcal{F}}$ be a generalized semi–Dirichlet form (cf. (2.1)), $(G_\alpha)_{\alpha>0}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ the associated strongly continuous contraction resolvents. Denote by $(T_t)_{t\ge0}$ and $(\widehat{T}_t)_{t\ge0}$ the corresponding strongly continuous contraction semigroups. $(T_t)_{t\ge0}$ is sub–Markovian since $(G_\alpha)_{\alpha>0}$ is sub–Markovian. Note that $(\widehat{G}_\alpha)_{\alpha>0}$ and $(\widehat{T}_t)_{t\ge0}$ are positivity preserving.

Definition 1. Let $\alpha > 0$. An element $u \in \mathcal{H}$ is called *α–excessive* (resp. *α–coexcessive*) if $\beta G_{\beta+\alpha}u \le u$ (resp. $\beta\widehat{G}_{\beta+\alpha}u \le u$) for all $\beta \ge 0$.

Since $(G_\alpha)_{\alpha>0}$ is positivity preserving we have that

$$u \geq \beta G_{\beta+\alpha} u = \beta \left(G_\alpha u - \beta G_\alpha G_{\beta+\alpha} u\right) = \beta G_\alpha \left(u - \beta G_{\beta+\alpha} u\right) \geq 0,$$

i.e. every α–excessive function is positive. Similarly we can show that every α–coexcessive function is positive. Denote by $\mathcal{P}$ (resp. $\widehat{\mathcal{P}}$) the set of all 1–excessive (resp. 1–coexcessive) elements in $\mathcal{V}$.

As in the theory of Dirichlet forms we can prove:

Proposition 2. *For $u \in \mathcal{V}$ and $\alpha > 0$ the following statements are equivalent:*

(i) u is α–excessive.

(ii) $\mathcal{E}_\alpha(u,v) \geq 0$ $\forall v \in \hat{\mathcal{F}}$ with $v \geq 0$.

Proof. (i) $\Rightarrow$ (ii) $\mathcal{E}_\alpha(u,v) = \lim_{\beta\to\infty} \mathcal{E}_\alpha(\beta G_{\beta+\alpha}u, v) = \lim_{\beta\to\infty} \beta(u - \beta G_{\beta+\alpha}u, v)_{\mathcal{H}} \geq 0$ for $v \in \widehat{\mathcal{F}}$ such that $v \geq 0$.

(ii) $\Rightarrow$ (i) Let $v \in \mathcal{H}$ such that $v \geq 0$. Then

$$\beta(u - \beta G_{\beta+\alpha}u, v)_{\mathcal{H}} = \beta(v - \beta \widehat{G}_{\beta+\alpha}v, u)_{\mathcal{H}} = \mathcal{E}_\alpha(u, \beta \widehat{G}_{\beta+\alpha} v) \geq 0$$

since $(\widehat{G}_\beta)_{\beta>0}$ is positivity preserving. $\square$

Note that Proposition 2 (ii) easily implies that $\mathcal{E}_\alpha(u,v) \geq 0$ $\forall v \in \mathcal{V}$ with $v \geq 0$, if u is an α–excessive function in $\mathcal{F}$.

Remark 3. For $u \in \mathcal{H}$ and $\alpha > 0$ the following statements are equivalent:

(i) u is α–excessive.

(ii) $e^{-\alpha t} T_t u \leq u$ $\forall t \geq 0$.

Proof. (i) $\Rightarrow$ (ii) Since $(\widehat{T}_t)_{t\geq 0}$ is positivity preserving we have for $v \in \mathcal{H}$ such that $v \geq 0$:

$$\widehat{G}_\alpha v - e^{-\alpha t} \widehat{T}_t \widehat{G}_\alpha v = \int_0^t e^{-\alpha s} \widehat{T}_s v \, ds \geq 0 \ ,$$

and therefore

$$\begin{aligned}
\left(\beta G_{\beta+\alpha}u - e^{-\alpha t} T_t \left(\beta G_{\beta+\alpha}u\right), v\right)_{\mathcal{H}} &= \left(\beta G_{\beta+\alpha}u, v - e^{-\alpha t}\widehat{T}_t v\right)_{\mathcal{H}} \\
&= \mathcal{E}_\alpha\left(\beta G_{\beta+\alpha}u, \widehat{G}_\alpha v - e^{-\alpha t}\widehat{T}_t \widehat{G}_\alpha v\right) \\
&= \beta\left(u - \beta G_{\beta+\alpha}u, \widehat{G}_\alpha v - e^{-\alpha t}\widehat{T}_t\widehat{G}_\alpha v\right)_{\mathcal{H}} \geq 0 \ .
\end{aligned}$$

Hence $e^{-\alpha t} T_t \left(\beta G_{\beta+\alpha}u\right) \leq \beta G_{\beta+\alpha}u \leq u$ $\forall \beta \geq 0$. Since $\lim_{\beta\to\infty} \beta G_{\beta+\alpha}u = u$ the last inequality implies (ii).

(ii) $\Rightarrow$ (i) Clearly $\beta G_{\beta+\alpha}u = \beta \int_0^\infty e^{-(\beta+\alpha)t} T_t u dt \leq \beta \int_0^\infty e^{-\beta t} u \, dt = u$. $\square$

Remark 4. Note that in the proof of Proposition 2 and Remark 3 we have only used the fact that $(G_\alpha)_{\alpha>0}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ are positivity preserving. Consequently, the corresponding statements for α–coexcessive functions are also true.

In order to define $\mathcal{E}$–nests and $\mathcal{E}$–exceptional sets we need to introduce the notion of 1–reduced functions. For $h \in \mathcal{H}$ let $\mathcal{L}_h := \{v \in \mathcal{H} \mid v \geq h\}$ and $\beta_h : \mathcal{H} \to \mathcal{H}$ defined by $\beta_h(v) := (v-h)^-$. For the proof of the next proposition we refer to [L, p. 232, Théorème 7.1 and Remarque 7.8] (cf. also [St2]).

Proposition 5. *For $\alpha > 0$ there exist $h_\alpha \in \mathcal{F}$ and $\hat{h}_\alpha \in \widehat{\mathcal{F}}$ such that $\mathcal{E}_1(h_\alpha, v) = \alpha(\beta_h(h_\alpha), v)_{\mathcal{H}}$ and $\mathcal{E}_1(v, \hat{h}_\alpha) = \alpha(\beta_h(\hat{h}_\alpha), v)_{\mathcal{H}}$ $\forall v \in \mathcal{V}$. h_α and $\hat{h}_\alpha$ are uniquely determined.*

For the proof of the next proposition we need the following

Lemma 6. *$u \in \mathcal{F}$ implies $u^+ \in \mathcal{V}$ and $\mathcal{E}(u, u^+) \geq 0$.*

Proof. Since $(G_\beta)_{\beta>0}$ is positivity preserving we know that

$$\beta\left(u^+ - \beta G_\beta u^+, u^-\right)_{\mathcal{H}} \leq 0 \text{ and } \beta\left(u^- - \beta G_\beta u^-, u^+\right)_{\mathcal{H}} \leq 0 \,.$$

Since βG_β is a contraction we obtain that

$$\beta\left(u^- - \beta G_\beta u^-, u^-\right)_{\mathcal{H}} \geq 0 \,.$$

Thus $\beta\left(u^+ - \beta G_\beta u^+, u^+\right)_{\mathcal{H}} \leq \beta\left(u - \beta G_\beta u, u\right)_{\mathcal{H}}$ and therefore

$$\sup_{\beta>0} \mathcal{A}\left(\beta G_\beta u^+, \beta G_\beta u^+\right) \leq \sup_{\beta>0} \mathcal{E}\left(\beta G_\beta u, \beta G_\beta u\right) < +\infty \,.$$

Hence $u^+ \in \mathcal{V}$ by [M/R, p. 21, Lemma 2.12] and

$$\mathcal{E}(u, u^+) = \lim_{\beta\to\infty} \mathcal{E}(\beta G_\beta u, u^+) = \lim_{\beta\to\infty} \beta(u - \beta G_\beta u, u^+)_{\mathcal{H}} \geq 0 \,. \quad \square$$

Since $\left(\hat{G}_\beta\right)_{\beta>0}$ is positivity preserving the corresponding dual statement is also true.

Proposition 7. *Suppose $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$ (resp. $\mathcal{L}_h \cap \widehat{\mathcal{F}} \neq \emptyset$). Then there exists $e_h \in \mathcal{L}_h \cap \mathcal{P}$ (resp. $\hat{e}_h \in \mathcal{L}_h \cap \widehat{\mathcal{P}}$) such that:*

(i) $e_h \leq u$ for all $u \in \mathcal{L}_h \cap \mathcal{P}$ $\left(\text{resp. } \hat{e}_h \leq u \text{ for all } u \in \mathcal{L}_h \cap \widehat{\mathcal{P}}\right)$.

(ii) $\mathcal{E}_1(e_h, v) \geq \mathcal{A}_1(e_h, e_h)$ $\forall v \in \mathcal{L}_h \cap \widehat{\mathcal{F}}$
(resp. $\mathcal{E}_1(v, \hat{e}_h) \geq \mathcal{A}_1(\hat{e}_h, \hat{e}_h)$ $\forall v \in \mathcal{L}_h \cap \mathcal{F}$).

Proof. Denote by h_α, $\alpha > 0$, the unique element in $\mathcal{F}$ such that $\mathcal{E}_1(h_\alpha, v) = \alpha(\beta_h(h_\alpha), v)_{\mathcal{H}} \forall v \in \mathcal{V}$.

1. Step: $h_{\alpha_1} \le h_{\alpha_2}$ for $\alpha_1 \le \alpha_2$.
Since

$$\begin{aligned}
0 =& \mathcal{E}_1\left(h_{\alpha_1} - h_{\alpha_2}, (h_{\alpha_1} - h_{\alpha_2})^+\right) - \alpha_1 \left(\beta_h(h_{\alpha_1}), (h_{\alpha_1} - h_{\alpha_2})^+\right)_{\mathcal{H}} \\
&+ \alpha_2 \left(\beta_h(h_{\alpha_2}), (h_{\alpha_1} - h_{\alpha_2})^+\right)_{\mathcal{H}} \\
=& \mathcal{E}_1\left(h_{\alpha_1} - h_{\alpha_2}, (h_{\alpha_1} - h_{\alpha_2})^+\right) - \alpha_1 \left(\beta_h(h_{\alpha_1}) - \beta_h(h_{\alpha_2}), (h_{\alpha_1} - h_{\alpha_2})^+\right)_{\mathcal{H}} \\
&+ (\alpha_2 - \alpha_1) \left(\beta_h(h_{\alpha_2}), (h_{\alpha_1} - h_{\alpha_2})^+\right)_{\mathcal{H}} \\
\ge& \| (h_{\alpha_1} - h_{\alpha_2})^+ \|_{\mathcal{H}}^2
\end{aligned}$$

by the last lemma, it follows $(h_{\alpha_1} - h_{\alpha_2})^+ = 0$, i.e. $h_{\alpha_1} \le h_{\alpha_2}$.

2. Step: $h_\alpha \le u$ for $u \in \mathcal{L}_h \cap \mathcal{P}$ and $\alpha > 0$.

$$\begin{aligned}
\| (h_\alpha - u)^+ \|_{\mathcal{H}}^2 =& \lim_{\beta\to\infty} \left(h_\alpha - \beta G_{\beta+1}u, (h_\alpha - \beta G_{\beta+1}u)^+\right)_{\mathcal{H}} \\
\le& \overline{\lim_{\beta\to\infty}} \mathcal{E}_1\left(h_\alpha - \beta G_{\beta+1}u, (h_\alpha - \beta G_{\beta+1}u)^+\right) \\
\le& \overline{\lim_{\beta\to\infty}} \mathcal{E}_1\left(h_\alpha, (h_\alpha - \beta G_{\beta+1}u)^+\right) \\
=& \overline{\lim_{\beta\to\infty}} \alpha \left(\beta_h(h_\alpha), (h_\alpha - \beta G_{\beta+1}u)^+\right)_{\mathcal{H}} \\
=& \alpha \left((h_\alpha - h)^-, (h_\alpha - u)^+\right)_{\mathcal{H}} = 0 \ .
\end{aligned}$$

Hence $(h_\alpha - u)^+ = 0$, i.e. $h_\alpha \le u$.

3. Step: $\sup\limits_{\alpha>0} \|h_\alpha\|_{\mathcal{V}} < +\infty$.
For $v \in \mathcal{L}_h \cap \mathcal{F}$ we have that

$$\begin{aligned}
\|v - h_\alpha\|_{\mathcal{V}}^2 \le& \mathcal{E}_1(v - h_\alpha, v - h_\alpha) = \mathcal{E}_1(v, v - h_\alpha) - \alpha\left(\beta_h(h_\alpha), v - h_\alpha\right)_{\mathcal{H}} \\
\le& \mathcal{E}_1(v, v - h_\alpha) \le 2K\|v\|_{\mathcal{F}}\|v - h_\alpha\|_{\mathcal{V}} \ .
\end{aligned}$$

Hence $\sup\limits_{\alpha>0} \|h_\alpha\|_{\mathcal{V}} \le \sup\limits_{\alpha>0} \|h_\alpha - v\|_{\mathcal{V}} + \|v\|_{\mathcal{V}} < +\infty$.

By the monotone convergence theorem there exists an element $e_h \in \mathcal{H}$ such that $h_\alpha \to e_h$ in $\mathcal{H}$. Consequently $e_h \in \mathcal{V}$ and $h_\alpha \to e_h$ weakly in $\mathcal{V}$ by [M/R, p. 21, Lemma 2.12]. Clearly $e_h \in \mathcal{P}$ since $h_\alpha \in \mathcal{P}$, and $e_h \le u$ for all $u \in \mathcal{L}_h \cap \mathcal{P}$ by the 2. Step. Since

$$\begin{aligned}
\left(\beta_h(e_h), v\right)_{\mathcal{H}} =& \lim_{\alpha\to\infty} \left(\beta_h(h_\alpha), v\right)_{\mathcal{H}} \\
=& \lim_{\alpha\to\infty} 1/\alpha \mathcal{E}_1(h_\alpha, v) \\
\le& \overline{\lim}_{\alpha\to\infty} 2K/\alpha \|h_\alpha\|_{\mathcal{V}} \|v\|_{\widehat{\mathcal{F}}} = 0
\end{aligned}$$

for all $v \in \widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}$ dense in $\mathcal{H}$ it follows $\beta_h(e_h) = 0$, i.e. $e_h \in \mathcal{L}_h$. For $v \in \mathcal{L}_h \cap \widehat{\mathcal{F}}$ we obtain that

$$\begin{aligned}
\mathcal{A}_1(e_h, e_h) &\leq \underline{\lim}_{\alpha\to\infty} \mathcal{A}_1(h_\alpha, h_\alpha) \leq \overline{\lim}_{\alpha\to\infty} \mathcal{E}_1(h_\alpha, h_\alpha) \\
&= \overline{\lim}_{\alpha\to\infty} \alpha\left(\beta_h(h_\alpha), h_\alpha\right)_{\mathcal{H}} \leq \overline{\lim}_{\alpha\to\infty} \alpha\left(\beta_h(h_\alpha), v\right)_{\mathcal{H}} \\
&= \overline{\lim}_{\alpha\to\infty} \mathcal{E}_1(h_\alpha, v) = \mathcal{E}_1(e_h, v) \ .
\end{aligned}$$

Since we have only used the fact that $(G_\alpha)_{\alpha>0}$ is positivity preserving the dual statements for $\hat{e}_h$ can be shown similarly. $\square$

Definition 8. e_h (resp. $\hat{e}_h$) is called the *1–reduced* (resp. *1–coreduced*) function of h.

From now on we assume that E is a Hausdorff topological space and $\mathcal{B}$ its Borel–σ–algebra. For an open subset $U \subset E$ and a function $f \in \mathcal{H}$ let $f_U := e_{(f\cdot 1_U)}$ if the 1–reduced function $e_{(f\cdot 1_U)}$ exists. Denote by f_U^α (resp. $\hat{f}_U^\alpha$) the function $(f\cdot 1_U)_\alpha$ (resp. $(\widehat{f\cdot 1_U})_\alpha$) (cf. Proposition 5). Note that in the proof of Proposition 7 we have shown that $f_U^\alpha \to f_U$ in $\mathcal{H}$ and weakly in $\mathcal{V}$ if $\mathcal{L}_{f\cdot 1_U} \cap \mathcal{F} \neq \emptyset$. It is easy to see that the corresponding statement for $\hat{f}_U^\alpha$ and $\hat{f}_U$ is also true.

Lemma 9. *For $U \subset E$ open it follows that*

$$\|f_U\|_{\mathcal{V}} \leq 3K\|f\|_{\mathcal{F}}\, \forall f \in \mathcal{F} \left(\text{ resp. } \|\hat{f}_U\|_{\mathcal{V}} \leq 3K\|\hat{f}\|_{\hat{\mathcal{F}}}\, \forall f \in \hat{\mathcal{F}} \right).$$

Proof. For $f \in \mathcal{F}$

$$\begin{aligned}
\mathcal{A}_1(f - f_U, f - f_U) &\leq \underline{\lim}_{\alpha\to\infty} \mathcal{A}_1(f - f_U^\alpha, f - f_U^\alpha) \\
&\leq \overline{\lim}_{\alpha\to\infty} \mathcal{E}_1(f - f_U^\alpha, f - f_U^\alpha) \\
&\leq \overline{\lim}_{\alpha\to\infty} \mathcal{E}_1(f, f - f_U^\alpha) \\
&\leq 2K\|f\|_{\mathcal{F}}\|f - f_U\|_{\mathcal{V}} \ .
\end{aligned}$$

Hence $\|f_U\|_{\mathcal{V}} \leq 3K\|f\|_{\mathcal{F}}$. The second statement can be proved similarly. $\square$

Definition 10. An increasing sequence of closed subsets $(F_k)_{k\geq 1}$ is called an *$\mathcal{E}$–nest*, if for every function $u \in \mathcal{P} \cap \mathcal{F}$ it follows that $u_{F_k^c} \to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$.

We define the notions *regular $\mathcal{E}$–nest* and *$\mathcal{E}$–exceptional set* analogous to [M/R, Chapter III]: An $\mathcal{E}$–nest $(F_k)_{k\geq 0}$ is called regular if for all $k \geq 1$, $U \subset E$, U open, $m(U \cap F_k) = 0$ implies that $U \subset F_k^c$. A subset $N \subset E$ is called $\mathcal{E}$–exceptional if there is an $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ such that $N \subset \bigcap_{k\geq 1} E \setminus F_k$. We say that a property of points in E holds $\mathcal{E}$–quasi everywhere ($\mathcal{E}$–q.e.) if it holds outside some $\mathcal{E}$–exceptional set.

For $A \subset E$ closed and a linear subspace $W \subset L^2(E, \mathcal{B}, m)$ let

$$W_A := \{u \in W \mid u = 0 \quad m - \text{a.e. on } E \setminus A\}.$$

Remark 11. (i) If $(F_k)_{k\geq 1}$ is an $\mathcal{E}$–nest then $\bigcup_{k\geq 1}\mathcal{V}_{F_k}$ is dense in $\mathcal{V}$. Consequently every $\mathcal{E}$–nest is an $\mathcal{A}$–nest in the sense of [M/R, p. 75, Definition 21.] if $(\mathcal{A},\mathcal{V})$ in (2.1) is a Dirichlet form.

Proof. Since every function $G_1h, h\geq 0$, is 1–excessive, the linear span of all 1–excessive functions $u\in\mathcal{P}\cap\mathcal{F}$ is dense in $\mathcal{F}$. Because of $u_{F_k^c}\to 0$ weakly in $\mathcal{V}$, the Cesaro mean $w_N := \frac{1}{N}\sum_{\ell=1}^N u_{F_{k_\ell}^c}$ of a subsequence converges strongly to 0 in $\mathcal{V}$. Since $u - w_N \in \bigcup\limits_{k\geq 1}\mathcal{V}_{F_k^c}$ the assertion follows.

(ii) If $(F_k)_{k\geq 1}$ is an $\mathcal{E}$–nest then $u_{F_k^c}\to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$ for every $u\in\mathcal{P}$ with $\mathcal{L}_u\cap\mathcal{F}\neq\emptyset$.

Proof. **1. Step:** First let $u\in\mathcal{F}$ and $(h_n)_{n\geq 1}\subset\mathcal{H}$ be a sequence such that $G_1(h_n)\to u$ in $\mathcal{F}$. Since $u\leq u-G_1(h_n)+G_1(h_n^+)$ we obtain that $u_{F_k^c}\leq (u-G_1(h_n))_{F_k^c}+G_1(h_n^+)_{F_k^c}$ and therefore

$$\begin{aligned}\|u_{F_k^c}\|_{\mathcal{H}} &\leq \|\left(u-G_1\left(h_n\right)\right)_{F_k^c}\|_{\mathcal{H}} + \|G_1\left(h_n^+\right)_{F_k^c}\|_{\mathcal{H}}\\ &\leq 3K\|u-G_1(h_n)\|_{\mathcal{F}} + \|G_1(h_n^+)_{F_k^c}\|_{\mathcal{H}}\ .\end{aligned}$$

It follows $u_{F_k^c}\to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$ since $\sup\limits_{k\geq 1}\|u_{F_k^c}\|_{\mathcal{V}}<\infty$ by Lemma 9.

2. Step: Let $u\in\mathcal{P}$ with $\mathcal{L}_u\cap\mathcal{F}\neq\emptyset$ and $f\in\mathcal{L}_u\cap\mathcal{F}$. Since $f_{F_k^c}\to 0$ in $\mathcal{H}$ and $u_{F_k^c}\leq f_{F_k^c}$ we obtain that $u_{F_k^c}\to 0$ in $\mathcal{H}$. Using the inequality

$$\begin{aligned}\mathcal{A}_1\left(f-u_{F_k^c}, f-u_{F_k^c}\right) &= \lim_{\beta\to\infty}\mathcal{A}_1\left(f-\beta G_{\beta+1}u_{F_k^c}, f-\beta G_{\beta+1}u_{F_k^c}\right)\\ &\leq \varlimsup_{\beta\to\infty}\mathcal{E}_1\left(f-\beta G_{\beta+1}u_{F_k^c}, f-\beta G_{\beta+1}u_{F_k^c}\right)\\ &\leq \varlimsup_{\beta\to\infty}\mathcal{E}_1\left(f, f-\beta G_{\beta+1}u_{F_k^c}\right)\\ &= \mathcal{E}_1\left(f, f-u_{F_k^c}\right)\leq 2K\|f\|_{\mathcal{F}}\|f-u_{F_k^c}\|_{\mathcal{V}}\end{aligned}$$

we see that $\sup\limits_{k\geq 1}\|f-u_{F_k^c}\|_{\mathcal{V}}<\infty$ and thus $u_{F_k^c}\to 0$ weakly in $\mathcal{V}$. $\square$

We want to construct a Choquet capacity whose zero sets are exactly the $\mathcal{E}$–exceptional sets. To this end take $\varphi\in\mathcal{H}$ such that $0<\varphi\leq 1$. Let $h:=G_1\varphi$ and $\hat{g}:=\widehat{G}_1\varphi$. Since $\mathcal{L}_h\cap\mathcal{F}\neq\emptyset$ the 1–reduced function h_U for all open subsets $U\subset E$ is well–defined.

Definition 12. For $U\subset E$ open let $\mathrm{Cap}_\varphi(U):=(h_U,\varphi)_{\mathcal{H}}$ and for arbitrary $A\subset E$ let $\mathrm{Cap}_\varphi(A):=\inf\limits_{\substack{A\subset U\\ U\text{open}}}\mathrm{Cap}_\varphi(U)$.

Remark 13. $\mathrm{Cap}_\varphi(A)=0$ for $A\in\mathcal{B}(E)$ implies $m(A)=0$ since

$$0=\inf_{\substack{A\subset U\\ U\text{open}}}(h_U,\varphi)\geq(1_A\cdot h,\varphi)_{\mathcal{H}}=\left(\widehat{G}_1\left(1_A\varphi\right),\varphi\right)_{\mathcal{H}}$$

which implies $\widehat{G}_1\left(1_A\varphi\right)=0$, thus $1_A\varphi=0$ and consequently $m(A)=0$.

Lemma 14. *(i) If $(u_n)_{n\geq 1}$ is an increasing sequence of open subsets then*

$$\mathrm{Cap}_\varphi(\bigcup_{n\geq 1} U_n) = \sup_{n\geq 1} \mathrm{Cap}_\varphi(U_n).$$

(ii) For $U, V \subset E$ open we have

$$\mathrm{Cap}_\varphi(U\cap V) + \mathrm{Cap}_\varphi(U\cup V) \leq \mathrm{Cap}_\varphi(U) + \mathrm{Cap}_\varphi(V)\ .$$

("strong subadditivity").

Proof. (i) Let $U := \bigcup_{n\geq 1} U_n$. Obviously $h_{U_n} \leq h_{U_{n+1}}$ and therefore by the monotone convergence theorem $h_{U_n} \to h_\infty$ in $\mathcal{H}$ for some $h_\infty \in \mathcal{H}$. Since $\sup_{n\geq 1} \|h_{U_n}\|_{\mathcal{V}} < \infty$ by Lemma 9 we obtain that $h_\infty \in \mathcal{V}$ and $h_{U_n} \to h_\infty$ weakly in $\mathcal{V}$. Obviously $h_\infty \in \mathcal{L}_{h\cdot 1_U} \cap \mathcal{P}$, so $h_U \leq h_\infty$. Hence $\mathrm{Cap}_\varphi(\bigcup_{n\geq 1} U_n) = (h_U, \varphi)_{\mathcal{H}} \leq (h_\infty, \varphi)_{\mathcal{H}} = \sup_{n\geq 1} \mathrm{Cap}_\varphi(U_n)$. Since Cap_φ is monotone the converse inequality is clear.

(ii) Since $h_{U\cap V} \leq h_U \wedge h_V$ and $h_{U\cup V} = h_U$ on U and $h_{U\cup V} = h_V$ on V it follows that

$$\begin{aligned}
\mathrm{Cap}_\varphi(U\cap V) + \mathrm{Cap}_\varphi(U\cup V) &= \mathcal{E}_1(h_{U\cap V} + h_{U\cup V}, \hat{g}) \\
&= \lim_{\alpha\to\infty} \mathcal{E}_1(h^\alpha_{U\cap V} + h^\alpha_{U\cup V}, \hat{g}) \\
&= \lim_{\alpha\to\infty} \mathcal{E}_1(h^\alpha_{U\cap V} + h^\alpha_{U\cup V}, \hat{g}_{U\cup V}) \\
&= \lim_{\alpha\to\infty} \lim_{\beta\to\infty} \mathcal{E}_1(h^\alpha_{U\cap V} + h^\alpha_{U\cup V}, \hat{g}^\beta_{U\cup V}) \\
&\leq \overline{\lim_{\alpha\to\infty}}\, \overline{\lim_{\beta\to\infty}}\, \mathcal{E}_1(h_U + h_V, \hat{g}^\beta_{U\cup V}) \\
&\leq \overline{\lim_{\alpha\to\infty}}\, \overline{\lim_{\beta\to\infty}}\, \mathcal{E}_1(h_U + h_V, \hat{g}) \\
&= \mathrm{Cap}_\varphi(U) + \mathrm{Cap}_\varphi(V)\ . \quad \square
\end{aligned}$$

With the last lemma the following proposition is an easy consequence (cf. [M/R, p.77, Theorem 2.8]).

Proposition 15. Cap_φ *is a Choquet capacity, i.e. it has the following properties:*

(i) If $(A_n)_{n\geq 1}$ is an increasing sequence of arbitrary subsets of E then

$$\mathrm{Cap}_\varphi(\bigcup_{n\geq 1} A_n) = \sup_{n\geq 1} \mathrm{Cap}_\varphi(A_n).$$

(ii) If $(K_n)_{n\geq 1}$ is a decreasing sequence of compact subsets of E then

$$\mathrm{Cap}_\varphi(\bigcap_{n\geq 1} K_n) = \inf_{n\geq 1} \mathrm{Cap}_\varphi(K_n).$$

Remark 16. We will need the following fact several times. Let $f \in \mathcal{F} \cap \mathcal{P}$, $\hat{f} \in \widehat{\mathcal{F}} \cap \widehat{\mathcal{P}}$ and $U \subset E$ open. Then $\mathcal{E}_1(f_U, \hat{f}) = \mathcal{E}_1(f, \hat{f}_U)$.

Proof.

$$\begin{aligned} \mathcal{E}_1(f_U, \hat{f}) &= \lim_{\alpha\to\infty} \alpha((f_U^\alpha - f1_U)^-, \hat{f})_{\mathcal{H}} \\ &= \lim_{\alpha\to\infty} \alpha((f_U^\alpha - f1_U)^-, \hat{f}_U)_{\mathcal{H}} \\ &= \lim_{\alpha\to\infty} \mathcal{E}_1(f_U^\alpha, \hat{f}_U) \leq \mathcal{E}_1(f, \hat{f}_U)\,. \end{aligned}$$

Since the converse inequality can be shown similarly the proof is complete. □

Proposition 17. *Let $(F_k)_{h\geq 1}$ be an increasing sequence of closed subsets. Then $(F_k)_{k\geq 1}$ is an $\mathcal{E}$–nest if and only if $\lim\limits_{k\to\infty} \mathrm{Cap}_\varphi(F_k^c) = 0$.*

Proof. Suppose $(F_k)_{k\geq 1}$ is an $\mathcal{E}$–nest. Then $h_{F_k^c} \to 0$ in $\mathcal{H}$ and it follows $\mathrm{Cap}_\varphi(F_k^c) \to 0$, $k \to \infty$.

On the other hand suppose $\lim\limits_{k\to\infty} \mathrm{Cap}_\varphi(F_k^c) = 0$. Since $(\hat{g}_{F_k^c})_{k\geq 1}$ is decreasing, there is some $\hat{g}_\infty \in \mathcal{H}$ such that $\hat{g}_{F_k^c} \to \hat{g}_\infty$ in $\mathcal{H}$. By the last remark

$$\begin{aligned} (\varphi, \hat{g}_\infty) &= \lim_{k\to\infty} (\varphi, \hat{g}_{F_k^c})_{\mathcal{H}} = \lim_{k\to\infty} \mathcal{E}_1(h, \hat{g}_{F_k^c}) \\ &= \lim_{k\to\infty} \mathcal{E}_1(h_{F_k^c}, \hat{g}) = \lim_{k\to\infty} \mathrm{Cap}_\varphi(F_k^c) = 0 \end{aligned}$$

and therefore $\hat{g}_\infty = 0$. Since $\sup_{k\geq 1} \|\hat{g}_{F_k^c}\|_{\mathcal{V}} < \infty$ by Lemma 9 we conclude that $\hat{g}_{F_k^c} \to 0$ weakly in $\mathcal{V}$.

Let $f \in \mathcal{F} \cap \mathcal{P}$ be arbitrary. In the same manner as above we can show that there is a function $f_\infty \in \mathcal{V}$ such that $f_{F_k^c} \to f_\infty$ in $\mathcal{H}$ and weakly in $\mathcal{V}$. Since

$$\begin{aligned} (f_\infty, \varphi)_{\mathcal{H}} =& \mathcal{E}_1(f_\infty, \hat{g}) = \lim_{k\to\infty} \mathcal{E}_1(f_{F_k^c}, \hat{g}) \\ &= \lim_{k\to\infty} \mathcal{E}_1(f, \hat{g}_{F_k^c}) = 0 \end{aligned}$$

we have that $f_\infty = 0$ from which the assertion follows. □

Remark 18. (i) Let $(F_k)_{k\geq 1}$ be an increasing sequence of closed subsets. $(F_k)_{k\geq 1}$ is an $\mathcal{E}$–nest, if $\bigcup\limits_{k\geq 1} \mathcal{F}_{F_k}$ is dense in $\mathcal{F}$.

Proof. In the same way as in the proof of the last proposition there exist $\hat{g}_\infty \in \mathcal{V}$ such that $\hat{g}_{F_k^c} \to \hat{g}_\infty$ in $\mathcal{H}$ and weakly in $\mathcal{V}$. Clearly, for $f \in \mathcal{F}_{F_k}$ and $\ell \geq k$ we have

$$\mathcal{E}_1(f, \hat{g}_{F_\ell^c}) = \lim_{\alpha\to\infty} \mathcal{E}_1(f, \hat{g}_{F_\ell^c}^\alpha) = \lim_{\alpha\to\infty} \alpha((\hat{g}_{F_\ell^c}^\alpha - \hat{g}1_{F_\ell^c})^-, f) = 0$$

and therefore $\mathcal{E}_1(f,\hat g_\infty)=\lim_{\ell\to\infty}\mathcal{E}_1(f,\hat g_{F_\ell^c})=0\ \forall f\in\bigcup_{k\geq 1}\mathcal{F}_{F_k}$. It follows $\hat g_\infty=0$ since $\bigcup\limits_{k\geq 1}\mathcal{F}_{F_k}$ is dense in $\mathcal{F}$. Thus

$$\begin{aligned}\lim_{k\to\infty}\mathrm{Cap}_\varphi(F_k^c)&=\lim_{k\to\infty}\mathcal{E}_1(h_{F_k^c},\hat g)\\&=\lim_{k\to\infty}\mathcal{E}_1(h,\hat g_{F_k^c})=0\ .\quad\square\end{aligned}$$

(ii) Suppose that the co–form $\widehat{\mathcal{E}}$ is also a generalized semi–Dirichlet form. Then $\mathrm{Cap}_\varphi(U)=\mathcal{E}_1(h_U,\hat g)=\mathcal{E}_1(h,\hat g_U)=\widehat{\mathrm{Cap}}_\varphi(U)$. Thus $(F_k)_{k\geq1}$ is an $\mathcal{E}$–nest if and only if it is an $\widehat{\mathcal{E}}$–nest and consequently the $\mathcal{E}$–exceptional and $\widehat{\mathcal{E}}$–exceptional sets coincide.

Definition 19. A function f defined up to some $\mathcal{E}$–exceptional set $A\subset E$ is called *$\mathcal{E}$–quasi–continuous* ($\mathcal{E}$–q.c.) (resp. *$\mathcal{E}$–quasi–lower–semicontinuous* ($\mathcal{E}$–q.l.s.c.)) if there is some $\mathcal{E}$–nest $(F_k)_{k\geq1}$ such that $\bigcup\limits_{k\geq1}F_k\subset E\setminus A$ and $f_{|F_k}$ is continuous (resp. lower semicontinuous) $\forall k$.

For an $\mathcal{E}$–nest $(F_k)_{k\geq1}$ let

$$\begin{aligned}C(\{F_k\})=&\{f:A\to\mathbb{R}\mid\bigcup_{k\geq1}F_k\subset A\subset E,\, f_{|F_k}\text{ is continuous }\forall k\}\\C_l(\{F_k\})=&\{f:A\to\mathbb{R}\mid\bigcup_{k\geq1}F_k\subset A\subset E,\, f_{|F_k}\text{ is lower semicontinuous }\forall k\}\ .\end{aligned}$$

We can derive the following Chebychev–type inequalities:

Proposition 20. *For $u\in\mathcal{F}$ (resp. $u\in\mathcal{P}$) with $\mathcal{E}$–q.c. (resp. $\mathcal{E}$–q.l.s.c) m–version $\tilde u$ and $\lambda>0$*

$$\mathrm{Cap}_\varphi(\{|\tilde u|>\lambda\})\leq(6K/\lambda)\ \|\varphi\|_{\mathcal{H}}\|u\|_{\mathcal{F}}\tag{3.1}$$

$$(\text{resp. }\mathrm{Cap}_\varphi(\{\tilde u>\lambda\})\leq(1/\lambda)\ \|\varphi\|_{\mathcal{H}}\|u\|_{\mathcal{H}})\ \ .\tag{3.2}$$

Proof. Suppose $\tilde u\in C(\{F_k\})$ (resp. $C_l(\{F_k\})$). Then $\{|\tilde u|>\lambda\}\cup F_k^c$ (resp. $\{\tilde u>\lambda\}\cup F_k^c$) is open and

$$h\leq e_{\frac{u}{\lambda}}+e_{-\frac{u}{\lambda}}+h_{F_k^c}\text{ on }\{|\tilde u|>\lambda\}\cup F_k^c$$

$$(\text{resp. }h\leq\frac{u}{\lambda}+h_{F_k^c}\text{ on }\{\tilde u>\lambda\}\cup F_k^c)\,\forall k.$$

Hence

$$\begin{aligned}\mathrm{Cap}_\varphi(\{|\tilde u|>\lambda\})\leq&(e_{\frac{u}{\lambda}}+e_{-\frac{u}{\lambda}}+h_{F_k^c},\varphi)_{\mathcal{H}}\\\leq&(6K/\lambda)\|u\|_{\mathcal{F}}\|\varphi\|_{\mathcal{H}}+\ \mathrm{Cap}_\varphi(F_k^c)\\(\text{resp. }\mathrm{Cap}_\varphi(\{\tilde u>\lambda\})\leq&(u/\lambda+h_{F_k^c},\varphi)_{\mathcal{H}}\\\leq&(1/\lambda)\|u\|_{\mathcal{H}}\|\varphi\|_{\mathcal{H}}+\mathrm{Cap}_\varphi(F_k^c))\,\forall k.\end{aligned}$$

$\square$

Proposition 21. *(i) Let $u_n \in \mathcal{F}$ with $\mathcal{E}$–q.c. m–version $\tilde{u}_n$ such that $u_n \to u$ in $\mathcal{F}$. Then there exists a subsequence $(\tilde{u}_{n_k})_{k\geq 1}$ and an $\mathcal{E}$–q.c. m–version $\tilde{u}$ of u such that $\tilde{u}_{n_k} \to \tilde{u}$ $\mathcal{E}$–quasi–uniformly (i.e. there is an $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ such that $\tilde{u}_{n_k} \to \tilde{u}$ uniformly on $F_k \, \forall k$).*

(ii) Let $(u_n) \subset \mathcal{P}$ with $\mathcal{E}$–q.l.s.c. m–versions $\tilde{u}_n$ such that $u_n \to 0$ in $\mathcal{H}$. Then there exists a subsequence $(\tilde{u}_{n_k})_{k\geq 1}$ such that $\overline{\lim\limits_{k\to\infty}}\, \tilde{u}_{n_k} \leq 0$ $\mathcal{E}$–q.e.

Proof. Due to (3.1) the proof of (i) can be done as in [M/R, p. 82, Proposition 3.5].

(ii) By (3.2) we can find a subsequence $(u_{n_k})_{k\geq 1}$ with $\mathrm{Cap}_\varphi(\{\tilde{u}_{n_k} > 2^{-k}\}) \leq 2^{-k}$. Let $u_\infty := \overline{\lim\limits_{k\to\infty}}\, \tilde{u}_{n_k}$. Since for $n \geq \ell$

$$\mathrm{Cap}_\varphi\left(\bigcup_{m\geq n} \{\tilde{u}_{k_m} > 2^{-\ell}\}\right) \leq \sum_{m\geq n} \mathrm{Cap}_\varphi\left(\{\tilde{u}_{k_m} > 2^{-m}\}\right) \leq 2^{-(n-1)}$$

it follows that

$$\begin{aligned}\mathrm{Cap}_\varphi\left(\{u_\infty > 2^{-\ell}\}\right) &= \mathrm{Cap}_\varphi\left(\bigcap_{n\geq 1}\bigcup_{m\geq n} \{\tilde{u}_{k_m} > 2^{-\ell}\}\right)\\ &\leq \inf_{n\geq 1} \mathrm{Cap}_\varphi\left(\bigcup_{m\geq n} \{\tilde{u}_{k_m} > 2^{-\ell}\}\right)\\ &\leq \inf_{n\geq \ell} \mathrm{Cap}_\varphi\left(\bigcup_{m\geq n} \{\tilde{u}_{k_m} > 2^{-\ell}\}\right) = 0\ .\end{aligned}$$

Consequently $\mathrm{Cap}_\varphi(\{u_\infty > 0\}) = \sup\limits_{\ell\geq 1} \mathrm{Cap}_\varphi(\{u_\infty > 2^{-\ell}\}) = 0.$ □

IV. Quasi–Regularity and construction of associated processes

Throughout this section let E be a Hausdorff topological space and assume that its Borel σ–algebra $\mathcal{B}(E)$ is generated by the set $C(E)$ of all continuous functions on E. Let m be a σ–finite measure on $(E, \mathcal{B}(E))$ such that $\mathcal{H} = L^2(E, m)$ is a separable (real) Hilbert space. Let $\mathcal{E}$ be a generalized semi–Dirichlet form with domain $\mathcal{F} \times \mathcal{V} \cup \mathcal{V} \times \hat{\mathcal{F}}$ (cf. (2.1)) and denote with $(G_\alpha)_{\alpha>0}$ and $(\hat{G}_\alpha)_{\alpha>0}$ the corresponding strongly continuous contraction resolvents.

Definition 1. The generalized semi–Dirichlet form $\mathcal{E}$ is called *quasi–regular* if:

(i) There exists an $\mathcal{E}$–nest $(E_k)_{k\geq 1}$ consisting of compact sets.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$–q.c. m–versions.

(iii) There exists $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having $\mathcal{E}$–q.c. m–versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$–exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

Remark 2. Let $\mathcal{E}$ be quasi–regular.
(i) As in [M/R, p. 101, Remark 3.2] we can show that for every $\mathcal{E}$–nest $(\widetilde{E}_k)_{k\geq 1}$ of compact sets there is an $\mathcal{E}$–nest $(E_k)_{k\geq 1}$ of compact metrizable sets E_k with $E_k \subset \widetilde{E}_k$.

(ii) Every $u \in \mathcal{F}$ has an $\mathcal{E}$–q.c. m–version $\tilde{u}$. This is an easy consequence of Proposition III.21 and Definition 1(ii).

(iii) There exist a countable subset $D_0^+ \subset \mathcal{F}$ consisting of bounded 1–excessive functions, such that $D_0^+ - D_0^+$ is a dense subset of $\mathcal{F}$ and an $\mathcal{E}$–exceptional subset $N \subset E$ and $\mathcal{E}$–q.c. m–versions $\tilde{u}$ for $u \in D_0^+$ such that $\widetilde{D}_0^+ = \{\tilde{u} \mid u \in D_0^+\}$ separates the points of $E \setminus N$.

We want to take a look at some examples for quasi–regular generalized semi–Dirichlet forms.

Definition 3. We call a Hilbert space $\mathcal{W} \subset L^2(E,m)$ a *quasi–regular semi–Dirichlet space* if there is a quasi–regular semi–Dirichlet form $(\mathcal{E},\mathcal{W})$ such that the $\widetilde{\mathcal{E}}^{1/2}$–Norm is equivalent to the norm on $\mathcal{W}$.

Proposition 4. *The spaces $H^{1,2}(\mathbb{R},H)\cap\mathcal{V}$ in Example II.11 (iii) and $D\left((\frac{d}{dt})^\alpha,\mathcal{H}\right)$ $\cap\mathcal{V}$ in Example II.11 (iv) are quasi–regular semi–Dirichlet spaces if the space V is a quasi–regular semi–Dirichlet space.*

Due to Remark III.18 we obtain the following

Corollary 5. *The generalized semi–Dirichlet forms of Examples II.11(iii) and (iv) are quasi–regular if V is a quasi–regular semi–Dirichlet space.*

Proof *(of Proposition 4).* We first look at Example II.11 (iii): Let

$$\mathcal{E}'(u,v) := \left(\frac{du}{dt},\frac{dv}{dt}\right)_{\mathcal{H}} + \mathcal{A}(u,v);\ u,v \in D(\mathcal{E}') := H^{1,2}(\mathbb{R},H) \cap \mathcal{V}\ .$$

Clearly $(\mathcal{E}',D(\mathcal{E}'))$ is a semi–Dirichlet form on $\mathcal{H}$. We will show that $(\mathcal{E}',D(\mathcal{E}'))$ is quasi–regular. Let $(E_n)_n$ be a V–nest and $\widetilde{D} :=$ span $\{fu | f \in C_0^1(\mathbb{R}),\ u \in \bigcup_{n\geq 1} V_{E_n}\}$, where $C_0^1(\mathbb{R})$ denotes the space of continuously differentiable functions on $\mathbb{R}$ with compact support. Then $\widetilde{D} \subset \mathcal{V}$ dense, $U_t(\widetilde{D}) \subset \widetilde{D}$ $\forall t \geq 0$ and $\widetilde{D} \subset H^{1,2}(\mathbb{R},V)$. Thus $\widetilde{D} \subset H^{1,2}(\mathbb{R},V)$ dense by [Re/Si II, p. 241, Theorem X.49]. Since $\widetilde{D} \subset \bigcup_{n\geq 1} D(\mathcal{E}')_{K_n}$, where $K_n := [-n,n] \times E_n$, we have that $(K_n)_{n\geq 1}$ is an $\mathcal{E}'$–nest. Hence

(4.1) If $(E_n)_{n\geq 1}$ is a V–nest of compact sets, $(K_n)_{n\geq 1}$ is an $\mathcal{E}'$–nest of compact sets.

(4.2) If $\tilde{u} \in C(\{E_n\})$ is the V–q.c. μ–version of an element $u \in V$, and $f \in C_0^1(\mathbb{R})$, then $f \cdot \tilde{u} \in C(\{K_n\})$ is the $\mathcal{E}'$–q.c. $\lambda \otimes \mu$–version of $f \cdot u \in D(\mathcal{E}')$.

(4.3) If $(\tilde{u}_n) \subset C(\{E_n\})$ separates the points of $\bigcup\limits_{n\geq 1} E_n$ such that for all $x \in \bigcup\limits_{n\geq 1} E_n$ there is some $\tilde{u}_n$ with $\tilde{u}_n(x) \neq 0$ and $(f_n)_{n\geq 1} \subset C_0^1(\mathbb{R})$ separates the points of $\mathbb{R}$ such that for all $s, t \in \mathbb{R}$ there is some f_n with $f_n(s) = f_n(t) \neq 0$, then $(f_n \tilde{u}_m)_{n,m\geq 1}$ separates the points of $\bigcup\limits_{n\geq 1} K_n$.

Clearly, (4.1)–(4.3) imply that $(\mathcal{E}', D(\mathcal{E}'))$ is quasi–regular.

Now, consider Example II.11 (iv): Let

$$\mathcal{E}'(u,v) := \left((\frac{d}{dt})^\alpha u, (\frac{d}{dt})^\alpha v \right)_{\mathcal{H}} + \mathcal{A}(u,v);\ u, v \in D(\mathcal{E}') := D\left((\frac{d}{dt})^\alpha, \mathcal{H} \right) \cap \mathcal{V}\,.$$

Note that

$$D\left((\frac{d}{dt})^\alpha, \mathcal{H} \right) = \left\{ u \in L^2\left(\mathbb{R}^{d+1}, dt \otimes dx\right) \mid \int_{\mathbb{R}^{d+1}} |\hat{u}|^2(t,x)|t|^{2\alpha} d(t,x) < \infty \right\}$$

(cf. [St2, p. 58 f]) and

$$\left((\frac{d}{dt})^\alpha u, (\frac{d}{dt})^\alpha u \right)_{\mathcal{H}} = \int_{\mathbb{R}^{d+1}} |\hat{u}|^2(t,x)|t|^{2\alpha} d(t,x)\ .$$

Let $D_1 :=$ span $\{fu \mid f \in H^\alpha, u \in V\}$, where

$$H^\alpha := \left\{ f \in L^2(\mathbb{R}, dt) \mid \int_{\mathbb{R}} |\hat{f}|^2(t)|t|^{2\alpha} dt < \infty \right\}\ .$$

Denote by $(\tilde{U}_t)_{t\geq 0}$ the contraction semigroup on $L^2(\mathbb{R}, dt)$ corresponding to $\frac{d}{dt}$ in $\mathbb{R}$, and by $(\tilde{U}_t^\alpha)_{t\geq 0}$ the contraction semigroup subordinated to $(\tilde{U}_t)_{t\geq 0}$ by means of $(\eta_t^\alpha)_{t\geq 0}$. Since $\widehat{\tilde{U}_t^\alpha f}(u) = \hat{f}(u)\exp(-t(iu)^\alpha)$ it follows

$$\int_{\mathbb{R}} |\widehat{\tilde{U}_t^\alpha f}|^2(u)|u|^{2\alpha}\, du \leq \int_{\mathbb{R}} |\hat{f}|^2(u)|u|^{2\alpha}\, du$$

where we used $Re(iu)^\alpha \geq 0$.

Therefore $\tilde{U}_t^\alpha(H^\alpha) \subset H^\alpha$ and $U_t^\alpha(D_1) \subset D_1$ since $U_t^\alpha(fu) = (\tilde{U}_t^\alpha f)u$. From [Re/Si II, p. 241, Theorem X. 49] we obtain that $D_1 \subset D\left((\frac{d}{dt})^\alpha, \mathcal{V}\right)$ dense.

Now let $(E_n)_{n\geq 0}$ be a V–nest. Note that

$$\int_{\mathbb{R}^{d+1}} \left|\widehat{fu}\right|^2 (t,x)\, |t|^{2\alpha}\ d(t,x) \leq\ \text{const}\ \|f\|_{H^\alpha}^2 \|u\|_{L^2(\mathbb{R}^d,dx)}^2$$

for $f \in H^\alpha$ and $u \in L^2(\mathbb{R}^d, dx)$. Since $C_0^\infty(\mathbb{R}) \subset H^\alpha$ dense (cf. [Wl, p. 96, Satz I.5.1]) it is now easy to see that $D_2 :=$ span $\{fu | f \in C_0^\infty(\mathbb{R}), u \in \bigcup_{n\geq 1} V_{E_n}\} \subset D\left((\frac{d}{dt})^\alpha, \mathcal{H}\right) \cap \mathcal{V}$ dense.

Consequently $(K_n)_{n\geq 1}$ is an $\mathcal{E}'$–nest, where $K_n := [-n, n] \times E_n$ and the assertion follows as in the case of Example II.11(iii). $\square$

We shall give some examples for time dependent Dirichlet forms on infinite–dimensional state spaces (for the terminology we refer to [M/R, Section II.3]).

Examples 6. (i) Let $\widetilde{E}$ be a separable real Banach space, μ some finite positive measure on $\mathcal{B}(\widetilde{E})$ such that supp $\mu = \widetilde{E}$. Let $K_0 \subset \widetilde{E}$ be some countable set of well–μ–admissible elements with

$$\exists c > 0 \text{ such that } \sum_{k\in K_0} {}_{\widetilde{E}'}\langle \ell, k\rangle_{\widetilde{E}}^2 \leq c^2 \|\ell\|_{\widetilde{E}'}^2 \ \forall \ell \in \widetilde{E}' \ . \tag{4.4}$$

It is well–known that the form

$$D(u,v) := \sum_{k\in K_0} \int_{\widetilde{E}} \frac{\partial u}{\partial k}\frac{\partial v}{\partial k}\, d\mu;\ u, v \in \mathcal{F}C_b^\infty ,$$

where $\mathcal{F}C_b^\infty := \{f(\ell_1, \ldots, \ell_m) | m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), \ell_1, \ldots, \ell_m \in \widetilde{E}'\}$, and $\frac{\partial u}{\partial k}(z) := \frac{d}{ds}u(z+sk)|_{s=0}$, is closable on $L^2(\widetilde{E}, \mu)$ and its closure $(D, \widetilde{V})$ is a Dirichlet form on $L^2(\widetilde{E}, \mu)$. Consider some product measurable function $\varphi : \mathbb{R} \times \widetilde{E} \to \mathbb{R}$ such that

$$\begin{aligned} &\varphi(t_0, \cdot) \in \widetilde{V} \text{ with } \varphi(t_0, \cdot) > 0 \text{ for some } t_0 \in \mathbb{R} \text{ and} \\ &\qquad \varphi(t_0, \cdot)\beta_k \in L^2(\widetilde{E}, \mu)\ \forall k \in K_0 \ , \end{aligned} \tag{4.5}$$

where $\beta_k \in L^2(\widetilde{E}, \mu)$ denotes the logarithmic derivative in direction $k \in K_0$. Assume there is a positive constant $\tilde{c}$ with

$$\tilde{c}^{-1}\varphi(t, \cdot) \leq \varphi(t_0, \cdot) \leq \tilde{c}\,\varphi(t, \cdot)\ \forall t \ . \tag{4.6}$$

It follows that the form

$$\mathcal{E}^{(t)}(u,v) := \sum_{k\in K_0} \int_{\widetilde{E}} \frac{\partial u}{\partial k}\frac{\partial v}{\partial k}\varphi^2(t, \cdot) d\mu;\ u, v \in \mathcal{F}C_b^\infty ,$$

is closable on $H = L^2(\widetilde{E}, \varphi^2(t_0, \cdot)\mu)$ for $t = t_0$ (and hence all t) and its closure $(\mathcal{E}^{(t)}, V)$ is (by (4.4)) a quasi–regular Dirichlet form on H. Hence, the space V is a quasi–regular Dirichlet space, and the corresponding time dependent Dirichlet form is a quasi–regular generalized Dirichlet form.

(ii) Assume next, that we are given a Hilbert space $\widetilde{H}$ such that $(\widetilde{E}, \widetilde{H}, \mu)$ is an abstract Wiener space and K_0 an orthonormal basis of $\widetilde{H}$ consisting of well–μ–admissible elements in $\widetilde{E}$. Let $\varphi : \mathbb{R} \times \widetilde{E} \to \mathbb{R}$ be a product–measurable function satisfying (4.5) and (4.6), where $\widetilde{V}$ is now the domain of the closure of the form $\int_{\widetilde{E}} \langle \nabla u, \nabla v \rangle_{\widetilde{H}} \, d\mu$; $u, v \in \mathcal{F}C_b^\infty$ in $L^2(\widetilde{E}, \mu)$. It is well–known that the form

$$D^{(t)}(u,v) := \int_{\widetilde{E}} \langle \nabla u, \nabla v \rangle_{\widetilde{H}} \varphi^2(t,\cdot) d\mu; \ u, v \in \mathcal{F}C_b^\infty \ ,$$

is closable for $t = t_0$ (and hence every t) and the closure $(D^{(t)}, V)$ a quasi–regular Dirichlet form, since (4.4) is trivially satisfied. Here $\nabla u(z)$ is defined through $\langle \nabla u(z), h \rangle_H = \frac{\partial u}{\partial h}(z)$.

Let $b, d \in L^\infty(\mathbb{R} \times \widetilde{E} \to H, \varphi^2 dt \otimes d\mu)$, $c \in L^\infty(\mathbb{R} \times \widetilde{E}, \varphi^2 dt \otimes d\mu)$ such that

$$\int \left(\langle d(t,\cdot), \nabla u \rangle_{\widetilde{H}} + c(t,\cdot) u \right) \varphi^2(t,\cdot) d\mu, \int \left(\langle b(t,\cdot), \nabla u \rangle_{\widetilde{H}} + c(t,\cdot) u \right) \varphi^2(t,\cdot) d\mu \geq 0$$

for $u \in \mathcal{F}C_b^\infty$, $u \geq 0$.

Then the form

$$\begin{aligned} \mathcal{E}^{(t)}(u,v) = & \int_{\widetilde{E}} \langle \nabla u, \nabla v \rangle_{\widetilde{H}} \varphi^2(t,\cdot) d\mu + \int_{\widetilde{E}} u \cdot \langle d(t,\cdot), \nabla v \rangle_{\widetilde{H}} \varphi^2(t,\cdot) d\mu \\ & + \int_{\widetilde{E}} \langle b(t,\cdot), \nabla u \rangle_{\widetilde{H}} v \varphi^2(t,\cdot) d\mu + \int_{\widetilde{E}} uvc(t,\cdot) \varphi^2(t,\cdot) d\mu \ ; \\ & u, v \in \mathcal{F}C_b^\infty \end{aligned}$$

is closable, since

$$D^{(t)}(u,u) \leq \mathcal{E}^{(t)}(u,u), \ u \in \mathcal{F}C_b^\infty \ , \tag{4.7}$$

and

$$|\mathcal{E}^{(t)}(u,v)| \leq (1 + \|b\|_\infty + \|d\|_\infty + \|c\|_\infty) D_1^{(t)}(u,u)^{1/2} D_1^{(t)}(v,v)^{1/2} \ . \tag{4.8}$$

Clearly, using (4.7) and (4.8), the closure is a quasi–regular Dirichlet form on V. Hence, the corresponding time dependent Dirichlet form is a quasi regular generalized Dirichlet form.

For the statement of the main theorem we recall some definitions (cf. [M/R, Section IV.1]). Adjoin Δ as an isolated point to E and let $E_\Delta = E \cup \{\Delta\}$. As usual we extend every function f on E to E_Δ by setting $f(\Delta) := 0$. Let $\mathbb{M} = (\Omega, \mathcal{F}, (Y_t)_{t \geq 0}, (P_x)_{x \in E_\Delta})$ be a right process with state space E and life time ζ. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the corresponding natural filtration. For an arbitrary probability measure μ on $(E_\Delta, \mathcal{B}(E_\Delta))$ let $P_\mu := \int P_x \, \mu(dx)$ and for a sub–σ–algebra $\mathcal{A} \subset \mathcal{F}$ denote by $\mathcal{A}^{P_\mu}$ its P_μ–completion in $\mathcal{F}$. For a subset $A \in \mathcal{B}(E)$ let $\sigma_A := \inf\{t > 0 \mid Y_t \in A\}$ be the first hitting time.

Definition 7.

(i) $\mathbb{M}$ is called *μ–tight*, if there exists an increasing sequence $(K_n)_{n\geq 1}$ of compact metrizable sets in E such that

$$P_\mu\left[\lim_{n\to\infty}\sigma_{E\setminus K_n}<\zeta\right]=0.$$

(ii) $\mathbb{M}$ is called *μ–special standard*, if

(a) $Y_{t-}:=\lim_{\substack{s\uparrow t\\ s<t}}Y_s$ exists in E for all $t\in(0,\zeta)$ P_μ–a.s.

(b) $\lim_{n\to\infty}Y_{\tau_n}=Y_\tau$ P_μ–a.s. on $\{\tau<\zeta\}$ and Y_τ is $\bigvee_{n\geq 1}\mathcal{F}_{\tau_n}^{P_\mu}$–measurable for every increasing sequence $(\tau_n)_{n\geq 1}$ of $(\mathcal{F}_t)_{t\geq 0}$–stopping times with limit τ.

(iii) $\mathbb{M}$ is called *special standard*, if $\mathbb{M}$ is μ–special standard for all probability measures μ on $(E_\Delta,\mathcal{B}(E_\Delta))$.

Theorem 8. *Let $\mathcal{E}$ be a quasi–regular generalized semi–Dirichlet form. Then there exists an m–tight special standard process $\mathbb{M}=(\Omega,\mathcal{M},(Y_t)_{t\geq 0},(P_z)_{z\in E_\Delta})$ with state space E which is associated with the form $\mathcal{E}$ in the following sense: for all $\alpha>0$ and $f\in\mathcal{B}_b(E)\cap\mathcal{H}$*

$$E_\cdot\left[\int_0^\infty e^{-\alpha t}f(Y_t)dt\right]$$

is an $\mathcal{E}$–q.c. m–version of $G_\alpha f$.

Remark 9. Suppose that the co–form $\widehat{\mathcal{E}}$ is a quasi–regular generalized semi–Dirichlet form. Then there exists also an m–tight special standard process $\hat{\mathbb{M}}=(\widehat{\Omega},\widehat{\mathcal{M}},(\widehat{Y}_t)_{t\geq 0},(\widehat{P}_z)_{z\in E_\Delta})$ such that $\widehat{E}_\cdot\left[\int_0^\infty e^{-\alpha t}f(\widehat{Y}_t)dt\right]$ is an $\mathcal{E}$–q.c. m–version of $\widehat{G}_\alpha f$ for $\alpha>0$ and $f\in\mathcal{B}_b(E)\cap\mathcal{H}$, since the $\mathcal{E}$–exceptional and $\widehat{\mathcal{E}}$–exceptional sets coincide. $\hat{\mathbb{M}}$ is then in duality with $\mathbb{M}$ w.r.t. m.

The rest of this paper is devoted to the proof of Theorem 8. The construction of an associated process will be essentially the same as in [M/R, Section IV.3] (cf. also [A/M/R]). Therefore only the aspects different to the situation in [M/R] will be presented in detail.

Let $(E_k)_{k\geq 1}$ be an $\mathcal{E}$–nest of compact metrizable sets and $Y:=\bigcup_{k\geq 1}E_k$. By [Sch, p. 102, Corollary 2], Y is a Lusin space and by [A/M, p. 397, Theorem 4.2] we have

Proposition 10. *For $\alpha>0$ there exists a kernel $\widetilde{R}_\alpha$ from $(E,\mathcal{B}(E))$ to $(Y,\mathcal{B}(Y))$ with the following properties:*

(i) $\widetilde{R}_\alpha f$ is an $\mathcal{E}$–q.c. m–version of $G_\alpha f$ $\forall f\in\mathcal{H}$.

(ii) $\alpha\widetilde{R}_\alpha(z,Y)\leq 1$ $\forall z\in E$.

The kernel $\widetilde{R}_\alpha$ is unique in the following sense: If K is another kernel from $(E, \mathcal{B}(E))$ to $(Y, \mathcal{B}(Y))$ satisfying (i) and (ii) then $K(z,\cdot) = \widetilde{R}_\alpha(z,\cdot)\,\mathcal{E}$–q.e.

Let $\varphi \in \mathcal{H}$ be such that $0 < \varphi \le 1$ and $h := \widetilde{R}_1\varphi$.
If u is a measurable m–version of an element in $\mathcal{P}$ we can define some $\mathcal{E}$–q.l.s.c. regularization by

$$\tilde{u} := \sup_{n\ge 1} n\widetilde{R}_{n+1}u \ .$$

Assume that u is $\mathcal{E}$–q.c. Obviously $\tilde{u} \le u\,\mathcal{E}$–q.e. In order to get an associated process without branching points it is now necessary to find "enough" 1–excessive functions u with $\tilde{u} = u\,\mathcal{E}$–q.e. Under Assumption D3 (resp. D3') this is possible by the next

Proposition 11. *Let u be an $\mathcal{E}$-q.c. m–version of some element in $\mathcal{P}$ with $\mathcal{L}_u \cap \mathcal{F} \cap \mathcal{P} \neq \emptyset$. Suppose there is a sequence $(u_n)_{n\ge 1}$ of $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{Y}$ such that $u_n \to u$ $\mathcal{E}$–quasi uniformly. Then $u = \tilde{u}$ $\mathcal{E}$–q.e.*

For the proof of Proposition 11 we need two lemmas:

Lemma 12. *Let u, v be $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{Y}$. Then for any sequence $(\alpha_n)_{n\ge 1} \subset \mathbb{Q}^+$ with $\lim\limits_{n\to\infty} \alpha_n = +\infty$ there is a subsequence $(\alpha_{n_\ell})_{\ell\ge 1}$ with $\lim\limits_{\ell\to\infty} \alpha_{n_\ell}\widetilde{R}_{\alpha_{n_\ell}+1} u \wedge v^+ = u \wedge v^+$ $\mathcal{E}$–q.e.*

Proof. The proof is obvious due to Proposition III.7 (iii) and Proposition III.21 (i) if D3 holds, since $u \wedge v^+$ is an $\mathcal{E}$–q.c. m-version of some element in $\mathcal{Y}$. In the case of D3$'$ let $M := \|u\|_\infty \vee \|v\|_\infty$. By the Weierstrass Polynomial Approximation Theorem (cf. [Al, p. 206]) there is a polynomial P_ε on $\mathbb{R}^2$ for $\varepsilon > 0$ with $|x \wedge y^+ - P_\varepsilon(x,y)| < \varepsilon$ for $x, y \in [-M, M]$. We may assume $P_\varepsilon(0) = 0$, hence $f_\varepsilon := P_\varepsilon(u,v)$ is the $\mathcal{E}$–q.c. m-version of some element in $\mathcal{Y}$. Due to Proposition III.21 (i) and the usual diagonal argument we can find a sub–sequence $(\alpha_{n_\ell})_{\ell\ge 1}$ and an $\mathcal{E}$–exceptional set N such that

$$\lim_{\ell\to\infty} \alpha_{n_\ell}\tilde{R}_{\alpha_{n_\ell}+1} f_{1/k}(x) = f_{1/k}(x)\ \forall k \ge 1,\ x \in E \setminus N\ .$$

Hence $\lim\limits_{\ell\to\infty} \alpha_{n_\ell}\widetilde{R}_{\alpha_{n_\ell}+1} u \wedge v^+(x) = u \wedge v^+(x)\ \forall x \in E \setminus N.$ $\square$

Lemma 13. *Let u be some $\mathcal{E}$–q.c. m–version of an element in $\mathcal{F}$ and v be some $\mathcal{E}$–q.c. m–version of an element in $\mathcal{Y}$. Then for any sequence $(\alpha_n)_{n\ge 1} \subset \mathbb{Q}^+$ with $\lim\limits_{n\to\infty} \alpha_n = +\infty$ there is a subsequence $(\alpha_{n_\ell})_{\ell\ge 1}$ with*

$$\lim_{\ell\to\infty} \alpha_{n_\ell}\widetilde{R}_{\alpha_{n_\ell}+1} u \wedge v^+ = u \wedge v^+ \ \mathcal{E}\text{–q.e.}$$

Proof. Due to Remark 2 and Proposition III.21 (i) there exist $\mathcal{E}$–q.c. m–versions u_n of elements in $\mathcal{Y}$ such that $u_n \to u$ in $\mathcal{F}$ and $\mathcal{E}$–quasi uniformly. Hence $v_n :=$

$e_{u-u_n}+e_{u_n-u} \to 0$ in $\mathcal{V}$ by Lemma III.9. Denote by $\tilde{v}_n$ some $\mathcal{E}$–q.l.s.c. regularization of v_n. By Proposition III.21 (ii) there is a subsequence $(n_k)_{k\geq 1} \subset \mathbb{N}$ with

$$\limsup_{k\to\infty} \tilde{v}_{n_k}(x) \leq 0 \ \mathcal{E}\text{–q.e.}$$

Since $\tilde{v}_{n_k} \geq 0$ $\mathcal{E}$–q.e. there exists some $\mathcal{E}$–exceptional set N_1 such that

$$\lim_{k\to\infty} \tilde{v}_{n_k}(x) = 0 \ \forall x \in E \setminus N_1 \,.$$

Note that $|\alpha\widetilde{R}_{\alpha+1}u \wedge v^+ - \alpha\widetilde{R}_{\alpha+1}u_n \wedge v^+| \leq \alpha\widetilde{R}_{\alpha+1}|u-u_n| \leq \tilde{v}_n$ $\mathcal{E}$–q.e. Hence we can find an $\mathcal{E}$–exceptional set $N_2 \supset N_1$ so that

$$|m\widetilde{R}_{m+1}u \wedge v^+(x) - m\widetilde{R}_{m+1}u_n \wedge v^+(x)| \leq \tilde{v}_n(x) \qquad \text{and}$$
$$|u \wedge v^+(x) - u_n \wedge v^+(x)| \leq \tilde{v}_n(x) \ \forall n,m \geq 1,\ x \in E \setminus N_2 \,.$$

By Lemma 12 and the usual diagonal argument we obtain a subsequence $(\alpha_{n_\ell})_{\ell\geq 1}$ and some $\mathcal{E}$–exceptional set $N_3 \supset N_2$ with

$$\lim_{\ell\to\infty} \alpha_{n_\ell}\widetilde{R}_{\alpha_{n_\ell}+1}u_m \wedge v^+(x) = u_m \wedge v^+(x) \ \forall m \geq 1,\ x \in E \setminus N_3 \,.$$

Clearly $\lim_{\ell\to\infty} \alpha_{n_\ell}\widetilde{R}_{\alpha_{n_\ell}+1}u \wedge v^+(x) = u \wedge v^+(x) \ \forall x \in E \setminus N_3$ which implies the assertion. $\square$

Proof*(of Proposition 11).* We only have to show $\sup_{n\geq 1} n\widetilde{R}_{n+1}u \geq u$ $\mathcal{E}$–q.e. By quasi–regularity of $\mathcal{E}$ we can find an $\mathcal{E}$–nest $(F_k'')_{k\geq 1}$ of compact metrizable sets with $\{h\} \cup \{u_n | n \in \mathbb{N}\} \subset C(\{F_k''\})$ and $u_n \to u$ uniformly on F_k'', $k \geq 1$. As in [M/R, p. 82, Proposition 3.6] we can find an $\mathcal{E}$–nest $(F_k')_{k\geq 1}$ with $F_k' \subset F_k''$ and $h \geq 1/k$ on F_k', $k \geq 1$, and subsequently a regular $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ with $F_k \subset F_k'$ as in [M/R, p. 83, Proposition 3.7]. For every k there is some $n_k \in \mathbb{N}$ such that $|u_{n_k} - u| \leq (1/k)h$ on F_k, hence we may assume $|u_k - u| \leq (1/k)h$ on F_k. Let f be an $\mathcal{E}$–q.c. m-version of some element in $\mathcal{L}_u \cap \mathcal{F} \cap \mathcal{P}$. Clearly there is some $\mathcal{E}$–exceptional set N_1 with $\lim_{n\to\infty} f \wedge u_n^+(x) = u(x) \ \forall x \in E \setminus N_1$. Note that $v_k := (1/k)h + f_{F_k^c} \to 0$ in $\mathcal{H}$. By Proposition III.21 (ii) there is a subsequence $(n_k)_{k\geq 1} \subset \mathbb{N}$ and an $\mathcal{E}$–exceptional set $N_2 \supset N_1$ with

$$\lim_{k\to\infty} \tilde{v}_{n_k}(x) = 0 \ \forall x \in E \setminus N_2$$

since $\tilde{v}_k \geq 0$ $\mathcal{E}$–q.e. Note that $|\alpha\widetilde{R}_{\alpha+1}f \wedge u_n^+ - \alpha\widetilde{R}_{\alpha+1}u| \leq \alpha\widetilde{R}_{\alpha+1}|f \wedge u_n^+ - u| \leq \tilde{v}_n$ $\mathcal{E}$–q.e. Hence there is some $\mathcal{E}$–exceptional set $N_3 \supset N_2$ so that

$$|m\widetilde{R}_{m+1}f \wedge u_n^+(x) - m\widetilde{R}_{m+1}u(x)| \leq \tilde{v}_n(x)$$
$$\text{and } |f \wedge u_n^+(x) - u(x)| \leq \tilde{v}_n(x) \ \forall n,m \geq 1,\ x \in E \setminus N_3.$$

By Lemma 13 and the usual diagonal argument we can find a sequence $(n_\ell)_{\ell\geq 1} \subset \mathbb{N}$ with $\lim_{\ell\to\infty} n_\ell = +\infty$ and an $\mathcal{E}$–exceptional set $N_4 \supset N_3$ with

$$\lim_{\ell\to\infty} n_\ell\widetilde{R}_{n_\ell+1}f \wedge u_k^+(x) = f \wedge u_k^+(x) \ \forall k \geq 1, x \in E \setminus N_4.$$

Hence $\lim_{\ell\to\infty} n_\ell\widetilde{R}_{n_\ell+1}u(x) = u(x) \ \forall x \in E \setminus N_4$ and thus

$$\sup_{n\geq 1} n\widetilde{R}_{n+1}u(x) \geq \lim_{\ell\to\infty} n_\ell\widetilde{R}_{n_\ell+1}u(x) = u(x) \ \forall x \in E \setminus N_4$$

which implies the assertion. $\square$

Construction of a Ray resolvent on some compactification of a subset of E

Let $\mathcal{U}_0$ be a countable family of open sets in E containing $E \setminus E_k$, $k \geq 1$, such that $\{U \cap E_k \mid U \in \mathcal{U}_0\}$ forms a base for the relative topology on E_k $\forall k$ and let $\mathcal{U}_1 = \{U = \bigcup_{i=1}^{n} U_i \mid U_i \in \mathcal{U}_0\} \cup \{E\}$. Let $\widetilde{D}_0^+$ be a countable subset of $\mathcal{E}$–q.c. m–versions of bounded, 1–excessive functions as in Remark 2 (iii). Let $\varphi \in \mathcal{H}$ be such that $0 < \varphi \leq 1$ and $h := \tilde{R}_1 \varphi$.

Define J_0 to be the smallest family of $\mathcal{E}$–q.c. functions such that

(4.9) $\widetilde{D}_0^+ \subset J_0, h \in J_0$.

(4.10) $\widetilde{R}_\alpha u_U \in J_0, \tilde{R}_\alpha u \in J_0$ for $u \in J_0, U \in \mathcal{U}_1, \alpha \in \mathbb{Q}^+$.

(4.11) $u \wedge v, u \wedge 1, (u+1) \wedge v \in J_0$ for $u, v \in J_0$.

(4.12) J_0 is a $\mathbb{Q}^+$–cone, i.e. $c_1 u + c_2 v \in J_0$ for $u, v \in J_0, c_1, c_2 \in \mathbb{Q}^+$.

Clearly, J_0 is countable and we have

Proposition 14. *For every $u \in J_0$, $\mathcal{L}_u \cap \mathcal{F} \cap \mathcal{P} \neq \emptyset$ and there is a sequence $(f_n)_{n\geq 1}$ of $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{Y}$ with $f_n \to u$ $\mathcal{E}$–quasi uniformly.*

Proof. The assertion is obvious for any $u \in J_0$ which is the m–version of some element in $\mathcal{F}$. Therefore it remains to show that if $u, v \in J_0$, $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ are $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{Y}$ with $f_n \to u$ and $g_n \to v$ $\mathcal{E}$–quasi uniformly, there is a sequence $(h_n)_{n\geq 1}$ of $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{Y}$ such that $h_n \to u \wedge v$ (resp. $u \wedge 1$, resp. $(u+1) \wedge v$) $\mathcal{E}$–quasi uniformly. This is obvious if $\mathcal{E}$ satisfies D3 since we can take $h_n = g_n \wedge f_n$ (resp. $g_n \wedge 1$, resp. $(g_n + 1) \wedge f_n$). In the case of D3$'$ there are $\mathcal{E}$–q.c. m–versions h_n of elements in $\mathcal{Y}$ such that $|h_n - g_n \wedge f_n| \leq 1/n$ (resp. $|h_n - f_n \wedge 1| \leq 1/n$, resp. $|h_n - (f_n+1) \wedge g_n| \leq 1/n$) $\mathcal{E}$–q.e. (cf. the proof of Lemma 12). Since $f_n \wedge g_n \to u \wedge v$ (resp. $f_n \wedge 1 \to u \wedge 1$, resp. $(f_n+1) \wedge g_n \to (u+1) \wedge v$) $\mathcal{E}$–quasi uniformly, the assertion follows. $\square$

Proposition 15. *There is a regular $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ of compact metrizable sets $F_k \subset E_k$ such that*

(i) $J_0 \subset C(\{F_k\})$.

(ii) $\inf\limits_{x \in F_k} h(x) > 0$ $\forall k$.

(iii) $\lim\limits_{\substack{\alpha \to \infty \\ \alpha \in \mathbb{Q}^+}} \alpha \tilde{R}_{\alpha+1} u(x) = u(x)$ $\forall u \in J_0, x \in Y_1 := \bigcup\limits_{k \geq 1} F_k$.

(iv) J_0 separates the points of Y_1.

Proof. J_0 separates the points up to some $\mathcal{E}$–exceptional set N_1. Since J_0 is countable there is some $\mathcal{E}$–exceptional set N_2 such that $\lim_{\substack{\alpha\to\infty \\ \alpha\in\mathbb{Q}^+}} \alpha\tilde{R}_{\alpha+1}u(x) = u(x)\ \forall x \in E \setminus N_2$ by Proposition 11 and Proposition 14. Hence there is an $\mathcal{E}$–nest $(F''_k)_{k\geq 1}$ of compact sets with $\bigcup_{k\geq 1} F''_k \subset E \setminus N_2$ and $F''_k \subset E_k$, $k \in \mathbb{N}$ and $J_0 \subset C(\{F''_k\})$. As in [M/R, p. 82, Proposition 3.6] we can find an $\mathcal{E}$–nest $(F'_k)_{k\geq 1}$ with $F'_k \subset F''_k$ and $h \geq 1/k$ on F'_k $\forall k$ and subsequently a regular $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ with $F_k \subset F'_k$ as in [M/R, p. 83, Proposition 3.7]. $\square$

In the same way as in [M/R, p. 107, Lemma 3.11] we can find some set $Y_2 \in \mathcal{B}(E), Y_2 \subset Y_1$ such that $E \setminus Y_2$ is $\mathcal{E}$–exceptional and $\widetilde{R}_\alpha(x, Y\setminus Y_2) = 0\ \forall x \in Y_2, \alpha \in \mathbb{Q}^+$.

Adjoin Δ as an isolated point to Y_2 and let $Y_\Delta := Y_2 \cup \{\Delta\}$. Each function f on Y_2 will be regarded as a function on Y_Δ by setting $f(\Delta) = 0$.

Define for $\alpha \in \mathbb{Q}^+$ and $A \in \mathcal{B}(Y_\Delta)$

$$R_\alpha(x, A) := \begin{cases} \widetilde{R}_\alpha(x, A\cap Y_2) + ((1/\alpha) - \widetilde{R}_\alpha(x, Y_2))\delta_\Delta(A) & \text{if } x \in Y_2 \\ (1/\alpha)\delta_\Delta(A) & \text{if } x = \Delta\ . \end{cases}$$

Then $(R_\alpha)_{\alpha\in\mathbb{Q}^+}$ is a Markovian resolvent of kernels on $(Y_\Delta, \mathcal{B}(Y_\Delta))$. Let $J := \{u + c\cdot 1\!\!1_{Y_\Delta} \mid u \in J_0, c \in \mathbb{Q}^+\}$. Then J is a $\mathbb{Q}^+$–cone, inf–stable and $R_\alpha u \in J$ for $u \in J$. J separates the points of Y_Δ, each function $u \in J$ is 1–supermedian w.r.t. $(R_\alpha)_{\alpha\in\mathbb{Q}^+}$ and $\lim_{\substack{\alpha\to\infty \\ \alpha\in\mathbb{Q}^+}} \alpha R_{\alpha+1}u(x) = u(x)\ \forall u \in J, x \in Y_\Delta$.

Let $J = \{u_n \mid n \in \mathbb{N}\}$ and define the metric

$$\rho(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|u_n(x) - u_n(y)|}{1 + |u_n(x) - u_n(y)|} \text{ on } Y_\Delta\ .$$

Denote with the prefix "ρ–" concepts which are meant relative to the topology induced by the metric ρ and with $\overline{Y}$ the completion of Y_Δ w.r.t. ρ. $\overline{Y}$ is a compact metric space and for every $u \in J$ there is a unique ρ–continuous extension $\overline{u}$. Let $\overline{J}$ be the set of all ρ-continuous extensions $\overline{u}$ of functions $u \in J$. Note that $\overline{J} - \overline{J}$ is dense in $C(\overline{Y})$ by the Stone–Weierstrass–Theorem since $\overline{J}$ separates the points of $\overline{Y}$ and $\overline{J} - \overline{J}$ is inf–stable.

For $\alpha \in \mathbb{Q}^+$ and $\overline{u} \in \overline{J}$ define $\overline{R}_\alpha\overline{u} := \overline{R_\alpha u}$ and extend $\overline{R}_\alpha$ on $\overline{J} - \overline{J}$ by linearity. Since $\|\overline{R}_\alpha\overline{u}\|_\infty = \|R_\alpha u\|_\infty \leq 1/\alpha\|u\|_\infty = 1/\alpha\|\overline{u}\|_\infty$, we can extend $\overline{R}_\alpha$ to a linear bounded positive operator on $C(\overline{Y})$. The family of operators $(\overline{R}_\alpha)_{\alpha\in\mathbb{Q}^+}$ satisfies the resolvent equation, so $\alpha \mapsto \overline{R}_\alpha f$ is locally uniformly continuous on $\mathbb{Q}^+$ and consequently we can define $\overline{R}_\alpha f := \lim_{\substack{\beta\to\alpha \\ \beta\in\mathbb{Q}^+}} \overline{R}_\beta f$ for arbitrary $\alpha > 0$. Denote by $(\overline{R}_\alpha)_{\alpha>0}$ the corresponding Markovian resolvent of kernels. Clearly $(\overline{R}_\alpha)_{\alpha>0}$ is a Ray resolvent.

Let $D := \{x \in \overline{Y} \mid \lim_{\alpha\to\infty} \alpha\overline{R}_\alpha f(x) = f(x),\ \forall f \in C(\overline{Y})\}$ be the set of non–branching points. By [G, Theorem 3.6] there exists a unique semigroup $(\overline{p}_t)_{t\geq 0}$ of Markovian kernels on $(\overline{Y}, \mathcal{B}(\overline{Y}))$ such that

$$\overline{R}_\alpha f = \int_0^\infty e^{-\alpha t}\overline{p}_t f\, dt\ \forall \alpha > 0,\ f \in C(\overline{Y})\ . \tag{4.13}$$

By [Sh, Theorem 9.13] there exists a right process $\overline{\mathbb{M}} = (\overline{\Omega}, \overline{\mathcal{F}}, (\overline{Y}_t)_{t\geq 0}, (\overline{P}_x)_{x\in D})$ with state space D and transition semigroup $(\overline{p}_t)_{t\geq 0}$ such that for all $\omega \in \overline{\Omega}, t \mapsto \overline{Y}_t(\omega)$ has left limits in $\overline{Y}$ for $t \in (0,\infty)$. Denote by $(\overline{\mathcal{F}}_t)_{t\geq 0}$ the natural filtration corresponding to the right process $\overline{\mathbb{M}}$ and by $(\overline{\theta}_t)_{t\geq 0}$ the corresponding familiy of shift operators. Since $\lim_{\alpha\to\infty} \alpha\overline{R}_\alpha f(x) = f(x)$ for all $f \in \overline{J} - \overline{J}$ and $x \in Y_\Delta$ it follows that $Y_\Delta \subset D$.

For a Borel subset $A \subset \overline{Y}$ let $\sigma_A := \inf\{t > 0 \mid \overline{Y}_t \in A\}$ be the first hitting time.

Proposition 16. *Let $F \subset Y_2$ be compact, $F \subset F_k$ for some $k, A := \overline{Y} \setminus F$ and $u \in J_0$. Then there exist an $\mathcal{E}$–exceptional set $N \subset E$ and an $\mathcal{E}$–q.l.s.c. regularization $\tilde{u}_{E\setminus F}$ of $u_{E\setminus F}$ such that*

$$\overline{E}_x\left[e^{-\sigma_A}\overline{u}(\overline{Y}_{\sigma_A})\right] \leq \tilde{u}_{E\setminus F}(x)\ \forall x \in Y_\Delta \setminus N\ .$$

Proof. Since $F \subset F_k$ for some k there is an increasing sequence $(U_n)_{n\geq 1}$ of open sets in $\mathcal{U}_1$ such that $\bigcup_{n\geq 1} U_n = E \setminus F =: U$. For $f = \sup_{\alpha\in\mathbb{Q}^+}\sup_{n\geq 1}\alpha\overline{R_{\alpha+1}u_{U_n}}$ we have that $\lim_{\substack{\alpha\to\infty\\ \alpha\in\mathbb{Q}^+}} \alpha\overline{R}_{\alpha+1}f(x) = f(x),\ \forall x \in \overline{Y}$. Note that

$$\overline{E}_x\left[\int_0^\infty e^{-\alpha t} g(\overline{Y}_t)\, dt\right] = R_\alpha(g_{|Y_\Delta})(x)\ \forall g \in \overline{J}, x \in Y_\Delta. \tag{4.14}$$

Since $\sigma(\overline{J}) = \mathcal{B}(\overline{Y})$, a monotone class argument implies that (4.14) holds for all bounded $\mathcal{B}(\overline{Y})$–measurable functions g. Consequently for $g := (\overline{u} - f)\mathbb{1}_A$ it follows that $\overline{R}_\alpha g(x) = R_\alpha(g_{|Y_\Delta})(x) = \widetilde{R}_\alpha(g_{|Y_2})(x)$ for $x \in Y_2$ since $g(\Delta) = 0$ and therefore $\overline{R}_\alpha g_{|Y_2} = 0$ $\mathcal{E}$–q.e. since $g_{|Y_2} = 0$ m–a.e. Clearly, $\overline{R}_\alpha g(\Delta) = R_\alpha(g_{|Y_\Delta})(\Delta) = 0$. Hence there is an $\mathcal{E}$–exceptional set N with $\overline{R}_\alpha g(x) = 0\, \forall \alpha \in \mathbb{Q}^+, x \in Y_\Delta \setminus N$.

Let $\mathbb{Q}^+ = \{t_n \mid n \in \mathbb{N}\}$ and $\sigma_n := \min\{t_i \mid i \leq n, \overline{Y}_{t_i} \in A\}$ (with the convention $\min\emptyset = +\infty$). Then

$$\overline{E}_x \int_0^\infty e^{-\alpha t}\overline{E}_{\overline{Y}_t}\left[e^{-\sigma_n} g(\overline{Y}_{\sigma_n})\right] dt = 0\ \forall x \in Y_\Delta \setminus N\ . \tag{4.15}$$

Since $\left(e^{-t}\alpha\overline{R_{\alpha+1}u_{U_n}}(\overline{Y}_t), \overline{\mathcal{F}}_t, \overline{P}_x\right)_{t\in[0,\infty]}$ (where we set $e^{-\infty}\alpha\overline{R_{\alpha+1}u_{U_n}}(\overline{Y}_\infty) = 0$) is a right continuous supermartingale for $x \in D$ we obtain by the optional sampling theorem

$$\overline{E}_x\left[e^{-\sigma_n}\alpha\overline{R_{\alpha+1}u_{U_m}}(\overline{Y}_{\sigma_n})\right] \leq \alpha\overline{R_{\alpha+1}u_{U_m}}(x)\ \forall x \in D\ .$$

It follows from the monotone convergence theorem

$$\overline{E}_x\left[e^{-\sigma_n} f(\overline{Y}_{\sigma_n})\right] \leq f(x)\ \forall x \in D\ . \tag{4.16}$$

Combining (4.15) and (4.16) we get

$$\overline{E}_x \int_0^\infty e^{-\alpha t}\overline{E}_{\overline{Y}_t}\left[e^{-\sigma_n}\overline{u}(\overline{Y}_{\sigma_n})\right] dt \leq \overline{R}_\alpha f(x)\ \forall x \in Y_\Delta \setminus N\ .$$

Clearly $\sigma_n \to \sigma_A$ for $n \to \infty$ and thus

$$\overline{E}_x \int_0^\infty e^{-\alpha t}\overline{E}_{\overline{Y}_t}\left[e^{-\sigma_A}\overline{u}(\overline{Y}_{\sigma_A})\right] dt \leq \overline{R}_\alpha f(x)\ \forall x \in Y_\Delta \setminus N\ .$$

By the right–continuity of $\overline{Y}_t$ it follows $\overline{Y}_{t+\sigma_A\circ\overline{\theta}_t} \to \overline{Y}_{\sigma_A}$ for $t \downarrow 0$ and thus $\overline{E}_x\left[e^{-\sigma_A\circ\overline{\theta}_t}\overline{u}(\overline{Y}_{t+\sigma_A\circ\overline{\theta}_t})\right] \to \overline{E}_x\left[e^{-\sigma_A}\overline{u}(\overline{Y}_{\sigma_A})\right]$ by Lebesgues theorem. Hence, for $x \in Y_\Delta \setminus N$,

$$\begin{aligned}
\overline{E}_x\left[e^{-\sigma_A}\overline{u}(\overline{Y}_{\sigma_A})\right] &= \lim_{\substack{\alpha\to\infty \\ \alpha\in\mathbb{Q}^+}} \int_0^\infty \alpha e^{-(\alpha+1)t}\overline{E}_x\left[e^{-\sigma_A\circ\overline{\theta}_t}\overline{u}(\overline{Y}_{t+\sigma_A\circ\overline{\theta}_t})\right] dt \\
&= \lim_{\substack{\alpha\to\infty \\ \alpha\in\mathbb{Q}^+}} \alpha\overline{E}_x \int_0^\infty e^{-(\alpha+1)t}\overline{E}_{\overline{Y}_t}\left[e^{-\sigma_A}\overline{u}(\overline{Y}_{\sigma_A})\right] dt \\
&\leq \lim_{\substack{\alpha\to\infty \\ \alpha\in\mathbb{Q}^+}} \alpha\overline{R}_{\alpha+1} f(x) = f(x)\ .
\end{aligned}$$

Since $f_{|Y_2}$ is an $\mathcal{E}$–q.l.s.c. regularization of u_U the assertion follows. $\square$

Let $\zeta := \sigma_{\overline{Y}\setminus Y_2}$. ζ is an $(\overline{\mathcal{F}}_t)_{t\geq 0}$-stopping time, since $Y_2 \in \mathcal{B}(\overline{Y})$ (cf. [M/R, p. 108, Lemma 3.13]). Note that the set

$$\Omega_0 := \{\omega \in \Omega \mid \inf_{s\leq t} e^{-s}\overline{u}(\overline{Y}_s(\omega)) > 0 \text{ or } e^{-t}\overline{u}(\overline{Y}_t)(\omega)) = 0\,\forall t \in \mathbb{Q}^+, \overline{u} \in \overline{J}\}$$

is $\overline{\mathcal{F}}$–measurable, shift–invariant, and satisfies $\overline{P}_x[\Omega_0] = 1$ for all $x \in D$ since $(e^{-t}\overline{u}(\overline{Y}_t), \overline{\mathcal{F}}_t, \overline{P}_x)_{t\in[0,\infty]}$ is a right–continuous non negative supermartingale for $x \in D$ (where we set $e^{-\infty}\overline{u}(\overline{Y}_\infty) := 0$).

Let $(K_n)_{n\geq 1}$ be an $\mathcal{E}$–nest of compact metrizable sets such that $K_n \subset F_n \cap Y_2$ and $\sigma := \sup_{n\geq 1} \sigma_{\overline{Y}\setminus K_n}$. Due to Proposition 16 and Proposition III.21 (ii) we can find an $\mathcal{E}$–exceptional set $N \in \mathcal{B}(Y_2)$ and

$$\inf_{n\geq 1} \overline{E}_x\left[e^{-\sigma_{\overline{Y}\setminus K_n}}\overline{u}(\overline{Y}_{\sigma_{\overline{Y}\setminus K_n}})\right] \leq \inf_{n\geq 1} \tilde{u}_{E\setminus K_n}(x) = 0\ \forall u \in J_0, x \in Y_\Delta \setminus N.$$

Therefore the set

$$\Omega_1 := \{\omega \in \Omega_0 \mid e^{-\sigma(\omega)}\overline{u}(\rho - \lim_{n\to\infty} \overline{Y}_{\sigma_{\overline{Y}\setminus K_n}}(\omega)) = 0\ \forall u \in J_0\}$$

is $\overline{\mathcal{F}}$–measurable, shift–invariant, and satisfies $\overline{P}_x[\Omega_1] = 1$ for all $x \in Y_\Delta \setminus N$.

Clearly $e^{-t}\overline{u}(\overline{Y}_t(\omega)) = 0$ for all $t \geq \sigma(\omega)$ and $\omega \in \Omega_1$. In particular $e^{-t}\overline{h}(\overline{Y}_t(\omega)) = 0$ for all $t \geq \sigma(\omega)$ and $\omega \in \Omega_1$ which implies that $\overline{Y}_t(\omega) \in \overline{Y} \setminus Y_2$, hence $\sigma(\omega) \geq \zeta(\omega)$. On the other hand clearly $\sigma_{\overline{Y}\setminus K_n}(\omega) \leq \zeta(\omega)$ and therefore $\zeta = \sigma$ on Ω_1. This implies

$$(4.17)\quad R_\alpha u(x) = \overline{R}_\alpha \overline{u}(x) = \overline{E}_x\left[\int_0^\zeta e^{-\alpha t}\overline{u}(\overline{Y}_t)\,dt\right] \;\forall x \in Y_\Delta \setminus N\; \alpha \in \mathbb{Q}^+,\, u \in J_0\,.$$

Since K_n is compact in Y_2, the trace topologies on K_n induced by $\overline{Y}$ and Y_2 coincide. Hence $\overline{Y}_t(\omega)$ is right–continuous on $[0, \zeta(\omega))$ and has left–limits on $(0, \zeta(\omega))$ in $\bigcup\limits_{n\geq 1} K_n$ w.r.t. the original topology for $\omega \in \Omega_1$.

Let $Z_1 := Y_2 \setminus N$. Repeating the argument above we find a decreasing sequence $(Z_n)_{n\geq 1} \subset \mathcal{B}(Y_2)$ and $(\Omega_n)_{n\geq 1} \subset \overline{\mathcal{F}}$ such that $E \setminus Z_n$ is $\mathcal{E}$–exceptional, $\overline{P}_x[\Omega_n] = 1\,\forall x \in Z_n \cup \{\Delta\}$, Ω_n is shift–invariant, and $\overline{Y}_t(\omega) \in Z_{n-1}$ for $t \in [0, \zeta(\omega))$ and $\omega \in \Omega_n$, and $\overline{Y}_{t-}(\omega) \in Z_{n-1}$ for $t \in (0, \zeta(\omega))$ and $\omega \in \Omega_n$. Hence for $S := \bigcap\limits_{n\geq 1} Z_n \in \mathcal{B}(Y_2)$, $S_\Delta := S \cup \{\Delta\}$, and $\Omega := \bigcap\limits_{n\geq 1} \Omega_n \in \overline{\mathcal{F}}$ it follows that

(4.18) $E \setminus S$ is $\mathcal{E}$–exceptional.

(4.19) $\overline{P}_x[\Omega] = 1\ \forall x \in S_\Delta := S \cup \{\Delta\}$.

(4.20) Ω is shift–invariant.

(4.21) $\overline{Y}_t(\omega)$ is right continuous and has left limits in S for $t \in]0, \zeta(\omega)[$ w.r.t. the topology on Y_2 induced by the original topology on E.

(4.22) $R_\alpha u(x) = \overline{E}_x \int_0^\zeta e^{-\alpha t}\overline{u}(\overline{Y}_t)dt\ \forall \alpha \in \mathbb{Q}^+, u \in J_0, x \in S_\Delta$.

(4.23) $\lim\limits_{n\to\infty} \sigma_{\overline{Y}\setminus K_n}(\omega) = \zeta(\omega)\ \forall \omega \in \Omega$.

(4.24) $\zeta(\omega) > 0$ for $\omega \in \{\overline{Y}_0 = x\} \cap \Omega$ and $x \in S$ and $\zeta(\omega) = 0$ for $\omega \in \{\overline{Y}_0 = \Delta\} \cap \Omega$.

The next lemma can be proved as in [M/R, p.114, Lemma 3.21].

Lemma 17. *Let $(\tau_n)_{n\geq 1}$ be an increasing sequence of $(\overline{\mathcal{F}}_t)_{t\geq 0}$–stopping times and $\tau := \sup_{n\geq 1}\tau_n$. Set $V(\omega) := \rho - \lim_{n\to\infty}\overline{Y}_{\tau_n}(\omega)$ if $\tau(\omega) < \infty$. Then*

$$\overline{P}_x\left[V \neq \overline{Y}_\tau,\, V \in Y_2,\, \tau < \infty\right] = 0\,\forall x \in S_\Delta.$$

Define $\mathbb{M}_S = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (P_x)_{x\in S_\Delta})$, where $\mathcal{F} := \overline{\mathcal{F}} \cap \Omega, \mathcal{F}_t := \overline{\mathcal{F}}_t \cap \Omega, P_x := \overline{P}_{x|\mathcal{F}}$ for $x \in S_\Delta$ and

$$Y_t(\omega) := \begin{cases} \overline{Y}_t(\omega) & \text{if } t \in [0, \zeta(\omega)[\\ \Delta & \text{if } t \geq \zeta(\omega)\ . \end{cases}$$

Lemma 18.

(i) $\mathbb{M}_S$ *is a special standard process with state space* S *and life time* ζ*.*

(ii) $E_x\left[\int_0^\infty e^{-\alpha t} f(Y_t)dt\right] = \widetilde{R}_\alpha f(x)\ \forall f \in \mathcal{B}_b(Y_2), x \in S, \alpha > 0.$

Proof. (i) It is easy to see that $\mathbb{M}_S$ is a right process with state space S and life time ζ, such that $Y_t(\omega)$ has left limits in S for $t \in (0, \zeta(\omega))\ \forall \omega \in \Omega$. Note that $\lim_{n\to\infty} Y_{\tau_n} = \rho - \lim_{n\to\infty} Y_{\tau_n} = Y_\tau$ on $\{\tau < \zeta\}$ P_x–a.s. for every $x \in S_\Delta$ by Lemma 17 for any increasing sequence $(\tau_n)_{n\geq 1}$ of $(\mathcal{F}_t)_{t\geq 0}$–stopping times with limit τ. As a consequence we obtain that $\mathbb{M}_S$ is quasi–left continuous up to ζ. Let

$$V(\omega) := \begin{cases} \rho - \lim_{n\to\infty} Y_{\tau_n}(\omega) & \text{if } \omega \in \{\rho - \lim_{n\to\infty} Y_{\tau_n} \in S,\ \tau < \infty\} \\ \Delta & \text{if } \omega \in \{\rho - \lim_{n\to\infty} Y_{\tau_n} \notin S\} \cup \{\tau = \infty\}. \end{cases}$$

Since $\mathbb{M}_S$ is quasi–left continuous up to ζ we conclude

$$\begin{aligned} P_x\left[V \neq Y_\tau\right] &= P_x\left[V \neq Y_\tau,\ \tau = \zeta < \infty\right] \\ &= P_x\left[\rho - \lim_{n\to\infty} Y_{\tau_n} \neq Y_\tau,\ \rho - \lim_{n\to\infty} Y_{\tau_n} \in S,\ \tau = \zeta < \infty\right] \\ &\leq P_x\left[\rho - \lim_{n\to\infty} Y_{\tau_n} \neq Y_\tau,\ \rho - \lim_{n\to\infty} Y_{\tau_n} \in S,\ \tau < \infty\right] = 0 \end{aligned}$$

by Lemma 17 again. Therefore Y_τ is $\bigvee_{n\geq 1} \mathcal{F}_{\tau_n}^{P_x}$–measurable since V is $\bigvee_{n\geq 1} \mathcal{F}_{\tau_n}$–measurable. It is easy to see that this implies that $\mathbb{M}_S$ is special.

(ii) Note that for $\alpha \in \mathbb{Q}^+$, $f \in J_0$, $x \in S_\Delta$ we have by (4.22) that

$$E_x\left[\int_0^\infty e^{-\alpha t} f(Y_t)\, dt\right] = R_\alpha f(x) = \tilde{R}_\alpha f(x). \tag{4.25}$$

This equality holds in particular for $f = \sup_{n\geq 1}(nh \wedge 1) = \mathbb{1}_{Y_2}$. By Lebesgues Theorem (4.25) also holds for all $\alpha > 0$. Since Y_2 is the Borel subspace of a Lusin space and J_0 separates the points of Y_2 it follows $\sigma(J_0) = \mathcal{B}(Y_2)$ by [Sch, p.108, Lemma 18]. Hence monotone class arguments imply that (4.25) is true for all $f \in \mathcal{B}_b(Y_2)$ and $\alpha > 0$. $\square$

Denote by $\mathbb{M}$ the trivial extension of $\mathbb{M}_S$ to E. Then $\mathbb{M}$ is special standard again and m–tight by (4.23). This completes the proof of Theorem 8.

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