The chromatic numbers of distance graphs

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Abstract

Suppose D is a subset of all positive integers Z. The distance graph $G(\mathbb{Z}, D)$ is the graph with vertex set Z and two vertices x, y are adjacent if $|x - y| \in D$. This paper studies the chromatic number of $G(\mathbb{Z}, D)$. We shall concentrate on the case that $|D| = 3$ and give a complete classification of those $D = \{a, b, c\}$ for which b is a multiple of a and $\chi(G(\mathbb{Z}, D)) = 3$.

1 Introduction

Let $\mathbb{Z}$ be the set of all integers. For a subset D of $\mathbb{Z}$, the distance graph $G(\mathbb{Z}, D)$ is the graph with vertex set $\mathbb{Z}$ in which two vertices x, y are adjacent if $|x-y| \in D$. Distance graphs were introduced and studied by Eggleton, Erdös and Skilton [4]. They were motivated by the well known plane colouring problem: What is the least number of colours which can be used to colour all points of the Euclidean plane so that vertices of unit distance are coloured with distinct colours. Moser and Moser [8] showed that four colours are necessary, and Hadwiger, Debrunner and Klee [7] showed that seven colours are sufficient. Essentially no further progress has been made on the problem. The plane colouring problem is naturally generalized to higher dimension spaces. The gap between the known upper and lower bounds of the corresponding chromatic number increases as the dimension increases [3]. On the other hand, the corresponding problem for one dimensional space, i.e., for the real line, is trivial. It is easy to see that this problem is equivalent to find the chromatic number of the distance graph $G(\mathbb{Z}, D)$ for which $|D| = 1$. Obviously, $\chi(G(\mathbb{Z}, D)) = 2$ if $|D| = 1$.

Eggleton, Erdös and Skilton [4, 5] studied, among other things, the chromatic number of $G(\mathbb{Z}, D)$ for special classes of sets D. They proved that if D is the set of all primes, then $\chi(G(\mathbb{Z}, D)) = 4$; if D contains $2, 3$ and a pair of twin primes, then $\chi(G(\mathbb{Z}, D)) = 4$; if $|D| = 2$ and it contains two coprime integers, then $\chi(G(\mathbb{Z}, D)) = 3$ if the two integers are of opposite parity and $\chi(G(\mathbb{Z}, D)) = 2$ if the two integers are odd. Using a computer, they also calculated the chromatic number and the smallest period for a periodic colouring of graphs $G(\mathbb{Z}, D)$ for which $D = \{2, 3, p\}$ where p is a prime number less than 1000. There are some other special sets D for which the chromatic number $\chi(G(\mathbb{Z}, D))$ was determined, [4, 5, 6].

The chromatic number of distance graphs $G(\mathbb{Z}, D)$ was also studied by Chen, Chang and Huang [2]. Our present paper is motivated by the results and a conjecture in [2]. Chen, Chang and Huang [2] concentrated on the case that $|D| = 3$. They observed in [2] $\chi(G(\mathbb{Z}, D)) \leq |D| + 1$ for any finite set D. Thus $\chi(G(\mathbb{Z}, D)) \leq 4$ if $|D| = 3$. Also if $D = \{a, b, c\}$ and $\gcd(a, b, c) = 1$ (which is the only case of interest), then $\chi(G(\mathbb{Z}, D)) = 2$ if and only if all three integers a, b, c are odd. Therefore the only difficulty is to decide for which D $\chi(G(\mathbb{Z}, D)) = 3$ and for which D $\chi(G(\mathbb{Z}, D)) = 4$. For the remaining part of this paper, we assume that $D = \{a, b, c\}$, $a < b < c$ and that $\gcd(a, b, c) = 1$. The following results were obtained in [2]:

If $c = a + b$, then $\chi(G(\mathbb{Z}, D)) = 4$ when $a \not\equiv b(\text{mod}3)$ and $\chi(G(\mathbb{Z}, D)) = 3$ when $a \equiv b(\text{mod}3)$; If $a = 1, b = 2$, then $\chi(G(\mathbb{Z}, D)) = 4$ when $c \equiv 0(\text{mod}3)$ and $\chi(G(\mathbb{Z}, D)) = 3$ when $c \not\equiv 0(\text{mod}3)$; If $a \geq 2, b = a + 1$, then $\chi(G(\mathbb{Z}, D)) = 3$ when $c \neq a + b$.

They conjectured that $\chi(G(\mathbb{Z}, D)) = 4$ if and only if either $c = a + b$ and $a \not\equiv b(\text{mod}3)$ or $a = 1, b = 2$ and $c \equiv 0(\text{mod}3)$.

In this paper we also study the chromatic number of graphs $G(\mathbb{Z}, D)$ with $|D| = 3$. We take the above results of [2] as the starting point. Our results are in support of the above conjecture. In particular we shall prove that the conjecture is true if b is a multiple of a. In Section 2, we present a general method of colouring distance graphs $G(\mathbb{Z}, D)$ with $|D| = 3$. The method is simple, and the idea is to color the whole real line $\mathbb{R}$ instead of $\mathbb{Z}$ in a periodic way using certain parameters. It turns out many graphs $G(\mathbb{Z}, D)$

with $|D| = 3$ can be coloured by this method. In particular all previous known graphs $G(\mathbb{Z}, D)$ with $|D| = 3$ with $\chi(G) = 3$ can be coloured by this method. Section 2 also contains some easy applications of this colouring method. In Sections 3, we consider those graphs $G(\mathbb{Z}, D)$ with $D = \{a, b, c\}$ for which b is a multiple of a. By applying the general colouring method presented in Section 2, we determine the chromatic numbers of all those graphs $G(\mathbb{Z}, D)$ with $D = \{a, ma, c\}$. In Section 4, we present some other partial results on the chromatic numbers of $G(\mathbb{Z}, D))$ for which $|D| = 3$.

2 A general colouring method

We present in this section a general colouring method which will be used in the next two sections. The colouring is indeed very simple. The idea is to colour the whole real line $\mathbb{R}$ instead of colouring only the integers $\mathbb{Z}$. This makes the colouring easier and clearer. Some complicated colouring rules used in other papers can be very simply described by this method.

Let $r > 0$ be a real number. We partition the real line $\mathbb{R}$ into half open intervals I_i of length r, where $I_i = [ir, (i+1)r)$ $(i = 0, \pm 1, \cdots,)$. Then we call the colouring Δ which colours the ith interval with colour $i (\mathrm{mod} n)$ the *r periodic n-colouring* of $\mathbb{R}$.

Theorem 1 *Suppose $D = \{a, b, c\}$ and $a < b < c$. If there are three integers n_1, n_2, n_3 such that $n_i \equiv 2 (\mathrm{mod} 3)$, and $[(n_1 - 1)/a, n_1/a] \cap [(n_2 - 1)/b, n_2/b] \cap [(n_3 - 1)/c, n_3/c] \neq \emptyset$, then $\chi(G(\mathbb{Z}, D)) \leq 3$.*

Proof. Suppose $t \in [(n_1 - 1)/a, n_1/a] \cap [(n_2 - 1)/b, n_2/b] \cap [(n_3 - 1)/c, n_3/c]$ and $n_i \equiv 2 (\mathrm{mod} 3)$ for $i = 1, 2, 3$. Let $r = 1/t$. We shall show that the r periodic 3-colouring f is a proper 3-colouring of G. Let $i, j \in V(G)$ for which $i \sim j$. Suppose $i < j$. Then $j - i \in \{a, b, c\}$. Without loss of generality, we may assume that $j - i = a$. Since $t = 1/r \in [(n_1 - 1)/a, n_1/a]$, we have $n_1 - 1 \leq a/r \leq n_1$. It follows from the definition of r periodic 3-colouring that $f(j) - f(i)$ is equivalent to either $\lfloor a/r \rfloor$ or $\lceil a/r \rceil (\mathrm{mod} 3)$. Therefore $f(i) \neq f(j)$. ∎

Corollary 1 *Suppose $D = \{a, b, c\}$ and $a < b < c$. If there exist integers $k \not\equiv 0(\mathrm{mod}3)$ and $p, q \equiv 2(\mathrm{mod}3)$ such that $p - 1 \le bk/c \le p$ and $q - 1 \le ak/c \le q$, then $\chi(G(\mathbb{Z}, D)) \le 3$.*

Proof. Suppose k, p, q are integers such that $k \not\equiv 0(\mathrm{mod}3)$, $p, q \equiv 2(\mathrm{mod}3)$ and $p - 1 \le bk/c \le p$ and $q - 1 \le ak/c \le q$. Let $n_1 = q, n_2 = p$ and let $n_3 = k$ or $k - 1$ whichever is equivalent to $2(\mathrm{mod}3)$. It is straight forward to verify that the conditions of Theorem 1 are satisfied. ∎

We shall show in the remaining part of this paper that many graphs $G(\mathbb{Z}, D)$ can be 3-coloured as described in Theorem 1, by proving the existence of the parameters k, p and q. It may be true that all graphs $G(\mathbb{Z}, D)$ for which $|D| = 3$ and $\chi(G) = 3$ can be coloured by the general colouring method of Theorem 1. The difficulties lie in the establishment of the existence of the parameters n_1, n_2, n_3. We first derive some easy results in the this section to illustrate the ideas.

Corollary 2 *If $c \le 2a$, then $\chi(G(\mathbb{Z}, D)) \le 3$.*

Proof. Let $k = p = q = 2$. Then $1 = p - 1 \le kb/c = 2b/c < 2 = p$ and $1 = q - 1 \le ka/c = 2a/c < 2 = q$. Therefore the conditions of Corollary 1 are satisfied. ∎

Corollary 3 *Suppose $a = 3^r a'$ and $b = 3^s b'$ and $a', b' \not\equiv 0(\mathrm{mod}3)$. If $c \ge 2b$, $r = s$ and $a' \equiv b'(\mathrm{mod}3)$, then $\chi(G(\mathbb{Z}, D)) \le 3$.*

Proof. Since $a' \equiv b' \not\equiv 0(\mathrm{mod}3)$, either $a' \equiv b' \equiv 1(\mathrm{mod}3)$, or $2a' \equiv 2b' \equiv 1(\mathrm{mod}3)$. In the former case, assume that $a' = 3m + 1, b' = 3n + 1$ and in the latter case, assume that $2a' = 3m + 1, 2b' = 3n + 1$. Consider the closed interval $[(3n+1)\frac{c}{b}, (3n+2)\frac{c}{b}]$ of $\mathbb{R}$. It has length $\frac{c}{b} \ge 2$. Therefore it contains two consecutive integers. Thus it contains an integer $k \not\equiv 0(\mathrm{mod}3)$. Let $p = 3n+2, q = 3m+2$. Then $(p-1)\frac{c}{b} \le k \le p\frac{c}{b}$, i.e., $p-1 \le \frac{kb}{c} \le p$. Multiply all the three terms of the above inequality by $a/b = a'/b' = (3m+1)/(3n+1)$, we obtain $q-1 \le \frac{ka}{c} \le q$. Therefore the conditions of Corollary 1 are satisfied. ∎

We note that in the proof of Corollary 3, we concluded that a certain interval contains an integer k for which $k \not\equiv 0(\mathrm{mod}3)$ by using the fact that the specified interval(s) has length at least 2. All the requirements on the integers a, b, c are used to derive the fact that this certain interval(s) has length at least 2. However, having length at least 2 is certainly not a necessary condition for an interval to contain an integer k for which $k \not\equiv 0(\text{mod}3)$. In the next section we shall use the structure of the appropriate families of intervals to establish the existence of the parameters k, p, q which satisfy the condition of Corollary 1.

3 The case that b is a multiple of a

In this section, we consider the case that b is a multiple of a. We shall classify all those $D = \{a, b, c\}$ for which $\chi(G(\mathbb{Z}, D)) = 3$. To be precise, we shall prove the following:

Theorem 2 *Suppose $b = ma$ is a multiple of a, and that $D = \{a, b, c\}$. Then $\chi(G(\mathbb{Z}, D)) \leq 3$ provided that $c \neq a + b$ and $\{a, b\} \neq \{1, 2\}$.*

Note that the cases $c = a + b$ and the case $\{a, b\} = \{1, 2\}$ were solved in [2] (cf. Section 1). Also note that $\chi(G(\mathbb{Z}, D)) = 2$ only if D contains no even integers (under the assumption that $\gcd(a, b, c) = 1$). Therefore we do have a complete classification of all those $D = \{a, b, c\}$ for which b is a multiple of a and $\chi(G(\mathbb{Z}, D)) = 3$.

First we treat the special case that $b = 2a$. If $a|c$, then since $\gcd(a, b, c) = 1$ we have $a = 1, b = 2$. The corresponding results for this case were obtained in [2] (cf. Section 1). Thus we assume that c is not a multiple of a.

Lemma 1 *If $b = 2a$ and c is not a multiple of a, then $\chi(G(C, D)) = 3$.*

Proof. Since $\gcd(a, b, c) = 1$ and $a|b$, we know that $\gcd(a, c) = 1$. Suppose $c = ka + r$ where $0 < r < a$. Then $\gcd(a, r) = 1$. For $i = 0, 1, \cdots, a-1$, let t_i be the integer for which $0 \leq t_i \leq a-1$ and $t_i \equiv ir(\text{mod}a)$. Since $\gcd(a, r) = 1$, we conclude that $\{t_0, t_1, \cdots, t_{a-1}\} = \{0, 1, 2, \cdots, a-1\}$. We define a colouring $f : \{t_0, t_1, \cdots, t_{a-1}\} \mapsto \{0, 1, 2\}$ as follows: First let $f(t_0) = 0$. For $j = 1, 2, 3, \cdots, a-1$, we define $f(t_j)$ recursively so that $f(t_j) \equiv f(t_{j-1}) + 1 - (c - t_j + t_{j-1})/a(\text{mod}3)$. Note that $c - t_j + t_{j-1} \equiv c - jr + (j-1)r \equiv$

$c-r \equiv 0(\text{mod}a)$. Indeed, it is not difficult to see that $t_{j-1}-t_j = -r$ or $a-r$, depending on whether $t_j > t_{j-1}$ or $t_j < t_{j-1}$. Thus $(c-t_j+t_{j-1})/a$ is equal to either k or $k+1$, depending on whether $t_j > t_{j-1}$ or $t_j < t_{j-1}$.

Now we extend f to a colouring of $\mathbb{Z}$ as follows: For any $n \in \mathbb{Z}$, let $i \in \{0,1,2,\cdots,a-1\} = \{t_0,t_1,\cdots,t_{a-1}\}$ be the unique integer for which $n-i \equiv 0(\text{mod}a)$. Then we define $f(n) \in \{0,1,2\}$ so that $f(n) \equiv f(i)+(n-i)/a(\text{mod}3)$. We shall show that f is a proper 3-colouring of G.

Suppose $u,v \in \mathbb{Z}$, $u > v$ and $u \sim v$. Then $u-v$ is equal to a,b or c. If $u-v=a$ then $f(u) \equiv f(v)+1(\text{mod}3)$ by definition. If $u-v=b=2a$ then $f(u) \equiv f(v)+2(\text{mod}3)$ by definition. Assume that $u-v=c$. Let $i \in \{0,1,2,\cdots,a-1\}$ be the integer for which $i \equiv u(\text{mod}a)$; and let $j \in \{0,1,2,\cdots,a-1\}$ be the integer for which $j \equiv v(\text{mod}a)$. Suppose $j=t_s$. Then it follows from the fact $u-v=c$ that $i=t_{s+1}$. Therefore $f(i)=f(j)+1-(c+j-i)/a$. Thus $f(u) \equiv f(i)+(u-i)/a = f(j)+1-(c+j-i)/a+(u-i)/a = f(j)+1+(u-c-j)/a = f(j)+1+(v-j)/a \equiv f(v)+1(\text{mod}3)$. Thus f is indeed a proper colouring of G. ∎

For the remaining part of this section, we assume that $D=\{a,b,c\}$, $b=ma$ and $m \geq 3$. Moreover the case $c=a+b$ was solved in [2]. Thus we further assume that $c \neq a+b$. We shall prove that $\chi(G(\mathbb{Z},D)) \leq 3$ for all such graphs $G(\mathbb{Z},D)$. The proof is divided into three lemmas.

Lemma 2 *If* $m \equiv 0(\text{mod}3)$*, then* $\chi(G(\mathbb{Z},D)) \leq 3$.

Proof. First we assume that $\frac{mc}{b}$ is not an integer. For $i=0,1,\cdots$, let I_i be the closed interval $[((1+3i)m+1)\frac{c}{b},((1+3i)m+2)\frac{c}{b}]$ of $\mathbb{R}$. We claim that the union $\cup_{i=0}^{\infty} I_i$ contains an integer k for which $k \not\equiv 0(\text{mod}3)$.

Let C be the circle obtained from the interval $[0,3]$ of $\mathbb{R}$ by identifying 0 and 3. Let $g: R \mapsto C$ be the map for which $g(x) \equiv x(\text{mod}3)$. If the above claim is not true, then for each i the image $g(I_i)$ of I_i is contained in the open arc $(2,1)$ of C (which contains 0). Since $\frac{mc}{b}$ is not an integer, $g(I_i)$ and $g(I_{i+1})$ do not coincide, as the distance between the initial point of I_i and the initial point of I_{i+1} is $3m\frac{c}{b}$ which is not a multiple of 3. Thus $g(I_{i+1})$ is a shift of $g(I_i)$ of distance r for some nonzero real number r. Moreover r is independent of i. (With a little patience and care, one can determine r in terms of a,b,c. However it suffices for us to know that $r \neq 0$). Since $\frac{c}{b} > 1$

and the arc $(2,1)$ has length 2, the shift distance r must be less than 1, for otherwise $g(I_i)$ and $g(I_{i+1})$ can not be both contained in the open arc $(2,1)$. However this implies that $\cup_{i=0}^{\infty} g(I_i)$ covers the whole circle C, contrary to the assumption that $g(I_i)$ is contained in the open arc $(2,1)$ for all i. Thus the claim is proved.

Suppose $k \in [((1+3i)m+1)\frac{c}{b}, ((1+3i)m+2)\frac{c}{b}]$ is an integer for which $k \not\equiv 0(\mathrm{mod}3)$. Let $p = (1+3i)m+2$ and $q = 2+3i$. Then $(p-1)\frac{c}{b} \leq k \leq p\frac{c}{b}$ and $(q-1)\frac{c}{a} = (q-1)\frac{b}{a}\frac{c}{b}(1+3i)m\frac{c}{b} \leq k \leq ((1+3i)m+2)\frac{c}{b} \leq (2+3i)\frac{b}{a}\frac{c}{b} = q\frac{c}{a}$. Therefore $p-1 \leq \frac{kb}{c} \leq p$ and $q-1 \leq \frac{ka}{c} \leq q$. Hence the condition of Corollary 1 is satisfied by the three integers k, p, q. Therefore $\chi(G) \leq 3$.

Now we assume that $\frac{mc}{b} = s$ is an integer. Since $c > b$ we have $s > m$. If $s = m+1$ then $b+a = b+\frac{b}{m} = c$, contrary to our assumption that $b+a \neq c$. Thus $s \geq m+2$.

For $i = 0, 1, \cdots, \frac{m}{3}-1$, let I_i be the closed interval $[(m+(1+3i))\frac{c}{b}, (m+(2+3i))\frac{c}{b}]$ of $\mathbb{R}$. We claim that the union $\cup_{i=0}^{\frac{m}{3}-1} I_i$ contains an integer k for which $k \not\equiv 0(\mathrm{mod}3)$.

We assume to the contrary that the claim is false, i.e., all integers contained in the intervals I_i $(i = 0, 1, \cdots, \frac{m}{3}-1)$ are multiples of 3. Then each interval I_i $(i = 0, 1, \cdots, \frac{m}{3}-1)$ contains at most one integer. Note that each interval I_i has length $\frac{c}{b} = \frac{s}{m} > 1$, and hence contains at least one integer, we conclude that each interval contains exactly one integer. This implies that each interval has length less than 2, and hence $s < 2m$. Thus the integer contained in I_0 is $\lceil (m+1)\frac{s}{m} \rceil = s+2$. Since the claim is assumed to be false, $s+2 \equiv 0(\mathrm{mod}3)$. Moreover $\frac{s}{m} < \frac{3}{2}$, for otherwise we would have $s+3 \leq (m+2)\frac{s}{m}$ which implies $s+3 \in I_0$ and hence I_0 contains more than one integer.

For $j = 1, 2, \cdots, \frac{m}{3}-1$, let $P(j)$ be the following statement:

$$s+1+3j < (m+(1+3j))\frac{s}{m} \leq s+2+3j \ \text{ and } \ \frac{s}{m} \leq \frac{2+3j}{1+3j}$$

We now prove that under the assumption that the claim is false, the statements $P(j)$ hold for $j = 1, 2, \cdots, \frac{m}{3}-1$. We shall prove this by induction on j.

Since $2 \le \frac{s}{m} < \frac{3}{2}$, we have $s+4 < (m+4)\frac{s}{m} < s+6$. Thus $\lceil (m+4)\frac{s}{m} \rceil = s+5$ or $s+6$. However if $\lceil (m+4)\frac{s}{m} \rceil = s+6$, the claim would be true, because $s+6 \equiv 1(\text{mod}3)$. Thus $s+4 < (m+4)\frac{s}{m} \le s+5$, which implies that $\frac{4s}{m} \le 5$ and hence $\frac{s}{m} \le \frac{5}{4}$. Therefore $P(1)$ is true.

Suppose that $2 \le j \le \frac{m}{3} - 1$ and that the $P(j-1)$ hold and that the claim is false. We shall prove that $P(j)$ holds.

Since $P(j-1)$ holds, we have $\frac{s}{m} \le \frac{3j-1}{3j-2}$. Therefore $3j+1 < (1+3j)\frac{s}{m} \le (1+3j)\frac{3j-1}{3j-2} = \frac{9j^2-1}{3j-2} < 3j+3$. Hence $\lceil (m+1+3j)\frac{s}{m} \rceil = s+3j+2$, or $s+3j+3$. Since $\lceil (m+1+3j)\frac{s}{m} \rceil \in I_j$ and $3j+3 \equiv 1(\text{mod}3)$, we conclude that $\lceil (m+1+3j)\frac{s}{m} \rceil = s+3j+2$ (for otherwise the claim is true). Therefore $s+1+3j < (m+(1+3j))\frac{s}{m} \le s+2+3j$. This implies that $\frac{(1+3j)s}{m} \le 3j+2$. Thus $P(j)$ holds.

Since $P(\frac{m}{3}-1)$ holds, we have $\frac{s}{m} \le \frac{2+3(\frac{m}{3}-1)}{1+3(\frac{m}{3}-1)} = \frac{m-1}{m-2}$. However, we observed at the begining of the proof that $s \ge m+2$. Thus $\frac{m+2}{m} \le \frac{m-1}{m-2}$. Recall that $m \equiv 0(\text{mod}3)$, we see that the inequality is satisfied only if $m = 3$. Moreover s must be either 5 or 6. However, in the seventh previous paragraph, we concluded (under the assumption that the claim is false) that $s+2 \equiv 0(\text{mod}3)$, which is an obvious contradiction. Therefore the claim is proved, i.e., the union $\cup_{i=0}^{\frac{m}{3}-1} I_i$ contains an integer k for which $k \not\equiv 0(\text{mod}3)$.

Suppose $k \in I_i = [(m+(1+3i))\frac{c}{b}, (m+(2+3i))\frac{c}{b}]$ is an integer for which $k \not\equiv 0(\text{mod}3)$. Let $p = m+2+3i$ and let $q = 2$. It can be verified that the triple k, p, q satisfy the conditions of Corollary 1. (A similar detailed verification can be found in the previous paragraph.) Therefore $\chi(G) \le 3$. ∎

Lemma 3 *If* $m \equiv 1(\text{mod}3)$*, then* $\chi(G(\mathbb{Z}, D)) \le 3$.

Proof. The proof of this lemma is similar to, but simpler than, that of Lemma 2. We will only point out the differences. First we assume that $\frac{mc}{b}$ is not an integer. For $i = 0, 1, \cdots$, let I_i be the closed interval $[((1+3i)m)\frac{c}{b}, ((1+3i)m+1)\frac{c}{b}]$ of $\mathbb{R}$. With the same argument as the corresponding part of the proof of Lemma 2, we can show that the union $\cup_{i=0}^{\infty} I_i$ contains an integer k for which $k \not\equiv 0(\text{mod}3)$.

Suppose $k \in [((1+3i)m)\frac{c}{b}, ((1+3i)m+1)\frac{c}{b}]$ is an integer for which $k \not\equiv 0(\text{mod}3)$. Let $p = (1+3i)m+1$ and $q = 2+3i$. Then the three integers

k, p, q satisfy the conditions of Corollary 1. Therefore $\chi(G) \leq 3$.

Now we assume that $\frac{mc}{b} = s$ is an integer. Then the interval $[\frac{mc}{b}, \frac{(m+1)c}{b}]$ contains at least two integers. Hence $[\frac{mc}{b}, \frac{(m+1)c}{b}]$ contains an integer k for which $k \not\equiv 0(\text{mod}3)$. Let $p = m+1, q = 2$. It is again straight forward to verify that the three integers k, p, q satisfy the conditions of Corollary 1, and hence $\chi(G) \leq 3$. ∎

Lemma 4 *If $m \equiv 2(\text{mod}3)$, then $\chi(G(\mathbb{Z}, D)) \leq 3$.*

Proof. Again the proof is similar to that of Lemma 2, and we only point out the differences. If $\frac{mc}{b}$ is not an integer, then for $i = 0, 1, \cdots$, let I_i be the closed interval $[((1+3i)m+2)\frac{c}{b}, ((1+3i)m+3)\frac{c}{b}]$ of $\mathbb{R}$. With the same argument as the corresponding part of the proof of Lemma 2, we can show that the union $\cup_{i=0}^{\infty} I_i$ contains an integer k for which $k \not\equiv 0(\text{mod}3)$. Suppose $k \in [((1+3i)m+2)\frac{c}{b}, ((1+3i)m+3)\frac{c}{b}]$ is an integer for which $k \not\equiv 0(\text{mod}3)$. Let $p = (1+3i)m+3$ and $q = 2+3i$. Then the three integers k, p, q satisfy the condition of Corollary 1. Therefore $\chi(G) \leq 3$.

Now we assume that $\frac{mc}{b} = s$ is an integer. If $s \not\equiv 0(\text{mod}3)$, then let $k = s, p = m, q = 2$. It is easy to verify that k, p, q satisfy the conditions of Corollary 1. Therefore $\chi(G) \leq 3$. Thus we assume now that $s \equiv 0(\text{mod}3)$. We also have $s \geq m+2$, for otherwise we would have $c = a+b$.

For $i = 0, 1, \cdots, \frac{m-5}{3}$, let I_i be the closed interval $[(m+3i+2)\frac{c}{b}, (m+3i+3)\frac{c}{b}]$ of $\mathbb{R}$. We claim that the union $\cup_{i=0}^{\frac{m-5}{3}} I_i$ contains an integer k for which $k \not\equiv 0(\text{mod}3)$.

The proof of this claim is again similar to the corresponding part of the proof of Lemma 1. We assume to the contrary that all integers contained in the intervals I_i $(i = 0, 1, \cdots, \frac{m-5}{3})$ are multiples of 3. Then each interval I_i $(i = 0, 1, \cdots, \frac{m-5}{3})$ contains exactly one integer, and hence $s < 2m$.

First we consider the interval $I_0 = [(m+2)\frac{c}{b}, (m+3)\frac{c}{b}]$. The integer contained in I_0 is $\lceil (m+2)\frac{s}{m} \rceil$. Since $m < s < 2m$, $s+2 < (m+2)\frac{s}{m} < s+4$. Therefore $\lceil (m+2)\frac{s}{m} \rceil$ is equal to either $s+3$ or $s+4$. Since $s \equiv 0(\text{mod}3)$ and the integer contained in I_0 is a multiple of 3, we conclude that $\lceil (m+2)\frac{s}{m} \rceil = s+3$. This implies that $\frac{s}{m} \leq \frac{3}{2}$. Since $m \geq 5$, $s \geq m+2$ and $s \equiv 0(\text{mod}3)$, we conclude that $s \geq 9$. However this implies that $\frac{9}{m} \leq \frac{3}{2}$ and hence $m \geq 6$. Since $m \equiv 2(\text{mod}3)$, we have $m \geq 8$. Thus $\frac{m-5}{3} \geq 1$.

For $j = 0, 1, \cdots, \frac{m-5}{3}$, we shall prove by induction on j that $\frac{s}{m} \le \frac{3j+3}{3j+2}$ and $s+3j+2 < (m+2+3j)\frac{s}{m} \le s+3j+3$. The paragraph above showed that this is true for $j = 0$. Suppose that $j \ge 1$ and that the above statement is true for $j-1$. Then $s+3j+2 < (m+2+3j)\frac{s}{m} \le s+(2+3j)\frac{3j}{3j-1} < 3j+5$. Therefore $\lceil (m+2+3j)\frac{s}{m} \rceil$ is equal to $s+3j+3, s+3j+4$ or $s+3j+5$. However $\lceil (m+2+3j)\frac{s}{m} \rceil \in I_j$ and hence is a multiple of 3. Therefore $\lceil (m+2+3j)\frac{s}{m} \rceil = s+3j+3$. This implies that $(m+2+3j)\frac{s}{m} \le s+3j+3$ and $\frac{s}{m} \le \frac{3j+3}{3j+2}$. Thus the statement is true for j.

Let $j = \frac{m-5}{3}$. We then have $\frac{s}{m} \le \frac{m-2}{m-3}$. Since $s \ge m+2$, we must have $m = 5$ and $s = 7$. However, $s \equiv 0 (\mathrm{mod} 3)$ by assumption. This contradiction completes the proof of the claim that the union $\cup_{i=0}^{\frac{m-5}{3}} I_i$ contains an integer k for which $k \not\equiv 0 (\mathrm{mod} 3)$.

Suppose $k \in I_j$ for some $0 \le j \le \frac{m-5}{3}$. Then let $p = m+3j+3$ and let $q = 2$. It can be verified that k, p, q satisfy the conditions of Corollary 1. This completes the proof of Lemma 4, as well as the proof of Theorem 2. ∎

4 Some other cases

This section presents more cases for which the conjecture of Chang [2] (cf. Section 1) is true. We shall also use the general periodic colourings stated in Corollary 1.

Theorem 3 *Let $k = \lceil c/a \rceil$ if $\lceil c/a \rceil \not\equiv 0 (\mathrm{mod} 3)$, and let $k = \lceil c/a \rceil + 1$ otherwise. If $2c/b \ge k$ then $\chi(G(\mathbb{Z}, D)) \le 3$.*

Proof. Let $p = q = 2$. Then $q-1 = p-1 = 1 \le ka/c \le kb/c \le 2 = p = q$. Thus k, p, q satisfy the conditions of Corollary 1. ∎

The following corollary is Theorem 10 of [2], which generalized Theorem 16 of [5]:

Corollary 4 *If $a \ge 2$, $b = a+1$ and $c \ne 2a+1$, then $\chi(G(\mathbb{Z}, D)) = 3$.*

Proof. Since a, b are of opposite parity, we have $\chi(G(\mathbb{Z}, D)) \ge 3$. Thus we only need to show that $\chi(G(\mathbb{Z}, D)) \le 3$. If $c \le 2a$ then the result follows from Theorem 2. Thus we assume that $c \geq 2a+2$. If $\lceil c/a \rceil \not\equiv 0 (\text{mod} 3)$, then let $k = \lceil c/a \rceil$. We have $k \leq c/a + 1$. Since $a \geq 2$, we have $a^2 - a - 2 \geq 0$. Thus $a^2 + a \leq (2a+2)(a-1) \leq c(a-1)$. This implies that $c/a + 1 \leq 2c/(a+1) = 2c/b$. Therefore $k \leq c/a + 1 \leq 2c/b$, and $\chi(G(\mathbb{Z}, D)) \leq 3$ by Theorem 3.

Assume in the following that $\lceil c/a \rceil \equiv 0 (\text{mod} 3)$. Let $k = \lceil c/a \rceil + 1$. If $\lceil c/a \rceil = 3$ then $k = 4 = 2(2a+2)/(a+1) \leq 2c/b$, hence $\chi(G(\mathbb{Z}, D)) \leq 3$ by Theorem 3.

Suppose that $\lceil c/a \rceil \equiv 0 (\text{mod} 3)$ and $\lceil c/a \rceil \geq 6$. Then $c \geq 2a(a+1)/(a-1)$, provided that $(a, b, c) \neq (2, 3, 11)$. It can be derived from this inequality that $c/a + 2 \leq 2c/(a+1)$. Therefore $k \leq c/a + 2 \leq 2c/b$, and hence $\chi(G(\mathbb{Z}, D)) \leq 3$ by Theorem 3, provided that $(a, b, c) \neq (2, 3, 11)$. The case $(a, b, c) = (2, 3, 11)$ needs special treatment. In this case, we let $k = 26, p = 8, q = 5$, then the condition of Corollary 1 is satisfied, hence $\chi(G(\mathbb{Z}, D)) = 3$. ∎

Theorem 2 asserts that if $c \leq 2a$ then $\chi(G(\mathbb{Z}, D)) \leq 3$. Intuitively this means that if c is small (compared to a) then $\chi(G(\mathbb{Z}, D)) \leq 3$. Our next result more or less means that if c is big then $\chi(G(\mathbb{Z}, D)) \leq 3$.

Theorem 4 *Let $s = b/\gcd(a, b)$. If $s \geq 3bc/(a(c-2b))$ and $c \geq 2b$, then $\chi(G(\mathbb{Z}, D)) \leq 3$.*

Proof. Since $s = b/\gcd(a, b)$, we have $a/b = \frac{t}{s}$ for some integer t such that $\gcd(s, t) = 1$. Thus there are positive integers u, v such that $ut - vs = 1$. We shall make use of the fact later in the proof that $ua/b = ut/s = v + 1/s$.

Let $\alpha = a/(3b) - 1/3$, and $\beta = a(c-2b)/(3bc)$. For $i = 0, 1, \cdots$, let I_i be the closed interval $[\alpha + iua/b, \alpha + \beta + iua/b]$. Since each of the intervals I_i has length $\beta \geq \frac{1}{s}$, and that each I_i is the translation of I_{i-1} of a distance $v + \frac{1}{s}$, where v is an integer, we conclude that one of the interval contains an integer. Assume that m is an integer contained in I_i. Let $n = ui$. Then $\alpha + na/b \leq m \leq \alpha + \beta + na/b$. Replace in this inequality α and β by $a/(3b) - 1/3$ and $a(c-2b)/(3bc)$ respectively, and simplify it, we obtain

$$(1+3n)\frac{c}{b} \leq (1+3m)\frac{c}{a} \leq (1+3n)\frac{c}{b} + \frac{c}{b} - 2.$$

We now consider the closed interval $[(1+3m)c/a, (2+3n)c/b]$. It has length $(2+3n)c/b - (1+3m)c/a \geq (2+3n)c/b - (1+3n)c/b + c/b - 2 = 2$. Therefore it contains an integer k for which $k \not\equiv 0 (\mathrm{mod} 3)$. Let $p = 3n+2$ and let $q = 3m+2$. We now verify that k, p, q satisfy the conditions of Corollary 1. By the choice of k and m, n, we have $(1+3n)\frac{c}{b} \leq (1+3m)c/a \leq k \leq (2+3n)c/b$. Moreover, $(2+3n)c/b = (1+3n)c/b + c/b \leq (1+3m)c/a + c/b \leq (1+3m)c/a + c/a = (2+3m)c/a$. It then easily follows that $p - 1 = 1 + 3n \leq \frac{kb}{c} \leq p = 2 + 3n$ and $q - 1 = 1 + 3m \leq \frac{ka}{c} \leq q = 2 + 3m$. ∎

Corollary 5 *If* $gcd(a, b) = 1$, $a \geq 4$ *and* $c \geq 2ab/(a-3)$, *then* $\chi(G(\mathbb{Z}, D)) \leq 3$.

Proof. In this case $s = b/\mathrm{gcd}(a, b) = b$ and $c \geq 2ab/(a-3)$ implies that $c > 2b$ and $b \geq 3bc/(a(c-2b))$. ∎

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