Bounds for Determinants of Perturbed M-Matrices

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Abstract

A sharp upper bound for the relative error of the determinant of an M-matrix in terms of the relative errors of the matrix entries is given. It improves and completes a bound derived by Jungong and Jiang in [2].

1 Introduction, formulation of the result

Let A be an $n \times n$ M-matrix,

$$A = D - N = (a_{ij}), \tag{1}$$

where $D = diag(a_{ii}), N \geq 0$ with zero diagonal. Then

$$\gamma = \rho(D^{-1}N) < 1. \tag{2}$$

Here ρ denotes the Perron root of the nonnegative matrix $D^{-1}N$. We consider perturbations δA of A with small relative errors $\epsilon > 0$, i.e.

$$|\delta A| \leq \epsilon |A| = \epsilon(D + N). \tag{3}$$

Here we have used the notation $|B|$ for the matrix with the entries $|b_{ij}|$ if $B = (b_{ij})$. In the paper [2] the authors derive bounds for the relative error of the inverse of A in terms of γ and ϵ. They describe a function $c(\gamma, \epsilon)$ satisfying

$$|A^{-1} - (A + \delta A)^{-1}| \leq (c(\gamma, \epsilon) + O(\epsilon^2))A^{-1}. \tag{4}$$

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In the course of the proof a bound for the relative error of the determinant is needed. It is shown there that

$$\frac{\det(A+\epsilon|A|)}{\det(A)} \leq 1+\frac{n\epsilon}{1-\gamma}+O(\epsilon^2). \tag{5}$$

In this note we improve this result and give *sharp* bounds, also depending on γ and ϵ only.

For stating our result in a short way we introduce the function

$$b(\gamma,t) = 1+2t\frac{1+\gamma^2}{1-\gamma^2}+t^2 \tag{6}$$

and observe that this function is positive for $t > -\frac{1-\gamma}{1+\gamma}$.
The main result of this note is the following

Theorem 1 *Let $A = D-N$, where $D = diag(a_{ii})$, be an $n \times n$ M-matrix, $\gamma = \rho(D^{-1}N) < 1$, $\epsilon > 0$ and $|\delta A| \leq \epsilon|A|$. If $\epsilon < \frac{1-\gamma}{1+\gamma}$ then for n even*

$$b(\gamma,-\epsilon)^{n/2} \leq \frac{\det(A+\delta A)}{\det(A)} \leq b(\gamma,\epsilon)^{n/2} \tag{7}$$

while for n odd

$$(1-\epsilon)b(\gamma,-\epsilon)^{(n-1)/2} \leq \frac{\det(A+\delta A)}{\det(A)} \leq (1+\epsilon)b(\gamma,\epsilon)^{(n-1)/2} \tag{8}$$

The bounds are sharp.

2 Proof

The proof of Theorem 1 will be an easy consequence of the following two results which, though serving mainly as lemmata for Theorem 1, are interesting in themselves and hence are also addressed as Theorems.

Theorem 2 *Under the assumptions of Theorem 1 let $F = \epsilon|A|$. If $\epsilon < 1$ then for n even*

$$\frac{\det(A+F)}{\det(A)} \leq b(\gamma,\epsilon)^{n/2} \tag{9}$$

while for n odd

$$\frac{\det(A+F)}{\det(A)} \leq (1+\epsilon)b(\gamma,\epsilon)^{(n-1)/2} \tag{10}$$

holds.
This bound is sharp for all n.

Lower bounds are provided by

Theorem 3 *Under the assumptions of Theorem 1 let $F = \epsilon|A|$. If $\epsilon < \frac{1-\gamma}{1+\gamma}$ then for n even*

$$\frac{\det(A-F)}{\det(A)} \geq b(\gamma, -\epsilon)^{n/2} \tag{11}$$

and for n odd

$$\frac{\det(A-F)}{\det(A)} \geq (1-\epsilon) b(\gamma, -\epsilon)^{(n-1)/2}. \tag{12}$$

The bound is sharp for all n.

We start with the proof of Theorem 2. As in [2] we get after introducing the function

$$f(x_1, \ldots, x_n) = \Pi_{j=1}^{n}(1 + \epsilon(1 + x_j)(1 - x_j)^{-1}) \tag{13}$$

and simple manipulations the following relation

$$\frac{\det(A+F)}{\det(A)} = f(\lambda_1, \ldots, \lambda_n), \tag{14}$$

where $\lambda_j, j = 1, \ldots, n$ are the eigenvalues of $D^{-1}N$. The vector of eigenvalues belongs obviously to the set $\underset{\sim}{M} \in C^n$ of vectors $x = (x_1, \ldots, x_n)$ satisfying the following properties: $\Sigma_{i=1}^{n} x_i = 0, |x_i| \leq \gamma, i = 1, \ldots, n$ and with any x_i also $\overline{x_i}$ is an entry of the vector.
We claim now that the maximum of f over the set $\mathcal{M}$ is given by the right hand side of (9) and (10) respectively.
We introduce the function

$$\phi(s) = 1 + \epsilon \frac{1+s}{1-s} = (1-\epsilon)(1 + \frac{\delta}{1-s}), \tag{15}$$

where $\delta = 2\epsilon(1-\epsilon)^{-1} > 0$. We can easily find that for $|s| < 1$ we have

$$\phi(s)\phi(\bar{s}) \leq (\phi(\Re s))^2.$$

This implies that there is a real vector $\mu = (\mu_j)$ in $\mathcal{M}$ maximizing the function f over $\mathcal{M}$.
We observe that the logarithmic derivative of ϕ, namely

$$\frac{\phi'(s)}{\phi(s)} = \frac{\delta}{(1-s)(\delta + 1 - s)}$$

is monotone increasing for real $s < 1$. This implies that for real $a \leq b$ and $t > 0$ the inequality

$$\log(\phi(a-t)\phi(b+t)) > \log(\phi(a)\phi(b))$$

and hence

$$\phi(a-t)\phi(b+t) > \phi(a)\phi(b)$$

holds. From this we get that at most one entry of the maximizing vector μ is in $(-\gamma, \gamma)$. The only possibility is that for even $n = 2k$ we have k entries γ and k entries $-\gamma$, while for $n = 2k+1$ odd we have one entry 0 and k entries γ and $-\gamma$ resp. Inserting this in (14) we get (9) and (10).
Let

$$C = \begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}.$$

The M-matrix

$$A = I_n - diag(C, \ldots, C, 0) \tag{16}$$

satisfies (10) with equality for odd n, while

$$A = I_n - diag(C, \ldots, C) \tag{17}$$

is an even dimensional A satisfying (9). This finishes the proof of Theorem 2.

The most easy **proof of Theorem 3** can be given by applying Theorem 2 to the matrix $A - F$ instead of A. By the assumption on ϵ we have that $A - F$ is an M-matrix. Observe that then $A + F$ has to be replaced by $(1 - \epsilon^2)A$. After some manipulations we get the inequalities of Theorem 3. One could as well follow the steps of the proof of Theorem 2 accordingly. Equality holds for the same matrices (16) and (17).

Proof of Theorem 1 We recall the fact that for M-matrices A, B the relation $A \leq B$ implies $\det(A) \leq \det(B)$, see e.g. [1], p.117. Then using the obvious inequalities

$$A - F \leq A + \delta A \leq A + F$$

and Theorems 2 and 3, we get Theorem 1.

3 Final remarks

For completeness we remark that Theorem 1 leads immediately to a bound for the relative error of the determinant, namely

$$|\frac{\det(A+\delta A)-\det(A)}{\det(A)}| \leq b(\gamma,\epsilon)^{n/2}-1 = \frac{1+\gamma^2}{1-\gamma^2}n\epsilon + O(\epsilon^2).$$

It should be noted that here the convexity of the function $\epsilon \to b(\gamma,\epsilon)^{n/2}$ in the interval $(-1,1)$ is used.

In [2] the authors prove also bounds for the adjoint matrix,

$$|\operatorname{adj}(A+F)| \geq (1-\epsilon)^{n-1}|\operatorname{adj}(A)|$$

and

$$|\operatorname{adj}(A-F)| \leq (1+\epsilon)^{n-1}|\operatorname{adj}(A)|.$$

If we combine this with our results on the determinants and use the well-known relation $A^{-1} = (\det(A))^{-1}\operatorname{adj}(A)$, we get an improvement of the relative bound (4) for the inverse of an M-matrix. We refrain from elaborating this point, as it is described already in [2].

References

[1] R. Horn and C. Johnson **Topics in Matrix Analysis** *Cambridge University Press 1991*

[2] Xue Jungong and Erxiong Jiang, **Entrywise relative Perturbation Theory for nonsingular M-Matrice and Applications** *BIT-Nordisk Tidskrift for Informationsbehandlingen , accepted*