# On the dimensions of anisotropic quadratic forms with trivial invariants

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**Abstract.** Let $WF$ denote the Witt ring of a field $F$ of characteristic $\neq 2$ and let $I^nF$ denote the $n$-th power of the ideal $IF$ of even-dimensional forms in $WF$. The Arason-Pfister Hauptsatz states that if $0 \neq \varphi \in I^nF$ is anisotropic then $\dim \varphi \geq 2^n$. Pfister also showed that if $\varphi \in I^3F$ is anisotropic and $\dim \varphi > 8$ then $\dim \varphi \geq 12$. We extend this result to $I^4F$ and show that if $\varphi \in I^4F$ is anisotropic and $\dim \varphi > 16$ then $\dim \varphi \geq 24$ and we provide some results on anisotropic 24-dimensional forms in $I^4F$.

## 1. Introduction

Let $WF$ denote the Witt ring of a field $F$ of characteristic $\neq 2$ and let $I^nF$ denote the $n$-th power of the ideal $IF$ of even-dimensional forms in $WF$. An important problem in the algebraic theory of quadratic forms is the characterization of those forms which lie in $I^nF$. It is well-known and easy to show that $I^nF$ is additively generated by $n$-fold Pfister forms $\langle\langle a_1, \cdots, a_n\rangle\rangle = \langle 1, a_1\rangle \otimes \cdots \langle 1, a_n\rangle$. We will denote the set of all forms over $F$ which are isometric (resp. similar) to $n$-fold Pfister forms by $P_nF$ (resp. $GP_nF$). One possible approach to characterize forms in $I^nF$ is the use of invariants, for example the signed discriminant of a form $\varphi$ which is denoted by $d_\pm\varphi$ and which is an element in $\dot{F}/\dot{F}^2$ (here, $\dot{F} = F \setminus \{0\}$), or the Clifford invariant $c(\varphi)$ which lies in ${}_2BrF$, the 2-torsion part of the Brauer group of $F$. One readily checks that $I^2F = \{\varphi \in IF \,|\, d_\pm\varphi = 1\}$. A consequence of Merkuriev's celebrated theorem in [M] is that $I^3F = \{\varphi \in I^2F \,|\, c(\varphi) = 1\}$.

In order to extend these results it turned out to be useful to apply cohomological methods which was first done systematically by Arason in [A]. Let $H^nF$ denote the Galois cohomology group $H^n(\mathrm{Gal}(F_s/F), \mathbb{Z}/2)$ where $F_s$ is a separable closure of $F$. Arason showed that there are well-defined maps $P_nF \longrightarrow H^nF$ which, for $n \leq 3$, extend to well-defined homomorphisms

$$\bar{e}^n_F \,:\, I^nF/I^{n+1}F \longrightarrow H^nF\ .$$

Jacob and Rost [JR] and Szyjewski [Sz] have proved that Arason's map extends to a well-defined homomorphism $\bar{e}^n_F$ as above also for $n = 4$. In fact, one easily sees that $\bar{e}^0_F$ is nothing but the map given by dimension modulo 2 using $\mathbb{Z}/2 \simeq H^0F$, and $\bar{e}^1_F$ is induced by the signed discriminant via $\dot{F}/\dot{F}^2 \simeq H^1F$. $\bar{e}^2_F$ is induced by the Clifford invariant map using the fact that ${}_2BrF \simeq H^2F$. An easy check shows that $\bar{e}_F^0$ and $\bar{e}_F^1$ are isomorphisms. It is Merkuriev's theorem which says that $\bar{e}_F^2$ is an isomorphism as well. Merkuriev and Suslin [MSu] and independently Rost [R1] showed that $\bar{e}_F^3$ is also an isomorphism. The conjecture is that Arason's map extends to an isomorphism $\bar{e}_F^n$ for all $n$.

These maps $\bar{e}_F^n$ (provided they exist) characterize in a certain sense forms in $I^nF$ up to Witt equivalence. In this paper, however, we are interested in "structure" or "isometry" results about forms in $I^nF$. What we mean by results of this type is illustrated best by the following famous theorem which is due to Arason and Pfister [AP] and thus aptly named the "Arason-Pfister Hauptsatz" (we will refer to it later on simply by APH).

**Arason-Pfister Hauptsatz** *(Cf. [AP, Hauptsatz and Korollar 3].) Let $0 \neq \varphi \in I^nF$ be anisotropic. Then $\dim\varphi \geq 2^n$. If $\dim\varphi = 2^n$ then $\varphi \in GP_nF$.*

Before Merkuriev obtained his results on $\bar{e}_F^2$, and even before the above Hauptsatz, Pfister, in his attempt to classify forms in $I^3F$, proved the following (see also [S, Ch. 2, Theorem 14.4]).

**Pfister's Theorem** *(Cf. [P, Satz 14 and Zusatz].) Let $\varphi$ be an even-dimensional form with $\dim\varphi \leq 12$, $d_{\pm}\varphi = 1$ and $c(\varphi) = 1$. Then $\varphi \in I^3F$. More precisely, the following holds.*

(i) *If $\dim\varphi \leq 6$ then $\varphi = 0$ in $WF$, i.e. $\varphi$ is hyperbolic.*

(ii) *If $\dim\varphi = 8$ then $\varphi \in GP_3F$.*

(iii) *If $\dim\varphi = 10$ then $\varphi \simeq \pi \perp \mathbb{H}$ with $\pi \in GP_3F$ (here, $\mathbb{H} \simeq \langle 1, -1\rangle$ denotes the hyperbolic plane). In particular, $\varphi$ is isotropic.*

(iv) *If $\dim\varphi = 12$ then there exist $r, s, t, u, v, w \in \dot{F}$ such that*

$$\varphi \simeq r\langle\langle s\rangle\rangle \otimes \langle t, u, tu, -v, -w, -vw\rangle$$

*(or, in $WF$, $\varphi = r(\langle\langle s, t, u\rangle\rangle - \langle\langle s, v, w\rangle\rangle)$).*

**Remark 1.1** (i) Pfister himself formulated part (iv) of the above theorem differently. It is not difficult to deduce the above statement from Pfister's original version. We will give an alternative proof of our version (and of part (iii) above) in Theorem 2.8.

(ii) Parts (i) and (ii) of the above theorem are essentially the Arason-Pfister Hauptsatz for $n = 3$.

Part (iii) of Pfister's theorem states that there are no anisotropic 10-dimensional forms in $I^3F$. This raises the question whether there is an analogous result for $n \geq 4$. Are all forms in $I^nF$ of dimension $2^n + 2$ for $n \geq 4$ isotropic? One can go even further. For all $n \geq 2$ there exists a field $F$ with an anisotropic form of dimension $2^n + 2^{n-1}$ in $I^nF$. This can easily be achieved "generically". Let $F = F_0(X_1, \cdots, X_{n-2}, Y_1, Y_2, Z_1, Z_2)$ be the rational function field in the variables $X_i, Y_j, Z_k$ over an arbitrary field $F_0$, and let $\varphi \simeq \langle\langle X_1, \cdots, X_{n-2}\rangle\rangle \otimes \langle Y_1, Y_2, Y_1Y_2, -Z_1, -Z_2, -Z_1Z_2\rangle$. In $WF$ we have $\varphi = \langle\langle X_1, \cdots, X_{n-2}, Y_1, Y_2\rangle\rangle - \langle\langle X_1, \cdots, X_{n-2}, Z_1, Z_2\rangle\rangle$ and it follows that $\varphi \in I^nF$. The fact that $\varphi$ is anisotropic follows readily from Springer's theorem (cf. [S, Ch. 6, Corollary 2.6]). Moreover, if $\varphi \in I^nF$ is anisotropic with $\dim\varphi > 2^n$ and $\varphi = \rho_1 + \rho_2$ in $WF$ where $\rho_1, \rho_2 \in GP_nF$, then $\dim\varphi \geq 2^n + 2^{n-1}$ as follows from Lemma 2.3. These examples and facts together with Pfister's theorem lead quite naturally to the following conjecture (see also [Ka, Conjecture 9]).

**Conjecture 1** *Let $\varphi \in I^nF$ be anisotropic and $\dim\varphi > 2^n$. Then $\dim\varphi \geq 2^n + 2^{n-1}$.*

It should be remarked that this conjecture has implications regarding the classification of so-called good forms of height 2. For more on this we refer the reader to Kahn's paper [Ka] (in particular, Proposition 4.3 and Théorème 4.2 in that paper).

The objective of this paper is to prove that this conjecture is true for $n = 4$.

**Main Theorem** *Let $\varphi \in I^4F$ be anisotropic and $\dim\varphi > 16$. Then $\dim\varphi \geq 24$.*

In the next section we will provide several lemmas which will be needed for our proof of the Main Theorem. We will also give alternative proofs of parts (iii) and (iv) of Pfister's Theorem. In section 3 we will prove the Main Theorem, and in section 4 we will investigate 24-dimensional forms in $I^4F$.

Notations and terminology are borrowed from Lam's book [L] and Scharlau's book [S]. If $\varphi$ and $\psi$ are forms over $F$ then $\varphi \simeq \psi$ means that these forms are isometric, whereas $\varphi = \psi$ denotes equivalence of these forms in $WF$. We say that $\varphi$ is similar to $\psi$ if $\varphi \simeq a\psi$ for some $a \in \dot{F}$, and we write $\varphi \subset \psi$ if $\varphi$ is a subform of $\psi$, i.e., there exists a forms $\tau$ such that $\psi \simeq \varphi \perp \tau$. The Witt index of $\varphi$ is denoted by $i_W(\varphi)$, and the anisotropic part of $\varphi$ by $\varphi_{\mathrm{an}}$, so that we have $\varphi \simeq (i \times \mathbb{H}) \perp \varphi_{\mathrm{an}}$ where $i = i_W(\varphi)$. $\varphi$ is called a Pfister neighbor if there exist $a \in \dot{F}$ and $\pi \in P_nF$ for some $n$ such that $\varphi \subset a\pi$ and $2\dim\varphi > \dim\pi$. We then say that $\varphi$ is a Pfister neighbor of $\pi$. If $K/F$ is a field extension we write $\varphi_K$ instead of $\varphi \otimes K$, and the kernel of the natural ring homomorphism $WF \to WK$ is denoted by $W(K/F)$. The function field $F(\varphi)$ of a form $\varphi$ over $F$ ($\dim\varphi \geq 2$, $\varphi \not\simeq \mathbb{H}$) is the function field of the projective variety $\varphi = 0$. For basic facts about Pfister forms and function fields of quadratic forms which we will use frequently but which we will not mention explicitly, we refer the reader to [L] and [S].

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## 2. Some lemmas and Pfister's theorem

Let us first collect some basic facts (without proof) on function fields of quadratic forms and on "linkage" of Pfister forms. Recall that a field extension $K/F$ is called *excellent* if for any $\varphi \in WF$ there exists $\psi \in WF$ such that $(\varphi_K)_{\mathrm{an}} \simeq \psi_K$, i.e. the anisotropic part of $\varphi$ over $K$ is defined over $F$.

**Lemma 2.1** *Let $\pi \in P_nF$ be anisotropic ($n \geq 1$) and let $K = F(\pi)$. Let $\varphi \in WF$ be anisotropic.*

(i) *(Cf. [S, Ch. 2, Lemma 5.1].) Suppose $n = 1$, i.e. $\pi \simeq \langle\langle a\rangle\rangle$ for some $a \in \dot{F}$ and $K = F(\pi) = F(\sqrt{-a})$. If $i_W(\varphi_K) = i \geq 1$ then there exists $\tau \in WF$, $\dim\tau = i$, such that $\pi \otimes \tau \subset \varphi$.*

(ii) *(Cf. [S, Ch. 4, Theorem 5.4].) Let $n \geq 1$. If $\varphi \in W(K/F)$ then there exists $\tau \in WF$ such that $\varphi \simeq \pi \otimes \tau$. In particular, $\varphi \in I^nF$ and $\dim\varphi \equiv 0 \pmod{2^n}$. Furthermore, $\varphi \in I^{n+1}F$ iff $\dim\tau$ is even iff $\dim\varphi \equiv 0 \pmod{2^{n+1}}$.*

(iii) *(Cf. [ELW, Appendix II by Arason].) Let $n = 1$ or $n = 2$. Then $K/F$ is excellent.*

**Remark 2.2** In this lemma, (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii) for $n = 1$. (i) generally fails for $n \geq 2$. (iii) generally fails for $n \geq 3$ as has been shown by Izhboldin [I]. Arason's original proof of excellence in the case $n = 2$ uses algebraic geometry. A more recent and elementary proof can be found in [R2].

**Lemma 2.3** *(Cf. [EL, Theorem 4.5].) Let $\sigma \in P_nF$ and $\pi \in P_mF$ be anisotropic with $m \leq n$. Let $a, b \in \dot{F}$. Then $i := i_W(a\sigma \perp b\pi) = 0$ or $2^r$ for some integer $r$ with $0 \leq r \leq m$. Furthermore, $i \geq 1$ iff there exists $x \in \dot{F}$ such that $(a\sigma \perp b\pi)_{\mathrm{an}} \simeq x(\sigma \perp -\pi)_{\mathrm{an}}$. Let $i_W(\sigma \perp -\pi) = 2^r \geq 1$. Then there exist $\alpha \in P_rF$, $\sigma_1 \in P_{n-r}F$, and $\pi_1 \in P_{m-r}F$ such that $\sigma \simeq \alpha \otimes \sigma_1$ and $\pi \simeq \alpha \otimes \pi_1$.*

**Corollary 2.4** *Let $\rho$, $\sigma \in P_nF$ and $\tau \in P_mF$, $m, n \geq 1$. Let $K = F(\tau)$ and suppose that $\sigma_K \simeq \rho_K$. Then the following holds.*

(i) *If $m > n$ then $\rho \simeq \sigma$.*

(ii) *If $m = n$ then there exist $\pi \in P_{n-1}F$, $\rho_1$, $\sigma_1 \in P_1F$ such that $\rho \simeq \pi \otimes \rho_1$ and $\sigma \simeq \pi \otimes \sigma_1$.*

(iii) *If $m < n$ then there exist $\pi \in P_mF$, $\rho_1$, $\sigma_1 \in P_{n-m}F$ such that $\rho \simeq \pi \otimes \rho_1$ and $\sigma \simeq \pi \otimes \sigma_1$.*

Proof. The cases where one of the three Pfister forms is isotropic and hence hyperbolic are quite straightforward and left to the reader. So let us assume that all three Pfister forms ar anisotropic. Let $\psi \simeq (\rho \perp -\sigma)_{\mathrm{an}}$. Note that $\psi \in I^nF$, $\psi_K = 0$, and that $\dim \psi \leq 2^{n+1} - 2$ as both $\rho$ and $\sigma$ represent 1. If $\psi = 0$ then $\rho \simeq \sigma$ and there is nothing to show. So let us assume that $\psi \neq 0$. Since $\psi \in W(K/F)$ and by Lemma 2.3(ii) there exists $\eta \in WF$, $\dim \eta \geq 1$, such that $\psi \simeq \tau \otimes \eta$. Necessarily $\dim \tau = 2^m \leq 2^n$ and thus $m \leq n$. This implies (i). Now if $m = n$ then we must have $\dim \psi = 2^n$ and hence $i_W(\rho \perp -\sigma) = 2^{n-1}$ and (ii) follows from Lemma 2.3. If $m < n$ then in order for $\psi \simeq \tau \otimes \eta$ to be in $I^nF$ we must have $\dim \eta$ to be even. Since $\dim \psi < 2^{n+1}$ this implies that $\dim \eta \leq 2^{n+1-m} - 2$ and thus $\dim \psi \leq 2^{n+1} - 2^{m+1}$ or $i_W(\rho \perp -\sigma) \geq 2^m$. Again, Lemma 2.3 implies (iii). $\square$

The next three lemmas are due to Rost and they will play crucial roles in what follows.

**Lemma 2.5** *(Rost) Let $\varphi \in WF$ and $\pi \in P_2F$ both be anisotropic. Let $L/F$ be a quadratic field extension and $K = F(\pi)$. Suppose that $\varphi_L = \pi_L = 0$ and that $\varphi_K$ is isotropic. Then there exists $x \in \dot{F}$ such that $x\pi \subset \varphi$.*

Proof. Let $L = F(\sqrt{-a})$. By Lemma 2.1(ii), there exists $b \in \dot{F}$ such that $\pi \simeq \langle\langle a, b\rangle\rangle$. Also, as $\varphi_K$ is isotropic and as $K/F$ is excellent by Lemma 2.1(iii), there exists $\psi \in WF$, $\dim \psi < \dim \varphi$, such that $(\varphi_K)_{\mathrm{an}} \simeq \psi_K$. In particular, $(\varphi \perp -\psi)_{\mathrm{an}} \in W(K/F)$ and thus, by Lemma 2.1(ii), $(\varphi \perp -\psi)_{\mathrm{an}} \simeq \langle\langle a, b\rangle\rangle \otimes \tau$ for some $\tau \in WF$ with $\dim \tau \geq 1$ as $\varphi$ is anisotropic and $\dim \varphi > \dim \psi$. Note that $\varphi_L = 0$ by assumption and that clearly also $(\langle\langle a, b\rangle\rangle \otimes \tau)_L = 0$. Hence, $\psi_L = 0$ and there exists $\sigma \in WF$ such that $\psi \simeq \langle\langle a\rangle\rangle \otimes \sigma$. In $WF$ we have $\varphi = \psi \perp \langle\langle a, b\rangle\rangle \otimes \tau$. Now $\varphi$ is anisotropic and we therefore get that $\dim \varphi \leq \dim \psi + \dim(\langle\langle a, b\rangle\rangle \otimes \tau)$ and hence

$$m = \dim \psi + \dim(\langle\langle a, b\rangle\rangle \otimes \tau) - \dim \varphi \geq 0\ .$$

We use induction on $m$ to show that $\varphi$ contains a subform similar to $\langle\langle a, b\rangle\rangle$. If $m = 0$ then $\varphi \simeq \psi \perp \langle\langle a, b\rangle\rangle \otimes \tau$ and the claim follows because $\dim \tau \geq 1$.

If $m > 0$ and using the fact that $\psi \simeq \langle\langle a\rangle\rangle \otimes \sigma$, we have that $\langle\langle a\rangle\rangle \otimes \sigma \perp \langle\langle a,b\rangle\rangle \otimes \tau$ is isotropic and there exists $r \in \dot{F}$ such that $r$ is represented by $\langle\langle a\rangle\rangle \otimes \sigma$ and $-r$ is represented by $\langle\langle a,b\rangle\rangle \otimes \tau$. Hence, by [EL, Theorem 1.4], we may assume that $\sigma \simeq \langle r\rangle \perp \sigma'$ and $\tau \simeq \langle -r\rangle \perp \tau'$ for suitable forms $\sigma'$, $\tau' \in WF$. Thus, in $WF$,

$$\begin{aligned}
\psi \perp -r\langle\langle a,b\rangle\rangle &= \langle\langle a\rangle\rangle \otimes \sigma \perp -r\langle\langle a,b\rangle\rangle \\
&= r\langle\langle a\rangle\rangle \perp \langle\langle a\rangle\rangle \otimes \sigma' \perp -r\langle\langle a,b\rangle\rangle \\
&= -rb\langle\langle a\rangle\rangle \perp \langle\langle a\rangle\rangle \otimes \sigma' \\
&= \langle\langle a\rangle\rangle \otimes (\langle -rb\rangle \perp \sigma')\ .
\end{aligned}$$

Let $\tilde{\sigma} \simeq \langle -rb\rangle \perp \sigma'$ and $\tilde{\psi} \simeq \langle\langle a\rangle\rangle \otimes \tilde{\sigma}$. Then $\dim \psi = \dim \tilde{\psi}$ and $\tilde{\psi}_K = (\psi \perp -r\langle\langle a,b\rangle\rangle)_K = \psi_K$. Hence, $(\varphi_K)_{\mathrm{an}} \simeq \tilde{\psi}_K$. Also,

$$\begin{aligned}
\varphi &= \psi \perp \langle\langle a,b\rangle\rangle \otimes \tau \\
&= \psi \perp -r\langle\langle a,b\rangle\rangle \perp \langle\langle a,b\rangle\rangle \otimes \tau' \\
&= \tilde{\psi} \perp \langle\langle a,b\rangle\rangle \otimes \tau'\ .
\end{aligned}$$

Now $\dim \tau' = \dim \tau - 1$ and therefore

$$0 \leq \dim \tilde{\psi} + \dim(\langle\langle a,b\rangle\rangle \otimes \tau') - \dim \varphi < m\ .$$

By induction, $\varphi$ contains a subform similar to $\langle\langle a,b\rangle\rangle$. □

The next two lemmas have also been proved by Rost as we already remarked, however only in the case $n = 4$. His proofs carry over to arbitrary $n \geq 3$ resp. $n \geq 4$ without major changes.

**Lemma 2.6** *(Rost) Let $n \geq 3$ and let $\varphi$ be anisotropic with $\dim \varphi = 2^n$. Suppose that $\varphi \equiv \pi \pmod{I^nF}$ for some $\pi \in P_{n-1}F$. Then there exists $a \in \dot{F}$ such that for $L = F(\sqrt{-a}) = F(\langle\langle a\rangle\rangle)$ we have $\varphi_L = \pi_L = 0$.*

Proof. Note that by APH $\varphi_L = 0$ implies that $\pi_L = 0$. If $\pi$ is isotropic and hence hyperbolic then $\varphi \in I^nF$ and by APH we have that $\varphi \in GP_nF$. For any quadratic extension $L/F$ with $\varphi_L$ isotropic we then get that $\varphi_L = 0$ and we are done. So let us assume that $\pi$ is anisotropic.

Let $r, e \in \dot{F}$ such that $r\langle 1, e\rangle \subset \varphi$ and let $E = F(\sqrt{-e})$. Clearly, $\varphi_E$ is isotropic and $\varphi_{E(\pi)} \equiv 0 \pmod{I^nE(\pi)}$. By APH, this implies that $\varphi_{E(\pi)} = 0$. Since $\dim(\varphi_E)_{\mathrm{an}} < 2^n$, it follows from Lemma 2.1(ii) that either $\varphi_E = 0$, in which case we are done by putting $a = e$, or $(\varphi_E)_{\mathrm{an}} \simeq x\pi_E$ for some $x \in \dot{E}$ (in fact one can choose $x \in \dot{F}$ by the excellence of $E/F$, see [ELW, Proposition 2.11], but we don't need this at this point). In this case, we have $i_W(\varphi_E) = 2^{n-2}$ and by Lemma 2.1(i) there exist $\mu$, $\psi \in WF$, $\dim \mu = 2^{n-2}$, $\dim \psi = 2^{n-1}$, such that $\varphi \simeq \langle\langle e\rangle\rangle \otimes \mu \perp \psi$. Note that $\dim \mu$ is even as $n \geq 3$. Hence, $\varphi$, $\langle\langle e\rangle\rangle \otimes \mu$ and thus also $\psi$ are in $I^2F$. Let $s \in \dot{F}$ be represented by $\psi$. As $\varphi \equiv \pi \equiv s\pi \pmod{I^nF}$ we get that

$$(\varphi \perp -s\pi)_E \equiv (\psi \perp -s\pi)_E \equiv 0 \pmod{I^nE}\ .$$

Since $\psi \perp -s\pi$ is isotropic by our choice of $s$, and since $\dim(\psi \perp -s\pi) = 2^n$, APH implies that $(\psi \perp -s\pi)_E = 0$. Hence, by Lemma 2.1(ii), there exists $\tau \in WF$ such that $(\psi \perp -s\pi)_{\mathrm{an}} \simeq \langle\langle e\rangle\rangle \otimes \tau$ with $\dim \tau < 2^{n-1}$ and even because $\psi \perp -s\pi \in I^2F$

and $\dim(\psi \perp -s\pi)_{\rm an} < 2^n$. Thus, $\dim(\psi \perp -s\pi)_{\rm an} = 2\dim\tau \leq 2^n - 4$. This implies that $\psi$ and $s\pi$ have at least a 2-dimensional subform in common, say, $t\langle 1, a\rangle$ with $t, a \in \dot{F}$. Let $L = F(\sqrt{-a})$. Clearly, $\pi_L$ is isotropic and hence hyperbolic. Thus, $\varphi_L \equiv 0 \pmod{I^n L}$. But $\dim\varphi = 2^n$ and $\varphi_L$ is isotropic because its subform $\psi_L$ is isotropic. Therefore, by APH, we have $\varphi_L = 0$. $\square$

**Lemma 2.7** *(Rost) Let $n \geq 4$ and let $\varphi$ be anisotropic with $\dim\varphi = 2^n$. Suppose that $\varphi \equiv \pi \pmod{I^n F}$ for some $\pi \in P_{n-1}F$. Then there exist $u, v \in \dot{F}$ such that for $K = F(\langle\langle u, v\rangle\rangle)$ we have $\varphi_K = \pi_K = 0$.*

Proof. By Lemma 2.6, there exist $a \in \dot{F}$, $\alpha \in WF$, such that $\varphi \simeq \langle\langle a\rangle\rangle \otimes \alpha$. Let $r, b \in \dot{F}$, $\alpha' \in WF$ such that $\alpha \simeq r\langle 1, b\rangle \perp \alpha'$. Thus, $\varphi \simeq r\langle\langle a, b\rangle\rangle \perp \varphi'$ with $\varphi' \simeq \langle\langle a\rangle\rangle \otimes \alpha'$. Let $E = F(\langle\langle a, b\rangle\rangle)$. Note that $\varphi_E$ is isotropic. As is the proof of Lemma 2.6, we distinguish two cases. Either $\varphi_E = 0$ and we are done with $\langle\langle u, v\rangle\rangle \simeq \langle\langle a, b\rangle\rangle$. Or $(\varphi_E)_{\rm an} \simeq x\pi_E$ for some $x \in \dot{E}$. Now in $WE$ we have $\varphi'_E = (\varphi_E)_{\rm an}$. With $\dim(\varphi_E)_{\rm an} = 2^{n-1}$ and $\dim\varphi' = 2^n - 4$ it follows that $\varphi'_E$ is isotropic because $n \geq 4$. Also, over $L = F(\sqrt{-a})$ we have that $\varphi'_L = \langle\langle a, b\rangle\rangle_L = 0$. Hence, by Lemma 2.5, $\varphi'$ contains a subform similar to $\langle\langle a, b\rangle\rangle$. By repeating this argument (if $n \geq 5$), we conclude that there exist $\beta, \psi \in WF$ with $\dim\beta = 2^{n-3}$ and $\dim\psi = 2^{n-1}$ such that $\varphi \simeq \langle\langle a, b\rangle\rangle \otimes \beta \perp \psi$.

We now proceed similarly as in the proof of the previous lemma. $\dim\beta$ is even because $n \geq 4$. Hence, $\langle\langle a, b\rangle\rangle \otimes \beta$, $\varphi \in I^3F$ and thus also $\psi \in I^3F$. Let $s \in \dot{F}$ be represented by $\psi$. It follows that $(\psi \perp -s\pi)_E = 0$ because $(\psi \perp -s\pi)_E \equiv 0 \pmod{I^n E}$, $\dim(\psi \perp -s\pi) = 2^n$ and $\psi \perp -s\pi$ is isotropic. Hence, by Lemma 2.1(ii), $(\psi \perp -s\pi)_{\rm an} \simeq \langle\langle a, b\rangle\rangle \otimes \tau$ for some $\tau \in WF$ with $\dim\tau$ even and $< 2^{n-2}$. Therefore, $\dim(\psi \perp -s\pi)_{\rm an} = 4\dim\tau \leq 2^n - 8$ and it follows that $\psi$ and $s\pi$ have at least a 4-dimensional subform in common. Suppose $\langle u, v, uv\rangle$ is similar to a subform of both $\psi$ and $s\pi$. Then, over $K = F(\langle\langle u, v\rangle\rangle)$, $\langle u, v, uv\rangle_K$ is isotropic. Hence, $\pi_K$ is isotropic and therefore hyperbolic which implies that $\varphi_K \equiv 0 \pmod{I^n K}$. But $\dim\varphi = 2^n$ and $\varphi_K$ is isotropic because its subform $\psi_K$ is isotropic. By APH, this yields that $\varphi_K = 0$. $\square$

Let us now give our versions of the proof of Pfister's result on forms of dimension 10 and 12 in $I^3F$. Other than in Pfister's original proof where he actually characterizes forms of dimension 10 and 12 with trivial discriminant and Clifford invariant, we will a priori assume that our forms are in $I^3F$ and the Clifford invariant will not appear at all in our proof.

**Theorem 2.8** *Let $\varphi \in I^3F$.*

(i) *If $\dim\varphi = 10$ then $\varphi$ is isotropic, i.e. $\varphi \simeq \pi \perp \mathbb{H}$ for some $\pi \in GP_3F$.*

(ii) *If $\dim\varphi = 12$ then there exist $r, s, t, u, v, w \in \dot{F}$ such that $\varphi = r(\langle\langle s, t, u\rangle\rangle - \langle\langle s, v, w\rangle\rangle)$ in $WF$.*

Proof. (i) Let $\varphi \in I^3F$, $\dim\varphi = 10$, and suppose that $\varphi$ is anisotropic. We may assume that $\varphi \simeq \langle 1, x, y\rangle \perp \varphi'$ with $x, y \in \dot{F}$. Hence,

$$\varphi - \langle\langle x, y\rangle\rangle \equiv \varphi' \perp \langle -xy\rangle \equiv -\langle\langle x, y\rangle\rangle \equiv \langle\langle x, y\rangle\rangle \pmod{I^3F}.$$

Let $\tilde\varphi \simeq \varphi' \perp \langle -xy\rangle$. Note that $\dim\tilde\varphi_{\rm an} = 6$ or $8$, and that for $K = F(\langle\langle x, y\rangle\rangle)$ we have $\tilde\varphi_K \in I^3K$. If $\dim\tilde\varphi_{\rm an} = 6$ this would imply by APH that $\tilde\varphi_K = 0$ and thus $\tilde\varphi_{\rm an} \simeq \langle\langle x, y\rangle\rangle \otimes \tau$ by Lemma 2.1(ii). This in turn yields $\dim\tilde\varphi_{\rm an} \equiv 0 \pmod 4$, a contradiction. Hence, $\tilde{\varphi}$ is anisotropic. By Lemma 2.6, there exists $a \in \dot{F}$ such that for $L = F(\sqrt{-a})$ we have $\tilde{\varphi}_L = \langle\langle x,y\rangle\rangle_L = 0$. Since $\varphi = \tilde{\varphi} + \langle\langle x,y\rangle\rangle$ in $WF$, we also get that $\varphi_L = 0$. Thus, by Lemma 2.1(ii), there exists $\sigma \in WF$, $\dim\sigma = 5$, such that $\varphi \simeq \langle\langle a\rangle\rangle \otimes \sigma$. But then $\varphi \notin I^3F$, as remarked also in Lemma 2.1(ii), a contradiction. Hence, $\varphi$ must be isotropic. It readily follows from APH that $\varphi \simeq \pi \perp \mathbb{H}$ for some $\pi \in GP_3F$.

(ii) Let now $\varphi \in I^3F$ and $\dim\varphi = 12$. If $\varphi$ is isotropic then $\dim\varphi_{\mathrm{an}} \leq 8$ by part (i) and therefore $\varphi = \pi$ in $WF$ for some $\pi \in GP_3F$. To write $\varphi$ as in the statement of part (ii), we may choose the second 3-fold Pfister form to be hyperbolic. We may therefore assume that $\varphi$ is anisotropic and that possibly after scaling we have $\varphi \simeq \langle 1,x\rangle \perp \varphi'$ for some $x \in \dot{F}$. Let $L = F(\sqrt{-x})$. Then $\varphi_L = \varphi'_L \in I^3L$. As $\dim\varphi' = 10$ it follows from part (i) that $\varphi'_L$ is isotropic. Thus, by Lemma 2.1(i), there exists $y \in \dot{F}$ such that $y\langle 1,x\rangle \subset \varphi'$. Let now $\varphi'' \in WF$ and $z \in \dot{F}$ such that $\varphi \simeq \langle\langle x,y\rangle\rangle \perp \langle z\rangle \perp \varphi''$. For dimension reasons, $\varphi \perp -\langle\langle x,y,z\rangle\rangle \in WF$ is not hyperbolic. Hence,

$$0 \neq \varphi - \langle\langle x,y,z\rangle\rangle = \varphi'' - z\langle x,y,xy\rangle \in I^3F \ .$$

Let $\tilde{\varphi} \simeq (\varphi'' \perp -z\langle x,y,xy\rangle)_{\mathrm{an}}$. Since $\dim\tilde{\varphi} \leq \dim\varphi'' + 3 = 10$ and $\tilde{\varphi} \in I^3F$, it follows from part (i) and APH that $\tilde{\varphi} \in GP_3F$. Thus, $\varphi = \tilde{\varphi} + \langle\langle x,y,z\rangle\rangle$, and the existence of $r,s,t,u,v,w \in \dot{F}$ as in the statement of part (ii) follows readily from Lemma 2.3. □

## 3. Proof of the Main Theorem

Before we turn to the proof of the Main Theorem, we will provide a lemma which is crucial in what will follow and which is a consequence of Pfister's result on forms in $I^3F$. Part (ii) of this lemma also appears implicitly in [H, Theorem 4.1].

**Lemma 3.1** *Let $\varphi \in WF$ be anisotropic with $\dim\varphi = 8$. Let $d \neq 1$ in $\dot{F}/\dot{F}^2$ and $L = F(\sqrt{d})$. Suppose that $\varphi_L \in GP_3L$.*

(i) *If $d_{\pm}\varphi = 1$ in $\dot{F}/\dot{F}^2$ then there exist $r,s,t,u,v,w \in \dot{F}$ such that $\varphi = r(\langle\langle s,t,u\rangle\rangle - \langle\langle s,v\rangle\rangle)$ in $WF$, i.e. $\varphi \simeq r\langle\langle s\rangle\rangle \otimes \langle t,u,tu,-v\rangle$, and $\langle\langle s,v\rangle\rangle \simeq \langle\langle -d,-w\rangle\rangle$.*

(ii) *If $d_{\pm}\varphi \neq 1$ in $\dot{F}/\dot{F}^2$ then there exist $r \in \dot{F}$ and $\pi \simeq \langle 1\rangle \perp \pi' \in P_3F$ such that $\varphi \simeq r(\langle d\rangle \perp \pi')$.*

Proof. (i) Since $\varphi_L \in GP_3L$ we have $c(\varphi_L) = 1$ in $BrL$, i.e. the Clifford invariant of $\varphi$ splits over the quadratic extension $L = F(\sqrt{d})$. It is well-known that in this case $c(\varphi)$ can be represented by a single quaternion algebra of the form $(d,w)_F$ for some $w \in \dot{F}$ (if $c(\varphi)$ is already split, we may choose $w = 1$). It is also well-known that $c(\langle\langle -d,-w\rangle\rangle) = (d,w)_F$ in $BrF$. Thus, by Pfister's theorem (note that $d_{\pm}\varphi = 1$ and $\dim\varphi + \dim\langle\langle -d,-w\rangle\rangle = 12$) we have $\varphi \equiv \langle\langle -d,-w\rangle\rangle \pmod{I^3F}$. Thus, by Lemma 2.6 and Lemma 2.3(ii), there exist $r,s,t,u,v \in \dot{F}$ such that $\varphi \simeq r\langle\langle s\rangle\rangle \otimes \langle t,u,tu,-v\rangle$. In $WF$ we then have $\varphi = r(\langle\langle s,t,u\rangle\rangle - \langle\langle s,v\rangle\rangle)$ and therefore,

$$\varphi \equiv r(\langle\langle s,t,u\rangle\rangle - \langle\langle s,v\rangle\rangle) \equiv -r\langle\langle s,v\rangle\rangle \equiv \langle\langle s,v\rangle\rangle \equiv \langle\langle -d,-w\rangle\rangle \pmod{I^3F} \ .$$

APH readily implies that $\langle\langle s,v\rangle\rangle \simeq \langle\langle -d,-w\rangle\rangle$.

(ii) As $d_{\pm}\varphi \neq 1$ in $\dot{F}/\dot{F}^2$ but $d_{\pm}\varphi_L = 1$ in $\dot{L}/\dot{L}^2$ because $\varphi_L \in GP_3L$, we must have $d_{\pm}\varphi = d$ in $\dot{F}/\dot{F}^2$. After scaling, we may assume that $\varphi \simeq \varphi' \perp \langle d\rangle$. Let $\tilde{\varphi} \simeq \varphi' \perp \langle 1\rangle$. Then $d_{\pm}\tilde{\varphi} = 1$ in $\dot{F}/\dot{F}^2$ and clearly $\varphi_L \simeq \tilde{\varphi}_L \in GP_3L$. If $\tilde{\varphi}$ were isotropic then

$\varphi_L \simeq \tilde{\varphi}_L$ would be hyperbolic and by Lemma 2.1(ii) we would have $\varphi \simeq \langle\langle -d\rangle\rangle \otimes \tau$ for some $\tau \in WF$, $\dim\tau = 4$. In particular, $d_\pm\varphi = 1$, a contradiction. Hence, $\tilde{\varphi}$ is anisotropic, $d_\pm\tilde{\varphi} = 1$ and $\tilde{\varphi}_L \in GP_3L$. By part (i), there exists $w \in \dot{F}$ such that $\tilde{\varphi} \equiv \langle\langle -d, -w\rangle\rangle \pmod{I^3F}$. Hence, with $\tilde{\varphi} = \varphi + \langle 1, -d\rangle$ in $WF$, we get

$$\tilde{\varphi} \equiv \varphi + \langle 1, -d\rangle \equiv \langle\langle -d, -w\rangle\rangle \equiv \langle 1, -d\rangle - w\langle 1, -d\rangle \pmod{I^3F}\ .$$

Thus, $\varphi \equiv -w\langle 1, -d\rangle \pmod{I^3F}$ or $\varphi \perp w\langle 1, -d\rangle \in I^3F$. But $\dim(\varphi \perp w\langle 1, -d\rangle) = 10$ and by Pfister's theorem there exits $\rho \in GP_3F$ such that $\varphi \perp w\langle 1, -d\rangle = \rho$ in $WF$ or $\varphi = \rho - w\langle 1, -d\rangle$. The existence of $r \in \dot{F}$ and $\pi \in P_3F$ as in the statement of part (ii) follows readily from Lemma 2.3. □

The next lemma will lay the foundation for the proof of the Main Theorem.

**Lemma 3.2** *Let $\varphi \in WF$ be anisotropic with $\dim\varphi = 16$. Suppose there exists $a \neq 1$ in $\dot{F}/\dot{F}^2$ such that for $E = F(\sqrt{a})$ we have $\varphi_E \in GP_4E$. Then there exists $d \in \dot{F}$ such that for $L = F(\sqrt{d})$ we have*

(i) $i_W(\varphi_L) \geq 4$ *if $d_\pm\varphi = 1$ in $\dot{F}/\dot{F}^2$,*

(ii) $i_W(\varphi_L) \geq 3$ *if $d_\pm\varphi \neq 1$ in $\dot{F}/\dot{F}^2$.*

Proof. (i) If $\varphi_E$ is isotropic then $\varphi_E \in GP_4E$ is hyperbolic and we are done by putting $L = E$. So we may assume that $\varphi_E$ is anisotropic. Choose $x \in \dot{F}$ such that $\langle 1, x\rangle$ is similar to a subform on $\varphi$ and let $K = F(\sqrt{-x})$. Obviously, $\varphi_K$ is isotropic. If $i_W(\varphi_K) \geq 4$ we are done by putting $L = K$. So let us assume that $i_W(\varphi_K) \leq 3$. Now $K(\sqrt{a}) = E(\sqrt{-x})$ and we have that $\varphi_{E(\sqrt{-x})} \in GP_4E(\sqrt{-x})$ is isotropic and hence hyperbolic. Thus, by Lemma 2.1(ii), there exists $\tau \in WK$ such that $(\varphi_K)_{\mathrm{an}} \simeq \langle 1, -a\rangle \otimes \tau$. $\dim\tau$ must be even because $\varphi \in I^2F$, and by assumption we have $1 \leq i_W(\varphi_K) \leq 3$. This forces $i_W(\varphi_K) = 2$. By Lemma 2.1(i) and possibly after scaling, we may assume that there exist $y, z \in \dot{F}$ and $\varphi' \in WF$ such that $\varphi \simeq \langle\langle x, y\rangle\rangle \perp \langle z\rangle \perp \varphi'$. Consider $\varphi \perp -\langle\langle x, y, z\rangle\rangle$. The above shows that $i_W(\varphi \perp -\langle\langle x, y, z\rangle\rangle) \geq 5$. Suppose $i_W(\varphi \perp -\langle\langle x, y, z\rangle\rangle) \geq 6$. Then $\varphi$ contains a 6-dimensional Pfister neighbor $\eta$ of $\langle\langle x, y, z\rangle\rangle$ as a subform. It is well-known and easy to show (see, e.g., [K, p. 10]) that then, with $M = F(\sqrt{d_\pm\eta})$, one has $\eta_M = 0$. In particular, $i_W(\varphi_M) \geq 3$, and since $i_W(\varphi_M)$ has to be even by the same argument as above, we must have $i_W(\varphi_M) \geq 4$ and we are done by putting $M = L$. Hence, suppose that $i_W(\varphi \perp -\langle\langle x, y, z\rangle\rangle) = 5$. This implies that

$$\tilde{\varphi} \simeq (\varphi \perp -\langle\langle x, y, z\rangle\rangle)_{\mathrm{an}} \simeq \varphi' \perp -z\langle x, y, xy\rangle$$

and $\dim\tilde{\varphi} = 14$. Now $\varphi_E$ is anisotropic and an easy dimension count shows that therefore $\tilde{\varphi}_E \neq 0$. Furthermore, $\varphi_E \in GP_4E$ and hence,

$$\tilde{\varphi}_E \equiv \varphi_E - \langle\langle x, y, z\rangle\rangle \equiv \langle\langle x, y, z\rangle\rangle \pmod{I^4E}\ .$$

For $N = E(\langle\langle x, y, z\rangle\rangle)$, we have that $\tilde{\varphi}_N \in I^4N$ and APH implies that $\tilde{\varphi}_N = 0$. But $\tilde{\varphi}_E \neq 0$ and $\dim\tilde{\varphi} = 14$. By Lemma 2.1(ii), there exists $w \in \dot{E}$ such that $(\tilde{\varphi}_E)_{\mathrm{an}} \simeq w\langle\langle x, y, z\rangle\rangle_E$. By Lemma 2.1(i), there exist forms $\alpha, \psi \in WF$ with $\dim\alpha = 3$, $\dim\psi = 8$, such that $\tilde{\varphi} \simeq \langle\langle -a\rangle\rangle \otimes \alpha \perp \psi$, and it is obvious that $\psi_E \simeq (\tilde{\varphi}_E)_{\mathrm{an}} \simeq w\langle\langle x, y, z\rangle\rangle_E \in GP_3E$. Also, $d_\pm\tilde{\varphi} = d_\pm(\varphi \perp -\langle\langle x, y, z\rangle\rangle) = 1$ because $d_\pm\varphi = 1$, and $d_\pm(\langle\langle -a\rangle\rangle \otimes \alpha) = a$ because $\dim\alpha$ is odd. Hence, $d_\pm\psi = a$. By Lemma 3.1(ii), there exist $\pi \simeq \langle 1\rangle \perp \pi' \in P_3F$ and $r \in \dot{F}$ such that $\psi \simeq r(\langle a\rangle \perp \pi')$. Note that $r\pi_E \simeq w\langle\langle x, y, z\rangle\rangle_E$ which implies that $\pi_E \simeq \langle\langle x,y,z\rangle\rangle_E$. Thus, by Corollary 2.4, there exists $d \in \dot{F}$ such that $\pi \simeq \langle\langle -d,\cdots\rangle\rangle$ and $\langle\langle x,y,z\rangle\rangle \simeq \langle\langle -d,\cdots\rangle\rangle$. In particular, for $L = F(\sqrt{d})$ we have that $\pi_L = \langle\langle x,y,z\rangle\rangle_L = 0$. Hence, with $\tilde{\alpha} \simeq \alpha \perp \langle -r\rangle$ and $\psi = -r\langle 1,-a\rangle + r\pi$ in $WF$, we get $\tilde{\varphi} = \langle\langle -a\rangle\rangle \otimes \tilde{\alpha} + r\pi$ in $WF$. Therefore,

$$\varphi_L = \varphi_L - \langle\langle x,y,z\rangle\rangle_L = \tilde{\varphi}_L = (\langle\langle -a\rangle\rangle \otimes \tilde{\alpha})_L + r\pi_L = (\langle\langle -a\rangle\rangle \otimes \tilde{\alpha})_L$$

in $WL$. Now $\dim(\langle\langle -a\rangle\rangle \otimes \tilde{\alpha}) = 8$ and $\dim\varphi = 16$, and we immediately get that $i_W(\varphi_L) \geq 4$.

(ii) Since $d_\pm\varphi \neq 1$ in $\dot{F}/\dot{F}^2$ and $d_\pm\varphi_E = 1$ in $\dot{E}/\dot{E}^2$, we necessarily have that $d_\pm\varphi = a$ in $\dot{F}/\dot{F}^2$. After scaling, we may assume that $\varphi \simeq \varphi' \perp \langle a\rangle$. Let $\tilde{\varphi} \simeq \varphi' \perp \langle 1\rangle$. By the above, $d_\pm\tilde{\varphi} = 1$ and $\varphi_E \simeq \tilde{\varphi}_E \in GP_4E$. Note that $\tilde{\varphi}_E$ is anisotropic. For otherwise, $\varphi_E = \tilde{\varphi}_E = 0$ and thus, by Lemma 2.1(ii), there exists $\tau \in WF$, $\dim\tau = 8$, such that $\varphi \simeq \langle\langle -a\rangle\rangle \otimes \tau$. In particular, $d_\pm\varphi = 1$, a contradiction. Part (i) now implies that there exists $d \in \dot{F}$ such that for $L = F(\sqrt{d})$ we have $\tilde{\varphi}_L \simeq \gamma \perp (4\times\mathbb{H})$ for some $\gamma \in WL$, $\dim\gamma = 4$. (Actually, $\gamma$ is defined over $F$ as $L/F$ is excellent, but we don't need this here.) Therefore,

$$\varphi_L = \tilde{\varphi}_L - \langle 1,-a\rangle_L = \gamma \perp -\langle 1,-a\rangle_L$$

in $WL$. Now $\dim\varphi = 16$ and $\dim(\gamma \perp -\langle 1,-a\rangle_L) = 10$, and it follows that $i_W(\varphi_L) \geq 3$. □

We now come to the proof of the Main Theorem. We will split it up into three propositions dealing with forms in $I^4F$ of dimension 18, 20, and 22, respectively.

**Proposition 3.3** *Let $\varphi \in I^4F$ with $\dim\varphi = 18$. Then $\varphi$ is isotropic.*

Proof. Suppose $\varphi$ is anisotropic. After scaling, we may write $\varphi \simeq \tilde{\varphi} \perp \langle 1,-a\rangle$ for some $a \in \dot{F}$ and some $\tilde{\varphi} \in WF$, $\dim\tilde{\varphi} = 16$. Let $E = F(\sqrt{a})$. Then $\varphi_E = \tilde{\varphi}_E \in I^4E$ and thus, by APH, $\tilde{\varphi}_E \in GP_4E$. By Lemma 3.2, there exists $d \in \dot{F}$ ($d \notin \dot{F}^2$) such that for $L = F(\sqrt{d})$ we have $i_W(\tilde{\varphi}_L) \geq 3$ and therefore also $i_W(\varphi_L) \geq 3$, i.e. $\dim(\varphi_L)_{\mathrm{an}} \leq 12$. But $\varphi_L \in I^4L$, and by APH we get that $\varphi_L = 0$. Lemma 2.1(ii) then yields the existence of some $\tau \in WF$ such that $\varphi \simeq \langle\langle -d\rangle\rangle \otimes \tau$. We have that $\dim\tau = 9$ which implies that $d_\pm\varphi = d \neq 1$ in $\dot{F}/\dot{F}^2$, clearly a contradiction to $\varphi \in I^4F$. □

**Proposition 3.4** *Let $\varphi \in I^4F$ with $\dim\varphi = 20$. Then $\varphi$ is isotropic.*

Proof. Suppose $\varphi$ is anisotropic. Let $a \in \dot{F}$ such that $\langle 1,-a\rangle$ is similar to a subform of $\varphi$ and let $E = F(\sqrt{a})$. Then $\dim(\varphi_E)_{\mathrm{an}} \leq 18$. Since $\varphi_E \in I^4E$ and by the 18-dimensional case, we have $\dim(\varphi_E)_{\mathrm{an}} \leq 16$. Hence, by Lemma 2.1(i), there exist $\alpha$, $\tilde{\varphi} \in WF$ with $\dim\alpha = 2$ and $\dim\tilde{\varphi} = 16$ such that $\varphi \simeq \langle\langle -a\rangle\rangle \otimes \alpha \perp \tilde{\varphi}$. Now $\varphi_E = \tilde{\varphi}_E$ in $WE$ and thus, by APH, $\tilde{\varphi}_E \in GP_4E$. By Lemma 3.2, there exists $x \in \dot{F}$ such that for $L = F(\sqrt{-x})$ we have $i_W(\tilde{\varphi}_L) \geq 3$ and hence $i_W(\varphi_L) \geq 3$. By Lemma 2.1(i) and possibly after scaling, there exist $y,z \in \dot{F}$ and $\varphi' \in WF$, $\dim\varphi' = 14$, such that $\varphi \simeq \langle\langle x\rangle\rangle \otimes \langle y,z,yz\rangle \perp \varphi'$. Let $\pi \simeq \langle\langle x,y,z\rangle\rangle$ and $\psi \simeq (\varphi \perp -\pi)_{\mathrm{an}}$. In $WF$, we have $\psi = -\langle\langle x\rangle\rangle \perp \varphi'$ and thus, $16 \geq \dim\psi \geq \dim\varphi - \dim\pi = 12$. Furthermore, $\psi \equiv \varphi - \pi \equiv \pi \pmod{I^4F}$ and therefore $\psi_{F(\pi)} \in I^4F(\pi)$. Hence, if $\dim\psi < 16$ then by APH we have $\psi_{F(\pi)} = 0$ in $WF(\pi)$ and thus, by Lemma 2.1(ii), $\dim\psi \equiv 0 \pmod 8$, a contradiction. Thus, $\dim\psi = 16$. By Lemma 2.7, there exists $\sigma \in P_2F$

such that for $K = F(\sigma)$ we have $\psi_K = \pi_K = 0$. Hence, $\varphi_K = \psi_K + \pi_K = 0$ in $WK$ and by Lemma 2.1(ii), there exists $\tau \in WF$, $\dim \tau = 5$, such that $\varphi \simeq \sigma \otimes \tau$. But then $\varphi \notin I^3F$ as $\dim \tau$ is odd, a contradiction to $\varphi \in I^4F$. $\square$

**Proposition 3.5** *Let $\varphi \in I^4F$ with $\dim \varphi = 22$. Then $\varphi$ is isotropic.*

Proof. Suppose $\varphi$ is anisotropic. Let $a \in \dot{F}$ such that $\langle 1, -a\rangle$ is similar to a subform of $\varphi$ and let $E = F(\sqrt{a})$. Then $\dim(\varphi_E)_{\rm an} \leq 20$. Since $\varphi_E \in I^4E$ and by the 18- and 20-dimensional cases, we have $\dim(\varphi_E)_{\rm an} \leq 16$. If $\dim(\varphi_E)_{\rm an} < 16$ then $\varphi_E = 0$ by APH. Therefore, there exists $\tau \in WF$, $\dim \tau = 11$, such that $\varphi \simeq \langle\langle -a\rangle\rangle \otimes \tau$. But then $\varphi \notin I^2F$ because $\dim \tau$ is odd, a contradiction to $\varphi \in I^4F$. So $\dim(\varphi_E)_{\rm an} = 16$ and by Lemma 2.1(i), there exist $\alpha$, $\psi \in WF$, $\dim \alpha = 3$, $\dim \psi = 16$, such that $\varphi \simeq \langle\langle -a\rangle\rangle \otimes \alpha \perp \psi$ and $(\varphi_E)_{\rm an} \simeq \psi_E \in I^4E$. In particular, $\psi_E \in GP_4E$ by APH and $d_\pm \psi = a$ in $\dot{F}/\dot{F}^2$. By Lemma 3.2, there exists $x \in \dot{F}$ such that for $L = F(\sqrt{-x})$ we have $i_W(\varphi_L) \geq 3$. By Lemma 2.1(i) and after possibly scaling, there exist $y, z \in \dot{F}$ and $\psi' \in WF$, $\dim \psi' = 10$, such that $\psi \simeq \langle\langle x\rangle\rangle \otimes \langle 1, x, y\rangle \perp \psi'$. Note that $\psi$ represents 1 ($\langle\langle x\rangle\rangle \simeq \langle 1, x\rangle$!) and that therefore we have in fact $\psi_E \in P_4E$. Since $\psi$ contains the Pfister neighbor $\langle\langle x\rangle\rangle \otimes \langle 1, x, y\rangle$ of $\langle\langle x, y, z\rangle\rangle$, it follows that $\psi_E \in P_4E$ becomes isotropic and hence hyperbolic over $E(\langle\langle x, y, z\rangle\rangle)$. This implies readily that there exists $w \in \dot{E}$ such that $\psi_E \simeq \langle\langle x, y, z, w\rangle\rangle$. (In fact, $w$ can be chosen so as to lie in $\dot{F}$ by the excellence of $E/F$, cf. [ELW, Proposition 2.11], but we don't need this here.)

Let $\tilde{\varphi} \simeq (\varphi \perp -\langle\langle x, y, z\rangle\rangle)_{\rm an}$ and $\tilde{\psi} \simeq (\psi \perp -\langle\langle x, y, z\rangle\rangle)_{\rm an}$. In $WF$ we have $\tilde{\psi} = -yz\langle\langle x\rangle\rangle \perp \psi'$ and $\tilde{\varphi} = \langle\langle -a\rangle\rangle \otimes \alpha \perp \tilde{\psi}$. In particular, $12 \geq \dim \tilde{\psi} \geq \dim \psi - \dim \langle\langle x, y, z\rangle\rangle = 8$. Note that $\dim \tilde{\psi}$ is even. If $\dim \tilde{\psi} \leq 10$ then $\dim \tilde{\varphi} \leq 16$ and we have $\tilde{\varphi} \equiv \varphi \perp -\langle\langle x, y, z\rangle\rangle \equiv \langle\langle x, y, z\rangle\rangle \pmod{I^4F}$. Thus, there exists a quadratic extension $L = F(\sqrt{d})$ such that $\langle\langle x, y, z\rangle\rangle_L = \tilde{\varphi}_L = 0$. If $\dim \tilde{\varphi} = 16$ this follows from Lemma 2.6, and if $\dim \tilde{\varphi} < 16$ we may choose $d = -x$ which immediately implies $\langle\langle x, y, z\rangle\rangle_L = 0$ and therefore, by APH, also $\tilde{\varphi}_L = 0$. But then $\varphi_L = \tilde{\varphi}_L + \langle\langle x, y, z\rangle\rangle_L = 0$ in $WL$. This leads to a contradiction in the same way as at the beginning of this proof where we showed that $\varphi_E \neq 0$.

It remains to consider the case $\dim \tilde{\psi} = 12$. Now in $WE$ we have

$$\tilde{\psi}_E = \psi_E - \langle\langle x, y, z\rangle\rangle_E = \langle\langle x, y, z, w\rangle\rangle_E - \langle\langle x, y, z\rangle\rangle_E = w\langle\langle x, y, z\rangle\rangle_E \ .$$

Hence, by Lemma 2.1(i), there exists $\beta$, $\gamma \in WF$, $\dim \beta = 2$, $\dim \gamma = 8$, such that $\tilde{\psi} \simeq \langle\langle -a\rangle\rangle \otimes \beta \perp \gamma$ and $\gamma_E \simeq w\langle\langle x, y, z\rangle\rangle_E \in GP_3E$. Note that $d_\pm \gamma = d_\pm \tilde{\psi} = d_\pm \psi = a$ in $\dot{F}/\dot{F}^2$. Hence, by Lemma 3.1(ii), there exist $r \in \dot{F}$ and $\pi \simeq \langle 1\rangle \perp \pi' \in P_3F$ such that $\gamma \simeq r(\langle a\rangle \perp \pi')$. Hence, $\gamma_E \simeq r\pi_E \simeq w\langle\langle x, y, z\rangle\rangle_E$, i.e. $\pi_E$ and $\langle\langle x, y, z\rangle\rangle_E$ are similar and hence isometric, and by Corollary 2.4 there exists $d \in \dot{F}$ such that $\pi \simeq \langle\langle -d, \cdots\rangle\rangle$ and $\langle\langle x, y, z\rangle\rangle \simeq \langle\langle -d, \cdots\rangle\rangle$. Over $L = F(\sqrt{d})$ we therefore have $\pi_L = \langle\langle x, y, z\rangle\rangle_L = 0$. Therefore, using $\gamma = -r\langle 1, -a\rangle + r\pi$ is $WF$, the following holds in $WL$:

$$\begin{array}{rcl}
\varphi_L & = & (\langle\langle -a\rangle\rangle \otimes \alpha)_L + \psi_L \\
 & = & (\langle\langle -a\rangle\rangle \otimes \alpha)_L + \psi_L - \langle\langle x, y, z\rangle\rangle_L \\
 & = & (\langle\langle -a\rangle\rangle \otimes \alpha)_L + \tilde{\psi}_L \\
 & = & (\langle\langle -a\rangle\rangle \otimes \alpha)_L + (\langle\langle -a\rangle\rangle \otimes \beta)_L + \gamma_L \\
 & = & (\langle\langle -a\rangle\rangle \otimes \alpha)_L + (\langle\langle -a\rangle\rangle \otimes \beta)_L - r\langle\langle -a\rangle\rangle_L + r\pi_L \\
 & = & (\langle\langle -a\rangle\rangle \otimes (\alpha \perp \beta \perp \langle -r\rangle))_L \ .
\end{array}$$

The right hand side has dimension $2(\dim\alpha + \dim\beta + 1) = 12$. As $\varphi_L \in I^4L$ and by APH, we get that $\varphi_L = 0$, i.e. $\varphi$ becomes hyperbolic over the quadratic extension $L = F(\sqrt{d})$ which leads to the same contradiction as before. $\square$

The Main Theorem now follows by combining Propositions 3.3, 3.4, and 3.5.

**Remark 3.6** (i) The non-existence of anisotropic forms of dimension 10 in $I^3F$ can also be proved using facts about algebraic groups as was done by Tits [T, Theorem 2(ii) and Remark 4.4.3(c)]. It is not clear whether Conjecture 1 can be proved for $n \geq 4$ using arguments of that type.

(ii) In [ELW, Theorem 2.15], it was proved that if for all fields $F$ and all $\varphi \in P_nF$ we have that the function field $F(\varphi)$ is always excellent over $F$, then Conjecture 1 follows for $n$. However, Izhboldin [I] proved recently that if $n \geq 3$ and $\varphi \in P_nF$ then $F(\varphi)/F$ is *not* excellent in general. Thus, the arguments in [ELW] could not even be employed to prove Pfister's result on 10-dimensional forms in $I^3F$.

## 4. Some remarks on 24-dimensional forms

In view of Pfister's result on 12-dimensional forms in $I^3F$ (see also Theorem 2.8) and the example we gave in the introduction of a field $F$ with an anisotropic form of dimension $2^n + 2^{n-1}$ in $I^nF$, it seems natural to conjecture the following.

**Conjecture 2** *Let $n \geq 2$ and $\varphi \in I^nF$. If $\dim\varphi = 2^n + 2^{n-1}$ then there exist $\pi \in P_{n-2}F$ and $r, s, t, u, v \in \dot{F}$ such that $\varphi = r(\pi \otimes \langle\langle s,t\rangle\rangle - \pi \otimes \langle\langle u,v\rangle\rangle)$ in $WF$, i.e. $\varphi \simeq r\pi \otimes \langle s, t, st, -u, -v, -uv\rangle$.*

Forms in $I^2F$ of dimension 6 are commonly referred to as Albert forms. Rephrasing the above conjecture, is it true that if $\varphi \in I^nF$ and $\dim\varphi = 2^n + 2^{n-1}$, then there always exist $\pi \in P_{n-2}F$ and an Albert form $\alpha$ such that $\varphi \simeq \pi \otimes \alpha$? For $n = 2$ this is trivially true, and for $n = 3$ this is a consequence of Pfister's theorem. In the case $n = 4$ and in view of the Main Theorem, we may clearly restrict ourselves to anisotropic forms $\varphi$ of dimension 24. For if $\varphi$ is isotropic then $\dim\varphi_{\mathrm{an}} \leq 16$ and by APH there exists $\rho \in GP_4F$ such that $\varphi = \rho$ in $WF$. Thus, if we want to write $\varphi$ as in the conjecture, we may choose $r, s, t \in \dot{F}$ and $\pi \in P_{n-2}F$ such that $\rho \simeq r\pi \otimes \langle\langle s,t\rangle\rangle$ and by putting $u = -1$ we get that $\pi \otimes \langle\langle u,v\rangle\rangle = 0$ in $WF$.

We have not been able to verify this conjecture for $n = 4$, i.e. for 24-dimensional forms in $I^4F$. However, we do have some partial results which may give some insight into whether the conjecture is true for $n = 4$.

**Proposition 4.1** *Let $\varphi \in I^4F$ be anisotropic with $\dim\varphi = 24$. Then the following statements are equivalent.*

(i) *There exist $\pi \in P_2F$ and an Albert form $\alpha$ such that $\varphi \simeq \pi \otimes \alpha$.*
(ii) *There exist $\rho_1, \rho_2 \in GP_4F$ such that $\varphi = \rho_1 + \rho_2$ in $WF$.*
(iii) *There exists a quadratic extension $L = F(\sqrt{d})$ such that $\varphi_L = 0$ in $WF$.*
(iv) *There exists a quadratic extension $L = F(\sqrt{d})$ such that $i_W(\varphi_L) \geq 5$.*
(v) *There exists $\sigma \in GP_3F$ such that $\sigma \subset \varphi$.*
(vi) *There exists a 9-dimensional Pfister neighbor $\chi$ in $WF$ such that $\chi \subset \varphi$.*
(vii) *There exists a 7-dimensional Pfister neighbor $\psi$ in $WF$ such that $\psi \subset \varphi$.*

P r o o f. It is rather obvious that (i) implies (ii). The converse follows readily from Lemma 2.3.

Trivially, (iii) implies (iv). Conversely, if $i_W(\varphi_L) \geq 5$ for some quadratic extension $L/F$ then $\dim(\varphi_L)_{\mathrm{an}} \leq 14$. Now $\varphi_L \in I^4L$ and the Main Theorem yields $\varphi_L = 0$.

(i)$\Rightarrow$(iii). Let $d \in \dot{F}$ be such that for $L = F(\sqrt{d})$ we have that $\pi_L$ is isotropic and hence hyperbolic. This clearly implies that $\varphi_L$ is hyperbolic as well.

(iii)$\Rightarrow$(v). Since $\varphi_L = 0$ and by Lemma 2.1(ii), there exists $\tau \in WF$ such that $\varphi \simeq \langle\langle -d\rangle\rangle \otimes \tau$. After possibly scaling, we may assume that $\tau \simeq \langle\langle t\rangle\rangle \perp \tau'$ for suitable $t \in \dot{F}$ and $\tau' \in WF$. Let $\varphi' \simeq \langle\langle -d\rangle\rangle \otimes \tau'$ so that $\varphi \simeq \langle\langle -d, t\rangle\rangle \perp \varphi'$, and let $K = F(\langle\langle -d, t\rangle\rangle)$. Now $\varphi_K = \varphi'_K \in I^4K$. As $\dim \varphi' = 20$, it follows from the Main Theorem that $\varphi'_K$ is isotropic. But $\varphi'_L = \langle\langle -d, t\rangle\rangle_L = 0$. Hence, by Lemma 2.5, there exists $x \in \dot{F}$ such that $x\langle\langle -d, t\rangle\rangle \subset \varphi'$. Thus, $\langle\langle -d, t, x\rangle\rangle \subset \varphi$.

(v)$\Rightarrow$(vi). Let $\sigma \in GP_3F$ with $\sigma \subset \varphi$. Write $\sigma \simeq s\tilde{\sigma}$ with $\tilde{\sigma} \in P_3F$ and suitable $s \in \dot{F}$. Let $r \in \dot{F}$ and $\varphi' \in WF$ such that $\varphi \simeq \sigma \perp \langle r\rangle \perp \varphi'$. Clearly, $\varphi$ contains the 9-dimensional Pfister neighbor $\sigma \perp \langle r\rangle$ of $\tilde{\sigma} \otimes \langle\langle rs\rangle\rangle \in P_4F$.

(vi)$\Rightarrow$(ii). Let $\chi \subset \varphi$ be a 9-dimensional Pfister neighbor and let $\rho_1 \in GP_4F$ such that $\chi \subset \rho_1$. Let $\varphi', \eta \in WF$ be such that $\varphi \simeq \chi \perp \varphi'$ and $\rho_1 \simeq \chi \perp \eta$. We have that $\varphi \perp -\rho_1 = \varphi' \perp -\eta \in I^4F$ and $\dim(\varphi' \perp -\eta) = 22$. By the Main Theorem and APH, there exists $\rho_2 \in GP_4F$ such that $\varphi \perp -\rho_1 = \varphi' \perp -\eta = \rho_2$ and therefore $\varphi = \rho_1 + \rho_2$ in $WF$.

This shows that the statements (i) through (vi) are equivalent. Clearly, (v) implies (vii) and it remains to show that (vii) implies any of the previous statements.

So let $\psi \subset \varphi$ be a 7-dimensional Pfister neighbor and let $\sigma \in GP_3F$, $s \in \dot{F}$ and $\varphi' \in WF$, $\dim \varphi' = 17$, such that $\sigma \simeq \psi \perp \langle -s\rangle$ and $\varphi \simeq \psi \perp \varphi'$. After possibly scaling $\varphi$ (and thus also $\psi$ and $\sigma$ accordingly), we may assume that $\sigma \in P_3F$. Let $\tilde{\varphi} \simeq \varphi' \perp \langle s\rangle$. Note that $\dim \tilde{\varphi} = 18$ and $\tilde{\varphi} = \varphi - \sigma$ in $WF$. If $\tilde{\varphi}$ is isotropic then one has that $\sigma \subset \varphi$ and we are in (v). So suppose that $\tilde{\varphi}$ is anisotropic. We have $\tilde{\varphi} \equiv \varphi - \sigma \equiv \sigma \pmod{I^4F}$. Let $a \in \dot{F}$ such that $\langle\langle a\rangle\rangle \subset \sigma$. Let $E = F(\sqrt{-a})$. Then $\sigma_E \in P_rE$ is isotropic and hence hyperbolic. Thus, $\tilde{\varphi}_E \equiv 0 \pmod{I^4E}$ and it follows from the Main Theorem that $\tilde{\varphi}_E$ is isotropic. By Lemma 2.1(i), there exists $x \in \dot{F}$ such that $x\langle\langle a\rangle\rangle \subset \tilde{\varphi}$. Let $\eta \simeq (\tilde{\varphi} \perp -x\sigma)_{\mathrm{an}}$. Since $\tilde{\varphi}$ and $x\sigma$ both contain $x\langle\langle a\rangle\rangle$ as a subform, and since $\dim \tilde{\varphi} = 18$ and $\dim \sigma = 8$, it readily follows that $\dim \eta \leq 22$. Now in $WF$ and with $\rho_1 \simeq \sigma \perp x\sigma \simeq \sigma \otimes \langle\langle x\rangle\rangle \in P_4F$, we have

$$\eta = \tilde{\varphi} - x\sigma = \varphi - \sigma - x\sigma = \varphi - \rho_1 \in I^4F\ .$$

The Main Theorem and APH yield that $\eta = \rho_2$ in $WF$ for some $\rho_2 \in GP_4F$. Hence, $\varphi = \eta + \rho_1 = \rho_1 + \rho_2$ in $WF$ and we are in (ii). $\square$

The previous proposition provides several necessary and sufficient conditions for a 24-dimensional form in $I^4F$ to be representable in $WF$ as a sum of two forms in $GP_4F$. The next result shows that three forms in $GP_4F$ will always suffice.

**Proposition 4.2** *Let $\varphi \in I^4F$ be anisotropic with $\dim \varphi = 24$. Then there exist $\pi \in P_2F$, an Albert form $\alpha \in I^2F$, and $\rho \in GP_4F$ such that $\varphi = \pi \otimes \alpha + \rho$ in $WF$.*

P r o o f. Let $d \in \dot{F}$ such that $\varphi_L$ is isotropic where $L = F(\sqrt{d})$. Since $\varphi_L \in I^4L$, it follows from the Main Theorem and APH that either $\varphi_L = 0$ in $WL$, or $\dim(\varphi_L)_{\mathrm{an}} = 16$. In the first case we are done by Proposition 4.1(iii) and by choosing $\rho$ to be hyperbolic.

So we may assume that we are in the second case. By Lemma 2.1(i), there exist $\delta$, $\tilde{\varphi} \in WF$ with $\dim \delta = 4$, $\dim \tilde{\varphi} = 16$, such that $\varphi \simeq \tilde{\varphi} \perp \langle\langle -d \rangle\rangle \otimes \delta$ and we have $\tilde{\varphi}_L \simeq (\varphi_L)_{\mathrm{an}} \in GP_4 L$. Note that $d_\pm \tilde{\varphi} = 1$ in $\dot{F}/\dot{F}^2$. Thus, by Lemma 3.2(i), there exist $e \in \dot{F}$, $\epsilon$, $\psi \in WF$ with $\dim \epsilon = 4$ and $\dim \psi = 8$ such that $\tilde{\varphi} \simeq \langle\langle -e \rangle\rangle \otimes \epsilon \perp \psi$. Hence,

$$\varphi \simeq \langle\langle -e \rangle\rangle \otimes \epsilon \perp \psi \perp \langle\langle -d \rangle\rangle \otimes \delta \ .$$

After scaling, we may assume that $\delta \simeq \langle 1, u, v, w \rangle$ for suitable $u, v, w \in \dot{F}$. Let $\sigma \simeq \langle\langle -d, u, v \rangle\rangle$. We then get $\varphi \perp -\sigma = \tilde{\varphi} \perp \langle\langle -d \rangle\rangle \otimes \langle w, -uv \rangle$. Suppose the right hand side is isotropic. Then $\dim(\varphi \perp -\sigma)_{\mathrm{an}} \leq 18$, i.e. $i_W(\varphi \perp -\sigma) \geq 7$ and we are done by Proposition 4.1(vii) because $\varphi$ contains a 7-dimensional Pfister neighbor of $\sigma$ in this case. We may therefore assume that

$$(\varphi \perp -\sigma)_{\mathrm{an}} \simeq \tilde{\varphi} \perp \langle\langle -d \rangle\rangle \otimes \langle w, -uv \rangle \simeq \langle\langle -e \rangle\rangle \otimes \epsilon \perp \psi \perp \langle\langle -d \rangle\rangle \otimes \langle w, -uv \rangle \ .$$

Let $\tilde{\psi} \simeq \psi \perp \langle\langle -d \rangle\rangle \otimes \langle w, -uv \rangle$. Then $\dim \tilde{\psi} = 12$ and, over $E = F(\sqrt{e})$, we have

$$\tilde{\psi}_E \equiv \varphi_E - \sigma_E \equiv \sigma_E \pmod{I^4 E} \ .$$

Hence, $\tilde{\psi}_{E(\sigma)} \equiv 0 \pmod{I^4 E(\sigma)}$ and APH implies that $\tilde{\psi}_{E(\sigma)} = 0$ in $WE$. By Lemma 2.1(ii), two cases can occur (recall that $\dim \tilde{\psi} = 12$!). Either $\tilde{\psi}_E = 0$ which then also implies that $\sigma_E = 0$ and hence that $\varphi_E = 0$ as well, so that we are done by Proposition 4.1(iii). Or $(\tilde{\psi}_E)_{\mathrm{an}} \simeq x\sigma_E$ for some $x \in \dot{E}$ (again, the excellence of $E/F$ implies that one may choose $x \in \dot{F}$ already). By Lemma 2.1(i), there exist $\epsilon_1$, $\mu \in WF$ with $\dim \epsilon_1 = 2$, $\dim \mu = 8$, such that $\tilde{\psi} \simeq \langle\langle -e \rangle\rangle \otimes \epsilon_1 \perp \mu$ and $\mu_E \simeq (\tilde{\psi}_E)_{\mathrm{an}} \simeq x\sigma_E \in GP_3 E$. Note that $d_\pm \tilde{\psi} = d_\pm(\varphi \perp -\sigma) = 1$ and $d_\pm(\langle\langle -e \rangle\rangle \otimes \epsilon_1) = 1$. Thus, $d_\pm \mu = 1$ and by Lemma 3.1(i), there exist $r \in \dot{F}$, $\epsilon_2 \in WF$ with $\dim \epsilon_2 = 2$, and $\beta \in P_3 F$ such that $\mu = r\beta + \langle\langle -e \rangle\rangle \otimes \epsilon_2$ in $WF$. In particular, $r\beta_E \simeq x\sigma_E$ and hence $\beta_E \simeq \sigma_E$. By Corollary 2.4, there exists $s \in \dot{F}$ such that $\beta \simeq \langle\langle s, \cdots \rangle\rangle$ and $\sigma \simeq \langle\langle s, \cdots \rangle\rangle$. It follows that $\beta_M = 0$ for $M = F(\sqrt{-s})$ which in turn yields $\mu_M = (\langle\langle -e \rangle\rangle \otimes \epsilon_2)_M$. Comparing dimensions, we see that $\mu_M$ is isotropic so that by Lemma 2.1(i) there exists $y \in \dot{F}$ with $y\langle\langle s \rangle\rangle \subset \mu$. We then have $\dim(\mu \perp -y\sigma)_{\mathrm{an}} \leq 12$, in particular we have that $\mu \perp -y\sigma$ is isotropic, and also

$$\mu_E \perp -y\sigma_E \simeq r\beta_E \perp -y\sigma_E \simeq r\sigma_E \perp -y\sigma_E \simeq (\sigma \otimes \langle r, -y \rangle)_E \in GP_4 E \ .$$

This together implies that $(\mu \perp -y\sigma)_E \in GP_4 E$ is isotropic and hence hyperbolic. By Lemma 2.1(ii) and the above dimension consideration, we have that $(\mu \perp -y\sigma)_{\mathrm{an}} \simeq \langle\langle -e \rangle\rangle \otimes \eta$ for some $\eta \in WF$ with $\dim \eta \leq 6$. Let $\rho \simeq \sigma \otimes \langle\langle y \rangle\rangle \in P_4 F$ and let $\gamma \simeq (\varphi \perp -\rho)_{\mathrm{an}} \in I^4 F$. If $\dim \gamma < 24$ then, by the Main Theorem and APH, $\gamma = \tau$ in $WF$ for some $\tau \in GP_4 F$ so that $\varphi = \rho + \tau$ in $WF$ and we are done. So suppose $\dim \gamma \geq 24$. In $WF$, we have

$$\begin{aligned} \gamma &= \varphi - \rho = \varphi - \sigma - y\sigma \\ &= \langle\langle -e \rangle\rangle \otimes \epsilon + \tilde{\psi} - y\sigma \\ &= \langle\langle -e \rangle\rangle \otimes \epsilon + \langle\langle -e \rangle\rangle \otimes \epsilon_1 + \mu - y\sigma \\ &= \langle\langle -e \rangle\rangle \otimes \epsilon + \langle\langle -e \rangle\rangle \otimes \epsilon_1 + \langle\langle -e \rangle\rangle \otimes \eta \ . \end{aligned}$$

The right hand side has dimension $2(\dim \epsilon + \dim \epsilon_1 + \dim \eta) \leq 2(4+2+6) = 24$. Hence, $\dim \gamma = 24$. Note also that $\gamma_E = 0$. By Proposition 4.1(iii), it follows that $\gamma \simeq \pi \otimes \alpha$ for some $\pi \in P_2 F$ and some Albert form $\alpha$. Therefore, $\varphi = \gamma + \rho = \pi \otimes \alpha + \rho$ in $WF$. $\square$

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