THE SEMI-SIMPLE ZETA FUNCTION OF SHIMURA VARIETIES ASSOCIATED TO QUATERNION ALGEBRAS

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Abstract

For the Shimura variety $Sh_{D,C}$ attached to a quaternion division algebra D over a totally real number field F and a sufficiently small open compact subgroup C of $D^\times(\mathbb{A}_f)$, and for every rational prime p which is unramified in F and such that $C = C_p \times C^p$ where C_p is a maximal compact subgroup of $D^\times(\mathbb{Q}_p)$ and C^p is a subgroup of $D^\times(\mathbb{A}_f^p)$, the semi-simple local zeta function $Z_p^{\mathrm{ss}}(Sh_{D,C}, s)$ is expressed as a product of integral powers of semi-simple local L–functions of automorphic representations of $D^\times(\mathbb{A})$. In an appendix a mistake occuring in the constructions of Langlands and Rapoport needed to state their conjecture on the reduction of a Shimura variety in the most general case is corrected. This conjecture is proved for $Sh_{D,C}$ up to a question concerning the reduction of abelian varieties of CM–type which is discussed in another appendix.

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Introduction

The question whether the Hasse-Weil zeta function of an algebraic variety over a number field — which is a priori defined in a certain right half-plane — admits a meromorphic continuation to the whole complex plane and satisfies a functional equation is one of the central problems in algebraic number theory. Conjecturally, these functions are products of integral powers of automorphic L–functions. Thus the problem is naturally divided into two different parts: the proof of this conjecture and the study of the analytic properties of automorphic L–functions. We restrict our attention to the first task, and since the general case appears to be inaccessible at the present time, we also restrict the class of varieties to be considered.

Concerning a *Shimura variety* which is associated to a connected algebraic group G over $\mathbb{Q}$ and a $G(\mathbb{R})$–conjugacy class X of homomorphisms $h\colon \mathbb{C}^\times \to G_{\mathbb{R}}$ of algebraic groups over $\mathbb{R}$ satisfying conditions (2.1.1.1–3) of [10], it seems reasonable to look for an expression of its zeta function in terms of L–functions of automorphic representations of $G(\mathbb{A})$. The study of Shimura varieties started with the work of Eichler, Shimura, Kuga, Sato and Ihara, who considered the curves corresponding to Gl_2 and its inner forms. After introducing L–groups and the principle of functoriality, Langlands [21], [22], [23] gave a conjectural description of the zeta function of a general Shimura variety. It turns out that in general — due to the effect of L*–indistinguishability* — one has to consider automorphic representations of other groups than G, as well. These are the *endoscopic groups* for G. In the case we will ultimately be concerned with this phenomenon does not play a role, however, so that we shall not go into further details at this point. Moreover, we assume that the derived group of G is anisotropic over $\mathbb{Q}$. This implies that the Shimura variety associated to G and X is a projective system $(\,Sh_C \mid C \subset G(\mathbb{A}_f)\,)$ of *projective* varieties over a number field $E(G,X) \subset \mathbb{C}$. Its complex points are given by

$$Sh_C(\mathbb{C}) = G(\mathbb{Q})\backslash\big(X \times G(\mathbb{A}_f)/C\big)\,.$$

On the projective limit over all open compact subgroups C of $G(\mathbb{A}_f)$ there is a natural continuous right action of $G(\mathbb{A}_f)$. If C is sufficiently small — as we will assume from now on — Sh_C is smooth. The cohomology groups $W_C^i = \mathrm{H}^i(Sh_C(\mathbb{C})\,,\mathbb{C})$ are in a natural way modules over the algebra $\mathcal{H}_C$ of C–bi-invariant functions in the Hecke algebra $\mathcal{H} = \mathcal{H}(G(\mathbb{A}_f))$, and we can consider their decomposition into isotypic components

$$W_C^i = \bigoplus_{\pi_f} W^i(\pi_f) \otimes \pi_f^C \quad \text{with} \quad W^i(\pi_f) = \mathrm{Hom}_{\mathcal{H}}(\pi_f\,, \varinjlim_C W_C^i)$$

where the sum is extended over the set of isomorphism classes of irreducible admissible representations of $G(\mathbb{A}_f)$ and π_f^C denotes the invariants of C in π_f.

On the other hand, if we fix a rational prime ℓ and a prime $\wp$ of $E(G,X)$ not dividing ℓ, and denote the localization of $E(G,X)$ at $\wp$ by E, then we have continuous representations ξ^i of the Galois group of E on the ℓ–adic cohomology groups $V_C^i = \mathrm{H}^i(Sh_C \otimes_{E(G,X)} \overline{E}, \mathbb{Q}_\ell)$. Let $W_E \subset \mathrm{Gal}(\overline{E}|E)$ denote the Weil group of E, I the inertia subgroup, Φ a geometric Frobenius element in W_E, and $\tau\colon I \to \mathbb{Q}_\ell$ a non-trivial continuous homomorphism. Then there is a unique nilpotent endomorphism N of V_C^i such that, for all γ in an appropriate open subgroup $I_0 \subseteq I$, $\xi^i(\gamma) = \exp(\tau(\gamma)N)$.

If we choose an embedding $\iota\colon \mathbb{Q}_\ell \to \mathbb{C}$, we obtain a canonical isomorphism $V_C^i \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong W_C^i$ which allows us to convert the ℓ–adic representation ξ^i into a (complex algebraic) representation ξ_ι^i of the Weil-Deligne group $W_E' = \mathbb{C} \rtimes W_E$ on W_C^i such that

$$\xi_\iota^i(x, \Phi^n\gamma) = \exp(N \otimes x)\left(\xi^i(\Phi^n\gamma)\exp(-\tau(\gamma)N) \otimes 1\right)$$

for all $x \in \mathbb{C}$, $n \in \mathbb{Z}$, and $\gamma \in I$. Obviously, these representations form a projective system with respect to C, and since the action of $G(\mathbb{A}_f)$ on the projective limit of the varieties Sh_C is defined over $E(G,X)$, this commutes with the action of $\mathcal{H}$. Hence we obtain a representation of W_E' on each $W^i(\pi_f)$ which is, up to isomorphism, independent of the choice of τ and Φ. Applying Matsushima's formula (cf. [3], Chapter VII, Theorem 3.2) these spaces can be decomposed into finite direct sums (with multiplicities) of Lie algebra cohomology groups

$$W^i(\pi_f) = \bigoplus_{\pi_\infty} m(\pi_f \otimes \pi_\infty)\, \mathrm{H}^i(\mathfrak{g}\,, C_\infty\,; \pi_\infty)$$

where $\mathfrak{g} = \mathrm{Lie}\, G(\mathbb{R})$ and C_∞ is the centralizer of h in $G(\mathbb{R})$. The sum is taken over all isomorphism classes of irreducible admissible representations of $Z(\mathbb{R})\backslash G(\mathbb{R})$ with trivial infinitesimal character, and $m(\pi_f \otimes \pi_\infty)$ is the multiplicity of $\pi_f \otimes \pi_\infty$ in the space $\mathcal{A}$ of automorphic forms on $G(\mathbb{Q})Z(\mathbb{R})\backslash G(\mathbb{A})$ where Z denotes the centre of G. In general, this sum may involve more than one non-vanishing term, so that the contribution of a single automorphic representation is not apparent, but for the Shimura varieties we consider in this paper, the situation is better.

Let $G = D^\times$ be the multiplicative group of a quaternion division algebra D over a totally real number field F. There is an obvious conjugacy class X as above (cf. §1), and the dimension d of the corresponding Shimura variety is equal to the number of infinite places of F at which D splits. Moreover, for every π_f, there is at most one π_∞ such that $\pi_f \otimes \pi_\infty$ is contained in $\mathcal{A}$, and in this case the multiplicity is 1. We assume that $d > 0$. Then, apart from the 1–dimensional π_∞ which have non-zero cohomology groups for even i, there is (up to isomorphism) only one representation π_∞° of $Z(\mathbb{R})\backslash G(\mathbb{R})$ with trivial infinitesimal character which can occur as factor at infinity of a constituent of $\mathcal{A}$, and this has non-zero cohomology only for $i = d$. If $\pi = \pi_f \otimes \pi_\infty$ is any irreducible constituent of $\mathcal{A}$, we define

$$W(\pi) = \begin{cases} \bigoplus_{i=0}^{d} W^{2i}(\pi_f) & \text{if } \dim \pi = 1 \\ W^d(\pi_f) & \text{if } \pi_\infty = \pi_\infty^\circ \\ 0 & \text{otherwise} \end{cases}$$

and $m(\pi, C) = m(\pi_\infty) \dim(\pi_f^C)$ where

$$m(\pi_\infty) = \begin{cases} 1 & \text{if } \dim \pi_\infty = 1 \\ (-1)^d & \text{if } \pi_\infty = \pi_\infty^\circ \\ 0 & \text{otherwise}\,. \end{cases}$$

Then we obtain

$$\bigoplus_{i \geqslant 0} (-1)^i V_C^i \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong \bigoplus_{\pi} m(\pi, C) W(\pi)$$

as virtual representations of W_E'. The sum on the right side is taken over all constituents of $\mathcal{A}$, and if $m(\pi, C) \neq 0$, then $W(\pi)$ has dimension 2^d. In particular, this yields the following expression for the local factor of the zeta function.

$$Z_\wp(Sh_C, s) = \prod_{\pi} L(W(\pi), s)^{m(\pi, C)} \tag{1}$$

where $L(W(\pi), s) = \det\bigl(1 - \Phi q^{-s} \bigm| W(\pi)^{I'}\bigr)^{-1}$. Here q denotes the number of elements in the residue class field $\kappa(\wp)$ of $\wp$, and $W(\pi)^{I'} \subseteq W(\pi)$ is the subspace of invariants with respect to $I' = \mathbb{C} \times I$.

Let us, for a moment, return to the case of an arbitrary reductive group G. Langlands [23] associates to the conjugacy class X a representation (r°, V°) of the L–group ${}^L G_{E(G,X)} = \widehat{G} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}|E(G, X))$. On the other hand, if p denotes the rational prime such that $\wp|p$, and π is an automorphic representation of $G(\mathbb{A})$, one would like to associate to its local factor at p, which is an irreducible admissible representation π_p of $G(\mathbb{Q}_p)$, an admissible homomorphism (cf. [2], Chapter III)

$$\varphi(\pi_p)\colon W'_{\mathbb{Q}_p} = \mathbb{C} \rtimes W_{\mathbb{Q}_p} \longrightarrow {}^L G_{\mathbb{Q}_p} = \widehat{G} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)\,.$$

If we fix a commutative diagram

$$\begin{tikzcd} E(G,X) \arrow[r, dotted] \arrow[d, dotted] & \overline{\mathbb{Q}} \subset \mathbb{C} \arrow[d] \\ E = E(G,X)_\wp \arrow[r] & \overline{\mathbb{Q}}_p \end{tikzcd}$$

where the dotted arrows indicate the natural inclusions, such a homomorphism would give rise to a representation $(r^\circ \circ \varphi(\pi_p)|_{W'_E}, V^\circ)$ of W'_E. Hence we could attach to π a local $L-$factor

$$L_\wp(\pi, r^\circ, s) = \det\big(1 - (r^\circ \circ \varphi(\pi_p))(\Phi)\, q^{-s} \mid (V^\circ)^{I'}\big)^{-1}.$$

In general, however, it is not known how to define $\varphi(\pi_p)$.

In the case $G = D^\times$ — which we will consider from now on — the local Langlands correspondence is established. Thus the local $L-$factors of an automorphic representation π are well-defined. Moreover, in this case $\dim V^\circ = 2^d$, so one might ask whether equation (1) remains true when $L(W(\pi), s)$ is replaced by $L_\wp(\pi, r^\circ, s - \frac{d}{2})$. The shift by half the dimension of the Shimura variety will clearly be necessary to obtain the same weights (at least conjecturally) on both sides. As Langlands has observed, the general answer will be negative due to the incorrect contribution of certain *anomalous* representations (cf. [23]).

In addition to this difficulty on the automorphic side, there is another serious difficulty on the geometric side which we must leave unsettled for now. If the variety under consideration has good reduction at the fixed place $\wp$, the corresponding local factor of its zeta function is given by the zeta function of the special fibre. Thus the problem to determine it is reduced to the calculation of the zeta function of a variety over a finite field, i.e. one has to find an expression for the number of points with values in every finite extension of this field. In the case of bad reduction, however, the local factor of the zeta function is not so obviously related to the information provided by the special fibre.

Therefore, in order to come to an assertion we are in a position to prove, we shall confine ourselves to a variant of this local factor, the *semi-simple local zeta function* introduced by Rapoport [34]. It is obtained by replacing the cohomology groups V_C^i occuring in the definition of the local factor of the Hasse-Weil zeta function by graded modules (associated to Galois invariant filtrations) on which the inertia subgroup I acts through a finite quotient. If I acts trivially on the sheaves of vanishing cycles on the special fibre (cf. [8], §2), this function can be identified with the alternating product of the zeta functions of these sheaves using the spectral sequence of vanishing cycles. In this case, in order to calculate the semi-simple local zeta function, one needs an appropriate description of the action of the Frobenius automorphism on the stalks of the sheaves of vanishing cycles at all points with values in an algebraic closure $\mathbb{F}$ of $\kappa(\wp)$.

Again the question arises whether equation (1) remains true when the left side is replaced by its semi-simple variant $Z_\wp^{\mathrm{ss}}(Sh_C, s)$ and $L(W(\pi), s)$ by the semi-simple version $L_\wp^{\mathrm{ss}}(\pi, r^\circ, s - \frac{d}{2})$ of the local $L-$factor attached to π which is obtained by replacing V° by a graded module on which I' acts via a finite quotient. The idea to consider this modified question is due to Rapoport (cf. [34], §2). In the present paper, we prove that the resulting equality holds for the Shimura variety attached to $D^\times$ and an open compact subgroup C at every place $\wp|p$ such that p is unramified in F and such that $C = C_p \times C^p$ where C_p is a maximal compact subgroup of $D^\times(\mathbb{Q}_p)$ and C^p is a sufficiently small open subgroup of $D^\times(\mathbb{A}_f^p)$ (cf. (6.1)). Moreover, we show that in this case the original question concerning the local factor of the Hasse-Weil zeta function has an affirmative answer if the map from the set of archimedean places of F to the set of primes of F dividing p which is induced by the fixed embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$ respects the subsets of places at which D splits (case of good reduction).

In two special cases this result has been obtained before. If D splits at all places of F dividing p, Sh_C has good reduction at the places above p. In this case, the semi-simple variants of the local factors of the zeta and L–functions coincide with their original versions, and we recover a result of Langlands [22] which is based on a conjectural description of the set of points with values in $\mathbb{F}$ together with the action of the Frobenius automorphism. This description was verified by Milne [27] (following ideas of Langlands) in the case of a totally indefinite quaternion algebra and, in general, by Zink and myself [36]. If, on the other hand, p stays prime in F and D is ramified at the corresponding place of F and split at all archimedean places of F, Sh_C is defined over $\mathbb{Q}$ and has bad reduction at p. This case has been treated by Rapoport [33] using a very precise description of a reduction at p given by Zink [43].

We shall now give a short survey of the contents of the present paper. Its first part comprising §§1–7 is devoted to the construction and detailed study of a reduction of the Shimura variety attached to $D^\times$ and C at a place $\wp$ as above. Apart from the totally indefinite case, this variety is not of PEL type. In §1 we recall the main idea of its construction (cf. [7], §6), namely the fact that there are embeddings as open and closed subschemes into Galois twists of various Shimura varieties which are coarse moduli schemes parameterizing abelian varieties with some additional structure. Moreover, although these embeddings are defined only over certain finite extensions of the original reflex field, for a given place $\wp$, the data G and h defining such a Shimura variety can be chosen so that this extension has an embedding into the local field E.

In §2 we extend the moduli problem to the category of O_E–schemes and show that it admits a coarse moduli scheme $\widetilde{Sh}_{G,C}$ which is proper over O_E. The main point is to provide a suitable replacement of the trace condition in [7] (4.11). In §3 we study the completion of the local ring of this moduli scheme with respect to the étale topology at a geometric point in $\widetilde{Sh}_{G,C}(\mathbb{F})$, and we show that the geometric special fibre is a union of smooth projective varieties which intersect transversally. The corresponding results of Zink [43] (§3) in the special case mentioned above have served as a guide to find the correct assertions.

In §4 we classify the set of $\mathbb{F}$–valued points of the moduli problem up to isogeny. At this, unless D is totally indefinite or $\widetilde{Sh}_{G,C}$ is smooth, there appear obstructions for the existence of those isogeny types which cannot be obtained by reducing special points of the general fibre. These obstructions which complicate the situation are discussed in a somewhat more general setting in Appendix A. Roughly speaking the problem is the following. Consider a complex abelian variety A_0 of CM–type which is defined over a number field $L \subset \mathbb{C}$, and has good reduction A over the algebraic closure of the residue class field of a prime of L dividing p. Then $H = \mathrm{H}_1(A_0, \mathbb{Q})$ admits a canonical embedding into the rational Tate module $\widehat{\mathrm{V}}^p(A)$. If an endomorphism of A does not come from an endomorphism of A_0, it does not respect H. The question arises whether there are other "compatibilities" between H and the action of $\mathrm{End}^\circ A$ on $\widehat{\mathrm{V}}^p(A)$. This question is made precise under certain restrictive assumptions in §A1. The discussion in §A2 and §A3 shows that an affirmative answer would ensure the existence of "all possible isogeny classes" in $\widetilde{Sh}_{G,C}(\mathbb{F})$.

In §5 we describe the set of points within a fixed isogeny class together with the action of the Frobenius automorphism over $\kappa(\wp)$ and with the information needed in §3 to determine the local structure at a given point. Using the embedding considered in §1 the results obtained for the Shimura variety attached to G can be translated into corresponding statements concerning the Shimura variety for $D^\times$. This is done in §6. In particular, we obtain a projective flat O_E–scheme $\widetilde{Sh}_{D,C}$ which satisfies the condition (PgD) in [35] (§3). Hence we can use the results of loc. cit. which turn the description of the local structure obtained in §3 into a description of the sheaves of vanishing cycles. At this point the main result of the first part is achieved. It is summarized in Theorem 6.6.

In the remainder of §6 we compare our description of the set of $\mathbb{F}$–valued points of $\widetilde{Sh}_{D,C}$ with the general conjecture of Langlands and Rapoport which is reviewed in Appendix B. Due to a mistake in the constructions in [24] preceding the statement of the conjecture, we have to give a new definition for a quasi-motivic groupoid replacing the definition of the quasi-motivic gerb (loc. cit.). Note that we cannot work with the pseudo-motivic gerb (for which the mistake does not occur) since our Shimura variety does not have rational weight unless D is totally indefinite. Moreover, we formulate the conjecture more precisely and under slightly weaker assumptions, so that it applies to our situation. It turns out that $\widetilde{Sh}_{D,C}$ has the properties predicted by this conjecture if and only if all of the obstructions for the existence of isogeny types mentioned above vanish.

The first part of the present paper is completed in §7 where the assertions of Theorem 6.6 are translated into a description of the set of points with values in any finite extension $\kappa\,|\,\kappa(\wp)$ together with the function provided by the alternating sum of the traces of the Frobenius automorphism over κ acting on the stalks of the sheaves of vanishing cycles.

In the second part which consists of §§ 8–11, we establish the expression for the semi-simple local zeta function in terms of semi-simple automorphic L–functions asserted above. In § 8 we use the description obtained in § 7 to derive a formula for the semi-simple traces which appear as coefficients in a power series expansion of the logarithm of the semi-simple local zeta function relating them to certain orbital and twisted orbital integrals. The procedure to obtain this formula is the same as that of Rapoport [33] which in turn is based on the method used by Kottwitz [18] in the case of a hyperspecial subgroup C_p.

In § 9 we compare the twisted orbital integrals occuring in this formula with the orbital integrals of certain explicitly defined functions in the Hecke algebra $\mathcal{H}(D^\times(\mathbb{Q}_p)//C_p)$. For this we use again ideas of Rapoport [33] (although some of the details in § 5 of loc. cit. are incorrect). In particular, we obtain a vanishing result for these twisted orbital integrals which implies that the intersection of $\widetilde{Sh}_{D,C}(\mathbb{F})$ with an isogeny class of the moduli problem considered in § 4 does not contribute to the semi-simple traces if it does not contain the reduction of a special point of the general fibre. In the special case treated in [33] this corresponds to the fact that only the supersingular isogeny class contributes to the semi-simple traces. An analogous fact was conjectured by Rapoport and proved by Waldspurger for the Shimura variety considered in [34] (conjecture 5.7 and the note on page 318). This observation eliminates the appearance of the obstructions mentioned above in the formula obtained in § 8.

Applying the Selberg trace formula and the comparison accomplished in § 9 to the expression obtained so far, we can identify the semi-simple traces with traces of certain functions in $C_c^\infty(D^\times(\mathbb{A})/F^\times(\mathbb{R}))$ acting on the space of automorphic forms on $D^\times F^\times(\mathbb{R})\backslash D^\times(\mathbb{A})$. This is done in § 10. Finally, in § 11 we introduce the semi-simple automorphic L–functions needed to deduce the main result which is summarized in Theorem 11.6.

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Part I

§ 1. The basic embedding

Let $D^\times$ denote the algebraic group over $\mathbb{Q}$ associated to a quaternion division algebra D over a totally real number field F, and let $\mathcal{Z}$ and $\mathcal{V}$ denote the sets of infinite places of F at which D is split or ramified, respectively. We assume that $\mathcal{Z}$ is not empty, and for each $v \in \mathcal{Z}$, we fix an embedding of $\mathbb{R}$–algebras $\mathbb{C} \subset D_v = D \otimes_{F,v} \mathbb{R}$. This defines a homomorphism of algebraic groups over $\mathbb{R}$

$$\begin{aligned} h_D \colon \mathbb{S} = \mathbb{C}^\times &\longrightarrow \prod_{v \in \mathcal{Z}} D_v^\times \subseteq (D \otimes_{\mathbb{Q}} \mathbb{R})^\times \\ z &\longmapsto (\, z \mid v \in \mathcal{Z}\,)\,. \end{aligned}$$

The conjugacy class X of h_D which does not depend on the choice of the embeddings $\mathbb{C} \subset D_v$ defines a Shimura variety Sh_D with a continuous right action of $D^\times(\mathbb{A}_f)$ over the totally real number field

$$E(D) = \mathbb{Q}\Big(\sum_{v \in \mathcal{Z}} v(f) \;\Big|\; f \in F \Big) \subset \mathbb{C}.$$

For every open compact subgroup $C \subset D^\times(\mathbb{A}_f)$, $Sh_{D,C} = Sh_D/C$ is a projective variety over $E(D)$, and

$$Sh_{D,C}(\mathbb{C}) = D^\times \backslash \big(X \times D^\times(\mathbb{A}_f)/C \big)$$

where $d \in D^\times$ acts on X as $h \mapsto dhd^{-1}$ and on the second factor as left multiplication by d. We recall the construction of $Sh_{D,C}$ in [7] (§ 6). Fix a totally imaginary quadratic extension $K|F$ with complex conjugation $^-$ and a subset $\mathcal{V}_K$ of $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$ such that restriction to $F \subset K$ induces a bijection $\mathcal{V}_K \to \mathcal{V}$, i.e. for every $v \in \mathcal{V}$, $K_v = K \otimes_{F,v} \mathbb{R}$ is identified with $\mathbb{C}$. Consider $K^\times$ as an algebraic group over $\mathbb{Q}$ together with the homomorphism of real algebraic groups

$$\begin{aligned} h_K \colon \mathbb{S} &\longrightarrow \prod_{v \in \mathcal{V}} K_v^\times \subset (K \otimes_{\mathbb{Q}} \mathbb{R})^\times \\ z &\longmapsto (\, z \mid v \in \mathcal{V}\,) \end{aligned}$$

determined by $\mathcal{V}_K$. It defines a Shimura variety Sh_K over a finite extension $E(K)$ of $E(D)$ in $\mathbb{C}$ with a continuous action of $K^\times(\mathbb{A}_f)$. For every open compact subgroup $C \subset K^\times(\mathbb{A}_f)$, $Sh_{K,C} = Sh_K/C$ is a finite reduced scheme over $E(K)$, and

$$Sh_{K,C}(\overline{\mathbb{Q}}) = K^\times(\mathbb{A}_f)/CK^\times$$

where $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}|E(K))$ factors through a finite abelian quotient which, by class field theory, is a quotient of $E(K)^\times(\mathbb{A})$. For this identification we follow the conventions of Deligne [10], i.e. the Artin reciprocity maps of local and global class field theory are normalized so that a local uniformizing parameter corresponds to a geometric Frobenius element. There is a homomorphism of algebraic groups over $\mathbb{Q}$, the reciprocity law of Sh_K

$$r_K\colon E(K)^\times \longrightarrow K^\times$$

such that the image in this quotient of an idèle with finite component $\xi \in E(K)^\times(\mathbb{A}_f)$ acts as multiplication by $r_K(\xi) \in K^\times(\mathbb{A}_f)$. $E(K)$ and r_K can be described as follows. Consider the homomorphism of complex algebraic groups

$$\begin{aligned} \mu_{K,\mathbb{C}}\colon \mathbb{G}_{m,\mathbb{C}} &\longrightarrow (K \otimes_{\mathbb{Q}} \mathbb{C})^\times \\ z &\longmapsto h_{K,\mathbb{C}}(z,1) \end{aligned}$$

where we have identified $\mathbb{S}_{\mathbb{C}}$ with $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ via the isomorphism of complex algebraic groups induced by the $\mathbb{C}$–algebra isomorphism

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\longrightarrow \mathbb{C} \times \mathbb{C} \\ x \otimes y &\longmapsto (xy, \overline{x}y)\,. \end{aligned}$$

$E(K)$ is the field of definition of $\mu_{K,\mathbb{C}}$, and r_K is the map

$$E(K)^\times \xrightarrow{\mathrm{Res}_{E(K)|\mathbb{Q}}(\mu_K)} (K \otimes_{\mathbb{Q}} E(K))^\times \xrightarrow{\mathrm{Nm}_{E(K)|\mathbb{Q}}} K^\times$$

where $\mathrm{Res}_{E(K)|\mathbb{Q}}$ denotes restriction of scalars and $\mathrm{Nm}_{E(K)|\mathbb{Q}}$ the norm map.

Lemma 1.1 *$\mathrm{Gal}(\overline{\mathbb{Q}}|E(K))$ is the stabilizer of $\mathcal{V}_K$ in $\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, and for every embedding φ of K into $\overline{\mathbb{Q}}$ and every $x \in E(K)^\times$,*

$$\varphi r_K(x) = \prod_{\xi \in \Xi(\varphi)} \xi(x)$$

where

$$\Xi(\varphi) = \{\, \xi \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \mid \varphi \in \xi\mathcal{V}_K \,\} / \mathrm{Gal}(\overline{\mathbb{Q}}|E(K))\,.$$

Proof. Straightforward. ∎

Let * denote the main involution on D, and consider $B = D \otimes_F K$ as a simple $D \otimes_F B$–module via

$$(d \otimes b)x = bxd^* \qquad \text{for all } d \in D \text{ and } b, x \in B\,.$$

We choose a positive involution $'$ on B extend it to an involution of $D \otimes_F B$ such that $(d \otimes b)' = d^* \otimes b'$ and fix a non-degenerate alternating $\mathbb{Q}$–bilinear form

$$\psi \colon B \times B \longrightarrow \mathbb{Q}$$

such that, for all $b \in D \otimes_F B$, $x, y \in B$,

$$\psi(bx, y) = \psi(x, b'y)\,.$$

The involution $'$ and the pairing ψ, up to a factor in $F^\times$, determine each other uniquely. The group of B–module automorphisms of B which preserve ψ up to a factor in $F^\times$ is the set of $\mathbb{Q}$–valued points of an algebraic group G over $\mathbb{Q}$ which occurs in the exact sequence of algebraic groups

$$\begin{aligned} &1 \longrightarrow F^\times \longrightarrow D^\times \times K^\times \xrightarrow{\tau} G \longrightarrow 1 \\ &\qquad\qquad f \longmapsto (f\,, f^{-1}) \\ &\qquad\qquad\qquad\qquad (d\,, k) \longmapsto (x \mapsto kxd^*) \end{aligned} \tag{1.2}$$

where τ is surjective on points with values in any field of characteristic 0. Let Y denote the conjugacy class of

$$h_G = \tau_{\mathbb{R}} \circ (h_D \times h_K) \colon \mathbb{S} \longrightarrow G_{\mathbb{R}}\,.$$

It is easily checked that, for an appropriate $f \in F^\times$, $\psi \circ (f \times h_G(i))$ is a positive definite symmetric bilinear form on $B \otimes_{\mathbb{Q}} \mathbb{R}$. According to Deligne [7] (§6), Y defines a Shimura variety Sh_G with a continuous right action of $G(\mathbb{A}_f)$ over $E(K)$. For every open compact subgroup $C \subset G(\mathbb{A}_f)$, $Sh_{G,C} = Sh_G/C$ is a projective variety over $E(K)$, and

$$Sh_{G,C}(\mathbb{C}) = G(\mathbb{Q}) \backslash \big(Y \times G(\mathbb{A}_f)/C\big)$$

where $g \in G(\mathbb{Q})$ acts on Y as $h \mapsto ghg^{-1}$ and on the second factor as left multiplication by g. The exact sequence (1.2) defines a $D^\times(\mathbb{A}_f) \times K^\times(\mathbb{A}_f)$–equivariant morphism over $E(K)$

$$\big(Sh_D \otimes_{E(D)} E(K)\big) \times Sh_K \longrightarrow Sh_G\,. \tag{1.3}$$

For every open compact subgroup $C \subset D^\times(\mathbb{A}_f)$, let

$$\begin{aligned} C_F &= C \cap F^\times(\mathbb{A}_f)\,; \\ N_C &= \min\{\, N > 0 \mid (1 + NO_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times \subseteq C_F \,\}\,; \\ C_K &= C_F(1 + N_C O_K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times\,; \\ \overline{C} &= \tau_{\mathbb{A}_f}(C \times C_K)\,. \end{aligned} \tag{1.4}$$

Lemma 1.5 *C_F, C_K, and $\overline{C}$ are open compact subgroups of $F^\times(\mathbb{A}_f)$, $K^\times(\mathbb{A}_f)$, and $G(\mathbb{A}_f)$, respectively, and the correspondences $C \mapsto C_F$, $C \mapsto C_K$, and $C \mapsto \overline{C}$ respect inclusions. Moreover,*

(i) $\overline{C} \cap K^\times(\mathbb{A}_f) = C_K$ *and* $\overline{C} \cap D^\times(\mathbb{A}_f) = C$ *(via τ);*

(ii) $C_K \cap F^\times(\mathbb{A}_f) = C_F$ *;*

(iii) $C_K K^\times \cap F^\times(\mathbb{A}_f) = C_F F^\times$ *if* $N_C > 2$.

Proof. Concerning (iii), it suffices to prove that, for any $N > 2$,

$$(1 + NO_K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times K^\times \cap F^\times(\mathbb{A}_f) \subseteq (1 + NO_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times F^\times .$$

The remaining assertions are obvious. Consider $x \in (1 + NO_K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times$ and $k \in K^\times$ with $xk \in F^\times(\mathbb{A}_f)$. This implies $\overline{x}x^{-1} = \overline{k}^{-1}k \in (1+NO_K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times \cap K^\times = (1+NO_K)^\times$ and $\mathrm{Nm}_{K|F}(\overline{k}^{-1}k) = 1$. Hence $\overline{k}^{-1}k$ is a root of unity which is congruent to 1 modulo N, i.e. $\overline{k}^{-1}k = 1$ (cf. [31], Lemma on page 207), hence $k \in F^\times$ and

$$x \in (1 + NO_K \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times \cap F^\times(\mathbb{A}_f) = (1 + NO_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times . \qquad \blacksquare$$

Using this lemma, we see that, for any $C \subset D^\times(\mathbb{A}_f)$ with $N_C > 2$, (1.3) defines a Galois covering

$$\big(Sh_{D,C} \otimes_{E(D)} E(K)\big) \times Sh_{K,C_K} \longrightarrow Sh_{G,\overline{C}}$$

with Galois group $F^\times(\mathbb{A}_f)/C_F F^\times$. After an appropriate Galois twist, this yields a constant Galois covering (with the same group)

$$\big(Sh_{D,C} \otimes_{E(D)} E(K)\big) \times K^\times(\mathbb{A}_f)/C_K K^\times \longrightarrow M_{\overline{C}} \tag{1.6}$$

where the projective variety $M_{\overline{C}}$ over $E(K)$ is defined as follows. τ identifies $K^\times$ with the centre of G and C_K with a subgroup of $\overline{C}$, hence defines an action of $K^\times(\mathbb{A}_f)/C_K K^\times$ on $Sh_{G,\overline{C}}$. Let L_C denote the finite abelian extension of $E(K)$ such that

$$r_K \colon E(K)^\times(\mathbb{A}_f) \longrightarrow K^\times(\mathbb{A}_f)$$

induces an injection

$$r_{K,C} \colon \mathrm{Gal}(L_C | E(K)) \longrightarrow K^\times(\mathbb{A}_f)/C_K K^\times .$$

Then there is an isomorphism ϕ_C of L_C–schemes such that the diagram

$$\begin{array}{ccc} M_{\overline{C}} \otimes_{E(K)} L_C & \xrightarrow{\phi_C} & Sh_{G,\overline{C}} \otimes_{E(K)} L_C \\ \Big\downarrow{\scriptstyle \mathrm{id} \times \mathrm{Spec}(\xi)} & & \Big\downarrow{\scriptstyle r_{K,C}(\xi) \times \mathrm{Spec}(\xi)} \\ M_{\overline{C}} \otimes_{E(K)} L_C & \xrightarrow[\phi_C]{} & Sh_{G,\overline{C}} \otimes_{E(K)} L_C \end{array} \tag{1.7}$$

commutes for every $\xi \in \mathrm{Gal}(L_C | E(K))$. The existence and uniqueness of $M_{\overline{C}}$ follows by descent theory.

Projection onto the second factor in (1.6) defines a morphism

$$\pi_C \colon M_{\overline{C}} \longrightarrow K^\times(\mathbb{A}_f)/F^\times(\mathbb{A}_f)C_K K^\times \tag{1.8}$$

together with a natural isomorphism of $E(K)$–schemes

$$Sh_{D,C} \otimes_{E(D)} E(K) \longrightarrow \pi_C^{-1}(1) \subseteq M_{\overline{C}}\,. \tag{1.9}$$

Using (1.7), we get a continuous right action of $G(\mathbb{A}_f)$ on the projective limit of the varieties $M_{\overline{C}}$ with respect to the directed set of all open compact subgroups of $D^\times(\mathbb{A}_f)$. (1.9) defines a closed embedding

$$Sh_D \otimes_{E(D)} E(K) \longrightarrow \varprojlim_{C} M_{\overline{C}}$$

which is $D^\times(\mathbb{A}_f)$–equivariant with respect to τ. The isomorphism (1.9) plays an essential role in the construction and analysis of a reduction of $Sh_{D,C}$ since it reduces the problem to the corresponding problem for a Shimura variety of PEL type which will be treated in the following sections.

§2. A reduction of Sh_G

According to Deligne [7] (4.14), for every open compact subgroup $C \subset G(\mathbb{A}_f)$, $Sh_{G,C}$ is a coarse moduli scheme for a contravariant functor $\underline{M}_{G,C}$ on the category of $E(K)$–schemes which we recall after fixing some notation and conventions.

First of all, schemes which are not explicitly specified (e.g. Sh_D) are always *locally noetherian* and abelian schemes are always *projective* (in the sense of Grothendieck [16], 5.5). Let S be a connected scheme and p a rational prime. An isogeny of abelian schemes over S is called p–principal if its degree, i.e. the rank of its kernel which is a finite flat group scheme over S, is prime to p.

Consider an abelian scheme A over S, and let A_s denote the fibre of A over a geometric point s of S. If S is a $\mathbb{Z}_{(p)}$–scheme, $\widehat{\mathrm{T}}^p(A_s)$ denotes the product of the Tate modules $\mathrm{T}_\ell(A_s)$ for all primes $\ell \neq p$, and $\widehat{\mathrm{V}}^p(A_s) = \widehat{\mathrm{T}}^p(A_s) \otimes_{\mathbb{Z}} \mathbb{Q}$. For every polarization $\lambda \colon A \to \widehat{A}$,

$$E_s^\lambda \colon \widehat{\mathrm{T}}^p(A_s) \times \widehat{\mathrm{T}}^p(A_s) \longrightarrow \widehat{\mathbb{Z}}^p(1)_s$$

denotes the Riemann form of λ at the geometric point s. The fundamental group $\pi_1(S,s)$ acts on the free $\widehat{\mathbb{Z}}^p$–module $\widehat{\mathrm{T}}^p(A_s)$, and if s' is another geometric point of S, there are an isomorphism of topological groups $\gamma \colon \pi_1(S,s) \to \pi_1(S,s')$ and a $\pi_1(S,s)$–equivariant isomorphism (with respect to γ) $\widetilde{\gamma} \colon \widehat{\mathrm{T}}^p(A_s) \to \widehat{\mathrm{T}}^p(A_{s'})$ which are uniquely determined up to a transformation $(\gamma, \widetilde{\gamma}) \mapsto (\gamma \circ \mathrm{ad}\, \sigma, \widetilde{\gamma} \circ \sigma)$ with $\sigma \in \pi_1(S, s)$. If V is any $\mathbb{A}_f^p$–module equipped with an action of a group C, a *C–class of isomorphisms* $V \to \widehat{\mathrm{V}}^p(A)$ is an orbit of the action of C on the set of isomorphisms $V \to \widehat{\mathrm{V}}^p(A_s)$ which is invariant under the action of $\pi_1(S, s)$ for one (hence every) geometric point s of S. Similarly, we shall speak of a C–class of isomorphisms $T \to \widehat{\mathrm{T}}^p(A)$ if T is a $\widehat{\mathbb{Z}}^p$–module on which C acts. Finally, if S is a $\mathbb{Q}$–scheme, we can include the Tate module $\mathrm{T}_p(A)$ omitting the superscript p in all notations.

Returning to the situation in §1, we define ideals

$$\begin{aligned} O(\mathcal{Z}) &= \{\, x \in O_K \otimes_{\mathbb{Z}} O_{E(K)} \mid (\varphi \otimes \mathrm{id})(x) = 0 \text{ if } \varphi \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C}) \text{ and } \varphi|_F \in \mathcal{V} \,\}; \\ O(\mathcal{V}_K) &= \{\, x \in O_K \otimes_{\mathbb{Z}} O_{E(K)} \mid (\varphi \otimes \mathrm{id})(x) = 0 \text{ if } \varphi \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C}) \smallsetminus \mathcal{V}_K \,\}; \\ \overline{O}(\mathcal{V}_K) &= \{\, x \in O_K \otimes_{\mathbb{Z}} O_{E(K)} \mid (\overline{\varphi} \otimes \mathrm{id})(x) = 0 \text{ if } \varphi \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C}) \smallsetminus \mathcal{V}_K \,\}; \end{aligned}$$

where id denotes the given embedding of $E(K)$ into $\mathbb{C}$. Since $\mathrm{Gal}(\overline{\mathbb{Q}} | E(K))$ stabilizes $\mathcal{V}_K$ and $\mathcal{Z}$, there is a natural decomposition

$$O_K \otimes_{\mathbb{Z}} O_{E(K)} = O(\mathcal{Z}) \times O(\mathcal{V}_K) \times \overline{O}(\mathcal{V}_K).$$

If $\mathfrak{F}$ is a sheaf of $O_K \otimes_{\mathbb{Z}} \mathcal{O}_S$–modules on an $O_{E(K)}$–scheme S, this induces a corresponding decomposition

$$\mathfrak{F} = \mathfrak{F}(\mathcal{Z}) \times \mathfrak{F}(\mathcal{V}_K) \times \overline{\mathfrak{F}}(\mathcal{V}_K).$$

Let L be a totally imaginary quadratic extension of F which is contained in D.

Definition 2.1 A *sheaf of type* $(L, \mathcal{V}_K)$ on an $O_{E(K)}$–scheme S is a coherent sheaf of $\mathcal{O}_S$–modules $\mathfrak{F}$ together with a homomorphism

$$O_L \otimes_{O_F} O_K \longrightarrow \mathrm{End}_{\mathcal{O}_S} \mathfrak{F}$$

such that locally with respect to the flat topology on S the following conditions are satisfied:

$\mathfrak{F}$ is a free $O_L \otimes_{\mathbb{Z}} \mathcal{O}_S$–module;

$\mathfrak{F}(\mathcal{Z})$ is a free $O_L \otimes_{O_F} O(\mathcal{Z}) \otimes_{O_{E(K)}} \mathcal{O}_S$–module;

$\overline{\mathfrak{F}}(\mathcal{V}_K) = 0$.

Remark 2.2 For every $O_{E(K)}$–scheme S, there is a finite Galois covering $\pi \colon S' \to S$ such that, for every sheaf $\mathfrak{F}$ of type $(L, \mathcal{V}_K)$ on S, $\pi^*\mathfrak{F}$ satisfies the conditions in Definition 2.1 locally with respect to the Zariski topology on S'. If, in particular, S is an $O_{E(K)} \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$–scheme for a rational prime p which is unramified in L and K, then one can choose π to be étale, and if, in addition, S is simply connected, the flat topology in Definition 2.1 may be replaced by the Zariski topology.

If $\mathfrak{F}$ is a locally free $\mathcal{O}_S$–module together with an action of $O_L \otimes_{O_F} O_K$ and $U \subset S$ is a dense open subset such that $\mathfrak{F}|_U$ is a sheaf of type $(L, \mathcal{V}_K)$ on U, then $\mathfrak{F}$ is a sheaf of type $(L, \mathcal{V}_K)$ on S.

Finally, if $B \otimes_{\mathbb{Q}} \mathbb{R}$ is considered as a $\mathbb{C}$–vector space via h_G and is equipped with the action of $L \otimes_F K$ by left multiplication, it induces a sheaf of type $(L, \mathcal{V}_K)$ on $\operatorname{Spec}(\mathbb{C})$.

Definition 2.3 For every connected $E(K)$–scheme S, $\underline{M}_{G,C}(S)$ is the set of isomorphism classes of $(A, \iota, \Lambda, \overline{\eta})$ where

> A is an abelian scheme over S up to isogeny;
>
> $\iota\colon B \to \operatorname{End}^\circ A$ is an embedding such that $\operatorname{Lie} A$ is a sheaf of type $(L, \mathcal{V}_K)$;
>
> $\Lambda = \lambda \circ \iota(F^\times) \subset \operatorname{Hom}^\circ(A, \widehat{A})$ for a polarization λ of A;
>
> $\overline{\eta}$ is a C–class of $B \otimes_{\mathbb{Q}} \mathbb{A}_f$–module isomorphisms $B \otimes_{\mathbb{Q}} \mathbb{A}_f \to \widehat{\mathrm{V}}(A)$ such that, for a geometric point s of S, there is an element $f \in F \otimes_{\mathbb{Q}} \mathbb{A}_f(-1)_s$ satisfying $E_s^\lambda \circ (f\eta \times \eta) = \psi$.

For an arbitrary $E(K)$–scheme S, $\underline{M}_{G,C}(S)$ is the product of the sets corresponding to the connected components of S.

On the projective system of sets $\underline{M}_{G,C}(S)$ with respect to all open compact subgroups $C \subset G(\mathbb{A}_f)$, $g \in G(\mathbb{A}_f)$ acts as $(A, \iota, \Lambda, \overline{\eta}) \mapsto (A, \iota, \Lambda, \overline{\eta \circ g})$. It is easy to see that in characteristic zero the condition on $\operatorname{Lie} A$ in Definition 2.3 is equivalent to the trace condition in [7] (4.11). This is no longer true in positive characteristic.

Let us now fix a rational prime p, a prime $\wp$ of $E(D)$ dividing it, and a commutative diagram

$$\begin{tikzcd} E(D) \arrow[r, dotted] \arrow[d, dotted] & \overline{\mathbb{Q}} \subset \mathbb{C} \arrow[d] \\ E = E(D)_\wp \arrow[r] & \overline{\mathbb{Q}}_p \end{tikzcd} \tag{2.4}$$

where the dotted arrows indicate the natural inclusions. In particular, this defines a bijection

$$\operatorname{Hom}_{\mathbb{Q}}(F, \mathbb{R}) \longleftrightarrow \coprod_{i=1}^{m} \operatorname{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are all primes of F dividing p. We will abuse notation by using the same symbol for corresponding elements.

Remark 2.5 The image of $\mathcal{Z}$ with respect to the bijection above is, up to translation by an automorphism of $\overline{\mathbb{Q}}_p$, uniquely determined by the prime $\wp$ of $E(D)$. In particular, the number d_i of elements in $\mathcal{Z} \cap \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)$ does not depend on the choice of the diagram (2.4). On the other hand, for every $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$ and $\varphi' \in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)$, there is a prime $\wp$ and a diagram (2.4) such that φ and φ' correspond with respect to the resulting bijection. Hence, for every $i \in \{1, \dots, m\}$, there is a prime $\wp$ of $E(D)$ above p such that $d_i > 0$.

For every $i \in \{1, \dots, m\}$, we choose a prime $\mathfrak{q}_i$ of K dividing $\mathfrak{p}_i$, and we impose the following conditions on the choices of L, K, $\mathcal{V}_K$, and ψ:

(2.6) all $\mathfrak{p}_i$ split in K, and (2.4) defines an inclusion $\mathcal{V}_K \to \coprod_{i=1}^{m} \mathrm{Hom}_{\mathbb{Q}_p}(K_{\mathfrak{q}_i}, \overline{\mathbb{Q}}_p)$;

(2.7) if $\mathcal{V} = \varnothing$, K does not split D;

(2.8) all $\mathfrak{p}_i$ stay prime in L;

(2.9) there is an order $O_D \subset D$ which is maximal at $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ and contains O_L, and such that ψ restricts to a perfect pairing

$$\psi\colon O_B \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \times O_B \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}_{(p)}$$

where $O_B = O_D \otimes_{O_F} O_K$.

To see that O_D and ψ can be chosen as in (2.9), take any $\Pi_0 \in D$ such that $\Pi_0 l = l^* \Pi_0$ for all $l \in L$, and $\mathrm{ord}_{\mathfrak{p}_i}(\Pi_0^2) \in \{0, 1\}$ for all i. Put $O_D = O_L[\Pi_0]$, and fix a purely imaginary element α in the CM–field $L \otimes_F K \subset B$ such that $\mathcal{V}$ is the set of infinite places of F where $\alpha\alpha^* \in F^\times$ is negative. Then, for a sufficiently large integer n and $b = \Pi_0 + p^n \alpha$, $\overline{b}^* = -b$ and $\mathcal{V}$ is the set of infinite places of F at which $bb^* \in F^\times$ is negative. Now $\psi(b_1, b_2) = \mathrm{Tr}_{B|\mathbb{Q}}(b^{-1} b_1 \overline{b}_2^*)$ has the desired property. Moreover, if p is unramified in F, we may consider $\Pi = \Pi_0 l$ for an appropriate $l \in (O_L \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$, and $b_0 = b\Pi^{-1}$ to see that the positive involution $'$ induced by the pairing ψ just defined has the following additional property:

(2.10) there are elements $\Pi \in O_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $b_0 \in (O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ such that

$(b_0 \Pi)' = -b_0 \Pi$;

for all $l \in L$, $\Pi l = l^* \Pi$ and $(b_0 l)' = b_0 l$;

$\Pi^2 \in F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ has component p in $F_{\mathfrak{p}_i}$ if D is ramified at $\mathfrak{p}_i$, and component 1 otherwise.

It is an obvious consequence of (2.6) and Lemma 1.1 that (2.4) defines an embedding $E(K) \subset E$. Using the description above, we construct a reduction of $Sh_{G,C}$ which is defined over O_E if $C \subset G(\mathbb{A}_f)$ is a subgroup which satisfies the following condition:

(2.11) there is an order $O_B \subset B$ as in (2.9) such that $C = C_p \times C^p$ where $C_p = \{ g \in G(\mathbb{Q}_p) \mid g(O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p) = O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \}$, C^p is an open compact subgroup of $G(\mathbb{A}_f^p)$, and there is an integer $N > 2$ such that, for all $b \in O_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ and $g \in C$, $g(b) - b \in NO_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

Note that (2.6) and (2.9) imply that, for any $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$–module T which is free of rank 4 as $O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$–module and any perfect alternating $\mathbb{Z}_p$–bilinear form $\Phi\colon T \times T \to \mathbb{Z}_p$ which satisfies $\Phi(bx, y) = \Phi(x, b'y)$ for all $x, y \in T$, $b \in O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$, there is an isomorphism of $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$–modules $T \to O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$ which respects the bilinear forms up to a factor in $(O_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$. Therefore $\underline{M}_{G,C}$ extends to a contravariant functor $\underline{\mathcal{M}}_{G,C}$ on the category of O_E–schemes as follows.

Definition 2.12 For every connected O_E–scheme S, $\underline{\mathcal{M}}_{G,C}(S)$ is the set of isomorphism classes of $(A, \iota, \Lambda, \overline{\eta})$ where

A is an abelian scheme over S up to p–principal isogeny;

$\iota\colon O_B \to \operatorname{End}^p A = \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is an embedding such that $\operatorname{Lie} A$ is a sheaf of type $(L, \mathcal{V}_K)$;

$\Lambda = \lambda \circ \iota(F^\times) \subset \operatorname{Hom}^\circ(A, \widehat{A})$ for a p–principal polarization λ of A;

$\overline{\eta}$ is a C^p–class of $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–module isomorphisms $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \widehat{\mathrm{V}}^p(A)$ such that, for a geometric point s of S, there is an element $f \in F \otimes_{\mathbb{Q}} \mathbb{A}_f^p(-1)_s$ satisfying $E_s^\lambda \circ (f\eta \times \eta) = \psi$.

For an arbitrary O_E–scheme S, $\underline{\mathcal{M}}_{G,C}(S)$ is the product of the sets corresponding to the connected components of S.

This definition depends only on $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ which in turn is uniquely determined by the maximal compact subgroup $C_p \subset G(\mathbb{Q}_p)$. If we consider the projective limit with respect to all C which satisfy (2.11) and such that C_p is fixed, there is an obvious right action of $G(\mathbb{A}_f^p)$ defined by changing η. Moreover, since $K^\times(\mathbb{Q}_p)$ commutes with C_p, it acts on $\underline{M}_{G,C}$. This action can be extended to $\underline{\mathcal{M}}_{G,C}$ as follows. An element $k \in K^\times(\mathbb{Q}_p)$ sends a point $(A, \iota, \Lambda, \overline{\eta})$ to $(A, \iota, \Lambda, \overline{\eta k_0})$ for any $k_0 \in K^\times$ with $k_0 k \in (O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$.

Lemma 2.13 *Let $\mathfrak{a} = (A, \iota, \Lambda, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(S)$ for a connected S. Then*

$$\operatorname{Aut} \mathfrak{a} = F^\times \cap C$$

where the intersection is defined via the embedding $F^\times \subset K^\times \xrightarrow{\tau} G(\mathbb{Q}) \subset G(\mathbb{A}_f)$.

Proof. Obviously, ι defines an embedding of $F^\times \cap C = O_F^\times \cap C$ into $\operatorname{Aut}\mathfrak{a}$. For the proof of the opposite inclusion we may assume, by the Rigidity Lemma (cf. [30], Proposition 6.1), that S is the spectrum of an algebraically closed field. We fix $\eta \in \overline{\eta}$ and an abelian variety A in the p–principal isogeny class defined by $\mathfrak{a}$ such that η induces an isomorphism $O_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \to \widehat{\mathrm{T}}^p(A)$. In particular, $\operatorname{Aut}\mathfrak{a} \subseteq \operatorname{Aut} A$.

On the other hand, for every $\alpha \in \operatorname{Aut}\mathfrak{a}$, η identifies $\widehat{\mathrm{T}}^p(\alpha)$ with an element $b \in (1 + NO_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p)^\times$ which acts on $O_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$ as $x \mapsto xb$. Here $N > 2$ is an integer as in (2.11). Let $\lambda \in \Lambda$ be a p–principal polarization. Then $\widehat{\alpha} \circ \lambda \circ \alpha = \lambda \circ \iota(f)$ for an $f \in (O_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})^\times$. Since η respects the bilinear forms up to a factor in $F \otimes_{\mathbb{Q}} \mathbb{A}_f^p(-1)$, we obtain $b\overline{b}^* = f \in (1 + NO_F)^\times$ where * denotes the main involution and $^-$ the map $d \otimes k \mapsto d \otimes \overline{k}$ on $B = D \otimes_F K$. Thus $\beta = \iota(f)^{-1}\alpha^2 \in \operatorname{Aut} A$ acts trivially on the set of N–division points of A, and $\widehat{\beta} \circ \lambda \circ \beta = \lambda$. Hence Serre's Lemma (cf. [31], §21, Theorem 5) implies that $\beta = 1$, i.e. $b^2 = f$. In particular, $\overline{b}^* = b$ and

$$(b^* - b)\operatorname{Tr}_{B|K} b = (b^2)^* - b^2 = 0\,.$$

Let b_ℓ denote the component of b corresponding to a prime ℓ dividing N. We may assume that either $\ell > 2$ or N is divisible by 4. But then $\operatorname{Tr}_{B|K} b_\ell$ is contained in $(2 + NO_K \otimes_{\mathbb{Z}} \mathbb{Z}_\ell) \subset (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$, and we obtain $b_\ell = b_\ell^* = \overline{b}_\ell$, hence $b_\ell \in F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. This implies $\alpha \in \iota(F^\times)$, and the lemma is proved. ∎

Now we come to the main result of this section. Note that, since the moduli problem $\underline{\mathcal{M}}_{G,C}$ involves F–homogeneous polarizations, it is not representable unless $F = \mathbb{Q}$, so the following result is the best we can expect.

Proposition 2.14 *For every open compact subgroup $C \subset G(\mathbb{A}_f)$ which satisfies condition* (2.11), *there is a proper O_E–scheme $\widetilde{Sh}_{G,C}$ together with a functor morphism $\mu \colon \underline{\mathcal{M}}_{G,C} \to \widetilde{Sh}_{G,C}$ which is, up to unique isomorphism, uniquely determined by the following two properties:*

(i) *for every simply connected O_E–scheme S, the map $\mu \colon \underline{\mathcal{M}}_{G,C}(S) \to \widetilde{Sh}_{G,C}(S)$ is a bijection;*

(ii) *every functor morphism $\underline{\mathcal{M}}_{G,C} \to X$ into an O_E–scheme X factors uniquely through μ.*

Proof. Fix an order O_B and $N > 2$ as in (2.11), and a set of representatives $\delta_1, \dots, \delta_r$ for $(F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times / (O_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p)^\times (O_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})_+^\times$ (where the subscript denotes the subgroup of totally positive elements) such that, for every $j \in \{1, \dots, r\}$, $\psi \circ (\delta_j \times \mathrm{id})$ induces a pairing

$$\psi_j \colon O_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \;\times\; O_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \longrightarrow \widehat{\mathbb{Z}}^p\,. \tag{2.15}$$

We define contravariant functors $\underline{\mathcal{M}}^j_C$ on the category of O_E–schemes such that, for every connected S, $\underline{\mathcal{M}}^j_C(S)$ is the set of isomorphism classes of objects $(A, \iota, \lambda, \overline{\eta})$ where

> A is an abelian scheme over S;
>
> $\iota\colon O_B \to \operatorname{End} A$ is an embedding such that $\operatorname{Lie} A$ is a sheaf of type $(L, \mathcal{V}_K)$;
>
> λ is a p–principal polarization of A;
>
> $\overline{\eta}$ is a C^p–class of $O_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$–module isomorphisms $O_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p \to \widehat{\mathrm{T}}^p(A)$ such that, for a geometric point s of S, $O_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p(-1)_s$ is generated as an $O_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^p$–module by an element f which satisfies $E^\lambda_s \circ (f\eta \times \eta) = \psi_j$.

Condition (2.11) implies that $\overline{\eta}$ determines a unique level–N–structure on A. Moreover, for fixed $j \in \{1, \ldots, r\}$, the degree of the polarization λ is uniquely determined by the pairing (2.15). The methods of Mumford [30] (Chapter 7), therefore, provide quasi-projective O_E–schemes $\mathcal{M}^j_C$ which represent the functors $\underline{\mathcal{M}}^j_C$. Lemma 2.13 implies that the finite group $O^\times_{F+}/(F^\times \cap C)^2$ acts without fixed points on $\underline{\mathcal{M}}^j_C$ by changing λ. Since any two polarizations in a homogeneous polarization Λ differ by a a totally positive factor (cf. [43], Lemma 1.4), $\underline{\mathcal{M}}_{G,C}$ is the quotient of $\coprod_{j=1}^r \underline{\mathcal{M}}^j_C$ with respect to this action.

Lemma 2.16 *$\mathcal{M}^j_C$ is a proper O_E–scheme.*

Proof. We use a variant of the valuative criterion. Let R be an O_E–algebra which is a discrete valuation ring with perfect residue class field $\overline{R}$. Fix an element $\mathfrak{a} = (A, \iota, \lambda, \overline{\eta}) \in \underline{\mathcal{M}}^j_C(L)$ where L is the quotient field of R. We have to show that there is an $\mathfrak{a}' \in \underline{\mathcal{M}}^j_C(R')$ where R' is the integral closure of R in a suitable finite extension $L'|L$ such that $\mathfrak{a}$ and $\mathfrak{a}'$ have the same image in $\underline{\mathcal{M}}^j_C(L')$.

Let A' be the Néron model of A over R. Replacing L by a finite extension, we may assume that A has semi-stable reduction, i.e. the connected component of 1 in the special fibre of A' is an extension of an abelian variety A'' by a torus T. The universal property of the Néron model provides an embedding $\iota'\colon O_B \to \operatorname{End} A'$. Since $\operatorname{Hom}(T, A'') = 0$, this induces a map $O_B \to \operatorname{End} T$, and $\dim T \leqslant \dim A = 2[K:\mathbb{Q}]$ implies that either T is trivial or $B \cong \mathrm{M}_2(K)$. By (2.7), the latter case can only occur if $\mathcal{V} \neq \varnothing$. But since A' is smooth over R, $\operatorname{Lie} A'$ is a free R–module, hence if $\operatorname{Lie} A$ has type $(L, \mathcal{V}_K)$, the same applies to $\operatorname{Lie} A'$ (cf. Remark 2.2). On the other hand, $\operatorname{Lie} T$ is a free $O_K \otimes_{\mathbb{Z}} \overline{R}$–module, hence cannot be contained in a module of type $(L, \mathcal{V}_K)$ unless T is trivial or $\mathcal{V} = \varnothing$. Therefore, T has to be trivial, i.e. A' is an abelian scheme over R (cf. [38], §1, Lemma 3). Moreover, $\widehat{A}'$ is the Néron model of $\widehat{A}$, and the universal property provides a polarization λ' of A' such that $(A', \iota', \lambda', \overline{\eta}) \in \underline{\mathcal{M}}^j_C(R)$ has image $\mathfrak{a}$ in $\underline{\mathcal{M}}^j_C(L)$. ∎

Lemma 2.16 and the considerations preceding it imply that there is a projective O_E–scheme $\widetilde{Sh}_{G,C}$ and a finite étale Galois covering

$$\coprod_{j=1}^{r} \mathcal{M}_C^j \longrightarrow \widetilde{Sh}_{G,C}$$

with Galois group $O_{F+}^{\times}/(F^{\times} \cap C)^2$ which induces a morphism μ of functors with the desired properties. ∎

Remark 2.17 The proof of Proposition 2.14 shows that $\widetilde{Sh}_{G,C}$ represents the functor which is obtained from $\underline{\mathcal{M}}_{G,C}$ if in Definition 2.12 Λ is replaced by an element in the direct limit over the category (f-ét$/S$) of connected non-empty finite étale S–schemes T of the sets of all $\Lambda \subset \mathrm{Hom}^{\circ}(A_T, \widehat{A}_T)$ as in the definition which have the additional property that, for every pair of morphisms $\psi_1, \psi_2 \colon T' \to T$ in (f-ét$/S$), $\psi_1^*(\Lambda) = \psi_2^*(\Lambda) \subset \mathrm{Hom}^{\circ}(A_{T'}, \widehat{A}_{T'})$.

Corollary 2.18 *There is a natural $K^{\times}(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$–equivariant projective system of isomorphisms of varieties over E*

$$\widetilde{Sh}_{G,C} \otimes_{O_E} E \cong Sh_{G,C} \otimes_{E(K)} E$$

with respect to all C which satisfy (2.11) *and such that C_p is fixed.*

Proof. This follows from the uniqueness of a coarse moduli scheme. ∎

§ 3. The local structure of $\widetilde{Sh}_G$

From now on we assume that p is unramified in F. This implies, in particular, that $E|\mathbb{Q}_p$ is unramified. $\mathbb{Q}_p^{\mathrm{nr}}$ denotes the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$, $\mathbb{Z}_p^{\mathrm{nr}}$ the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p^{\mathrm{nr}}$, $\mathbb{F}$ its residue class field, and $\mathrm{W}(\mathbb{F})$ its completion, i.e. the Witt ring of $\mathbb{F}$. σ is the (absolute) Frobenius automorphism on $\mathrm{W}(\mathbb{F})$. Moreover, we assume that the involution $'$ on B satisfies condition (2.10).

Consider a geometric point $x \in \widetilde{Sh}_{G,C}(\mathbb{F})$, i.e. a closed point x together with an O_E–algebra homomorphism $\mathcal{O}_x \to \mathbb{F}$. The strict completion (i.e. the completion of the strict henselization) $\widehat{\mathcal{O}}_x$ of the local ring $\mathcal{O}_x$ is a complete local $\mathrm{W}(\mathbb{F})$–algebra with a natural action of the Galois group of the residue class field of x. For the application in § 6, we need a twisted version of this action. Note that the group $K^{\times}(\mathbb{A}_f)$ acts on $\widetilde{Sh}_{G,C}$ via a finite quotient. If $\kappa \subset \mathbb{F}$ is a finite extension of the residue class field of O_E, $\gamma \in \mathrm{Gal}(\mathbb{F}|\kappa)$ the corresponding Frobenius automorphism, and k is an element in $K^{\times}(\mathbb{A}_f)$ such that $\gamma(x) = k(x)$ (as geometric points), there is a unique *twisted action* of $\mathrm{Gal}(\mathbb{F}|\kappa)$ on $\widehat{\mathcal{O}}_x$ which coincides with the natural action on a sufficiently small subgroup of finite index and for which the following diagram is commutative.

$$\begin{tikzcd}
\widehat{\mathcal{O}}_x \arrow[rrr, "\gamma"] \arrow[dd] & & & \widehat{\mathcal{O}}_x \arrow[dd] \\
& \mathcal{O}_x \arrow[ul] \arrow[dl] \arrow[r, "k"] & \mathcal{O}_x \arrow[ur] \arrow[dr] & \\
\mathbb{F} \arrow[rrr, "\gamma"'] & & & \mathbb{F}
\end{tikzcd}$$

Let $\mathcal{Z}_0$ be the set of all $v \in \mathcal{Z}$ which correspond to an element of $\mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)$ for a prime $\mathfrak{p}_i$ at which D is split. For any number field Ω in which p is unramified, (2.4) induces a natural bijection

$$\mathrm{Hom}_{\mathbb{Q}}(\Omega, \mathbb{C}) \longleftrightarrow \mathrm{Hom}_{\mathbb{Z}}(O_\Omega, \mathrm{W}(\mathbb{F}))\,.$$

In particular, this defines an isomorphism

$$\prod_{\varphi} w(\varphi) \times \overline{w}(\varphi) : \; O_{L \otimes_F K} \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}) \longrightarrow \prod_{\varphi} \mathrm{W}(\mathbb{F}) \times \mathrm{W}(\mathbb{F}) \tag{3.1}$$

where, for every $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C})$, $w(\varphi) \in \mathrm{Hom}_{\mathbb{Q}}(L \otimes_F K, \mathbb{C})$ is the unique extension of φ such that $w(\varphi)|_K \in \mathrm{Hom}_{\mathbb{Q}_p}(K_{\mathfrak{q}_i}, \overline{\mathbb{Q}}_p)$ for a certain i, and $\overline{w}(\varphi)$ is the other extension. If $\mathfrak{F}$ is a sheaf of $O_{L \otimes_F K} \otimes_{\mathbb{Z}} \mathcal{O}_S$–modules on a $\mathrm{W}(\mathbb{F})$–scheme S, (3.1) induces a decomposition

$$\mathfrak{F} = \prod_{\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C})} (\mathfrak{F}_\varphi \times \overline{\mathfrak{F}}_\varphi) \tag{3.2}$$

where

$$\begin{aligned}
\mathfrak{F}(\mathcal{Z}) \;\; &= \prod_{\varphi|_F \in \mathcal{Z}} (\mathfrak{F}_\varphi \times \overline{\mathfrak{F}}_\varphi)\,; \\
\mathfrak{F}(\mathcal{V}_K) &= \prod_{\varphi|_F \in \mathcal{V}} \mathfrak{F}_\varphi\,; \\
\overline{\mathfrak{F}}(\mathcal{V}_K) &= \prod_{\varphi|_F \in \mathcal{V}} \overline{\mathfrak{F}}_\varphi\,.
\end{aligned}$$

Hence $\mathfrak{F}$ is of type $(L, \mathcal{V}_K)$ if and only if there is a non-negative integer n such that, for every $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C})$, either $\varphi|_F \in \mathcal{Z}$ and $\mathfrak{F}_\varphi$ and $\overline{\mathfrak{F}}_\varphi$ are locally free of rank n, or $\varphi|_F \in \mathcal{V}$, $\mathfrak{F}_\varphi$ is locally free of rank $2n$ and $\overline{\mathfrak{F}}_\varphi = 0$.

Definition 3.3 For a connected $\mathrm{W}(\mathbb{F})$–scheme S and $\mathfrak{a} = (A, \iota, \Lambda, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(S)$,

$$\begin{aligned}
\mathcal{Z}(\mathfrak{a}) \;\; &= \{\, \varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C}) \mid \varphi|_F \in \mathcal{Z} \text{ and } (\mathrm{Lie}\, A)_\varphi \text{ is } O_D\text{–invariant} \,\}\,; \\
\mathcal{Z}^+(\mathfrak{a}) &= \mathcal{Z}(\mathfrak{a}) \cap \mathcal{Z}(\mathfrak{a})^*\,; \\
\mathcal{Z}^-(\mathfrak{a}) &= \mathcal{Z}(\mathfrak{a}) \smallsetminus \mathcal{Z}(\mathfrak{a})^*\,.
\end{aligned}$$

Remark 3.4 If there is a $\varphi \in \mathcal{Z}(\mathfrak{a})$, S is of characteristic p, i.e. $p\mathcal{O}_S = 0$, and $\varphi|_F \in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)$ for a prime $\mathfrak{p}_i$ at which D is ramified. Conversely, if S is any irreducible and reduced $\mathbb{F}$–scheme, $\mathcal{Z}(\mathfrak{a}) \cup \mathcal{Z}(\mathfrak{a})^*$ is the set of all $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C})$ such that $\varphi|_F \in \mathcal{Z} \smallsetminus \mathcal{Z}_0$.

Let now $\mathfrak{a} = (A, \iota, \Lambda, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(\mathbb{F})$ be the element which corresponds to the geometric point x considered above. The main result of the present section is the following proposition which describes the structure of $\widetilde{Sh}_{G,C}$ in every infinitesimal neighborhood of this point.

Proposition 3.5 *There is an isomorphism of* $\mathrm{W}(\mathbb{F})$*–algebras*

$$\widehat{\mathcal{O}}_x \cong \mathrm{W}(\mathbb{F})[\![\, X_\varphi \mid \varphi \in \mathcal{Z}_0 \cup \mathcal{Z}(\mathfrak{a}) \,]\!] / (\, X_\varphi X_{\varphi^*} - p \mid \varphi \in \mathcal{Z}^+(\mathfrak{a}) \,)$$

such that the twisted action of $\mathrm{Gal}(\mathbb{F}|\kappa)$ *on* $\widehat{\mathcal{O}}_x$ *corresponds to the extension of the natural Galois action on* $\mathrm{W}(\mathbb{F})$ *given by* $\varrho(X_\varphi) = X_{\varrho\varphi}$ *for all* $\varrho \in \mathrm{Gal}(\mathbb{F}|\kappa)$ *and* $\varphi \in \mathcal{Z}_0 \cup \mathcal{Z}(\mathfrak{a})$. *For every local artinian* $\mathbb{F}$*–algebra* R_0 *with residue class field* $\mathbb{F}$ *and every homomorphism of local* $\mathrm{W}(\mathbb{F})$*–algebras* $f_0 \colon \widehat{\mathcal{O}}_x \to R_0$ *with corresponding element* $\mathfrak{a}_0 \in \underline{\mathcal{M}}_{G,C}(R_0)$,

$$\mathcal{Z}(\mathfrak{a}_0) = \mathcal{Z}^-(\mathfrak{a}) \cup \{\, \varphi \in \mathcal{Z}^+(\mathfrak{a}) \mid f_0(X_\varphi) = 0 \,\}\,.$$

Proof. Let $\widehat{\mathfrak{m}}_x$ denote the maximal ideal in $\widehat{\mathcal{O}}_x$. For every integer $i > 0$, $\widehat{\mathfrak{m}}_x^{i-1}/\widehat{\mathfrak{m}}_x^i$ is a finite dimensional $\mathbb{F}$–vector space on which $\mathrm{Gal}(\mathbb{F}|\kappa)$ acts compatibly with its action on $\mathbb{F}$. Hence it admits a basis consisting of Galois invariant elements (cf. [45], Lemma 6.25). Since $\widehat{\mathcal{O}}_x$ is complete and $\mathbb{F}$ is algebraically closed, this implies that every element of $\widehat{\mathcal{O}}_x/\widehat{\mathfrak{m}}_x^i$ which is invariant under the action of an integral power of γ can be lifted to an invariant element of $\widehat{\mathcal{O}}_x$, and that $\mathrm{Gal}(\mathbb{F}|\kappa)$ acts via a finite quotient on the set of elements in $\widehat{\mathcal{O}}_x$ which are fixed by a given non-zero power of γ. Hence it suffices to construct an isomorphism of $\mathrm{W}(\mathbb{F})$–algebras which is compatible with the actions of γ. To start with, we consider the Galois invariant subalgebra

$$\Gamma = \mathrm{W}(\mathbb{F})[\![\, X_\varphi \mid \varphi \in \mathcal{Z}^+(\mathfrak{a}) \,]\!] / (\, X_\varphi X_{\varphi^*} - p \mid \varphi \in \mathcal{Z}^+(\mathfrak{a}) \,)$$

of the $\mathrm{W}(\mathbb{F})$–algebra occuring in the statement of the proposition and construct a γ–equivariant local homomorphism $\Gamma \to \widehat{\mathcal{O}}_x$ of $\mathrm{W}(\mathbb{F})$–algebras. Then we shall show that this map is formally smooth and that the relative tangent space together with its Galois action has the required structure.

Since $\widehat{\mathcal{O}}_x$ is strictly henselian, the natural map $\mathcal{O}_x \to \widehat{\mathcal{O}}_x$ corresponds to an element $\widetilde{\mathfrak{a}} = (\widetilde{A}, \widetilde{\iota}, \widetilde{\Lambda}, \overline{\eta})$ in $\underline{\mathcal{M}}_{G,C}(\widehat{\mathcal{O}}_x)$ with image $\mathfrak{a}$ in $\underline{\mathcal{M}}_{G,C}(\mathbb{F})$ and such that there is an isomorphism $\theta \colon \gamma(\widetilde{\mathfrak{a}}) \to k(\widetilde{\mathfrak{a}})$. We fix elements $\Pi \in O_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $b_0 \in (O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$ as in (2.10). For any $\varphi \in \mathcal{Z}^+(\mathfrak{a})$, $(\mathrm{Lie}\, \widetilde{A})_\varphi$ is a free $\widehat{\mathcal{O}}_x$–module of rank 1, and θ induces a γ–linear isomorphism $(\mathrm{Lie}\, \widetilde{A})_\varphi \to (\mathrm{Lie}\, \widetilde{A})_{\gamma\varphi}$. Since $\widehat{\mathcal{O}}_x$ is complete and has algebraically closed residue class field, we may choose generators $t_\varphi \in (\mathrm{Lie}\,\widetilde{A})_\varphi$ such that, for all $\varphi \in \mathcal{Z}^+(\mathfrak{a})$, $\theta(t_\varphi) = t_{\gamma\varphi}$. The fact that $(\mathrm{Lie}\,A)_\varphi$ is O_D–invariant can now be expressed as $\Pi t_\varphi \in \widehat{\mathfrak{m}}_x(\mathrm{Lie}\,\widetilde{A})_{\varphi^*}$. Hence there is a unique $x_\varphi \in \widehat{\mathfrak{m}}_x$ such that $\Pi t_\varphi = x_\varphi t_{\varphi^*}$. These elements satisfy the relations $\gamma(x_\varphi) = x_{\gamma\varphi}$ and $x_\varphi x_{\varphi^*} = \varphi(\Pi^2) = p$. Thus we get the desired homomorphism

$$\begin{aligned} \Gamma &\longrightarrow \widehat{\mathcal{O}}_x \\ X_\varphi &\longmapsto x_\varphi\,. \end{aligned}$$

Consider now the Lie algebra of the universal extension of $A_n = \widetilde{A} \otimes_{\widehat{\mathcal{O}}_x} \widehat{\mathcal{O}}_x/\widehat{\mathfrak{m}}_x^n$ for every positive integer n, and let $\widetilde{M}$ denote the projective limit over all n. It is a free $\widehat{\mathcal{O}}_x$–module and there is a natural surjective homomorphism $\widetilde{M} \to \mathrm{Lie}\,\widetilde{A}$. Let $\widetilde{V}$ denote the kernel of this map. $\widetilde{\iota}$ defines on $\widetilde{M}$ the structure of an $O_B \otimes_{\mathbb{Z}} \widehat{\mathcal{O}}_x$–module and $\widetilde{V}$ becomes a submodule. Moreover, θ induces a γ–linear automorphism of $\widetilde{M}$ which respects $\widetilde{V}$ and commutes with the action of O_B. If we fix a p–principal polarization $\widetilde{\lambda} \in \widetilde{\Lambda}$, we get a perfect alternating bilinear form

$$\widetilde{\Phi}\colon \widetilde{M} \times \widetilde{M} \longrightarrow \widehat{\mathcal{O}}_x$$

such that $\widetilde{V}$ is a maximal isotropic subspace. Let $\widetilde{\Phi}' = \widetilde{\Phi} \circ (\mathrm{id} \times b_0)$. This is again a perfect alternating bilinear form on $\widetilde{M}$. Since $\widetilde{\Phi}$ induces the involution $'$ on $O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$, (2.10) implies that $\widetilde{\Phi}'$ has the following properties:

$$\begin{aligned} \widetilde{\Phi}' \circ (k \times \mathrm{id}) &= \widetilde{\Phi}' \circ (\mathrm{id} \times \overline{k}) \quad \text{for all } k \in O_K\,; \\ \widetilde{\Phi}' \circ (l \times \mathrm{id}) &= \widetilde{\Phi}' \circ (\mathrm{id} \times l) \quad \text{for all } l \in O_L\,; \\ \widetilde{\Phi}' \circ (\Pi \times \mathrm{id}) &= -\widetilde{\Phi}' \circ (\mathrm{id} \times \Pi)\,. \end{aligned}$$

Recall that we have decompositions (3.2) for $\widetilde{M}$ and $\widetilde{V}$. For $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C})$, $\widetilde{M}_\varphi$ has rank 2, and $\widetilde{V}_\varphi$ has rank 1 if $\varphi|_F \in \mathcal{Z}$ and vanishes otherwise. Π maps $\widetilde{M}_\varphi$ into $\widetilde{M}_{\varphi^*}$ and $\widetilde{V}_\varphi$ into $\widetilde{V}_{\varphi^*}$. If $\varphi|_F \in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)$ for a prime $\mathfrak{p}_i$ at which D is split, these maps are isomorphisms. In addition, the bilinear form $\widetilde{\Phi}'$ induces perfect pairings

$$\widetilde{\Phi}'\colon \widetilde{M}_\varphi \times \overline{\widetilde{M}}_\varphi \longrightarrow \widehat{\mathcal{O}}_x$$

such that $\widetilde{\Phi}'(\widetilde{V}_\varphi \times \overline{\widetilde{V}}_\varphi) = 0$. For each $\varphi \in \mathcal{Z}^+(\mathfrak{a})$, we fix an element $e_\varphi \in \widetilde{M}_\varphi$ which maps to $t_\varphi \in (\mathrm{Lie}\,\widetilde{A})_\varphi$.

Lemma 3.6 *For every $\varphi \in \mathcal{Z}^-(\mathfrak{a})$, $\Pi\widetilde{V}_\varphi = \widetilde{V}_{\varphi^*}$, and for every $\varphi \in \mathcal{Z}^+(\mathfrak{a})$, $\widetilde{V}_\varphi$ is generated as $\widehat{\mathcal{O}}_x$–module by $\Pi e_{\varphi^*} - x_{\varphi^*} e_\varphi$.*

Proof. Let N denote the Dieudonné module of the Barsotti–Tate group of A in the sense of Zink [46] (Part II, page 51), i.e. $N \cong N' \otimes_{\mathrm{W}(\mathbb{F}),\sigma} \mathrm{W}(\mathbb{F})$ where N' is the Dieudonné module of the Barsotti–Tate group of the dual of A in the sense of Demazure [12] (Chapter III, §8). Then we have canonical isomorphisms which give rise to a commutative diagram

$$\begin{array}{ccc} \widetilde{V} \otimes_{\widehat{\mathcal{O}}_x} \mathbb{F} & \cong & VN/pN \\ \bigcap & & \bigcap \\ \widetilde{M} \otimes_{\widehat{\mathcal{O}}_x} \mathbb{F} & \cong & N/pN \end{array}$$

and a decomposition (3.2) of N into free $\mathrm{W}(\mathbb{F})$–modules of rank 2 such that, for every $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L\,,\mathbb{C})$,

$$\begin{array}{ll} VN_{\sigma\varphi} = pN_\varphi & \text{if } \varphi|_F \in \mathcal{Z}\,; \\ pN_\varphi \subsetneqq VN_{\sigma\varphi} \subsetneqq N_\varphi & \text{if } \varphi|_F \in \mathcal{V}\,. \end{array}$$

Fix now any $\varphi \in \mathcal{Z}(\mathfrak{a})$. Then

$$\Pi N_\varphi = \begin{cases} pN_{\varphi^*} & \text{if } \Pi N_{\varphi^*} = N_\varphi\,; \\ VN_{\sigma\varphi^*} & \text{otherwise}\,. \end{cases}$$

But since $N_\varphi / V^i N_{\sigma^i\varphi}$ and $N_{\varphi^*}/V^i N_{\sigma^i\varphi^*}$ have the same length as $\mathrm{W}(\mathbb{F})$–modules and Π commutes with V, $\Pi N_{\varphi^*} = N_\varphi$ would imply $\Pi N_{\sigma^i\varphi^*} = N_{\sigma^i\varphi}$ for every positive integer i and, in particular, $\Pi N_\varphi = N_{\varphi^*}$. Hence $\Pi N_\varphi = VN_{\sigma\varphi^*}$.

Let us assume now that $\Pi\widetilde{V}_\varphi \neq \widetilde{V}_{\varphi^*}$. Since $\widetilde{V}_{\varphi^*}$ has rank 1, this is equivalent to $\Pi V N_{\sigma\varphi} \subseteq pN_{\varphi^*}$, hence $pN_\varphi \subsetneqq VN_{\sigma\varphi} \subseteq \Pi N_{\varphi^*} \subsetneqq N_\varphi$. But then $VN_{\sigma\varphi} = \Pi N_{\varphi^*}$, and therefore, $\varphi^* \in \mathcal{Z}(\mathfrak{a})$. This proves the first part of the lemma.

Consider, finally, $\varphi \in \mathcal{Z}^+(\mathfrak{a})$. By the definition of x_φ, $\Pi e_{\varphi^*} - x_{\varphi^*} e_\varphi \in \widetilde{V}_\varphi$. So it remains to show that the image of this element in $\widetilde{V}_\varphi \otimes_{\widehat{\mathcal{O}}_x} \mathbb{F}$ does not vanish. We have proved, already, that $\Pi N_\varphi = VN_{\sigma\varphi^*}$ and $\Pi N_{\varphi^*} = VN_{\sigma\varphi}$, hence Π induces an isomorphism

$$(\mathrm{Lie}\,\widetilde{A})_{\varphi^*} \otimes_{\widehat{\mathcal{O}}_x} \mathbb{F} \cong N_{\varphi^*}/VN_{\sigma\varphi^*} \xrightarrow{\ \Pi\ } VN_{\sigma\varphi}/pN_\varphi \cong \widetilde{V}_\varphi \otimes_{\widehat{\mathcal{O}}_x} \mathbb{F}$$

which maps the element $t_{\varphi^*} \otimes 1$ to $\Pi e_{\varphi^*} \otimes 1$. Since $x_{\varphi^*} \in \widehat{\mathfrak{m}}_x$, the second assertion of the lemma follows. ∎

For every $v \in \mathcal{Z}_0$, we choose a $\varphi_v \in \mathrm{Hom}_{\mathbb{Q}}(L\,,\mathbb{C})$ such that $\varphi_v|_F = v$. The set of all these φ_v is denoted by $\mathcal{Z}^\circ$. We show that the map $\Gamma \to \widehat{\mathcal{O}}_x$ is formally smooth.

Lemma 3.7 *Any surjective homomorphism $\alpha\colon R \to R_0$ of local artinian Γ–algebras with residue class field $\mathbb{F}$ such that $\mathfrak{m}\mathfrak{c} = 0$, where $\mathfrak{c}$ denotes the kernel of α and $\mathfrak{m}$ the maximal ideal of R, induces a surjective map*

$$\mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x\,, R) \longrightarrow \mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x\,, R_0)\,.$$

Proof. Fix an $f_0 \in \mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x, R_0)$, and let $\mathfrak{a}_0 = (A_0, \iota_0, \Lambda_0, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(R_0)$ denote the image of $\widetilde{\mathfrak{a}}$ and $\lambda_0 \in \Lambda_0$ the image of $\widetilde{\lambda}$ defined by f_0. It defines a perfect alternating bilinear form Φ on the Lie algebra M of the universal extension of A_0 over R, and we can again consider the perfect alternating form $\Phi' = \Phi \circ (\mathrm{id} \times b_0)$. Since M is an $O_B \otimes_{\mathbb{Z}} R$-module, it admits a natural decomposition (3.2). Let $V_0 \subset M_0 = M \otimes_R R_0$ denote the kernel of the natural surjection onto $\mathrm{Lie}\, A_0$ and Φ'_0 the form on M_0 induced by Φ'. Then there is a canonical isomorphism

$$(\widetilde{V} \subset \widetilde{M}, \widetilde{\Phi}') \otimes_{\widehat{\mathcal{O}}_x} R_0 \cong (V_0 \subset M_0, \Phi'_0).$$

For every $\varphi \in \mathcal{Z}^+(\mathfrak{a})$, we lift $e_\varphi \otimes 1 \in M_{0,\varphi}$ to an element $m_\varphi \in M_\varphi$ and define $V_\varphi = R(\Pi m_{\varphi^*} - X_{\varphi^*} m_\varphi) \subset M_\varphi$. Lemma 3.6 and the condition $\mathfrak{mc} = 0$ imply that V_φ lifts $V_{0,\varphi}$ and is independent of the choice of the elements m_φ.

By the criterion of Messing [26] (Chapter V, Theorem 1.10), the homomorphisms $f \in \mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x, R)$ such that $\alpha f = f_0$ are in a canonical bijective correspondence with the liftings of V_0 to a direct summand V of M which is isotropic with respect to Φ, invariant under the action of O_B, and such that, for every $\varphi \in \mathcal{Z}^+(\mathfrak{a})$, V_φ defined above is the component corresponding to φ in the decomposition (3.2) of V. Such a lifting is uniquely determined by the components V_φ for all $\varphi \in \mathcal{Z}^\circ \cup \mathcal{Z}^-(\mathfrak{a})$ since, for all $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C})$,

$$\overline{V}_\varphi = \{\, m \in \overline{M}_\varphi \mid \Phi'(m, V_\varphi) = 0 \,\},$$

and for all $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C}) \smallsetminus (\mathcal{Z}^\circ \cup \mathcal{Z}(\mathfrak{a}))$,

$$V_\varphi = \begin{cases} 0 & \text{if } \varphi|_F \in \mathcal{V}\,; \\ \Pi V_{\varphi^*} & \text{if } \varphi^* \in \mathcal{Z}^\circ \cup \mathcal{Z}^-(\mathfrak{a})\,. \end{cases}$$

Here we have used Lemma 3.6 again. Since, if $\varphi \in \mathcal{Z}^\circ \cup \mathcal{Z}^-(\mathfrak{a})$, we may take $V_\varphi = R m_\varphi$ for any $m_\varphi \in M_\varphi$ which lifts a generator of $V_{0,\varphi}$, the lemma is proved. ▪

We will now describe the relative tangent space

$$T_x = \widehat{\mathfrak{m}}_x / (\mathfrak{m}_\Gamma \widehat{\mathcal{O}}_x + \widehat{\mathfrak{m}}_x^2)$$

of the map $\Gamma \to \widehat{\mathcal{O}}_x$ where $\mathfrak{m}_\Gamma$ denotes the maximal ideal of Γ.

Lemma 3.8 *There is a basis $\{\, \xi_\varphi \mid \varphi \in \mathcal{Z}_0 \cup \mathcal{Z}^-(\mathfrak{a}) \,\}$ of the $\mathbb{F}$-vector space T_x such that the twisted action of $\gamma \in \mathrm{Gal}(\mathbb{F}|\kappa)$ is the γ-linear automorphism which satisfies $\gamma(\xi_\varphi) = \xi_{\gamma\varphi}$ for all φ.*

Proof. Consider $\mathbb{F}[\varepsilon]$ with $\varepsilon \neq 0$ and $\varepsilon^2 = 0$ as Γ-algebra via the projection onto $\mathbb{F} \subset \mathbb{F}[\varepsilon]$. There is a natural bijection

$$\mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x, \mathbb{F}[\varepsilon]) \longleftrightarrow \mathrm{Hom}_{\mathbb{F}}(T_x, \mathbb{F}).$$

Let M be the Lie algebra of the universal extension of A and $V \subset M$ the filtration defined by A. Since the projection $\mathbb{F}[\varepsilon] \to \mathbb{F}$ has a section, the Lie algebra of the universal extension of A over $\mathbb{F}[\varepsilon]$ is canonically isomorphic to $M \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$. In the proof of Lemma 3.7 we have constructed a natural bijection between $\mathrm{Hom}_\Gamma(\widehat{\mathcal{O}}_x, \mathbb{F}[\varepsilon])$ and the set of all collections $(V_\varphi^\varepsilon \mid \varphi \in \mathcal{Z}^\circ \cup \mathcal{Z}^-(\mathfrak{a}))$ where V_φ^ε is a direct summand of $M_\varphi \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$ which lifts V_φ. But such lifts bijectively correspond to the elements $\delta_\varphi \in \mathrm{Hom}_{\mathbb{F}}(V_\varphi, (\mathrm{Lie}\, A)_\varphi)$ via

$$\delta_\varphi \longmapsto V_\varphi^\varepsilon = \{\, x \otimes 1 + m \otimes \varepsilon \mid x \in V_\varphi, m \in M_\varphi \text{ a lift of } \delta_\varphi(x) \in (\mathrm{Lie}\, A)_\varphi \,\}\,.$$

For $v \in \mathcal{Z}_0$, we identify V_{φ_v} with $V_{\varphi_v^*}$ and $(\mathrm{Lie}\, A)_{\varphi_v}$ with $(\mathrm{Lie}\, A)_{\varphi_v^*}$ via Π and denote them by V_v and $(\mathrm{Lie}\, A)_v$, respectively.

Combining all the maps described above, we get a bijection

$$\mathrm{Hom}_{\mathbb{F}}(T_x, \mathbb{F}) \longleftrightarrow \prod_{\varphi \in \mathcal{Z}_0 \cup \mathcal{Z}^-(\mathfrak{a})} \mathrm{Hom}_{\mathbb{F}}(V_\varphi, (\mathrm{Lie}\, A)_\varphi)\,,$$

and it is not hard to see that this bijection is $\mathbb{F}$–linear and commutes with the actions of γ defined by $f \mapsto \gamma \circ f \circ \gamma^{-1}$ on the left side and $(\delta_\varphi) \mapsto (\delta'_\varphi)$ with $\delta'_\varphi = \theta \circ \delta_{\gamma^{-1}\varphi} \circ \theta^{-1}$ on the right side. Here θ is the map induced by an isomorphism $\theta\colon \gamma(\mathfrak{a}) \to k(\mathfrak{a})$. Since $\mathbb{F}$ is algebraically closed, we find non-zero elements δ_φ in $\mathrm{Hom}_{\mathbb{F}}(V_\varphi, (\mathrm{Lie}\, A)_\varphi)$ such that $(\delta_\varphi \mid \varphi \in \mathcal{Z}_0 \cup \mathcal{Z}^-(\mathfrak{a}))$ is invariant under γ. Then the basis of T_x which is dual to the basis of $\mathrm{Hom}_{\mathbb{F}}(T_x, \mathbb{F})$ determined by these elements has the desired property. ∎

Since the natural map $\widehat{\mathfrak{m}}_x/\widehat{\mathfrak{m}}_x^2 \to T_x$ is a Galois equivariant surjective homomorphism of finite-dimensional $\mathbb{F}$–vector spaces (with compatible Galois actions), it induces a surjective map between the sets of invariant elements with respect to any integral power of γ. Hence, using our remark at the beginning of this proof, we can lift the basis of T_x constructed in Lemma 3.8 to elements $x_\varphi \in \widehat{\mathfrak{m}}_x$ such that, for all $\varphi \in \mathcal{Z}_0 \cup \mathcal{Z}^-(\mathfrak{a})$, $\gamma(x_\varphi) = x_{\gamma\varphi}$. By Lemma 3.7 we know that $\widehat{\mathcal{O}}_x$ is the ring of formal power series in these elements with coefficients in Γ, hence we get the desired isomorphism

$$\begin{aligned} \mathrm{W}(\mathbb{F})[\![\, X_\varphi \mid \varphi \in \mathcal{Z}_0 \cup \mathcal{Z}(\mathfrak{a}) \,]\!]/(\, X_\varphi X_{\varphi^*} - p \mid \varphi \in \mathcal{Z}^+(\mathfrak{a})\,) &\longrightarrow \widehat{\mathcal{O}}_x \\ X_\varphi &\longmapsto x_\varphi \end{aligned}$$

of $\mathrm{W}(\mathbb{F})$–algebras.

Consider now a local artinian $\mathbb{F}$–algebra R_0 with residue class field $\mathbb{F}$ and fix a homomorphism $f_0\colon \widehat{\mathcal{O}}_x \to R_0$ of $\mathrm{W}(\mathbb{F})$–algebras. We want to compute $\mathcal{Z}(\mathfrak{a}_0)$ for the point $\mathfrak{a}_0 = (A_0, \iota_0, \Lambda_0, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(R_0)$ defined by f_0. Since $(\mathrm{Lie}\, A)_\varphi = (\mathrm{Lie}\, A_0)_\varphi \otimes_{R_0} \mathbb{F}$ for all $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C})$, we have an inclusion $\mathcal{Z}(\mathfrak{a}_0) \subseteq \mathcal{Z}(\mathfrak{a})$.

If $\varphi \in \mathcal{Z}^-(\mathfrak{a})$, Π induces a surjective map $(\operatorname{Lie} A_0)_{\varphi^*} \to (\operatorname{Lie} A_0)_\varphi$ since this holds after base change to the residue class field. On the other hand, $pR_0 = 0$, hence $\Pi(\operatorname{Lie} A_0)_\varphi = \Pi^2(\operatorname{Lie} A_0)_{\varphi^*} = 0$, i.e. $\varphi \in \mathcal{Z}(\mathfrak{a}_0)$.

It remains to decide whether, for a given $\varphi \in \mathcal{Z}^+(\mathfrak{a})$, the element $\Pi e_\varphi \otimes 1$ of $\widetilde{M}_{\varphi^*} \otimes_{\widehat{\mathcal{O}}_x} R_0$ is contained in $\widetilde{V}_{\varphi^*} \otimes_{\widehat{\mathcal{O}}_x} R_0$ which, by definition of x_φ, already contains $\Pi e_\varphi \otimes 1 - e_{\varphi^*} \otimes f_0(x_\varphi)$. Thus $\varphi \in \mathcal{Z}(\mathfrak{a}_0)$ if and only if $t_{\varphi^*} \otimes f_0(x_\varphi) = 0$ in $(\operatorname{Lie} A_0)_{\varphi^*} = (\operatorname{Lie} \widetilde{A})_{\varphi^*} \otimes_{\widehat{\mathcal{O}}_x} R_0$. Since $t_{\varphi^*} \otimes 1$ is a generator of the free R_0–module $(\operatorname{Lie} A_0)_{\varphi^*}$, this is equivalent to $f_0(x_\varphi) = 0$, and we obtain the description of $\mathcal{Z}(\mathfrak{a}_0)$ given in the claim of our proposition. ∎

Corollary 3.9 *$\widetilde{Sh}_{G,C}$ is smooth at x if and only if $\mathcal{Z}^+(\mathfrak{a}) = \varnothing$.*

Definition 3.10 A subset $Z \subseteq \operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C})$ is called *admissible* if

$$Z \cup Z^* = \{\, \varphi \in \operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C}) \mid \varphi|_F \in \mathcal{Z} \setminus \mathcal{Z}_0 \,\}.$$

For every such Z, let

$$d(Z) = \#\{\, v \in \mathcal{Z} \mid v = \varphi|_F \text{ for a } \varphi \in \operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C}) \setminus Z \,\}.$$

Z is called *minimal* if $Z \cap Z^* = \varnothing$, i.e. $d(Z) = \#\mathcal{Z}$.

Definition 3.11 For every admissible subset $Z \subseteq \operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C})$, $Sh_{G,C,Z}$ denotes the closed subscheme of $\widetilde{Sh}_{G,C} \otimes_{O_E} \mathbb{F}$ which is a coarse moduli scheme for the contravariant functor $\underline{M}_{G,C,Z}$ such that, for every connected $\mathbb{F}$–scheme S,

$$\underline{M}_{G,C,Z}(S) = \{\, \mathfrak{a} \in \underline{\mathcal{M}}_{G,C}(S) \mid Z \subseteq \mathcal{Z}(\mathfrak{a}) \,\}$$

and for an arbitrary $\mathbb{F}$–scheme S, $\underline{M}_{G,C,Z}(S)$ is the product of the sets corresponding to the connected components of S.

Remark 3.12 The scheme $Sh_{G,C,Z}$ is defined over the finite extension $\kappa(Z) \subset \mathbb{F}$ of the residue class field $\kappa(E)$ of E such that $\operatorname{Gal}(\mathbb{F}|\kappa(Z))$ is the stabilizer of Z for the natural action of $\operatorname{Gal}(\mathbb{F}|\kappa(E))$ on $\operatorname{Hom}_{\mathbb{Z}}(O_L, \mathrm{W}(\mathbb{F}))$.

Proposition 3.13 *The geometric special fibre of $\widetilde{Sh}_{G,C}$ is the union of its closed subschemes $Sh_{G,C,Z}$ where Z runs through the set of all minimal admissible subsets of $\operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C})$. For every admissible Z, $Sh_{G,C,Z}$ is a non-singular projective variety of pure dimension $d(Z)$ over $\mathbb{F}$, and if Z' is another admissible subset, $Sh_{G,C,Z} \cap Sh_{G,C,Z'} = Sh_{G,C,Z \cup Z'}$. Moreover, if x is a closed point of $Sh_{G,C,Z}$ corresponding to $\mathfrak{a} \in \underline{M}_{G,C,Z}(\mathbb{F})$ and $\widehat{\mathcal{O}}_x$ denotes the completion of the local ring of x in $\widetilde{Sh}_{G,C} \otimes_{O_E} \mathbb{F}$, then there is an isomorphism of $\mathbb{F}$–algebras*

$$\widehat{\mathcal{O}}_x \cong \mathbb{F}[\![\, X_\varphi \mid \varphi \in \mathcal{Z}_0 \cup \mathcal{Z}(\mathfrak{a}) \,]\!] / (\, X_\varphi X_{\varphi^*} \mid \varphi \in \mathcal{Z}^+(\mathfrak{a}) \,)$$

such that the ideal of $Sh_{G,C,Z}$ in $\widehat{\mathcal{O}}_x$ is generated by $\{\, X_\varphi \mid \varphi \in Z \cap \mathcal{Z}(\mathfrak{a})^* \,\}$.

Proof. In §5 (cf. Corollary 5.13) we will show that there is a point $\mathfrak{a}$ in $\underline{\mathcal{M}}_{G,C}(\mathbb{F})$ such that $\mathcal{Z}(\mathfrak{a})^* = \mathcal{Z}(\mathfrak{a})$. In particular, the subschemes defined above are not empty. Since, for arbitrary admissible subsets $Z \subseteq Z'$, $(Z' \cup Z'^*) \smallsetminus (Z \cap Z^*)$ is the disjoint union of the set $Z' \smallsetminus (Z \cap Z'^*)$ and its image under *, $d(Z)$ is equal to the number of elements in $(\mathcal{Z}_0 \cup Z') \smallsetminus (Z \cap Z'^*)$. Therefore, our claim is an immediate consequence of Proposition 3.5. ▪

Corollary 3.14 *$\widetilde{Sh}_{G,C}$ is smooth if and only if $\mathcal{Z} = \mathcal{Z}_0$.*

§4. Isogeny types of $\mathbb{F}$−valued points

Considering the same situation as in §3, we study the structure of the set $\underline{\mathcal{M}}_{G,C}(\mathbb{F})$. This task can be divided into two parts: a classification up to isogeny and a description of a single isogeny class. These are separately dealt with in the present and the following sections. Due to a technical difficulty, we shall several times refer to Appendix A where a related question is considered in a slightly more general situation. To start with, we fix an element $\mathfrak{a} = (A, \iota, \Lambda, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(\mathbb{F})$. Let σ denote the Frobenius automorphism on the quotient field $\mathcal{K}$ of $\mathrm{W}(\mathbb{F})$, M the covariant Dieudonné module of A, i.e. the Dieudonné module of the Barsotti−Tate group of the dual of A in the sense of Demazure [12] (Chapter III, §8), and $\mathcal{F}$ its Frobenius endomorphism. The natural isomorphism $K \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \prod_{i=1}^m (K_{\mathfrak{q}_i} \times K_{\overline{\mathfrak{q}}_i})$ induces a decomposition

$$N = M \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{i=1}^{m} (N_i \oplus \overline{N}_i)\,. \tag{4.1}$$

$(N_i, \mathcal{F})$ and $(\overline{N}_i, \mathcal{F})$ are σ−isocrystals of height $4f_i$ where $f_i = [\, F_{\mathfrak{p}_i} : \mathbb{Q}_p \,]$, and D acts on them. We fix a p−principal polarization $\lambda \in \Lambda$ and obtain a perfect alternating bilinear form Φ on M. As in §3, let $\Phi' = \Phi \circ (\mathrm{id} \times b_0)$. This is again a perfect alternating bilinear form on M. It has the following properties:

$$\begin{aligned}
&\Phi' \circ (\mathcal{F} \times \mathcal{F}) = p\,\sigma \circ \Phi'\,;\\
&\Phi' \circ (k \times \mathrm{id}) \ = \Phi' \circ (\mathrm{id} \times \overline{k}) \quad \text{for all } k \in O_K\,;\\
&\Phi' \circ (l \times \mathrm{id}) \ \ = \Phi' \circ (\mathrm{id} \times l) \quad \text{for all } l \in O_L\,;\\
&\Phi' \circ (\Pi \times \mathrm{id}) = -\Phi' \circ (\mathrm{id} \times \Pi)\,.
\end{aligned}$$

In particular, Φ' induces a D−linear isomorphism between $(\overline{N}_i, \mathcal{F})$ and the dual of $(N_i, \mathcal{F})$ if the action of an element $d \in D$ on $N_i^\vee = \mathrm{Hom}_{\mathcal{K}}(N_i\,, \mathcal{K})$ is given by $h \mapsto h \circ \widehat{d}$ with $\widehat{d} = \Pi d^* \Pi^{-1} \in D \otimes_{\mathbb{Q}} \mathbb{Q}_p$.

Recall that N has a decomposition (3.2) into 2−dimensional $\mathcal{K}$−vector spaces. The corresponding decomposition of $M \subset N$ defines, for every $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L\,, \mathbb{C})$, a lattice $M_\varphi \subset N_\varphi$. For each i, we fix a $\varphi_i \in \operatorname{Hom}_{\mathbb{Q}_p}(L_{\mathfrak{p}_i}, \mathcal{K}) \subseteq \operatorname{Hom}_{\mathbb{Q}}(L, \mathbb{C})$ where $L_{\mathfrak{p}_i} = L \otimes_F F_{\mathfrak{p}_i}$ and denote the restriction of the σ^{f_i}–linear automorphism $\Pi\mathcal{F}^{f_i}$ to N_{φ_i} by $\mathcal{F}_i$. $(N_{\varphi_i}, \mathcal{F}_i)$ is an effective isocrystal of height 2 which determines $(N_i, \mathcal{F})$ together with the action of D up to unique isomorphism. In particular, if ξ is a slope of $(N_{\varphi_i}, \mathcal{F}_i)$ which occurs with multiplicity $\mu \in \{1,2\}$, and δ_i is 1 if D is ramified at $\mathfrak{p}_i$, and 0 otherwise, then $(2\xi - \delta_i)/2f_i$ is a slope of $(N_i, \mathcal{F})$ which occurs with multiplicity $2f_i\mu$, hence $\xi \geqslant \delta_i/2$. The endomorphism ring of $(N_{\varphi_i}, \mathcal{F}_i)$ is the commutant of D in the endomorphism ring of $(N_i, \mathcal{F})$. Since $\mathcal{F}M/pM \cong \operatorname{Lie} A$ has type $(L, \mathcal{V}_K)$, we obtain

$$
\begin{aligned}
\mathcal{F}M_{\sigma^{-1}\varphi} &= M_\varphi && \text{if } \varphi|_F \in \mathcal{V}\,; \\
pM_\varphi \subsetneqq \mathcal{F}M_{\sigma^{-1}\varphi} &\subsetneqq M_\varphi && \text{if } \varphi|_F \in \mathcal{Z}\,;
\end{aligned} \tag{4.2}
$$

and if $\varphi \in \operatorname{Hom}_{\mathbb{Q}_p}(L_{\mathfrak{p}_i}, \mathcal{K})$,

$$[\, M_{\varphi^*} : \mathcal{F}^{f_i} M_\varphi \,] = d_i = [\, M_\varphi : \mathcal{F}^{f_i} M_{\varphi^*} \,]$$

where d_i is the number of elements in $\mathcal{Z} \cap \operatorname{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathcal{K})$, and where, for arbitrary lattices Γ, Γ' in a finite-dimensional $\mathcal{K}$–vector space,

$$[\, \Gamma : \Gamma' \,] = \operatorname{length}_{\mathrm{W}(\mathbb{F})}(\Gamma/\Gamma \cap \Gamma') - \operatorname{length}_{\mathrm{W}(\mathbb{F})}(\Gamma'/\Gamma \cap \Gamma')\,.$$

Since Π commutes with $\mathcal{F}$, this implies

$$
\begin{aligned}
\Pi M_{\varphi^*} &= M_\varphi && \text{if } D \text{ splits at } \mathfrak{p}_i\,; \\
pM_\varphi \subsetneqq \Pi M_{\varphi^*} &\subsetneqq M_\varphi && \text{if } D \text{ is ramified at } \mathfrak{p}_i\,.
\end{aligned} \tag{4.3}
$$

In particular, $(N_{\varphi_i}, \mathcal{F}_i)$ has dimension $d_i + \delta_i$.

Lemma 4.4 *There is a basis $\{n_1, n_2\}$ of N_{φ_i} such that one of the following assertions holds:*

(i) $\mathcal{F}_i n_1 = n_2$, $\mathcal{F}_i n_2 = p^{d_i+\delta_i} n_1$, *and* $\operatorname{End}(N_{\varphi_i}, \mathcal{F}_i)$ *is a quaternion algebra with invariant* $(d_i + \delta_i)/2$ *over* $F_{\mathfrak{p}_i}$*;*

(ii) $\mathcal{F}_i n_1 = p^{\varepsilon_1+\delta_i} n_1$, $\mathcal{F}_i n_2 = p^{\varepsilon_2+\delta_i} n_2$ *where* $\varepsilon_1 > \varepsilon_2$ *are non-negative integers with* $\varepsilon_1 + \varepsilon_2 = d_i - \delta_i$, *and* $\operatorname{End}(N_{\varphi_i}, \mathcal{F}_i) \cong F_{\mathfrak{p}_i} \times F_{\mathfrak{p}_i}$.

Proof. This follows immediately from the classification of isocrystals over an algebraically closed field (cf. [45], Chapter VI). ∎

We shall now determine the commutant B' of B in $\operatorname{End}^\circ A$. To this end, we choose a finite field $\mathbb{F}_q \subset \mathbb{F}$ such that the abelian variety A and all of its endomorphisms are defined over $\mathbb{F}_q$, and we denote the Frobenius endomorphism of A over $\mathbb{F}_q$ by π_A.

Proposition 4.5 *One of the following two assertions holds:*

(i) $B' \cong D' \otimes_F K$ *for the quaternion algebra* $D'|F$ *which is ramified at all infinite places, has the same invariants as* D *at all finite places not dividing* p*, and invariant* $(d_i + \delta_i)/2$ *at* $\mathfrak{p}_i$*. The Rosati involution induced by* λ *restricts to the main involution on* D'*. Moreover,* $\pi_A \in K$ *with* $\pi_A \overline{\pi}_A = q$*, and*

$$\mathrm{ord}_{\overline{\mathfrak{q}}_i}(\pi_A) = d_i \, \mathrm{ord}_p(q)/2f_i \, .$$

(ii) $B' \cong L' \otimes_F K$ *for a totally imaginary quadratic extension* $L'|F$ *which splits* D *at all finite places not dividing* p*. The Rosati involution induced by* λ *restricts to complex conjugation on* L'*. Moreover,* $\pi_A \in B' \smallsetminus K$ *with* $\pi_A \overline{\pi}_A = q$*, and for every prime* $\mathfrak{p}$ *of* L' *dividing* $\mathfrak{p}_i$*, there is a unique integer* $\varepsilon(\mathfrak{p}) \geqslant 0$ *such that*

$$\sum_{\mathfrak{p}\,|\,\mathfrak{p}_i} (\varepsilon(\mathfrak{p}) + \delta_i) = d_i + \delta_i \, ;$$

$$\mathrm{ord}_{\mathfrak{q}}(\pi_A) = (d_i + \varepsilon(\mathfrak{p}) - \varepsilon(\overline{\mathfrak{p}}))\, \mathrm{ord}_{\mathfrak{q}}(q)/2f_i$$

if $\mathfrak{q}$ *is a prime of* B' *dividing* $\mathfrak{p}$ *and* $\overline{\mathfrak{q}}_i$*. For at least one* $\mathfrak{p}$*,* $\varepsilon(\mathfrak{p}) \neq \varepsilon(\overline{\mathfrak{p}})$*.*

Proof. The orders of π_A at a prime of $K(\pi_A)$ dividing p can be expressed in terms of the slopes of the isocrystal $(N, \mathcal{F})$ using the Theorem of Manin (cf. [12], Chapter IV). As mentioned above, these slopes can easily be read off from the slopes of the isocrystals $(N_{\varphi_i}, \mathcal{F}_i)$. Hence our claim follows from Lemma 4.4 combined with Proposition A2.7 and Remark A2.8 in Appendix A. The existence of a prime $\mathfrak{p}$ such that $\varepsilon(\mathfrak{p}) \neq \varepsilon(\overline{\mathfrak{p}})$ is equivalent to the fact that no power of π_A is contained in K. This holds if $\pi_A \notin K$ since all endomorphisms of A are defined over $\mathbb{F}_q$. ∎

Definition 4.6 Let $\mathfrak{I}_D$ denote the set of isomorphism classes of pairs (L', ε) where $L'|F$ is a totally imaginary quadratic extension which splits D at all places not dividing p and ε is a map from the set of all primes of L' dividing p into the set of non-negative integers such that $\varepsilon(\mathfrak{p}) \neq \varepsilon(\overline{\mathfrak{p}})$ for at least one $\mathfrak{p}$, and

$$\sum_{\mathfrak{p}\,|\,\mathfrak{p}_i} (\varepsilon(\mathfrak{p}) + \delta_i) = d_i + \delta_i$$

for all i. For every $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$, we consider the algebraic group

$$H_{\mathfrak{t}} = \begin{cases} D'^{\times} & \text{if } \mathfrak{t} = D' \, ; \\ L'^{\times} & \text{if } \mathfrak{t} = (L', \varepsilon) \, ; \end{cases}$$

over $\mathbb{Q}$. Let $Z_{\mathfrak{t}}$ denote the centre of $H_{\mathfrak{t}}$. The element of $\{D'\} \cup \mathfrak{I}_D$ associated to $\mathfrak{a}$ in Proposition 4.5 is called the *isogeny type* of $\mathfrak{a}$.

Consider now any pair (L',ε) where either $L'=L$ and $\varepsilon(\mathfrak{p})=d_i$ if $\mathfrak{p}\,|\,\mathfrak{p}_i$, or $(L',\varepsilon)\in\mathfrak{I}_D$. Let $Q'=L'\otimes_F K$. For a suitable power q of p, we find a $\pi\in O_{Q'}$ such that $\pi\overline{\pi}=q$ where $^{-}$ denotes complex conjugation on the CM–algebra Q', and

$$\operatorname{ord}_{\mathfrak{q}}(\pi)=(d_i+\varepsilon(\mathfrak{p})-\varepsilon(\overline{\mathfrak{p}}))\operatorname{ord}_{\mathfrak{q}}(q)/2f_i$$

if $\mathfrak{q}$ is a prime of Q' dividing $\mathfrak{p}$ and $\overline{\mathfrak{q}}_i$. In fact, the finiteness of the ideal class group of Q' implies that there is a $\pi_0\in Q'^{\times}$ and a positive power q_0 of p such that, for every prime $\mathfrak{q}$ of Q',

$$\operatorname{ord}_{\mathfrak{q}}(\pi_0)=\begin{cases}(f_i-d_i+\varepsilon(\mathfrak{p})-\varepsilon(\overline{\mathfrak{p}}))\operatorname{ord}_{\mathfrak{q}}(q_0)/f_i & \text{if } \mathfrak{q}\,|\,\mathfrak{q}_i \text{ and } \mathfrak{q}\,|\,\mathfrak{p}\,;\\ 0 & \text{otherwise}\,.\end{cases}$$

But then we may take $q=q_0^2$ and $\pi=q_0\overline{\pi}_0^{-1}\pi_0$. According to Tate [41], there is an abelian variety A of dimension $[\,Q':\mathbb{Q}]$ over $\mathbb{F}$ together with an embedding $Q'\to\operatorname{End}^{\circ}A$ such that A is defined over $\mathbb{F}_q$, π is mapped to the Frobenius endomorphism of A over $\mathbb{F}_q$, and the commutant of Q' in $\operatorname{End}^{\circ}A$ is isomorphic to $L'\otimes_F B$. In particular, we have an embedding $\iota_K\colon K\to\operatorname{End}^{\circ}A$, and the pair (A,ι_K) is, up to isogeny, uniquely determined by (L',ε).

Let $Q=L\otimes_F K$, and fix a Q'–algebra embedding $L'\otimes_F B\to\operatorname{End}^{\circ}A$ and a polarization λ of A such that the CM–algebra $R=Q'\otimes_K Q\subset L'\otimes_F B$ is invariant with respect to the corresponding Rosati involution. The proof of Lemma A3.3 shows that there is a positive involution on B which leaves Q invariant. Since two positive involutions on $L'\otimes_F B$ which coincide on R differ by conjugation with a totally positive real element of $R^{\times}$ (cf. [44], Satz 1.1), we may assume that the Rosati involution leaves B invariant. The same argument shows that we may multiply λ by a suitable totally positive element of a totally real commutative subalgebra of B to get a polarization such that the corresponding Rosati involution induces complex conjugation on L' and the involution $'$ on B. In particular, (A,ι_K) satisfies condition (A2.2) in Appendix A.

Definition 4.7 For every pair (L',ε) as above, let

$$\varrho(L',\varepsilon)=\varrho(Q\,|\,K,Q'\,|\,K,\Phi)\in\{+1,-1\}$$

where the right side is explained in §A1 if Φ is any CM–type of $R=Q'\otimes_K Q$ such that $\{\,\varphi|_Q\mid\varphi\in\Phi\,\}$ is $\operatorname{Gal}(Q\,|\,K)$–invariant and, for every prime $\mathfrak{r}\,|\,\overline{\mathfrak{q}}_i$ of R,

$$\#\Phi_{\mathfrak{r}}=\begin{cases}[\,R_{\mathfrak{r}}:L'_{\mathfrak{p}}\,]\,d_i & \text{if } \overline{\mathfrak{p}}=\mathfrak{p}\,;\\ d_i+\varepsilon(\mathfrak{p})-\varepsilon(\overline{\mathfrak{p}}) & \text{otherwise}\,;\end{cases}$$

with $\Phi_{\mathfrak{r}}=\Phi\cap\operatorname{Hom}_{\mathbb{Q}_p}(R_{\mathfrak{r}},\overline{\mathbb{Q}}_p)$ and $\mathfrak{p}=\mathfrak{r}\cap L'$. Using this obstruction, we define

$$\mathfrak{T}_D=\{D'\}\cup\{\,(L',\varepsilon)\in\mathfrak{I}_D\mid\varrho(L',\varepsilon)=1\,\}\,.$$

Lemma 4.8 *$\varrho(L',\varepsilon)$ is well-defined, and if L' splits D or D is totally indefinite, then $\varrho(L',\varepsilon)=1$.*

Proof. The existence of an appropriate CM–type Φ is given by Lemma A3.4. In fact, using the formula of Shimura and Taniyama (cf. [41], Lemme 5) one checks that the conditions on Φ in Definition 4.7 are equivalent to the conditions in this lemma. That $\varrho(L',\varepsilon)$ does not depend on Φ is a consequence of Proposition A3.6.

If L' splits D, we fix a CM–type Φ° of Q' such that, if $\mathfrak{q}\,|\,\overline{\mathfrak{q}}_i$ is a prime of Q', the number of elements in $\Phi^\circ_{\mathfrak{q}} = \Phi \cap \mathrm{Hom}_{\mathbb{Q}_p}(Q'_{\mathfrak{q}}, \overline{\mathbb{Q}}_p)$ equals $\varepsilon(\mathfrak{q}\cap L')$. Then $\Phi = \{\, \varphi \in \mathrm{Hom}_{\mathbb{Q}}(R,\mathbb{C}) \mid \varphi|_{Q'} \in \Phi^\circ \,\}$ is a CM–type of R which satisfies the conditions required in the definition. Let A_0 be an abelian variety of CM–type (R,Φ) over $\mathbb{C}$. Since Φ comes from a CM–type of Q', the commutant of Q' in $\mathrm{End}^\circ A_0$ is isomorphic to $\mathrm{M}_2(Q') \cong L' \otimes_F B$. This implies that the quasi-isogeny considered in Lemma A1.3 can be lifted to A_0 and, therefore, that the element r_Π used in the definition of $\varrho(Q\,|\,K, Q'\,|\,K,\Phi)$ in §A1 is contained in $R^\times$. From this it is evident that $\varrho(Q\,|\,K, Q'\,|\,K,\Phi)=1$.

In the totally indefinite case, we use a CM–type Φ which comes from $L'\otimes_F Q_0$ where Q_0 denotes the maximal totally real subfield of Q. The quasi-isogeny in Lemma A1.3 and the isomorphism $H\cong R$ in §A1 may now be chosen such that $r_\Pi \in (L'\otimes_F Q_0 \otimes_{\mathbb{Q}} \widetilde{\mathbb{A}}_f)^\times$. But then the element x which occurs in the construction of $\varrho(Q\,|\,K, Q'\,|\,K,\Phi)$ is contained in $(Q_0\otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$, and $x^2 = x\overline{x} \in Q_+^\times$ yields $x\in Q_0^\times$. Hence q can be chosen in $Q_0^\times$, which implies our claim. ∎

Remark 4.9 Lemma 4.8 shows, in particular, that $\mathfrak{T}_D = \{D'\}\cup \mathfrak{I}_D$ if D is totally indefinite or if $\mathcal{Z}=\mathcal{Z}_0$, i.e. $\widetilde{Sh}_{G,C}$ is smooth.

Consider now a pair (L',ε) and the abelian variety (A,ι_K) associated to it above. Let

$$\mathfrak{t} = \begin{cases} D' & \text{if } L'=L\,; \\ (L',\varepsilon) & \text{otherwise}\,; \end{cases}$$

and put $A_{\mathfrak{t}} = A$. As an immediate consequence of Definition A3.2 and (A2.6) we obtain the equation $\alpha(A,\iota_K)=\beta(\psi)$, and Proposition A3.6 implies that (A,ι_K) satisfies the conditions in Proposition A2.7 if and only if $\mathfrak{t}\in\mathfrak{T}_D$. Hence, for every $\mathfrak{t}\in\mathfrak{T}_D$, we may fix

a K–algebra homomorphism $\iota_{\mathfrak{t}}\colon B\to \mathrm{End}^\circ A_{\mathfrak{t}}$;

a polarization $\lambda_{\mathfrak{t}}$ of $A_{\mathfrak{t}}$;

an isomorphism $\eta_{\mathfrak{t}}\colon B\otimes_{\mathbb{Q}}\mathbb{A}_f^p \to \widehat{\mathrm{V}}^p(A_{\mathfrak{t}})$ of $B\otimes_{\mathbb{Q}}\mathbb{A}_f^p$–modules;

such that there is an element $f \in F \otimes_{\mathbb{Q}} \mathbb{A}_f^p(-1)$ which satisfies $E_{\lambda_{\mathfrak{t}}} \circ (f\eta_{\mathfrak{t}} \times \eta_{\mathfrak{t}}) = \psi$. We identify the commutant of B in $\mathrm{End}^\circ A_{\mathfrak{t}}$ with $D' \otimes_F K$ or $L' \otimes_F K$, respectively, as in Remark A2.8. Recall that B has a natural structure of $D \otimes_F B$–module (cf. §1). We fix an $F \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–algebra homomorphism from $D' \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ or $L' \otimes_{\mathbb{Q}} \mathbb{A}_f^p$, respectively, into $D \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ and assume that $\eta_{\mathfrak{t}}$ is compatible with the corresponding actions of these algebras on $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ and $\widehat{\mathrm{V}}^p(A_{\mathfrak{t}})$. That this is always possible follows easily from the Theorem of Skolem and Noether.

Lemma 4.10 *Let $G_{\mathfrak{t}}$ denote the group of all $\beta \in \mathrm{End}^\circ A_{\mathfrak{t}}$ such that*

$$\begin{aligned} &\iota_{\mathfrak{t}}(b) \circ \beta = \beta \circ \iota_{\mathfrak{t}}(b) \quad \text{for all } b \in B\,; \\ &\widehat{\beta} \circ \lambda_{\mathfrak{t}} \circ \beta \in \lambda_{\mathfrak{t}} \circ \iota_{\mathfrak{t}}(F^\times)\,; \end{aligned}$$

considered as an algebraic group over $\mathbb{Q}$. There is an exact sequence

$$\begin{array}{c} 1 \longrightarrow F^\times \longrightarrow H_{\mathfrak{t}} \times K^\times \longrightarrow G_{\mathfrak{t}} \longrightarrow 1 \\ f \longmapsto (f\,, f^{-1}) \\ (h\,, k) \longmapsto h \otimes k\,. \end{array}$$

In particular, $G_{\mathfrak{t}}$ satisfies the Hasse principle for H^1, i.e. the natural map

$$\mathrm{H}^1(\mathbb{Q}, G_{\mathfrak{t}}) \longrightarrow \prod_{\text{all } \ell} \mathrm{H}^1(\mathbb{Q}_\ell\,, G_{\mathfrak{t}})\,,$$

where ℓ runs through the set of all places of $\mathbb{Q}$, is injective.

Proof. Since $\widehat{\beta} \circ \lambda_{\mathfrak{t}} \circ \beta \in \lambda_{\mathfrak{t}} \circ \iota_{\mathfrak{t}}(F^\times)$ if and only if $\beta'\beta \in \iota_{\mathfrak{t}}(F^\times)$ where $'$ denotes the Rosati involution induced by $\lambda_{\mathfrak{t}}$, the exact sequence is an obvious consequence of the construction above. It gives rise to a commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathbb{Q}, G_{\mathfrak{t}}) \arrow[r] \arrow[d] & \mathrm{H}^2(\mathbb{Q}, F^\times) \arrow[d] \\ \prod_{\text{all } \ell} \mathrm{H}^1(\mathbb{Q}_\ell\,, G_{\mathfrak{t}}) \arrow[r] & \prod_{\text{all } \ell} \mathrm{H}^2(\mathbb{Q}_\ell\,, F^\times) \end{tikzcd}$$

whose top row is injective since $\mathrm{H}^1(\mathbb{Q}, H_{\mathfrak{t}}) = \mathrm{H}^1(\mathbb{Q}, K^\times) = 0$ by Hilbert's Theorem 90. Since the right vertical map in this diagram is injective by class field theory, the second claim follows. ∎

Proposition 4.11 *The isogeny type of an element $\mathfrak{a} = (A, \iota, \Lambda, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(\mathbb{F})$ is contained in $\mathfrak{T}_D$. For any $\mathfrak{t} \in \mathfrak{T}_D$, there is an element $\beta \in \mathrm{Hom}^\circ(A\,, A_{\mathfrak{t}})$ such that*

$$\begin{aligned} &\iota_{\mathfrak{t}}(b) \circ \beta = \beta \circ \iota(b) \qquad \text{for all } b \in O_B\,; \\ &\widehat{\beta} \circ \lambda_{\mathfrak{t}} \circ \beta \in \Lambda\,; \end{aligned}$$

if and only if $\mathfrak{a}$ has isogeny type $\mathfrak{t}$.

Proof. Proposition 4.5 implies the equation $\alpha(A, \iota|_K) = \beta(\psi)$ where the expressions on both sides are given by Definition A3.2 and (A2.6), respectively. On the other hand, by Proposition A2.7, we have the equation $\beta(A, \iota|_K, B\,|\,K) = \beta(\psi)$. The first part of our claim is, therefore, a consequence of Proposition A3.6.

Assume now that there is a β as above. It induces an isomorphism of K–algebras between the commutants of B in $\mathrm{End}^\circ A$ and $\mathrm{End}^\circ A_{\mathfrak{t}}$, respectively, which respects the Rosati involutions. If β and all endomorphisms of A and $A_{\mathfrak{t}}$ are defined over $\mathbb{F}_q$, this isomorphism carries the Frobenius endomorphism π_A of A over $\mathbb{F}_q$ into the Frobenius endomorphism $\pi_{A_{\mathfrak{t}}}$ of $A_{\mathfrak{t}}$ over $\mathbb{F}_q$. Together with Proposition 4.5 this implies that $\mathfrak{a}$ has isogeny type $\mathfrak{t}$.

If, on the other hand, $\mathfrak{a}$ has isogeny type $\mathfrak{t}$, and all endomorphisms of A and $A_{\mathfrak{t}}$ are defined over $\mathbb{F}_q$, Proposition 4.5 gives a K–algebra isomorphism $K(\pi_A) \cong K(\pi_{A_{\mathfrak{t}}})$ such that π_A and $\pi_{A_{\mathfrak{t}}}$ have the same absolute values at all infinite places and the same order at all finite places, hence differ by a root of unity. Replacing q by a suitable positive power, we may assume that they are identified by this isomorphism. According to Tate [41], this implies that A and $A_{\mathfrak{t}}$ are isogenous. Applying the Theorem of Skolem and Noether, we find an isogeny $\beta\colon A \to A_{\mathfrak{t}}$ which commutes with the actions of B and the F–algebra D' or L', respectively, given by $\mathfrak{t}$.

We fix a polarization $\lambda \in \Lambda$ and consider $\alpha_0 = \lambda^{-1} \circ \widehat{\beta} \circ \lambda_{\mathfrak{t}} \circ \beta \in \mathrm{End}^\circ A$. Since λ and $\lambda_{\mathfrak{t}}$ are self-dual and induce the same involution on B, α_0 commutes with B, and $\alpha_0' = \alpha_0$. If $\mathfrak{t} = D'$, λ and $\lambda_{\mathfrak{t}}$ induce the same involution on D', hence α_0 commutes with D'. This implies $\alpha_0 \in F$, i.e. β has the required properties.

Assume now that $\mathfrak{t} = (L', \varepsilon)$. It remains to show that there is an $\alpha \in B' = L' \otimes_F K$ such that $\alpha' \alpha_0 \alpha \in F^\times$ since then $\beta\alpha$ has the required properties. Corollary 4.10 implies that it suffices to solve this problem locally. Since there is an isomorphism

$$B' \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong (L' \otimes_{\mathbb{Q}} \mathbb{Q}_p) \times (L' \otimes_{\mathbb{Q}} \mathbb{Q}_p)$$

with $(\alpha_1, \alpha_2)' = (\overline{\alpha}_2, \overline{\alpha}_1)$, we find an $\alpha \in B' \otimes_{\mathbb{Q}} \mathbb{Q}_p$ such that $\alpha' \alpha_0 \alpha \in F^\times(\mathbb{Q}_p)$, and since

$$B' \otimes_{\mathbb{Q}} \mathbb{R} \cong (L' \otimes_{\mathbb{Q}} \mathbb{R}) \times (L' \otimes_{\mathbb{Q}} \mathbb{R})$$

with $(\alpha_1, \alpha_2)' = (\overline{\alpha}_1, \overline{\alpha}_2)$ and α_0 is totally positive, there is an $\alpha \in B' \otimes_{\mathbb{Q}} \mathbb{R}$ such that $\alpha' \alpha_0 \alpha \in F^\times(\mathbb{R})$.

Consider now a $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–linear isomorphism $\eta\colon B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \widehat{\mathrm{V}}^p(A)$ and elements f and $f_{\mathfrak{t}} \in F \otimes_{\mathbb{Q}} \mathbb{A}_f^p(-1)$ such that

$$E^{\lambda_{\mathfrak{t}}} \circ (f_{\mathfrak{t}} \eta_{\mathfrak{t}} \times \eta_{\mathfrak{t}}) = \psi = E^{\lambda} \circ (f\eta \times \eta)\,.$$

We may assume that η is compatible with the actions of L' on both sides. In particular, $\eta^{-1}\beta^{-1}\eta_{\mathfrak{t}}$ is $L' \otimes_F B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–linear, hence there is an $\alpha \in B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ such that

$\eta_{\mathfrak{t}} = \beta\alpha\eta$. This yields

$$E^{\lambda_{\mathfrak{t}}} \circ (\beta\alpha \times \beta\alpha) = E^{\lambda} \circ (f_0 \times \mathrm{id})$$

where $f_0 \in F^{\times}(\mathbb{A}_f^p)$ with $f_0 f_{\mathfrak{t}} = f$. By definition of α_0, we have

$$E^{\lambda_{\mathfrak{t}}} \circ (\beta \times \beta) = E^{\lambda} \circ (\alpha_0 \times \mathrm{id}).$$

Both equations together imply $\alpha' \alpha_0 \alpha = f_0$, and this completes the proof. ∎

§5. The structure of an isogeny class

Before we can describe the set of $\mathbb{F}$–valued points of a given isogeny type, we need some more notation. We choose an order $O_B \subset B$ as in (2.9), $\Pi \in O_D \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $b_0 \in (O_B \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\times}$ as in (2.10), and recall that, for each $i \in \{1, \ldots, m\}$, we have fixed a $\varphi_i \in \mathrm{Hom}_{\mathbb{Q}_p}(L_{\mathfrak{p}_i}, \mathcal{K})$ where $L_{\mathfrak{p}_i} \subset D_{\mathfrak{p}_i}$ is an unramified quadratic extension of $F_{\mathfrak{p}_i}$.

Definition 5.1 Y_D denotes the set of all $d \in D^{\times}(\mathcal{K})$ such that $d^{-1} \in O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})$ and, for every $v \in \mathrm{Hom}(F, \mathcal{K}) \cong \mathrm{Hom}(F, \mathbb{R})$,

$$\mathrm{ord}_p\big((v \otimes \mathrm{id}_{\mathcal{K}})(\mathrm{Nm}_{D|F}(d))\big) = \begin{cases} -1 & \text{if } v \in \mathcal{Z}\,; \\ 0 & \text{if } v \in \mathcal{V}\,. \end{cases}$$

For every $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$, we choose an embedding $j_{\mathfrak{t}}$ of the $F \otimes_{\mathbb{Q}} \mathcal{K}$–algebra $D' \otimes_{\mathbb{Q}} \mathcal{K}$ if $\mathfrak{t} = D'$ or $L' \otimes_{\mathbb{Q}} \mathcal{K}$ if $\mathfrak{t} = (L', \varepsilon)$, respectively, into $D \otimes_{\mathbb{Q}} \mathcal{K}$ which commutes with the actions of σ^n where n denotes the least common multiple of the integers $2f_i$. Moreover, we consider the element $z_{\mathfrak{t}} \in Z_{\mathfrak{t}}(\mathbb{Q}_p)$ with components $p^{-d_i n/2f_i} \in F_{\mathfrak{p}_i}$ if $\mathfrak{t} = D'$, or $p^{-n(\mathfrak{p})} \in L'_{\mathfrak{p}}$, where $n(\mathfrak{p}) = (d_i + \varepsilon(\mathfrak{p}) - \varepsilon(\overline{\mathfrak{p}}))n/2f_i$ if $\mathfrak{t} = (L', \varepsilon)$ and $\mathfrak{p}$ divides $\mathfrak{p}_i$.

Lemma 5.2 *For every $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$, there is an element $d_{\mathfrak{t}} \in D^{\times}(\mathcal{K})$ such that*

$$j_{\mathfrak{t}}(\sigma(h)) = d_{\mathfrak{t}} \sigma(j_{\mathfrak{t}}(h)) d_{\mathfrak{t}}^{-1} \quad \textit{for all } h \in H_{\mathfrak{t}}(\mathcal{K})$$

and

$$d_{\mathfrak{t}} \sigma(d_{\mathfrak{t}}) \cdots \sigma^{n-1}(d_{\mathfrak{t}}) = j_{\mathfrak{t}}(z_{\mathfrak{t}}).$$

The σ–conjugacy class of $d_{\mathfrak{t}}$ is uniquely determined and contains an element of Y_D.

Proof. We consider the isomorphism of $\mathcal{K}$–algebras

$$\omega\colon D \otimes_{\mathbb{Q}} \mathcal{K} \longrightarrow \prod_{i=1}^{m} \prod_{j=0}^{f_i - 1} \mathrm{M}_2(\mathcal{K})$$

which is determined by the following conditions.

$$\omega(l)_{i,j} = \begin{pmatrix} \sigma^{-j}\varphi_i(l) & 0 \\ 0 & \sigma^{f_i-j}\varphi_i(l) \end{pmatrix} \qquad \text{for all } l \in L \subset D\,;$$
$$\omega(\Pi)_{i,j} = \begin{pmatrix} 0 & p^{\delta_i} \\ 1 & 0 \end{pmatrix}.$$

This isomorphism commutes with the actions of σ^n, and for all $d \in D \otimes_{\mathbb{Q}} \mathcal{K}$,

$$\omega(\sigma(d))_{i,j} = \begin{cases} \sigma(\omega(d)_{i,j+1}) & \text{if } j < f_i - 1\,; \\ \sigma(\omega(\Pi\, d\,\Pi^{-1})_{i,0}) & \text{if } j = f_i - 1\,. \end{cases}$$

An element $d \in (D \otimes_{\mathbb{Q}} \mathcal{K})^\times$ is contained in Y_D if and only if, for every pair (i,j) occurring in the above product, the matrix $\omega(d^{-1})_{i,j}$ is contained in

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{M}_2(\mathrm{W}(\mathbb{F})) \;\middle|\; \mathrm{ord}_p(\beta) \geqslant \delta_i \right\}$$

and its determinant has order 1 if $\sigma^{-j}\varphi_i|_F \in \mathcal{Z}$, and order 0 otherwise. In particular, there is an element $y_{\mathfrak{t}} \in Y_D$ such that, for all pairs (i,j) as above,

$$\omega(y_{\mathfrak{t}})_{i,j} = \begin{cases} 1 & \text{if } \sigma^{-j}\varphi_i|_F \in \mathcal{V}\,; \\ \begin{pmatrix} 0 & 1 \\ p^{-1} & 0 \end{pmatrix} & \text{if } \sigma^{-j}\varphi_i|_F \in \mathcal{Z} \text{ and the condition below is satisfied}\,; \\ \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} & \text{otherwise}\,; \end{cases}$$

where the condition referred to in the second case is satisfied if either $\mathfrak{t} = D'$ and D is ramified at $\mathfrak{p}_i$, or $\mathfrak{t} = (L', \varepsilon)$ and

$$\#\{\, l \in \{j+1, \dots, f_i - 1\} \mid \sigma^{-l}\varphi_i|_F \in \mathcal{Z} \,\} = \varepsilon(\mathfrak{p}) < \varepsilon(\overline{\mathfrak{p}})$$

for a prime $\mathfrak{p}$ of L' dividing $\mathfrak{p}_i$. A straightforward computation shows that the $\mathbb{Q}_p$-algebra

$$\{\, x \in \mathrm{M}_2(\mathcal{K}) \mid x_{\mathfrak{t},i}\,\sigma^{f_i}(x)\,x_{\mathfrak{t},i}^{-1} = x \,\},$$

with $x_{\mathfrak{t},i} = \omega(y_{\mathfrak{t}}\sigma(y_{\mathfrak{t}}) \cdots \sigma^{f_i-1}(y_{\mathfrak{t}})\,\Pi)_{i,0}$, is isomorphic to $L' \otimes_F F_{\mathfrak{p}_i}$ if $\mathfrak{t} = (L', \varepsilon)$ and $\varepsilon(\mathfrak{p}) < \varepsilon(\overline{\mathfrak{p}})$ for a prime $\mathfrak{p}$ of L' dividing $\mathfrak{p}_i$, and is a quaternion algebra with invariant $(d_i + \delta_i)/2$ over $F_{\mathfrak{p}_i}$ otherwise. Another simple computation gives

$$\omega(y_{\mathfrak{t}}\sigma(y_{\mathfrak{t}}) \cdots \sigma^{n-1}(y_{\mathfrak{t}}))_{i,0} = \begin{cases} \begin{pmatrix} p^{-n(\mathfrak{p})} & 0 \\ 0 & p^{-n(\overline{\mathfrak{p}})} \end{pmatrix} & \text{if } \mathfrak{t} = (L', \varepsilon) \text{ and } \mathfrak{p} \,|\, \mathfrak{p}_i \text{ with } \varepsilon(\mathfrak{p}) < \varepsilon(\overline{\mathfrak{p}})\,; \\ p^{-d_i n/2f_i} & \text{otherwise}\,; \end{cases}$$

and this element is contained in the $\mathbb{Q}_p$–algebra considered above. Using these two observations, we conclude that there is an $F\otimes_{\mathbb{Q}}\mathcal{K}$–algebra homomorphism $\omega_{\mathfrak{t}}$ of $D'\otimes_{\mathbb{Q}}\mathcal{K}$ or $L'\otimes_{\mathbb{Q}}\mathcal{K}$, respectively, into $\prod_{i=1}^{m}\prod_{j=0}^{f_i-1}\mathrm{M}_2(\mathcal{K})$ (with the algebra structure induced by ω) which commutes with σ^n and satisfies

$$\omega_{\mathfrak{t}}(\sigma(h))_{i,j}=\begin{cases}\omega(y_{\mathfrak{t}})_{i,j}\,\sigma(\omega_{\mathfrak{t}}(h)_{i,j+1})\,\omega(y_{\mathfrak{t}})_{i,j}^{-1} & \text{if } j<f_i-1\,;\\ \omega(y_{\mathfrak{t}}\Pi)_{i,j}\,\sigma(\omega_{\mathfrak{t}}(h)_{i,0})\,\omega(y_{\mathfrak{t}}\Pi)_{i,j}^{-1} & \text{if } j=f_i-1\,;\end{cases}$$

for all $h\in H_{\mathfrak{t}}(\mathcal{K})$, and

$$\omega(y_{\mathfrak{t}}\sigma(y_{\mathfrak{t}})\cdots\sigma^{n-1}(y_{\mathfrak{t}}))=\omega_{\mathfrak{t}}(z_{\mathfrak{t}})\,.$$

The Theorem of Skolem and Noether implies that there is a $d\in D^{\times}(\mathcal{K}_n)$, where $\mathcal{K}_n\,|\,\mathbb{Q}_p$ denotes the extension of degree n in $\mathcal{K}$, such that

$$\omega_{\mathfrak{t}}(h)=\omega(d\,j_{\mathfrak{t}}(h)\,d^{-1})\quad\text{for all } h\in H_{\mathfrak{t}}(\mathcal{K})\,.$$

But then $d_{\mathfrak{t}}=d^{-1}y_{\mathfrak{t}}\sigma(d)$ has the desired properties.

It remains to prove the uniqueness assertion. Since the σ–conjugacy class of the element just constructed is independent of $j_{\mathfrak{t}}$, it is sufficient to do this for fixed $j_{\mathfrak{t}}$, so that any other element satisfying the two conditions of the lemma has the form $d=j_{\mathfrak{t}}(z)\,d_{\mathfrak{t}}$ for a certain $z\in Z_{\mathfrak{t}}(\mathcal{K})$ with

$$z\,\sigma(z)\cdots\sigma^{n-1}(z)=1\,.$$

Since $\mathrm{H}^1(\mathcal{K}_n\,|\,\mathbb{Q}_p\,,Z_{\mathfrak{t}})=0$ by Hilbert's Theorem 90, the restriction map

$$\mathrm{H}^1(\langle\sigma\rangle\,,Z_{\mathfrak{t}}(\mathcal{K}))\longrightarrow\mathrm{H}^1(\langle\sigma^n\rangle\,,Z_{\mathfrak{t}}(\mathcal{K}))$$

is injective. This implies that there is a $z_0\in Z_{\mathfrak{t}}(\mathcal{K})$ such that $z=z_0^{-1}\sigma(z_0)$, hence $d=j_{\mathfrak{t}}(z_0)^{-1}d_{\mathfrak{t}}\sigma(j_{\mathfrak{t}}(z_0))$. ∎

For every $\mathfrak{t}\in\{D'\}\cup\mathfrak{I}_D$, we fix a $d_{\mathfrak{t}}$ as in Lemma 5.2 and consider the set

$$X_{\mathfrak{t}}=\bigl\{\,d\in D^{\times}(\mathcal{K})\;\big|\;\mathcal{F}_{\mathfrak{t}}(d)\in dY_D\,\bigr\}/\bigl(O_D\otimes_{\mathbb{Z}}\mathrm{W}(\mathbb{F})\bigr)^{\times}$$

where $\mathcal{F}_{\mathfrak{t}}(d)=d_{\mathfrak{t}}\sigma(d)$. Obviously, this set is not empty, and since Y_D is invariant with respect to σ^e where $e=[\,E:\mathbb{Q}]$, $\mathcal{F}_{\mathfrak{t}}^e$ induces an automorphism $\mathrm{Fr}_{\mathfrak{t}}$ of $X_{\mathfrak{t}}$. Moreover, $H_{\mathfrak{t}}(\mathbb{Q}_p)$ acts on $X_{\mathfrak{t}}$ by left multiplication via the embedding

$$H_{\mathfrak{t}}(\mathbb{Q}_p)\subset H_{\mathfrak{t}}(\mathcal{K})\longrightarrow D^{\times}(\mathcal{K})$$

induced by $j_{\mathfrak{t}}$. The set $X_{\mathfrak{t}}$ together with the actions of $H_{\mathfrak{t}}(\mathbb{Q}_p)$ and $\mathrm{Fr}_{\mathfrak{t}}$ is, up to isomorphism, independent of the choices of $j_{\mathfrak{t}}$ and $d_{\mathfrak{t}}$.

From now on we consider a fixed $\mathfrak{t} \in \mathfrak{T}_D$. For every $C \subset G(\mathbb{A}_f)$ satisfying (2.11), let $\underline{\mathcal{M}}_{G,C}(\mathfrak{t}) \subset \underline{\mathcal{M}}_{G,C}(\mathbb{F})$ denote the subset of elements of isogeny type $\mathfrak{t}$. $\eta_{\mathfrak{t}}$ defines an embedding

$$G_{\mathfrak{t}}(\mathbb{Q}) \subset G_{\mathfrak{t}}(\mathbb{A}_f^p) \longrightarrow G(\mathbb{A}_f^p) \tag{5.3}$$

such that the diagram

$$\begin{tikzcd} H_{\mathfrak{t}}(\mathbb{A}_f^p) \times K^\times(\mathbb{A}_f^p) \arrow[r] \arrow[d] & D^\times(\mathbb{A}_f^p) \times K^\times(\mathbb{A}_f^p) \arrow[d] \\ G_{\mathfrak{t}}(\mathbb{A}_f^p) \arrow[r] & G(\mathbb{A}_f^p) \end{tikzcd}$$

commutes. We consider the rational covariant Dieudonné module $(N, \mathcal{F})$ of $A_{\mathfrak{t}}$ together with the perfect pairing $\Phi'\colon N \times N \to \mathcal{K}$ induced by $\lambda_{\mathfrak{t}}$ and b_0 as in §4. There is a natural map

$$G_{\mathfrak{t}}(\mathbb{Q}) \subset G_{\mathfrak{t}}(\mathbb{Q}_p) \longrightarrow \mathrm{Aut}_B(N, \mathcal{F})\,. \tag{5.4}$$

As in (4.1), we have a natural decomposition $N = \bigoplus_{i=1}^m (N_i \oplus \overline{N}_i)$ such that the bilinear form Φ' identifies $\overline{N}_i$ with the dual of N_i. From (5.4) we obtain an embedding

$$H_{\mathfrak{t}}(\mathbb{Q}_p) \longrightarrow \mathrm{Aut}_D\Big(\bigoplus_{i=1}^m N_i\,,\, \mathcal{F}\Big)\,.$$

Lemma 5.5 *There is an isomorphism*

$$\eta_p\colon D \otimes_{\mathbb{Q}} \mathcal{K} \longrightarrow \bigoplus_{i=1}^m N_i$$

of $D \otimes_{\mathbb{Q}} \mathcal{K}$–modules such that the action of any element $h \in H_{\mathfrak{t}}(\mathcal{K})$ on $\bigoplus_{i=1}^m N_i$ corresponds to right multiplication by $j_{\mathfrak{t}}(h)^* \in D^\times(\mathcal{K})$, and $\mathcal{F}$ corresponds to the map $x \mapsto \sigma(x) d_{\mathfrak{t}}^{-1}$.

Proof. To start with, we fix an isomorphism $\bigoplus_{i=1}^m N_i \cong D \otimes_{\mathbb{Q}} \mathcal{K}$ of $D \otimes_{\mathbb{Q}} \mathcal{K}$–modules such that the action of $h \in H_{\mathfrak{t}}(\mathcal{K})$ on $\bigoplus_{i=1}^m N_i$ corresponds to right multiplication by $j_{\mathfrak{t}}(h)^*$. Let $d \in D^\times(\mathcal{K})$ be such that $\mathcal{F}$ corresponds to the map $x \mapsto \sigma(x) d^{-1}$. Since the actions of $\mathcal{F}$ and $H_{\mathfrak{t}}(\mathbb{Q}_p)$ on $\bigoplus_{i=1}^m N_i$ commute, we obtain

$$j_{\mathfrak{t}}(\sigma(h)) = d\sigma(j_{\mathfrak{t}}(h))\, d^{-1} \quad \text{for all } h \in H_{\mathfrak{t}}(\mathcal{K})\,,$$

hence $d = j_{\mathfrak{t}}(z) d_{\mathfrak{t}}$ for a certain $z \in Z_{\mathfrak{t}}(\mathcal{K})$. Let $z_1 = z\sigma(z) \cdots \sigma^{n-1}(z) z_{\mathfrak{t}}$, and let $^-$ denote complex conjugation on L' if $\mathfrak{t} = (L', \varepsilon)$, and the identity on F if $\mathfrak{t} = D'$. $z_{\mathfrak{t}}$ is defined such that $(\bigoplus_{i=1}^m N_i\,, \bar{z}_{\mathfrak{t}} \mathcal{F}^n)$ is an isotypic σ^n–isocrystal of slope 0. This follows easily from the construction of $A_{\mathfrak{t}}$ and the Theorem of Manin (cf. [12], Chapter V). Since the above isomorphism identifies $\bar{z}_1\mathcal{F}^n$ with the map $x \mapsto \sigma^n(x)$, $(\bigoplus_{i=1}^m N_i, \bar{z}_1\mathcal{F}^n)$ is an isotypic σ^n–isocrystal of slope 0, too. Moreover, since the local degree of L' or F, respectively, at any place above p divides n, σ^n respects the decomposition of $F\otimes_{\mathbb{Q}}\mathcal{K}$ or $L'\otimes_{\mathbb{Q}}\mathcal{K}$, respectively, into a product of fields. Hence $\mathcal{F}^n$ respects the corresponding decomposition of $\bigoplus_{i=1}^m N_i$. This implies that every component of z_1 in the above product of fields differs from the corresponding component of $z_{\mathfrak{t}}$ by a unit, i.e. $z\sigma(z)\cdots\sigma^{n-1}(z)$ is contained in the σ^n–conjugacy class of 1. As in the proof of Lemma 5.2, this implies the existence of an element $z_0 \in Z_{\mathfrak{t}}(\mathcal{K})$ such that $d = j_{\mathfrak{t}}(z_0)^{-1}d_{\mathfrak{t}}\sigma(j_{\mathfrak{t}}(z_0))$, and we may compose the isomorphism chosen above with the action of $\bar{z}_0$ to get another one which satisfies the required conditions. ▪

Let now $Y_{\mathfrak{t}}$ denote the set of all O_B–invariant lattices $M \subset N$ such that

$pM \subseteq \mathcal{F}M \subseteq M$;

$\mathcal{F}M/pM$ is an $O_L\otimes_{O_F} O_K \otimes_{\mathbb{Z}} \mathbb{F}$–module of type $(L, \mathcal{V}_K)$;

there is an $f \in F^{\times}(\mathbb{Q}_p)$ such that $\Phi' \circ (f \times \mathrm{id})\colon M \times M \to \mathrm{W}(\mathbb{F})$ is a perfect pairing.

Consider a point $\mathfrak{a} = (A, \iota, \Lambda, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(\mathfrak{t})$ and an isogeny β as in Proposition 4.11. It defines an embedding of the covariant Dieudonné module M of A into N and identifies $\widehat{\mathrm{V}}^p(A)$ with $\widehat{\mathrm{V}}^p(A_{\mathfrak{t}})$, hence determines a class $\eta_{\mathfrak{t}}^{-1}\circ\overline{\eta} \in G(\mathbb{A}_f^p)/C^p$. Thus we have an obvious bijection

$$\underline{\mathcal{M}}_{G,C}(\mathfrak{t}) \cong G_{\mathfrak{t}}(\mathbb{Q})\big\backslash\big(Y_{\mathfrak{t}} \times G(\mathbb{A}_f^p)/C^p\big) \tag{5.6}$$

where $\alpha \in G_{\mathfrak{t}}(\mathbb{Q})$ acts on $Y_{\mathfrak{t}}$ as $M \mapsto \alpha M$ via (5.4) and on $G(\mathbb{A}_f^p)$ as left multiplication via (5.3). The natural action of the Frobenius automorphism of $\mathbb{F}$ over the residue class field of O_E on $\underline{\mathcal{M}}_{G,C}(\mathbb{F})$ corresponds to the map induced by the endomorphism $M \mapsto p^{-e}\mathcal{F}^eM$ of $Y_{\mathfrak{t}}$. For every $M \in Y_{\mathfrak{t}}$, there is a decomposition

$$M = \bigoplus_{i=1}^m (p^{n_i}M_i \oplus M_i^{\vee}) \tag{5.7}$$

where $M_i \subset N_i$ is a lattice, $M_i^{\vee} \subset \overline{N}_i$ is the dual lattice (with respect to Φ'), and n_i is an integer.

Let $X'_{\mathfrak{t},i}$ denote the set of lattices $M_i = \bigoplus_{\varphi} M_{\varphi} \subset N_i$ where the sum is extended over all $\varphi \in \mathrm{Hom}_{\mathbb{Q}_p}(L_{\mathfrak{p}_i}, \mathcal{K})$, and $M_{\varphi} \subset N_{\varphi}$ are lattices which satisfy (4.2) and (4.3). On $X'_{\mathfrak{t}} = \prod_{i=1}^m X'_{\mathfrak{t},i}$ we consider the dual of the obvious action of $H_{\mathfrak{t}}(\mathbb{Q}_p)$, i.e. $h \in H_{\mathfrak{t}}(\mathbb{Q}_p)$ acts as

$$\bigoplus_{i=1}^m M_i \longmapsto \Big(h\bigoplus_{i=1}^m M_i^{\vee}\Big)^{\vee} = \overline{h}^{-1}\bigoplus_{i=1}^m M_i \tag{5.8}$$

where the last equality follows from the fact that the Rosati involution corresponding to $\lambda_{\mathfrak{t}}$ induces the main involution on D' or complex conjugation on L', respectively, which we have denoted by $^{-}$. Let

$$Y'_{\mathfrak{t}} = F^{\times}(\mathbb{Q}_p)\backslash\big(X'_{\mathfrak{t}} \times K^{\times}(\mathbb{Q}_p)/(O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\times}\big)$$

where an element $f \in F^{\times}(\mathbb{Q}_p)$ acts on $X'_{\mathfrak{t}}$ via its natural embedding into $H_{\mathfrak{t}}(\mathbb{Q}_p)$ and on $K^{\times}(\mathbb{Q}_p)$ as multiplication by f^{-1}. The exact sequence in Lemma 4.10 induces a right action of $G_{\mathfrak{t}}(\mathbb{Q}_p)$ on $Y'_{\mathfrak{t}}$. Let

$$e_i = \#\{\, j \in \{0, \dots, e-1\} \mid \sigma^{-j}\varphi_i|_F \in \mathcal{V} \,\} = e(1 - \frac{d_i}{f_i}),$$

and attach to every $M_i \in X'_{\mathfrak{t},i}$ the integer $n_i(M_i) = \big[\, M_{\varphi_i} : M^{\circ}_{\varphi_i} \,\big]$ where $M^{\circ}_{\varphi_i} \subset N_{\varphi_i}$ denotes any fixed lattice.

Lemma 5.9 *The map*

$$[\,(\, M_i \mid i = 1, \dots, m\,)\,, k\,] \longmapsto k \bigoplus_{i=1}^{m} \big(p^{n_i(M_i)} M_i \oplus M_i^{\vee}\big)$$

defines a $G_{\mathfrak{t}}(\mathbb{Q}_p)$–equivariant bijection $Y'_{\mathfrak{t}} \cong Y_{\mathfrak{t}}$ such that the action of the Frobenius automorphism of $\mathbb{F}$ over the residue class field of O_E on $Y_{\mathfrak{t}}$ corresponds to the map

$$[\,(\, M_i \mid i = 1, \dots, m\,)\,, k\,] \longmapsto [\,(\, \mathcal{F}^e M_i \mid i = 1, \dots, m\,)\,, k_D^{-1} k\,]$$

where, for every i, $k_D \in K^{\times}(\mathbb{Q}_p)$ has component p^{e_i} in $K^{\times}_{\mathfrak{q}_i}$ and component 1 in $K^{\times}_{\bar{\mathfrak{q}}_i}$.

Proof. Since $(\mathcal{F}^e M_i)^{\vee} = p^{-e}\mathcal{F}^e(M_i^{\vee})$, the lemma will follow from (5.7) and (5.8) if we show that the integers $n_i(M_i)$ satisfy

$$\begin{aligned} n_i(\overline{h}^{-1} M_i) - \mathrm{ord}_{\mathfrak{p}_i}(h\overline{h}) &= n_i(M_i) \quad \text{for all } h \in H_{\mathfrak{t}}(\mathbb{Q}_p)\,; \\ n_i(\mathcal{F}^e M_i) + e - e_i &= n_i(M_i)\,. \end{aligned}$$

But this is obvious since

$$\begin{aligned} n_i(\overline{h}^{-1} M_i) - n_i(M_i) &= \big[\, \overline{h}^{-1} M_{\varphi_i} : M_{\varphi_i} \,\big] \\ &= \mathrm{ord}_p \det(\overline{h}|_{N_{\varphi_i}}) = \mathrm{ord}_{\mathfrak{p}_i}(h\overline{h})\,, \end{aligned}$$

and by (4.2),

$$\begin{aligned} n_i(M_i) - n_i(\mathcal{F}^e M_i) &= \big[\, M_{\varphi_i} : \mathcal{F}^e M_{\sigma^{-e}\varphi_i} \,\big] \\ &= \#\{\, j \in \{0, \dots, e-1\} \mid \sigma^{-j}\varphi_i|_F \in \mathcal{Z} \,\} = \frac{ed_i}{f_i}\,. \end{aligned}$$

∎

Lemma 5.10 *For every isomorphism η_p as in Lemma 5.5, the map*

$$d \longmapsto \eta_p\big((O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))d^{-1}\big)$$

defines an $H_{\mathfrak{t}}(\mathbb{Q}_p)$–equivariant bijection $X_{\mathfrak{t}} \cong X'_{\mathfrak{t}}$ such that the action of $\mathrm{Fr}_{\mathfrak{t}}$ on $X_{\mathfrak{t}}$ corresponds to the automorphism of $X'_{\mathfrak{t}}$ induced by $\mathcal{F}^e$.

Proof. It is easy to see that the map

$$d \longmapsto M(d) = \eta_p\big((O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))d^{-1}\big)$$

identifies $D^\times(\mathcal{K})/(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))^\times$ with the set of all lattices $\bigoplus_\varphi M_\varphi$ where, for every $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{Q}}(L, \mathcal{K})$, $M_\varphi \subset N_\varphi$ is a lattice such that (4.3) holds. Obviously, this map satisfies

$$\begin{aligned} \mathcal{F}M(d) &= M(\mathcal{F}_{\mathfrak{t}}(d))\,; \\ \overline{h}^{-1}M(d) &= M(hd) \qquad \text{for all } h \in H_{\mathfrak{t}}(\mathbb{Q}_p)\,. \end{aligned}$$

Consider now any $d \in D^\times(\mathcal{K})$ and put $c = \mathcal{F}_{\mathfrak{t}}(d)^{-1}d$. Then $c \in O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})$ if and only if $\mathcal{F}M(d) \subseteq M(d)$. Assume this and let, for every φ, $l_\varphi \in O_L \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})$ denote the corresponding idempotent. Then

$$\begin{aligned} [\,M(d)_\varphi : \mathcal{F}M(d)_{\sigma^{-1}\varphi}\,] &= [\,l_\varphi(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})) : l_\varphi(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))c\,] \\ &= \mathrm{ord}_p\big((\varphi|_F \otimes \mathrm{id}_{\mathcal{K}})(\mathrm{Nm}_{D|F}(c))\big)\,, \end{aligned}$$

hence (4.2) holds if and only if $c^{-1} \in Y_D$ which proves our claim. ∎

Proposition 5.11 *There is a projective system of bijections*

$$\underline{\mathcal{M}}_{G,C}(\mathfrak{t}) \cong (H_{\mathfrak{t}}(\mathbb{Q}) \times K^\times)\Big\backslash\Big(F^\times(\mathbb{Q}_p)\backslash\big(X_{\mathfrak{t}} \times K^\times(\mathbb{Q}_p)/(O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times\big) \times G(\mathbb{A}_f^p)/C^p\Big)$$

such that the action of $K^\times(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ on the projective limit with respect to all C^p which satisfy (2.11) *corresponds to the obvious right multiplication, and the Frobenius automorphism of $\mathbb{F}$ over the residue class field of O_E acts via the endomorphism $\mathrm{Fr}_{\mathfrak{t}} \times k_D^{-1}$ of $X_{\mathfrak{t}} \times K^\times(\mathbb{Q}_p)$.*

Proof. Combine (5.6), Lemma 5.9, and Lemma 5.10. ∎

Recall now that we still have to fill a gap in the proof of Proposition 3.13. To this end, we consider, for every prime $\mathfrak{p}_i$ at which D is ramified, the residue class field κ_i of $D_{\mathfrak{p}_i}$. Condition (2.8) implies that the embedding $L \subset D$ identifies this field with the residue class field of $L \otimes_F F_{\mathfrak{p}_i}$, i.e. via the fixed embeddings (2.4), we may consider $\mathrm{Hom}_{\mathbb{F}_p}(\kappa_i, \mathbb{F})$ as a subset of $\mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C})$. With respect to these identifications let

$$\mathcal{Z}_1 = \coprod_{\substack{i=1 \\ \delta_i=1}}^{m} \{ \varphi \in \mathrm{Hom}_{\mathbb{F}_p}(\kappa_i, \mathbb{F}) \mid \varphi|_F \in \mathcal{Z} \}.$$

Then Definition 3.10 can be reformulated as follows. A subset $Z \subseteq \mathcal{Z}_1$ is admissible if $Z \cup Z^* = \mathcal{Z}_1$. For every such set, $d(Z) = \#\mathcal{Z}_0 + \#(\mathcal{Z}_1 \setminus Z)$, and Z is minimal if $Z \cap Z^* = \varnothing$. For every $\varphi \in \mathrm{Hom}_{\mathbb{F}_p}(\kappa_i, \mathbb{F})$, let $K(\varphi)$ denote the kernel of the $\mathrm{W}(\mathbb{F})$–algebra homomorphism

$$O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}) \longrightarrow \kappa_i \otimes_{\mathbb{F}_p} \mathbb{F} \longrightarrow \mathbb{F}$$

induced by φ.

Definition 5.12 For every $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$ and $d \in X_{\mathfrak{t}}$, let $\mathcal{Z}_{\mathfrak{t}}(d) = \mathcal{Z}(d^{-1}\mathcal{F}_{\mathfrak{t}}(d))$ where, for any $y \in Y_D$,

$$\mathcal{Z}(y) = \{ \varphi \in \mathcal{Z}_1 \mid y^{-1} \in K(\varphi) \}$$

is an admissible subset of $\mathcal{Z}_1$ which is invariant with respect to the action of $H_{\mathfrak{t}}(\mathbb{Q}_p)$ on $X_{\mathfrak{t}}$ and depends only on the $(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))^\times$–double coset of y in Y_D.

Corollary 5.13 *For all $d \in X_{\mathfrak{t}}$, $k \in K^\times(\mathbb{Q}_p)$, $g \in G(\mathbb{A}_f^p)$, if $\mathfrak{a} \in \underline{\mathcal{M}}_{G,C}(\mathfrak{t})$ is determined by (d, k, g) via the bijection in Proposition 5.11, then $\mathcal{Z}(\mathfrak{a}) = \mathcal{Z}_{\mathfrak{t}}(d)$. In particular, if $\mathfrak{t} = D'$, there is an element $\mathfrak{a} \in \underline{\mathcal{M}}_{G,C}(\mathfrak{t})$ such that $\mathcal{Z}(\mathfrak{a})^* = \mathcal{Z}(\mathfrak{a})$.*

Proof. Consider a $\varphi \in \mathrm{Hom}_{\mathbb{Q}}(L, \mathbb{C})$ such that $\varphi|_F \in \mathcal{Z} \setminus \mathcal{Z}_0$. By definition, φ is contained in $\mathcal{Z}(\mathfrak{a})$ if and only if the corresponding component of the Dieudonné module of A satisfies the equation $\Pi \mathcal{F} M_{\sigma^{-1}\varphi} = p M_{\varphi^*}$. Using the bijection in Lemma 5.10 and the fact that $\Pi^2 \in F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ has component p at the place $\mathfrak{p}_i$ determined by φ, this can be reformulated as

$$l_\varphi (O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})) \mathcal{F}_{\mathfrak{t}}(d)^{-1} = l_\varphi \Pi (O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})) d^{-1}$$

where $l_\varphi \in O_L \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})$ denotes the idempotent corresponding to φ. Obviously, this holds if and only if $\mathcal{F}_{\mathfrak{t}}(d)^{-1} d$ is contained in $K(\varphi)$. This proves the first assertion. For the second part we simply note that, if $\mathfrak{t} = D'$, the element $y_{\mathfrak{t}} \in Y_D$ constructed in the proof of Lemma 5.2 satisfies $\mathcal{Z}(y_{\mathfrak{t}}) = \mathcal{Z}_1$. Hence, for every $d \in D^\times(\mathcal{K})$ with $\mathcal{F}_{\mathfrak{t}}(d) = d y_{\mathfrak{t}}$, $\mathcal{Z}_{\mathfrak{t}}(d)^* = \mathcal{Z}_{\mathfrak{t}}(d)$ which completes the proof. ∎

§6. A reduction of Sh_D

Using the isomorphism (1.9), we construct a reduction of $Sh_{D,C}$ over O_E if the open subgroup $C \subset D^\times(\mathbb{A}_f)$ satisfies the following condition:

(6.1) $C = C_p \times C^p$ where C_p is a maximal compact subgroup of $D^\times(\mathbb{Q}_p)$, C^p is a subgroup of $D^\times(\mathbb{A}_f^p)$, and there are an O_F–order $O_D \subset D$ and an integer $N > 2$ such that $C \subseteq (1 + NO_D \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times$.

We fix C_p. This determines $O_D \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ uniquely, and we can choose L, K, $\mathcal{V}_K$, and ψ subject to the conditions (2.6), ..., (2.10). Consider C_F, N_C, C_K, and $\overline{C}$ as in (1.4). Condition (6.1) implies $N_C > 2$ and (2.11) for $\overline{C}$. Hence we can apply the results of the preceding sections. Let

$$Sh_D^p = \varprojlim_{C^p} Sh_{D,C_pC^p} \otimes_{E(D)} E\,,$$

and recall that we assume that p is unramified in F. The reciprocity law r_K of Sh_K gives rise to a continuous homomorphism

$$r_{K,\wp}\colon E^\times \longrightarrow K^\times(\mathbb{Q}_p)\,.$$

In particular, since $O_E^\times$ is compact, $r_{K,\wp}(O_E^\times) \subseteq (O_K \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times$.

Lemma 6.2 *The homomorphism $r_{K,\wp}$ sends p to the element $k_D \in K^\times(\mathbb{Q}_p)$ defined in Lemma 5.9.*

Proof. Lemma 1.1 implies that, for every $\varphi \in \operatorname{Hom}_{\mathbb{Q}}(K, \overline{\mathbb{Q}}_p)$, $\varphi r_{K,\wp}(x) = p^{n(\varphi)}$ where $n(\varphi)$ is the number of elements $\xi \in \operatorname{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)/\operatorname{Gal}(\overline{\mathbb{Q}}_p|E)$ such that φ is contained in $\xi\mathcal{V}_K$. But obviously,

$$n(\varphi) = \begin{cases} 0 & \text{if } \varphi \text{ factors through } K_{\overline{\mathfrak{q}}_i}\,; \\ e_i & \text{if } \varphi \text{ factors through } K_{\mathfrak{q}_i}\,; \end{cases}$$

which proves our claim. ∎

The element $k_D \in K^\times(\mathbb{Q}_p)$ acts on $\widetilde{Sh}_{G,\overline{C}}$. Let $\mathrm{Fr} \in \operatorname{Gal}(\mathbb{Q}_p^{\mathrm{nr}}|E)$ denote the Frobenius automorphism and $\mathcal{M}_{\overline{C}}$ the projective O_E–scheme which is, up to isomorphism, uniquely determined by the existence of an isomorphism ϕ_C of $\mathbb{Z}_p^{\mathrm{nr}}$–schemes such that the diagram

(6.3)
$$\begin{tikzcd} \mathcal{M}_{\overline{C}} \otimes_{O_E} \mathbb{Z}_p^{\mathrm{nr}} \arrow[r, "\phi_C"] \arrow[d, "\mathrm{id} \times \mathrm{Spec}(\mathrm{Fr})"'] & \widetilde{Sh}_{G,\overline{C}} \otimes_{O_E} \mathbb{Z}_p^{\mathrm{nr}} \arrow[d, "k_D^{-1} \times \mathrm{Spec}(\mathrm{Fr})"] \\ \mathcal{M}_{\overline{C}} \otimes_{O_E} \mathbb{Z}_p^{\mathrm{nr}} \arrow[r, "\phi_C"'] & \widetilde{Sh}_{G,\overline{C}} \otimes_{O_E} \mathbb{Z}_p^{\mathrm{nr}} \end{tikzcd}$$

commutes. It defines a continuous action of $K^\times(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ on the projective limit of $\mathcal{M}_{\overline{C}}$ where C^p runs through the set of all open compact subgroups of $D^\times(\mathbb{A}_f^p)$ which satisfy (6.1). Combining (1.7), (6.3), and Corollary 2.18, we get a natural isomorphism of varieties over E

$$\mathcal{M}_{\overline{C}} \otimes_{O_E} E \cong M_{\overline{C}} \otimes_{E(K)} E \tag{6.4}$$

which induces a $K^\times(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$–equivariant isomorphism of the corresponding projective limits.

Lemma 6.5 *Every noetherian local ring whose strict completion is isomorphic to*

$$R[\![X_1, \ldots, X_m, Y_1, \ldots, Y_n]\!]/(X_i Y_i - r_i \mid i = 1, \ldots, m)$$

where R is a complete discrete valuation ring and, for $i = 1, \ldots, m \leqslant n$, r_i is a non-zero element in its maximal ideal, is normal.

Proof. Since the natural map from a noetherian local ring into its strict completion is faithfully flat, and since the R–algebra in the statement of the lemma is an adic completion of the excellent ring

$$A = R[X_1, \ldots, X_m, Y_1, \ldots, Y_n]/(X_i Y_i - r_i \mid i = 1, \ldots, m),$$

it is sufficient to show that A is normal (cf. [25], (21.E) and (34.C)). We use the criterion of Serre (cf. loc. cit. (17.I)). Consider any prime ideal $\mathfrak{q}$ in A. Obviously, $A_\mathfrak{q}$ is regular unless, for some $i \leqslant m$, $\mathfrak{q}$ contains the images x_i and y_i of X_i and Y_i in A. But then it contains x_i or y_i for every $i \leqslant m$. Thus we may assume without loss of generality that $(x_1, \ldots, x_m, y_1) \subset \mathfrak{q}$, hence the height of $\mathfrak{q}$ is at least 2 and x_1, y_1 is a regular sequence in $A_\mathfrak{q}$. ∎

Proposition 3.5 and Lemma 6.5 imply that the local ring at every closed point of $\mathcal{M}_{\overline{C}}$ is a normal domain of characteristic 0, hence $\mathcal{M}_{\overline{C}}$ is normal and every connected component has a non-empty general fibre. In particular, $\mathcal{M}_{\overline{C}}$ is flat over O_E, and (1.8) extends uniquely to a $K^\times(\mathbb{A}_f)$–equivariant morphism of O_E–schemes

$$\pi_C \colon \mathcal{M}_{\overline{C}} \longrightarrow K^\times(\mathbb{A}_f)/F^\times(\mathbb{A}_f)C_K K^\times .$$

Thus we may define $\widetilde{Sh}_{D,C} = \pi_C^{-1}(1) \subseteq \mathcal{M}_{\overline{C}}$ and

$$\widetilde{Sh}_D^p = \varprojlim_{C^p} \widetilde{Sh}_{D,C_pC^p} .$$

For every $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$, we consider the $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$–set

$$S(\mathfrak{t}) = \varprojlim_{C^p} H_\mathfrak{t}(\mathbb{Q}) \backslash \big(X_\mathfrak{t} \times (D^\times(\mathbb{A}_f^p)/C^p) \big)$$

where $H_{\mathfrak{t}}(\mathbb{Q})$ acts diagonally via its embeddings into $H_{\mathfrak{t}}(\mathbb{Q}_p)$ and $H_{\mathfrak{t}}(\mathbb{A}_f^p) \subseteq D^\times(\mathbb{A}_f^p)$, and $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$ acts as the obvious right multiplication. Let $\mathrm{Fr}_{\mathfrak{t}}$ act on $S(\mathfrak{t})$ via its action on $X_{\mathfrak{t}}$.

In addition to the structure of the set of points in the special fibre of $\widetilde{Sh}_{D,C}$, we are interested in the sheaves of vanishing cycles $\mathrm{R}^k\Psi\mathbb{Q}_\ell$ for a prime $\ell \neq p$ (cf. [8]). Following Rapoport [33], we consider them as sets together with an action of the Galois group of E and an equivariant map (with respect to the natural projection from the Galois group of E onto the Galois group of the residue class field of O_E) to the set of $\mathbb{F}$–valued points of $\widetilde{Sh}_{D,C}$ whose fibres are $\mathbb{Q}_\ell$–vector spaces. These sets are defined as the disjoint union of the stalks of $\mathrm{R}^k\Psi\mathbb{Q}_\ell$ at all $\mathbb{F}$–valued points of $\widetilde{Sh}_{D,C}$. In the description below we use the following notation. The set $\mathcal{Z}$ will be considered as a subset of $\bigsqcup_{i=1}^m \mathrm{Hom}_{\mathbb{F}_p}(\kappa(\mathfrak{p}_i), \mathbb{F})$ where $\kappa(\mathfrak{p}_i)$ denotes the residue class field of F at $\mathfrak{p}_i$. For every $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$ and $d \in X_{\mathfrak{t}}$, let $R_{\mathfrak{t},d}$ denote the cokernel of the injection

$$\mathbb{Q}_\ell^{(\mathcal{Z} \smallsetminus \mathcal{Z}_0)} \longrightarrow \mathbb{Q}_\ell^{\mathcal{Z}_{\mathfrak{t}}(d)}$$

induced by the restriction map $\mathcal{Z}_{\mathfrak{t}}(d) \to \mathcal{Z}$. For every non-negative integer k, let

$$R^k_{\mathfrak{t},d} = \bigwedge\nolimits^k R_{\mathfrak{t},d}\,.$$

The bijections

$$\begin{aligned} \mathcal{Z}_{\mathfrak{t}}(d) &\longrightarrow \mathcal{Z}_{\mathfrak{t}}(\mathrm{Fr}_{\mathfrak{t}}(d)) \\ \varphi &\longmapsto \sigma^e\varphi \end{aligned}$$

induce isomorphisms $R^k_{\mathfrak{t},\mathrm{Fr}_{\mathfrak{t}}(d)} \to R^k_{\mathfrak{t},d}$. We compose these isomorphisms with multiplication by p^{ke} to define an automorphism $\mathrm{Fr}_{\mathfrak{t}}$ of

$$R^k_{\mathfrak{t}} = \bigsqcup_{d \in X_{\mathfrak{t}}} R^k_{\mathfrak{t},d}\,.$$

Moreover, since $\mathcal{Z}_{\mathfrak{t}}$ is invariant with respect to the action of $H_{\mathfrak{t}}(\mathbb{Q}_p)$, there is an obvious action of $H_{\mathfrak{t}}(\mathbb{Q}_p)$ on $R^k_{\mathfrak{t}}$ such that the natural map $R^k_{\mathfrak{t}} \to X_{\mathfrak{t}}$ is equivariant.

Theorem 6.6 *The O_E–scheme $\widetilde{Sh}^p_D$ has the following properties.*

(i) *For every C which satisfies (6.1), $\widetilde{Sh}_{D,C}$ is a normal projective flat O_E–scheme such that, for fixed C_p, $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$ acts continuously on $\widetilde{Sh}^p_D$, and there is an $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$–equivariant isomorphism of E–schemes*

$$\widetilde{Sh}^p_D \otimes_{O_E} E \cong Sh^p_D\,.$$

(ii) *There is an $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$–equivariant bijection*

$$\widetilde{Sh}{}^p_D(\mathbb{F}) \cong \bigsqcup_{\mathfrak{t}\in\mathfrak{T}_D} S(\mathfrak{t})$$

such that the action of the Frobenius automorphism over the residue class field of O_E corresponds to the actions of $\mathrm{Fr}_\mathfrak{t}$ on $S(\mathfrak{t})$ for all $\mathfrak{t}$.

(iii) *If $\widehat{\mathcal{O}}_x$ denotes the strict completion of the local ring at a closed point $x \in \widetilde{Sh}_{D,C}$ with respect to a fixed embedding $\kappa(x) \subset \mathbb{F}$, and $d \in X_\mathfrak{t}$ is the first component of a representative of this $\mathbb{F}$–valued point under the bijection in* (ii), *there is an isomorphism of* $\mathrm{W}(\mathbb{F})$*–algebras which identifies $\widehat{\mathcal{O}}_x$ with*

$$\mathrm{W}(\mathbb{F})[\![\, X_\varphi \mid \varphi \in \mathcal{Z}_0 \cup \mathcal{Z}_\mathfrak{t}(d)\,]\!]/(\, X_\varphi X_{\varphi^*} - p \mid \varphi \in \mathcal{Z}_\mathfrak{t}(d) \cap \mathcal{Z}_\mathfrak{t}(d)^*\,)$$

such that the action of $\varrho \in \mathrm{Gal}(\mathbb{F}|\kappa(x))$ on $\widehat{\mathcal{O}}_x$ corresponds to the automorphism which extends the natural action on $\mathrm{W}(\mathbb{F})$ *and maps X_φ to $X_{\varrho\varphi}$.*

(iv) *The geometric special fibre of $\widetilde{Sh}_{D,C}$ is the union of closed subschemes $Sh_{D,C,Z}$ where Z runs through the set of all minimal admissible subsets of $\mathcal{Z}_1$. Moreover, for every admissible set Z, there is a non-singular projective subvariety $Sh_{D,C,Z}$ of pure dimension $d(Z)$ over $\mathbb{F}$, and for every pair Z^1, Z^2 of admissible sets, $Sh_{D,C,Z^1} \cap Sh_{D,C,Z^2} = Sh_{D,C,Z^1\cup Z^2}$. The $\mathbb{F}$–valued point considered in* (iii) *is contained in $Sh_{D,C,Z}$ if and only if $Z \subseteq \mathcal{Z}_\mathfrak{t}(d)$. In this case the ideal of $Sh_{D,C,Z}$ in $\widehat{\mathcal{O}}_x$ is generated by p and $\{\, X_\varphi \mid \varphi \in Z \cap \mathcal{Z}_\mathfrak{t}(d)^*\,\}$.*

(v) *The action of the Galois group of E on the sheaves of vanishing cycles $\mathrm{R}^k\Psi\mathbb{Q}_\ell$ on the geometric special fibre of $\widetilde{Sh}_{D,C}$ factors through the Galois group of the residue class field. There are commutative diagrams*

$$\begin{array}{ccc}
\mathrm{R}^k\Psi\mathbb{Q}_\ell & \xrightarrow{\ \mu_C^k\ } & \displaystyle\bigsqcup_{\mathfrak{t}\in\mathfrak{T}_D} H_\mathfrak{t}(\mathbb{Q})\backslash\big(R_\mathfrak{t}^k \times D^\times(\mathbb{A}_f^p)/C^p\big) \\
\big\downarrow & & \big\downarrow \\
\widetilde{Sh}_{D,C}(\mathbb{F}) & \xrightarrow{\ \mu_C\ } & \displaystyle\bigsqcup_{\mathfrak{t}\in\mathfrak{T}_D} H_\mathfrak{t}(\mathbb{Q})\backslash\big(X_\mathfrak{t} \times D^\times(\mathbb{A}_f^p)/C^p\big)
\end{array}$$

where μ_C denotes the bijection induced by (ii), *the vertical maps are the natural projections, and μ_C^k is a bijection which identifies the action of the Frobenius automorphism over the residue class field of O_E on $\mathrm{R}^k\Psi\mathbb{Q}_\ell$ with the automorphism induced by the actions of $\mathrm{Fr}_\mathfrak{t}$ on $R_\mathfrak{t}^k$ for all $\mathfrak{t}$.*

Proof. (i) follows immediately from (6.4) and (1.9). For every subgroup C^p of $D^\times(\mathbb{A}_f^p)$ which satisfies (6.1), ϕ_C identifies $\mathcal{M}_{\overline{C}}(\mathbb{F})$ with $\underline{\mathcal{M}}_{G,\overline{C}}(\mathbb{F})$ such that the action of Fr on $\mathcal{M}_{\overline{C}}(\mathbb{F})$ corresponds to the product of the action of k_D^{-1} and the action of Fr on $\underline{\mathcal{M}}_{G,\overline{C}}(\mathbb{F})$. On the subsets $\underline{\mathcal{M}}_{G,\overline{C}}(\mathfrak{t})$ this corresponds to the map induced by the endomorphism $\mathrm{Fr}_{\mathfrak{t}}$ of $X_{\mathfrak{t}}$ under the bijection in Proposition 5.11. Using the exact sequence (1.2) and the definition of $\overline{C}$, we can rewrite this bijection as

$$\underline{\mathcal{M}}_{G,\overline{C}}(\mathfrak{t}) \cong F^\times(\mathbb{A}_f)\Big\backslash\Big(H_{\mathfrak{t}}(\mathbb{Q})\backslash\big(X_{\mathfrak{t}} \times D^\times(\mathbb{A}_f^p)/C^p\big) \;\times\; K^\times(\mathbb{A}_f)/C_K K^\times\Big)\,.$$

The map π_C defines a $K^\times(\mathbb{A}_f)$–equivariant map

$$\pi_{\mathfrak{t},C}\colon \underline{\mathcal{M}}_{G,\overline{C}}(\mathfrak{t}) \longrightarrow K^\times(\mathbb{A}_f)/F^\times(\mathbb{A}_f)C_K K^\times\,.$$

Since the projective limit of the natural maps

$$K^\times(\mathbb{A}_f)/C_K K^\times \longrightarrow K^\times(\mathbb{A}_f)/F^\times(\mathbb{A}_f)C_K K^\times$$

is obviously surjective, we can choose a projective system of sections ξ_C and define bijections

$$\begin{aligned} H_{\mathfrak{t}}(\mathbb{Q})\backslash\big(X_{\mathfrak{t}} \times D^\times(\mathbb{A}_f^p)/C^p\big) &\longrightarrow \pi_{\mathfrak{t},C}^{-1}(1) \subseteq \underline{\mathcal{M}}_{G,\overline{C}}(\mathfrak{t}) \\ P &\longmapsto \big(P\,,\,(\xi_C \circ \pi_{\mathfrak{t},C}(P,1))^{-1}\big)\,. \end{aligned}$$

They commute with the action of the Frobenius, form a projective system, and the limit is $F^\times(\mathbb{Q}_p)\times D^\times(\mathbb{A}_f^p)$–equivariant. Hence (ii) follows easily from the definition of $\widetilde{Sh}{}^p_D$, Proposition 4.11, and Proposition 5.11. (iii) and (iv) are obvious consequences of Proposition 3.5 and Proposition 3.13, respectively, and we can apply the results of [35] (§3) to deduce (v). ∎

Remark 6.7 Obviously, $\widetilde{Sh}_{D,C}$ is smooth if and only if, for all $i \in \{1,\ldots,m\}$ such that D is ramified at $\mathfrak{p}_i$, $d_i = 0$. By Remark 2.5 this condition depends only on the prime $\wp$ of $E(D)$ and it is satisfied for all primes of $E(D)$ above p if and only if $D \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{M}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$.

In the remainder of this section we compare our results with the general conjecture of Langlands and Rapoport. This will not be used in the sequel. So the reader is invited to skip to §7 now. In any case it will be necessary to study Appendix B before reading the following material.

To start with, recall that the data $D^\times$, h_D and $C_p \subset D^\times(\mathbb{Q}_p)$ defining the Shimura variety Sh^p_D determine a subgroup $C_p(\mathcal{K}) \subset D^\times(\mathcal{K})$ and a $C_p(\mathcal{K})$–bi-invariant subset $Y_{h_D} \subset D^\times(\mathcal{K})$ (cf. (B3.2) and (B3.4)). They can be identified as follows.

Lemma 6.8 *$C_p(\mathcal{K}) = (O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))^\times$ and $Y_{h_D} = Y_D$.*

Proof. For every $v \in \mathrm{Hom}(F, \mathcal{K})$, we consider the element

$$\gamma_v = \begin{cases} \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} & \text{if } v \text{ factors through } F_{\mathfrak{p}_i} \text{ such that } D \text{ is ramified at } \mathfrak{p}_i\,; \\ 1 & \text{otherwise}\,; \end{cases}$$

in $\mathrm{Gl}_2(\mathcal{K})$ and the homomorphism $\mu_v\colon \mathbb{G}_{m,\mathcal{K}} \to \mathrm{Gl}_{2,\mathcal{K}}$ of algebraic groups over $\mathcal{K}$ defined by

$$\mu_v(x) = \begin{cases} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} & \text{if } v \in \mathcal{Z}\,; \\ 1 & \text{if } v \in \mathcal{V}\,. \end{cases}$$

Moreover, we fix isomorphisms of $\mathrm{W}(\mathbb{F})$–algebras

$$O_D \otimes_{O_F, v} \mathrm{W}(\mathbb{F}) \cong \mathrm{M}_2(\mathrm{W}(\mathbb{F})) \cap \gamma_v\, \mathrm{M}_2(\mathrm{W}(\mathbb{F}))\gamma_v^{-1}\,.$$

This induces isomorphisms

$$D^\times(\mathcal{K}) \cong \prod_v \mathrm{Gl}_2(\mathcal{K})\,;$$

$$\mathcal{B}(D^\times, \mathcal{K}) \cong \prod_v \mathcal{B}(\mathrm{Gl}_2, \mathcal{K})\,;$$

and

$$\widetilde{X}(D^\times, \mathcal{K}) \cong \prod_v \widetilde{X}(\mathrm{Gl}_2, \mathcal{K})$$

where v runs through the set of all embeddings of F into $\mathcal{K}$, $\mathcal{B}$ denotes the Bruhat-Tits building (cf. [42]), and $\widetilde{X}$ is defined in (B3.3). Let $\mathfrak{F}_0 \subset \mathcal{B}(\mathrm{Gl}_2, \mathcal{K})$ denote the facet corresponding to the subgroup $\mathrm{Gl}_2(\mathrm{W}(\mathbb{F})) \subset \mathrm{Gl}_2(\mathcal{K})$. Then under the above isomorphism, the facet of $\mathcal{B}(D^\times, \mathbb{Q}_p) \subset \mathcal{B}(D^\times, \mathcal{K})$ determined by C_p is contained in $\prod_v \mathfrak{F}_v$, where $\mathfrak{F}_v = \mathfrak{F}_0$ if $\gamma_v = 1$, and where $\mathfrak{F}_v$ is a facet of dimension 2 and the facets of minimal dimension in the closure of $\mathfrak{F}_v$ are $\mathfrak{F}_0$ and $\gamma_v(\mathfrak{F}_0)$ otherwise. In particular, $C_p(\mathcal{K})$ is identified with

$$\prod_v \bigl(\mathrm{Gl}_2(\mathrm{W}(\mathbb{F})) \cap \gamma_v\, \mathrm{Gl}_2(\mathrm{W}(\mathbb{F}))\gamma_v^{-1}\bigr)\,.$$

This proves the first assertion of the lemma. Consider now the sets

$$Y_v = \{\, g \in \mathrm{Gl}_2(\mathcal{K}) \mid \mathrm{inv}(gx_0, x_0) = [\mu_v] \,\}$$

where $x_0 \in \mathfrak{F}_0$ is any point and $[\ ]$ denotes the conjugacy class considered as an element of $\widetilde{X}$ (cf. §B3). Under the above isomorphisms, $[\mu_{h_D}] \in \widetilde{X}(D^\times, \mathcal{K})$ has components $[\mu_v] \in \widetilde{X}(\mathrm{Gl}_2, \mathcal{K})$. Hence Y_{h_D} is identified with

$$\prod_v \bigl(Y_v \cap \gamma_v Y_v \gamma_v^{-1}\bigr)\,.$$

Since it is easy to see that

$$Y_v = \mathrm{Gl}_2(\mathrm{W}(\mathbb{F}))\,\mu_v(p)^{-1}\,\mathrm{Gl}_2(\mathrm{W}(\mathbb{F}))\,,$$

the second assertion of the lemma follows. ∎

Using Proposition B2.15, the set of equivalence classes of admissible morphisms from a fixed quasi-motivic groupoid $\mathfrak{Q}$ into the neutral groupoid $\mathfrak{G}_{D^\times}$ (cf. Definition B3.5) can be considered as a subset of the set $\mathcal{H}(D)$ introduced in Definition B2.14. On the other hand, there is an embedding

$$\{D'\} \cup \mathfrak{I}_D \longrightarrow \mathcal{H}(D) \tag{6.9}$$

which sends D' to the triple $(D', \varphi_\infty, \varphi_p)$ with

$$\varphi_\infty(v) = \begin{cases} -1 & \text{if } v \in \mathcal{Z}\,; \\ 0 & \text{if } v \in \mathcal{V}\,; \end{cases}$$
$$\varphi_p(\mathfrak{p}_i) = \frac{d_i}{2f_i} \quad \text{for all } i\,;$$

and (L', ε) to the triple $(L', \varphi_\infty, \varphi_p)$ with

$$\varphi_\infty(v) = \begin{cases} -1 & \text{if } v|_F \in \mathcal{Z}\,; \\ 0 & \text{if } v|_F \in \mathcal{V}\,; \end{cases}$$
$$\varphi_p(\mathfrak{p}) = \frac{d_i + \varepsilon(\mathfrak{p}) - \varepsilon(\overline{\mathfrak{p}})}{2f_i} \quad \text{if } \mathfrak{p}\,|\,\mathfrak{p}_i\,.$$

Proposition 6.10 *The preceding construction identifies $\{D'\} \cup \mathfrak{I}_D$ with the set of equivalence classes of admissible morphisms $\mathfrak{Q} \to \mathfrak{G}_{D^\times}$. If $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$ corresponds to $\varphi\colon \mathfrak{Q} \to \mathfrak{G}_{D^\times}$, there is an $F^\times(\mathbb{Q}_p) \times D^\times(\mathbb{A}_f^p)$–equivariant bijection $S(\mathfrak{t}) \cong S(\varphi)$ which identifies the actions of $\mathrm{Fr}_{\mathfrak{t}}$ and Φ.*

Proof. If $(A, \varphi_\infty, \varphi_p)$ is any element of $\mathcal{H}(D)$, then either A is a quaternion algebra over F which has the same invariants as D at all finite places not dividing p, or A a is commutative algebra of degree 2 over F which splits D at all finite places not dividing p. The remaining conditions in Definition B2.14 can be expressed as follows. In the quaternion algebra case, for all $v \in \mathcal{P}_\infty(F)$ and all $i \in \{1, \ldots, m\}$,

$$\mathrm{inv}_v(A) = \begin{cases} \dfrac{\varphi_\infty(v)}{2} & \text{if } v \in \mathcal{Z}\,; \\ \dfrac{\varphi_\infty(v)+1}{2} & \text{if } v \in \mathcal{V}\,; \end{cases}$$
$$\mathrm{inv}_{\mathfrak{p}_i}(A) = \frac{2\varphi_p(\mathfrak{p}_i) + \delta_i}{2}\,;$$

and

$$\sum_{v\in\mathcal{P}_\infty(F)} \varphi_\infty(v) + \sum_{i=1}^{m} 2f_i\varphi_p(\mathfrak{p}_i) = 0\,.$$

In the commutative case at least one of the maps φ_∞ and φ_p is not invariant with respect to the non-trivial F–algebra automorphism $\overline{}$ of A and, for every real place v and every prime $\mathfrak{p}\,|\,\mathfrak{p}_i$ of A,

$$\varphi_\infty(v) \in \begin{cases} 2\mathbb{Z} & \text{if } v|_F \in \mathcal{Z}\,; \\ 1+2\mathbb{Z} & \text{if } v|_F \in \mathcal{V}\,; \end{cases}$$

$$2f_i\varphi_p(\mathfrak{p}) \in \begin{cases} \mathbb{Z} & \text{if } \overline{\mathfrak{p}} = \mathfrak{p}\,; \\ \delta_i + 2\mathbb{Z} & \text{if } \overline{\mathfrak{p}} \neq \mathfrak{p}\,; \end{cases}$$

and

$$\sum_{v\in\mathcal{P}_\infty(A)} [\,A_v:\mathbb{R}\,]\varphi_\infty(v) + \sum_{\mathfrak{p}\in\mathcal{P}_p(A)} 2[\,A_\mathfrak{p}:\mathbb{Q}_p\,]\varphi_p(\mathfrak{p}) = 0\,.$$

Let us now translate the conditions in Definition B3.5 into properties of the corresponding triple $(A,\varphi_\infty,\varphi_p)$. The cocharacter of $F^\times$ induced by μ_{h_D} is given by the map

$$X^*(F^\times) = \bigoplus_{v\in\mathcal{P}_\infty(F)} \mathbb{Z}e_v \longrightarrow \mathbb{Z}$$

$$e_v \longmapsto \begin{cases} 1 & \text{if } v \in \mathcal{Z}\,; \\ 0 & \text{if } v \in \mathcal{V}\,. \end{cases}$$

By Remark B2.16, this implies that the morphism $\psi_{h_D}:\mathfrak{Q}\to\mathfrak{G}_{F^\times}$ corresponds to the triple $(F,\psi_\infty,\psi_p)\in\mathcal{H}(F)$ with

$$\psi_\infty(v) = \begin{cases} -2 & \text{if } v \in \mathcal{Z}\,; \\ 0 & \text{if } v \in \mathcal{V}\,; \end{cases}$$

$$\psi_p(\mathfrak{p}_i) = \frac{d_i}{f_i} \quad \text{for all } i\,.$$

Thus, using Remark B2.16, condition (i) in Definition B3.5 can be expressed as follows. Either $(A,\varphi_\infty,\varphi_p)$ is the element of $\mathcal{H}(D)$ corresponding to D' via the embedding (6.9), or A is commutative and, for $\ell\in\{\infty,p\}$ and $v\in\mathcal{P}_\ell(A)$,

$$\varphi_\ell(v) + \varphi_\ell(\overline{v}) = \psi_\ell(v|_F)\,.$$

Since $\mathrm{H}^1(\mathbb{Q}_\ell\,, D^\times) = 0$, proposition A.18 in [29] implies that condition (ii) in Definition B3.5 is automatically satisfied for all finite places $\ell \neq p$, and is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
(F \otimes_{\mathbb{Q}} \mathbb{R})^\times \arrow[r] & (Z \otimes_{\mathbb{Q}} \mathbb{R})^\times \arrow[d, equal] \\
\mathbb{G}_{m,\mathbb{R}} \arrow[u, "w_{h_D}"] \arrow[r] & \displaystyle\prod_{v \in \mathcal{P}_\infty(Z)} Z_v^\times \\
x \arrow[r, mapsto] & (\, x^{\varphi_\infty(v)} \mid v \in \mathcal{P}_\infty(Z)\,)\,,
\end{tikzcd}$$

where Z denotes the centre of A, for $\ell = \infty$. Obviously, this is satisfied in the quaternion algebra case, and is equivalent to

$$\varphi_\infty(v) = \begin{cases} -1 & \text{if } v|_F \in \mathcal{Z}\,; \\ \ \ 0 & \text{if } v|_F \in \mathcal{V}\,; \end{cases}$$

otherwise. In particular, in the commutative case $(A, \varphi_\infty, \varphi_p)$ is obtained as in the definition of the map (6.9) from a totally imaginary field L' and a map

$$\begin{aligned} \varepsilon\colon \mathcal{P}_p(L') &\longrightarrow \mathbb{Z} \\ \mathfrak{p} &\longmapsto \begin{cases} d_i & \text{if } \overline{\mathfrak{p}} = \mathfrak{p}\,|\,\mathfrak{p}_i\,; \\ \dfrac{2f_i\,\varphi_p(\mathfrak{p}) - \delta_i}{2} & \text{if } \overline{\mathfrak{p}} \neq \mathfrak{p}\,|\,\mathfrak{p}_i\,. \end{cases} \end{aligned}$$

Moreover, $(L', \varepsilon) \in \mathfrak{I}_D$ if and only if, for all $\mathfrak{p} \in \mathcal{P}_p(L')$, $\varphi_p(\mathfrak{p}) \geqslant 0$.

Via Proposition B1.12, the localization $\varphi(p) \circ \zeta_p\colon \mathfrak{G}_p \to \mathfrak{G}_{(D \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times}$ is, up to equivalence, determined by a sequence

$$\gamma = (\,\gamma_i \mid i = 1, \dots, m\,) \in \prod_{i=1}^{m} \mathcal{H}(D_{\mathfrak{p}_i})\,.$$

Using Remark B2.17, this can be expressed in terms of φ_p. In the quaternion algebra case we obtain

$$\gamma_i(u) = \begin{cases} 2 & \text{if } u = \dfrac{d_i}{2f_i}\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

In the commutative case we get the same result if, for a prime $\mathfrak{p}\,|\,\mathfrak{p}_i$, $\varphi_p(\overline{\mathfrak{p}}) = \varphi_p(\mathfrak{p})$. Otherwise we find

$$\gamma_i(u) = \begin{cases} 1 & \text{if } u = \varphi_p(\mathfrak{p}) \text{ for a } \mathfrak{p}\,|\,\mathfrak{p}_i\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Let $L' = A$ in the commutative case and $L' = L$ in the quaternion algebra case. Obviously, $L' \otimes_F F_{\mathfrak{p}_i}$ can be embedded as a maximal commutative subalgebra into the semi-simple $F_{\mathfrak{p}_i}$–algebra A_i associated to γ_i in (B1.6). Let n denote the least common multiple of $2f_1, \dots, 2f_m$, and consider any $F \otimes_{\mathbb{Q}} \mathcal{K}$–algebra embedding

$$j\colon L' \otimes_{\mathbb{Q}} \mathcal{K} \longrightarrow D \otimes_{\mathbb{Q}} \mathcal{K}$$

which commutes with the actions of σ^n. Then, by Proposition B1.12, $\mathrm{cls}(\varphi(p) \circ \zeta_p)$ is the σ–conjugacy class of an element $d \in D^\times(\mathcal{K})$ which satisfies

$$j(\sigma(l)) = d\sigma(j(l))d^{-1} \quad \text{for all } l \in L' \otimes_{\mathbb{Q}} \mathcal{K}$$

and

$$d\sigma(d) \cdots \sigma^{n-1}(d) = j(z_\gamma)$$

where $z_\gamma \in L' \otimes_{\mathbb{Q}} \mathbb{Q}_p$ has components $p^{-nd_i/2f_i} \in L' \otimes_F F_{\mathfrak{p}_i}$ in the quaternion algebra case, and components $p^{-n\varphi_p(\mathfrak{p})} \in L'_{\mathfrak{p}}$ otherwise. By Lemma 6.8, condition (iii) in Definition B3.5 holds if and only if this σ–conjugacy class contains an element of Y_D. By Lemma 5.2, this is true if $(A, \varphi_\infty, \varphi_p)$ comes from an element of $\{D'\} \cup \mathfrak{I}_D$. Thus, concerning the first assertion of the proposition, it remains to check that, in the commutative case, the existence of such an element implies that z_γ^{-1} is contained in $O_{L'} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. To this end, we consider the σ^{-n}–isocrystal $(N, \lambda) = (D \otimes_{\mathbb{Q}} \mathcal{K}, j(z_\gamma^{-1}) \circ \sigma^{-n})$ and fix an element $x \in D^\times(\mathcal{K})$ such that $y = x d\sigma(x)^{-1} \in Y_D$. Left multiplication by x induces an isomorphism

$$(N, \lambda) \longrightarrow (D \otimes_{\mathbb{Q}} \mathcal{K}, (y \circ \sigma)^{-1})\,.$$

Since, by definition of Y_D, y^{-1} is contained in $O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F})$, this shows that (N, λ) is effective. On the other hand, there is an isomorphism

$$(N, \lambda) \cong ((L' \otimes_{\mathbb{Q}} \mathcal{K})^2, z_\gamma^{-1} \circ \sigma^{-n})$$

which implies what we need.

The proof of the second assertion consists of two parts. For the first part, we fix an element $x \in X^p(\varphi) \subset D^\times(\overline{\mathbb{A}_f^p})$. The action of $A^\times = I_\varphi(\mathbb{Q})$ on $X^p(\varphi)$ is given by left multiplication via an $F \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$–algebra embedding of $A \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ into $D \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. Left multiplication by x^{-1}, therefore, identifies $X^p(\varphi)$ with the set $D^\times(\mathbb{A}_f^p)$ on which $D^\times(\mathbb{A}_f^p)$ acts by right multiplication and $A^\times$ by left multiplication via an $F \otimes_{\mathbb{Q}} \overline{\mathbb{A}_f^p}$–algebra embedding of $A \otimes_{\mathbb{Q}} \overline{\mathbb{A}_f^p}$ into $D \otimes_{\mathbb{Q}} \overline{\mathbb{A}_f^p}$. Since the image of $A^\times$ must be contained in $D^\times(\mathbb{A}_f^p)$, this map is induced by an $F \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–algebra embedding of $A \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ into $D \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. Replacing x by xd for an appropriate $d \in D^\times(\mathbb{A}_f^p)$, we can make the identification of $X^p(\varphi)$ with $D^\times(\mathbb{A}_f^p)$ compatible with the action of $I_\varphi(\mathbb{Q}) = H_{\mathfrak{t}}(\mathbb{Q})$.

For the second part, we consider the construction of the morphisms θ_i corresponding to $\gamma_i \in \mathcal{H}(D_{\mathfrak{p}_i})$ in §B1. There are $F_{\mathfrak{p}_i} \otimes_{\mathbb{Q}_p} \mathcal{K}$ – algebra embeddings

$$j_i \colon A_i \otimes_{\mathbb{Q}_p} \mathcal{K} \longrightarrow D_{\mathfrak{p}_i} \otimes_{\mathbb{Q}_p} \mathcal{K}$$

such that $\theta_i(d_\sigma) = x_i \in D^\times_{\mathfrak{p}_i}(\mathcal{K}_n)$ satisfies

$$j_i(\sigma(a)) = x_i \sigma(j_i(a)) x_i^{-1} \quad \text{for all } a \in A_i \otimes_{\mathbb{Q}_p} \mathcal{K}$$

and

$$x_i \sigma(x_i) \cdots \sigma^{n-1}(x_i) = j_i(z_{\gamma_i}) .$$

Let $\theta = \prod_{i=1}^m \theta_i$, and fix a $g_\theta \in \operatorname{Isom}(\varphi(p) \circ \zeta_p , \theta)$. Then the map

$$\begin{aligned} I_\varphi(\mathbb{Q}) &\longrightarrow J_\theta = \prod_{i=1}^m A_i^\times \\ a &\longmapsto g_\theta \, a \, g_\theta^{-1} \end{aligned}$$

is induced by an $F \otimes_{\mathbb{Q}} \mathbb{Q}_p$ – algebra embedding

$$j_0 \colon A \otimes_{\mathbb{Q}} \mathbb{Q}_p \longrightarrow \prod_{i=1}^m A_i .$$

Changing g_θ if necessary, we may assume that $j_0(z_{\mathfrak{t}}) = (\, z_{\gamma_i} \mid i = 1, \ldots , m \,)$. Hence we may further assume that $j_{\mathfrak{t}}$ is the embedding induced by j_0 and $\prod_{i=1}^m j_i$, and $d_{\mathfrak{t}} = (\, x_i \mid i = 1, \ldots , m \,)$. But then, by Lemma 6.8, $X_p(\varphi)$ and $X_{\mathfrak{t}}$ can be identified such that the actions of $F^\times(\mathbb{Q}_p)$ and $A^\times$ coincide and Φ corresponds to $\mathrm{Fr}_{\mathfrak{t}}$. This completes the proof of the proposition. ∎

Corollary 6.11 *$\widetilde{Sh}{}^p_D$ defines an admissible model for Sh^p_D in the sense of Definition B3.11 if and only if, for all $\mathfrak{t} \in \mathfrak{I}_D$, $\varrho(\mathfrak{t}) = 1$. (In particular, this holds if D is totally indefinite or, in general, if Question A1.6 has an affirmative answer.) $\widetilde{Sh}{}^p_D$ is smooth if and only if μ_{h_D} determines an unramified $D^\times(\overline{\mathbb{Q}}_p)$ – conjugacy class in the sense of Definition B3.10. In this case the condition above is satisfied and we obtain an admissible model.*

Proof. This follows from the preceding results and the simple fact that μ_{h_D} determines an unramified conjugacy class if and only if $\operatorname{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i} , \mathcal{K}) \subseteq \mathcal{V}$ for every prime $\mathfrak{p}_i$ at which D is ramified. ∎

§7. Points on $\widetilde{Sh}_D$ with values in finite fields

Our aim is to describe the set of points x on $\widetilde{Sh}_D$ with values in a finite extension of the residue class field of O_E together with the map

$$x \longmapsto \sum_{k=0}^{\infty}(-1)^k \operatorname{Tr}(\mathrm{Fr}^j \mid (\mathrm{R}^k\Psi\mathbb{Q}_\ell)_x) \tag{7.1}$$

where j denotes the degree of that extension. For this we apply the method of Kottwitz [18] (1.4). First we observe that every C which satisfies (6.1) also satisfies conditions (1.3.7) and (1.3.8) in [18]. Using our notation this is expressed as follows.

Lemma 7.2 *Let $N > 2$ be an integer which is not divisible by p and $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$.*

(i) *If the image of $h \in H_\mathfrak{t}(\mathbb{Q})$ in $D^\times(\mathbb{A}_f^p) \times D^\times(\mathcal{K})$ is conjugate to an element of $(1 + NO_D \otimes_\mathbb{Z} \widehat{\mathbb{Z}}^p)^\times \times (O_D \otimes_\mathbb{Z} \mathrm{W}(\mathbb{F}))^\times$, then $h \in F^\times$.*

(ii) *If $h, g \in H_\mathfrak{t}(\mathbb{Q})$ and $z = ghg^{-1}h^{-1} \in (1 + NO_F)^\times$, then $z = 1$.*

Proof. Consider h as in (i). If $\mathfrak{t} = D'$, we fix a quadratic extension $L'|F$ in D' containing h. The assumptions on h imply that $h \in (1 + NO_{L'})^\times$. Let $g = h^{-1}\overline{h}$ where $^-$ denotes complex conjugation in the CM–field L'. Then g is a root of unity since it is a unit in L' and all of its archimedean absolute values are 1. Hence g can only be congruent to 1 modulo N if $g = 1$ (cf. [31], Lemma on page 207).

If $\mathfrak{t} \neq D'$, (ii) is obvious since then $H_\mathfrak{t}$ is commutative. If $\mathfrak{t} = D'$, the assumption implies that $z^2 = \mathrm{Nm}_{D'|F}(z) = 1$. Since -1 is not congruent to 1 modulo N, this completes the proof. ∎

Fix now a positive integer j, an open subgroup $C \subset D^\times(\mathbb{A}_f)$ which satisfies (6.1), and $\mathfrak{t} \in \{D'\} \cup \mathfrak{I}_D$. Let $r = je$, and let $\mathfrak{R}(\mathfrak{t}) \subset H_\mathfrak{t}(\mathbb{Q})$ be a system of representatives for the set of conjugacy classes in $H_\mathfrak{t}(\mathbb{Q})/(F^\times \cap C)$. The method of Kottwitz mentioned above provides a natural bijection

$$\coprod_{h \in \mathfrak{R}(\mathfrak{t})} \Big(H_h(\mathbb{Q})\backslash\big(Y_p(h) \times Y^p(h)\big)\Big) \longrightarrow \{\, x \in S(\mathfrak{t})/C^p \mid \mathrm{Fr}_\mathfrak{t}^j(x) = x \,\}, \tag{7.3}$$

where H_h denotes the centralizer of h in $H_\mathfrak{t}$,

$$Y^p(h) = \{\, d \in D^\times(\mathbb{A}_f^p) \mid d^{-1}hd \in C^p \,\}/C^p,$$

and

$$\begin{aligned} Y_p(h) &= \{\, x \in X_\mathfrak{t} \mid \mathrm{Fr}_\mathfrak{t}^j(x) = hx \,\} \\ &= \{\, d \in D^\times(\mathcal{K}) \mid d^{-1}\mathcal{F}_\mathfrak{t}(d) \in Y_D \text{ and } \mathcal{F}_\mathfrak{t}^r(d) = j_\mathfrak{t}(h)d \,\}/(O_D \otimes_\mathbb{Z} \mathrm{W}(\mathbb{F}_{p^r}))^\times . \end{aligned}$$

For the last equation we have used the fact that every element of $(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))^\times$ can be written as $d^{-1}\sigma(d)$ for some $d \in (O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}))^\times$. This follows from a result of Greenberg (cf. [15], §3, Proposition 3). On these sets $H_h(\mathbb{Q})$ acts by left multiplication via the fixed embedding of $H_{\mathfrak{t}}(\mathbb{A}_f^p)$ into $D^\times(\mathbb{A}_f^p)$ and via the embedding $j_{\mathfrak{t}}$ of $H_{\mathfrak{t}}(\mathcal{K})$ into $D^\times(\mathcal{K})$, respectively.

Lemma 7.4 *For $h \in H_{\mathfrak{t}}(\mathbb{Q})$ and $d_0 \in D^\times(\mathcal{K})$ such that $\mathcal{F}_{\mathfrak{t}}^r(d_0) = j_{\mathfrak{t}}(h)d_0$, there are a unique $\delta \in D^\times(\mathcal{K}_r)$ such that $j_{\mathfrak{t}}(h)$ and $\delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ are conjugate and a commutative diagram*

$$\begin{tikzcd}
 & \{\, d \in D^\times(\mathcal{K}) \mid \mathcal{F}_{\mathfrak{t}}^r(d) = j_{\mathfrak{t}}(h)d \,\} \arrow[d] & d \arrow[d, mapsto] \\
D^\times(\mathcal{K}_r) \arrow[r] \arrow[ur, dotted] & D^\times(\mathcal{K}) & d^{-1}\mathcal{F}_{\mathfrak{t}}(d) \\
d \arrow[r, mapsto] & d^{-1}\delta\sigma(d) &
\end{tikzcd}$$

where the dotted arrow indicates a bijection which is defined as left multiplication by d_0. Moreover, either $\mathfrak{t} = D'$ or $\mathfrak{t} = (F(h), \varepsilon)$ for some ε. For every prime $\mathfrak{p} | \mathfrak{p}_i$ of $F(h)$,

$$\operatorname{ord}_{\mathfrak{p}}(h) = \begin{cases} \dfrac{-r\varepsilon(\mathfrak{p})}{f_i} & \text{if } \mathfrak{t} = (F(h), \varepsilon) \text{ and } \overline{\mathfrak{p}} \neq \mathfrak{p}\,; \\[2ex] \dfrac{-rd_i \operatorname{ord}_{\mathfrak{p}}(p)}{2f_i} & \text{otherwise}. \end{cases}$$

If the conjugacy class of $j_{\mathfrak{t}}(h)$ in $D^\times(\mathcal{K})$ contains an element of $D^\times(\mathbb{Q}_p)$, there is an F–algebra embedding $F(h) \subset D$.

Proof. We consider $\delta = d_0^{-1} d_{\mathfrak{t}} \sigma(d_0)$. No other choice can make the diagram above commutative. Using the properties of $d_{\mathfrak{t}}$ in Lemma 5.2, we obtain

$$\begin{aligned} \delta^{-1}\sigma^r(\delta) &= \sigma(d_0^{-1})\, d_{\mathfrak{t}}^{-1} j_{\mathfrak{t}}(h)^{-1} d_{\mathfrak{t}}\sigma(d_{\mathfrak{t}}) \cdots \sigma^r(d_{\mathfrak{t}})\sigma^{r+1}(d_0) \\ &= \sigma\big(d_0^{-1} j_{\mathfrak{t}}(h)^{-1} d_{\mathfrak{t}}\sigma(d_{\mathfrak{t}}) \cdots \sigma^{r-1}(d_{\mathfrak{t}})\sigma^r(d_0)\big) = 1\,, \end{aligned}$$

i.e. $\delta \in D^\times(\mathcal{K}_r)$, and

$$j_{\mathfrak{t}}(h^{-n} z_{\mathfrak{t}}^r) = j_{\mathfrak{t}}(h)^{-n} d_{\mathfrak{t}}\sigma(d_{\mathfrak{t}}) \cdots \sigma^{nr-1}(d_{\mathfrak{t}}) = d_0 \sigma^{nr}(d_0^{-1})\,.$$

Recall that either $\mathfrak{t} = (L', \varepsilon) \in \mathfrak{I}_D$ or $\mathfrak{t} = D'$. In the latter case let $L' = F(h)$. If we consider $N = D \otimes_{\mathbb{Q}} \mathcal{K}$ as an $L' \otimes_{\mathbb{Q}} \mathcal{K}$–module via

$$\beta \circ x = x\, j_{\mathfrak{t}}(\beta)^* \qquad \text{for all } \beta \in L' \otimes_{\mathbb{Q}} \mathcal{K} \text{ and } x \in N\,,$$

the last equation means that the isocrystal $(N, (\overline{h}^n\, \overline{z}_{\mathfrak{t}}^{-r}) \circ \sigma^{nr})$ is isotypic of slope 0 where $\overline{}$ is the identity map if $h \in F$ and the non-trivial automorphism of $L' | F$

otherwise. As in the proof of Lemma 5.5 this implies that $h^n z_{\mathfrak{t}}^{-r}$ is a unit in $L' \otimes_{\mathbb{Q}} \mathbb{Q}_p$. But since $z_{\mathfrak{t}}$ is explicitly given, this determines the orders of h at all places of L' above p and shows, in particular, that h cannot be contained in F unless $\mathfrak{t} = D'$. By definition of δ,

$$\delta\,\sigma(\delta)\cdots\sigma^{r-1}(\delta) = d_0^{-1} j_{\mathfrak{t}}(h)\, d_0\,,$$

and for all $d \in D^\times(\mathcal{K})$,

$$(j_{\mathfrak{t}}(h)\, d_0 d)^{-1} \mathcal{F}_{\mathfrak{t}}^r(d_0 d) = d^{-1}\sigma^r(d)\,.$$

Moreover, if $h \notin F$, $L' = F(h)$ is a totally imaginary quadratic extension of F which splits D at all places not dividing p. The assumption that $j_{\mathfrak{t}}(h)$ is conjugate to an element of $D^\times(\mathbb{Q}_p)$ implies that L' splits D at all places above p, hence can be embedded into D. ∎

Let $\mathfrak{R}'(\mathfrak{t}) \subset \mathfrak{R}(\mathfrak{t})$ denote the subset of those elements h which admit a d_0 as in Lemma 7.4. For every such h, we fix a d_0 and obtain an embedding

$$\begin{aligned} H_h(\mathbb{Q}) &\longrightarrow D^\times_{\delta\sigma}(\mathbb{Q}_p) = \{\, d \in D^\times(\mathcal{K}_r) \mid \delta\,\sigma(d) = d\,\delta \,\} \\ h' &\longmapsto d_0^{-1} j_{\mathfrak{t}}(h')\, d_0 \end{aligned}$$

and a corresponding $H_h(\mathbb{Q})$–equivariant bijection

$$Y'_p(\delta) = \{\, d \in D^\times(\mathcal{K}_r) \mid d^{-1}\delta\,\sigma(d) \in Y_D \,\}/(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}_{p^r}))^\times \longrightarrow Y_p(h)\,. \tag{7.5}$$

For every $h \in \mathfrak{R}(\mathfrak{t}) \smallsetminus \mathfrak{R}'(\mathfrak{t})$, $Y_p(h)$ is empty. Using Theorem 6.6(ii), (7.3) and (7.5), we conclude that (7.1) induces a function on $Y'_p(\delta) \times Y^p(h)$. Applying Theorem 6.6(v) we see that this function is induced by a map

$$\begin{aligned} Y'_p(\delta) &\longrightarrow \mathbb{Q} \\ d &\longmapsto \Phi(d^{-1}\delta\,\sigma(d)) \end{aligned}$$

where, for $d \in D^\times(\mathcal{K}_r)$,

$$\Phi(d) = \begin{cases} \sum\limits_{k=0}^{\infty} (-1)^k p^{kr}\, \mathrm{Tr}\,(\bigwedge^k \Theta_d) & \text{if } d \in Y_D\,; \\ 0 & \text{otherwise}\,; \end{cases}$$

and where, if T_d denotes the $\mathbb{Q}$–vector space which is obtained as the cokernel of the injection

$$\mathbb{Q}^{\mathcal{Z} \smallsetminus \mathcal{Z}_0} \longrightarrow \mathbb{Q}^{\mathcal{Z}(d)}$$

induced by the restriction map $\mathcal{Z}(d) \to \mathcal{Z}$, Θ_d is the automorphism of T_d induced by the permutation $\varphi \mapsto \sigma^r\varphi$ of $\mathcal{Z}(d)$. The function Φ can be computed explicitly using the following observation.

Lemma 7.6 *For every field K and every endomorphism Θ of a finite dimensional K–vector space T, let*

$$P_\Theta(X) = \sum_{k=0}^{\infty} \mathrm{Tr}\,(\textstyle\bigwedge^k \Theta)\,(-X)^k \in K[X]\,.$$

If $T_1 \subset T$ is a subspace which is stable with respect to Θ, and Θ_1 and Θ_2 are the endomorphisms of T_1 and $T_2 = T/T_1$ induced by Θ, then

$$P_\Theta(X) = P_{\Theta_1}(X)\,P_{\Theta_2}(X)\,.$$

Moreover, if T has a basis on which Θ acts by a cyclic permutation, then

$$P_\Theta(X) = 1 - X^{\dim T}\,.$$

Proof. Straightforward using an appropriate basis of T. ∎

Applying Lemma 7.6 to Θ_d, for $d \in Y_D \cap D^\times(\mathcal{K}_r)$, we obtain

$$P_{\Theta_d}(X) = \prod_{\substack{i=1\\ \delta_i=1}}^{m} \frac{\left(1 - X^{l'_i}\right)^{a'_i/l'_i}}{\left(1 - X^{l_i}\right)^{d_i/l_i}}$$

with $a'_i = \#\{\, \varphi \in \mathcal{Z}(d) \mid \varphi|_F \in \mathrm{Hom}_{\mathbb{F}_p}(\kappa(\mathfrak{p}_i)\,,\mathbb{F})\,\}$, $l'_i = \dfrac{2f_i}{(2f_i\,, r)}$, and $l_i = \dfrac{f_i}{(f_i\,, r)}$, hence

$$\Phi(d) = P_{\Theta_d}(p^r) = \prod_{\substack{i=1\\ \delta_i=1}}^{m} \Phi_i(d)$$

where

$$\Phi_i(d) = \begin{cases} \left(1 + p^{l_i r}\right)^{d_i/l_i} & \text{if } \mathrm{ord}_2(r) \leqslant \mathrm{ord}_2(f_i)\,; \\ \left(1 - p^{l_i r}\right)^{a_i/l_i} & \text{if } \mathrm{ord}_2(r) > \mathrm{ord}_2(f_i)\,; \end{cases}$$

with $a_i = a'_i - d_i$, i.e.

$$a_i = \#\left\{\, v \in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}\,,\mathcal{K}) \;\middle|\; \begin{array}{l} \text{the image of } (\Pi\, d)^{-1} \text{ in } D \otimes_{F,v} \mathcal{K} \\ \text{is contained in } O_D \otimes_{O_F,v} \mathrm{W}(\mathbb{F}) \end{array} \right\}.$$

Note that $a_i = d_i$ if $\mathrm{ord}_2(r) \leqslant \mathrm{ord}_2(f_i)$.

It is convenient to introduce the following notation before we summarize the preceding discussion in a proposition. For every $i \in \{1, \ldots, m\}$, let $F_i \subset F_{\mathfrak{p}_i}$ be the biggest subfield which can be embedded into $\mathcal{K}_r$, and $q_i = p^{r\,[F_{\mathfrak{p}_i}:F_i]}$. For every $v \in \mathrm{Hom}_{\mathbb{Q}_p}(F_i\,,\mathcal{K}_r)$, let $F^v = F_{\mathfrak{p}_i} \otimes_{F_i,v} \mathcal{K}_r$. Then there are natural isomorphisms

$$D_{\mathfrak{p}_i} \otimes_{\mathbb{Q}_p} \mathcal{K}_r \cong \prod_{v:\,F_i \to \mathcal{K}_r} D \otimes_F F^v\,,$$

and we define maps

$$\Phi_v \colon (D \otimes_F F^v)^\times \longrightarrow \mathbb{Q}$$

as follows. If v is the restriction of an element in $\mathcal{V}$, let Φ_v be the characteristic function of $(O_D \otimes_{O_F} O_{F^v})^\times$. Otherwise, we have to distinguish three cases. If D splits at $\mathfrak{p}_i$, Φ_v denotes the characteristic function of

$$\left\{ d \in (D \otimes_F F^v)^\times \;\middle|\; \begin{array}{l} d^{-1} \in O_D \otimes_{O_F} O_{F^v} \\ \operatorname{ord}_p(\mathrm{Nm}_{D|F}(d)) = -1 \end{array} \right\}.$$

If F^v does not split D, let Φ_v be $(1+q_i)$ times the characteristic function of $\Pi^{-1}(O_D \otimes_{O_F} O_{F^v})^\times$, and if D is ramified at $\mathfrak{p}_i$ and split by F^v, we define

$$\Phi_v(d) = \begin{cases} 1 - q_i & \text{if } d^{-1} \in \Pi(O_D \otimes_{O_F} O_{F^v})^\times\,; \\ 1 & \text{if } d^{-1} \in (O_D \otimes_{O_F} O_{F^v}) \smallsetminus \Pi(O_D \otimes_{O_F} O_{F^v})^\times \\ & \text{and } \operatorname{ord}_p(\mathrm{Nm}_{D|F}(d)) = -1\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

Proposition 7.7 *There is a bijection*

$$\widetilde{Sh}_{D,C}(\mathbb{F}_{p^r}) \cong \coprod_{\substack{\mathfrak{t} \in \mathfrak{T}_D \\ h \in \mathfrak{R}'(\mathfrak{t})}} \Big(H_h(\mathbb{Q}) \backslash \big(Y'_p(\delta) \times Y^p(h) \big) \Big)$$

where the σ-conjugacy class of $\delta \in D^\times(\mathcal{K}_r)$ is determined by the condition that $j_{\mathfrak{t}}(h)$ and $\delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ are conjugate. With respect to this bijection, the function (7.1) on the left side corresponds to a function on the right side which is induced by $H_h(\mathbb{Q})$-invariant maps

$$\begin{aligned} Y'_p(\delta) &\longrightarrow \mathbb{Q} \\ d &\longmapsto \Phi(d^{-1}\delta\sigma(d)) \end{aligned}$$

where

$$\Phi = \prod_{\substack{1 \leqslant i \leqslant m \\ v \colon F_i \to \mathcal{K}_r}} \Phi_v \colon D^\times(\mathcal{K}_r) \longrightarrow \mathbb{Q}$$

is an $(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}_{p^r}))^\times$-bi-invariant function with support in Y_D.

Proof. This is just a reformulation of the preceding material. ∎

Part II

§ 8. The semi-simple local zeta function

We begin this section with some general remarks and definitions. Consider a group G acting on a finite-dimensional vector space V over a field K of characteristic 0, and a normal subgroup $I \subset G$.

Definition 8.1 An ascending filtration $F^\bullet$ on V is called *admissible* (with respect to I) if it is G–invariant, separated and exhausting, and the action of I on $\mathrm{gr}^{F^\bullet} V$ factors through a finite group.

Lemma 8.2 *If V admits an admissible filtration $F^\bullet$, taking invariants with respect to I in $\mathrm{gr}^{F^\bullet} V$ defines a representation of the factor group G/I. The corresponding virtual representation (i.e. the corresponding element of the Grothendieck group of finite-dimensional representations of G/I over K) is independent of the choice of the filtration. In particular, if I acts trivially on $\mathrm{gr}^{F^\bullet} V$ for one admissible filtration $F^\bullet$, it does so for every other.*

Proof. Consider another admissible filtration $\widetilde{F}^\bullet$ and define, for every integer j,

$$F(j)^\bullet = F^\bullet \cap \widetilde{F}^{\bullet + j}\,.$$

This is an admissible filtration on V since

$$\mathrm{gr}_k^{F(j)^\bullet} V \subset \mathrm{gr}_k^{F^\bullet} V \oplus \mathrm{gr}_{k+j}^{\widetilde{F}^\bullet} V\,.$$

The kernel and cokernel of the natural G–equivariant homomorphism

$$\mathrm{gr}^{F(j)^\bullet} V \longrightarrow \mathrm{gr}^{F(j+1)^\bullet} V$$

are isomorphic G–modules. Since taking invariants of the action of a finite group on a vector space over a field of characteristic 0 is an exact functor, and since for $j \ll 0$, $F(j)^\bullet = \widetilde{F}^{\bullet+j}$, and for $j \gg 0$, $F(j)^\bullet = F^\bullet$, this proves the first part. The second part follows since I acts trivially on the graded module if and only if $\dim\bigl(\mathrm{gr}^{F^\bullet} V\bigr)^I = \dim V$. ∎

Assume now that G is the (absolute) Galois group of a non-archimedean local field E and $I \subset G$ is the inertia group. Let $\mathrm{Fr} \in G/I$ denote the (arithmetic) Frobenius automorphism. For every rational prime ℓ different from the residue characteristic of E and every continuous finite-dimensional representation V of G over $\mathbb{Q}_\ell$, there is a finite Galois extension $E'|E$ such that the inertia group I' of E' acts via an abelian group consisting of unipotent endomorphisms of V (cf. [38], Appendix). In particular, there is a basis of V with respect to which the elements of I' correspond to unipotent upper-triangular matrices. Since I' is a normal subgroup of G which has finite index in I, it follows easily that V admits an admissible filtration $F^\bullet$. We define the semi-simple L–function

$$L^{\mathrm{ss}}(V,T) = \det\bigl(1 - \mathrm{Fr}^{-1}\,T \bigm| (\mathrm{gr}^{F^\bullet} V)^I\bigr)^{-1}$$

and, for every positive integer j, the semi-simple trace

$$\mathrm{Tr}^{\mathrm{ss}}(V,j) = \mathrm{Tr}\bigl(\mathrm{Fr}^{-j} \bigm| (\mathrm{gr}^{F^\bullet} V)^I\bigr)$$

where the upper index I denotes taking invariants under I. Lemma 8.2 shows that this does not depend on the choice of $F^\bullet$.

Consider now a smooth (not necessarily connected) projective variety X over a number field. Let E be the localization of this field at a finite place $\wp$. Let q denote the cardinality of the residue class field of E, and fix a rational prime ℓ which does not divide q.

Definition 8.3 For every $i \in \{0,\ldots,2\dim X\}$, $V^i = \mathrm{H}^i(X_{\overline{E}},\mathbb{Q}_\ell)$ is a continuous finite-dimensional representation of the Galois group of E, and we define

$$\begin{aligned} Z^{\mathrm{ss}}_\wp(X,s) &= \prod_{i=0}^{2\dim X} L^{\mathrm{ss}}(V^i,q^{-s})^{(-1)^i} \in \mathbb{Q}_\ell(q^{-s}) \\ \mathrm{Tr}^{\mathrm{ss}}_\wp(X,j) &= \sum_{i=0}^{2\dim X} (-1)^i\,\mathrm{Tr}^{\mathrm{ss}}(V^i,j) \in \mathbb{Q}_\ell\,, \end{aligned}$$

where q^{-s} is considered as a formal variable.

These two expressions are related by the equation

$$\log Z^{\mathrm{ss}}_\wp(X,s) = \sum_{j=1}^{\infty} \mathrm{Tr}^{\mathrm{ss}}_\wp(X,j)\frac{q^{-js}}{j} \in \mathbb{Q}_\ell[\![\,q^{-s}\,]\!]\,. \tag{8.4}$$

If X has good reduction at $\wp$, $Z^{\mathrm{ss}}_\wp(X,s) = Z_\wp(X,s)$ is the local factor of the zeta function of X at $\wp$ since in this case I acts trivially on all V^i, but in general this is not true. For a more detailed discussion of this see [34] (§2).

Definition 8.5 For every smooth projective variety X over a number field K, let

$$Z^{\mathrm{ss}}_p(X,s) = \prod_{\wp\mid p} Z^{\mathrm{ss}}_\wp(X,s)$$

where the product is extended over all places of K dividing p.

Now we return to the situation in §6. Using Theorem 6.6 we see that the spectral sequence of vanishing cycles

$$E_2^{i,k} = \mathrm{H}^i(\widetilde{Sh}_{D,C} \otimes_{O_E} \mathbb{F}, \mathrm{R}^k\Psi\mathbb{Q}_\ell) \Longrightarrow \mathrm{H}^{i+k}(Sh_{D,C} \otimes_{E(D)} \overline{E}, \mathbb{Q}_\ell)$$

(cf. [8], §2) induces an admissible filtration on the cohomology groups on the right side and provides the following expression for the semi-simple traces

$$\mathrm{Tr}^{\mathrm{ss}}_{\wp}(Sh_{D,C}, j) = \sum_{i,k=0}^{\infty} (-1)^{i+k}\, \mathrm{Tr}\left(\mathrm{Fr}^{-j} \mid \mathrm{H}^i(\widetilde{Sh}_{D,C} \otimes_{O_E} \mathbb{F}, \mathrm{R}^k\Psi\mathbb{Q}_\ell)\right).$$

Recall that the action of Fr^{-1} on $\widetilde{Sh}_{D,C} \otimes_{O_E} \mathbb{F}$ which is defined via its action on $\mathbb{F}$ induces the same map on the cohomology groups as the automorphism of $\mathbb{F}$–schemes which corresponds to the natural action of Fr on the set of $\mathbb{F}$–valued points together with the action on $\mathrm{R}^k\Psi\mathbb{Q}_\ell$ which maps the stalk at $\mathrm{Fr}(x)$ to the stalk at x (cf. [28], Chapter VI, §13). These are the actions described in Theorem 6.6. Thus the Lefschetz fixed point formula (loc. cit. Theorem 13.4) yields

$$\mathrm{Tr}^{\mathrm{ss}}_{\wp}(Sh_{D,C}, j) = \sum_{k=0}^{\infty} (-1)^k \sum_x \mathrm{Tr}\,(\mathrm{Fr}^j \mid (\mathrm{R}^k\Psi\mathbb{Q}_\ell)_x)$$

where the second sum ranges over the set $\widetilde{Sh}_{D,C}(\mathbb{F}_{p^r})$ with $r = je$. Proposition 7.7 allows us to write this in the form

$$\mathrm{Tr}^{\mathrm{ss}}_{\wp}(Sh_{D,C}, j) = \sum_{\mathfrak{t}\in\mathfrak{T}_D} \sum_{h\in\mathfrak{R}'(\mathfrak{t})} \sum_{(d_1,d_2)} f^p(d_2^{-1}hd_2)\Phi(d_1^{-1}\delta\sigma(d_1)), \tag{8.6}$$

where f^p denotes the characteristic function of C^p on $D^\times(\mathbb{A}_f^p)$ and the third sum is extended over

$$(d_1, d_2) \in H_h(\mathbb{Q}) \Big\backslash \Big(D^\times(\mathcal{K}_r)/\big(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}_{p^r})\big)^\times \times D^\times(\mathbb{A}_f^p)/C^p\Big).$$

In particular, this shows that the semi-simple traces are rational integers which are independent of the chosen prime ℓ. Using the same argument as Kottwitz in [18] (1.5), we can transform this sum into an expression involving orbital and twisted orbital integrals. In fact, by Lemma 7.4 we know that either $\mathfrak{t} = D'$ or $\mathfrak{t} = (F(h), \varepsilon)$ which implies that the fixed embedding $H_\mathfrak{t}(\mathbb{A}_f^p) \to D^\times(\mathbb{A}_f^p)$ identifies $H_h(\mathbb{A}_f^p)$ with the centralizer $D_h^\times(\mathbb{A}_f^p)$ of (the image of) h in $D^\times(\mathbb{A}_f^p)$, and the element d_0 considered in Lemma 7.4 induces an isomorphism

$$\begin{aligned} H_h(\mathcal{K}) &\longrightarrow \{\, d \in D^\times(\mathcal{K}) \mid \delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)\, d = d\,\delta\sigma(\delta)\cdots\sigma^{r-1}(\delta) \,\} \\ h' &\longmapsto d_0^{-1} j_\mathfrak{t}(h')\, d_0 \end{aligned}$$

which identifies $H_h(\mathbb{Q}_p)$ with $D^\times_{\delta\sigma}(\mathbb{Q}_p) = \{\, d \in D^\times(\mathcal{K}_r) \mid \delta\sigma(d) = d\delta \,\}$. Hence, if we choose Haar measures on $D^\times_h(\mathbb{A}^p_f)$ and $D^\times_{\delta\sigma}(\mathbb{Q}_p)$, this induces a measure on $H_h(\mathbb{A}_f)$. We normalize the measure on $D^\times_{\delta\sigma}(\mathbb{Q}_p)$ such that the volume of an Iwahori subgroup is 1 and the measures on $D^\times(\mathbb{A}^p_f)$, $D^\times(\mathcal{K}_r)$, and $H_h(\mathbb{Q})C_F$ such that

$$\operatorname{vol}(C^p) = \operatorname{vol}\big((O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}_{p^r}))^\times\big) = \operatorname{vol}(C_F) = 1$$

and consider the (twisted) orbital integrals

$$\mathrm{O}_h(f^p) = \int\limits_{D^\times_h(\mathbb{A}^p_f)\backslash D^\times(\mathbb{A}^p_f)} f^p(g^{-1}hg)\, d\bar{g}$$

$$\mathrm{TO}_{\delta\sigma}(\Phi) = \int\limits_{D^\times_{\delta\sigma}(\mathbb{Q}_p)\backslash D^\times(\mathcal{K}_r)} \Phi(g^{-1}\delta\sigma(g))\, d\bar{g}\,.$$

As Kottwitz (loc. cit.) we derive from (8.6) the formula

$$\text{(8.7)} \qquad \operatorname{Tr}^{\mathrm{ss}}_{\wp}(Sh_{D,C}, j) = \sum_{\mathfrak{t}\in\mathfrak{T}_D} \sum_{h\in\mathfrak{R}'(\mathfrak{t})} \operatorname{vol}\big(H_h(\mathbb{Q})C_F\backslash H_h(\mathbb{A}_f)\big)\, \mathrm{O}_h(f^p)\, \mathrm{TO}_{\delta\sigma}(\Phi)\,.$$

§ 9. Orbital and twisted orbital integrals

In this section we consider the twisted orbital integrals occuring in the expression (8.7) for the semi-simple trace. These integrals can be written as a product of integrals corresponding to the primes $\mathfrak{p}_i$ of F which will be treated separately. Therefore, we switch now to a purely local situation and simplify the notation as follows. In this section F denotes a finite unramified extension of $\mathbb{Q}_p$, r is a positive integer, and $\mathcal{Z} \subseteq \operatorname{Hom}_{\mathbb{Q}_p}(F, \mathcal{K})$ is a subset which is invariant with respect to the action of σ^r. Let F_0 denote the biggest subfield of F which can be embedded into $\mathcal{K}_r$. Fix such an embedding v_0, and let q' denote the cardinality of the residue class field of $F' = F \otimes_{F_0, v_0} \mathcal{K}_r$. Let a_0 denote the number of embeddings of F_0 into $\mathcal{K}_r$ which can be extended to an element of $\mathcal{Z}$, and consider a quaternion algebra D over F. We fix a maximal order $O_D \subset D$ and denote its radical by P_D. Of course, O_D is unique if D is a division algebra, and it is unique up to conjugacy if $D \cong \mathrm{M}_2(F)$. There is a natural $\mathcal{K}_r$–algebra isomorphism

$$D \otimes_{\mathbb{Q}_p} \mathcal{K}_r \cong \prod_{v\colon F_0 \to \mathcal{K}_r} (D \otimes_{F_0, v} \mathcal{K}_r)$$

and we define a map

$$\Phi = \prod_v \Phi_v \colon D^\times(\mathcal{K}_r) \longrightarrow \mathbb{Q}$$

as in §7, i.e. Φ_v is the characteristic function of $\left(O_D \otimes_{O_{F_0},v} \mathrm{W}(\mathbb{F}_{p^r})\right)^\times$ if v cannot be extended to an element of $\mathcal{Z}$, and $\Phi_v = \Phi' \circ \sigma^i$ if $v = \sigma^{-i} v_0$ can be extended to an element of $\mathcal{Z}$, and where

$$\Phi' \colon (D \otimes_F F')^\times \longrightarrow \mathbb{Q}$$

is a $\mathrm{Gal}(F'|F)$–invariant map for which we distinguish the following three cases to define it. If $D \cong \mathrm{M}_2(F)$, Φ' is the characteristic function of

$$\left\{ d \in (D \otimes_F F')^\times \;\middle|\; \begin{array}{l} d^{-1} \in O_D \otimes_{O_F} O_{F'} \\ \mathrm{ord}_p(\mathrm{Nm}_{D|F}(d)) = -1 \end{array} \right\}.$$

If F' does not split D, let Φ' be $(1+q')$ times the characteristic function of this set, and if D is a division algebra which is split by F', we define

$$\Phi'(d) = \begin{cases} 1 - q' & \text{if } d^{-1} \in P_D \otimes_{O_F} O_{F'} \\ & \text{and } \mathrm{ord}_p(\mathrm{Nm}_{D|F}(d)) = -1\,; \\ 1 & \text{if } d^{-1} \in (O_D \otimes_{O_F} O_{F'}) \smallsetminus (P_D \otimes_{O_F} O_{F'}) \\ & \text{and } \mathrm{ord}_p(\mathrm{Nm}_{D|F}(d)) = -1\,; \\ 0 & \text{otherwise}. \end{cases}$$

We want to express the twisted orbital integrals $\mathrm{TO}_{\delta\sigma}(\Phi)$ in terms of orbital integrals $\mathrm{O}_\gamma(f)$ for a function f in the Hecke algebra $\mathcal{H}(D^\times /\!/ O_D^\times)$ where we choose the Haar measure on $D^\times$ such that $\mathrm{vol}(O_D^\times) = 1$. We define f as the a_0–th power (with respect to the convolution product) of another element $f' \in \mathcal{H}(D^\times /\!/ O_D^\times)$. Let $r' = [\,F' : F\,]$. If D is a division algebra, f' is $(1+q')$ times the characteristic function of the set of elements of order $-r'$ in $D^\times$, and if $D \cong \mathrm{M}_2(F)$, f' is the image of Φ' under the base change map

$$b \colon \mathcal{H}\big((D \otimes_F F')^\times /\!/ (O_D \otimes_{O_F} O_{F'})^\times\big) \longrightarrow \mathcal{H}(D^\times /\!/ O_D^\times)\,.$$

In both cases, we normalize the Haar measures on $D^\times(\mathcal{K}_r)$ and $(D \otimes_F F')^\times$ such that $\mathrm{vol}\big((O_D \otimes_{\mathbb{Z}_p} \mathrm{W}(\mathbb{F}_{p^r}))^\times\big) = \mathrm{vol}\big((O_D \otimes_{O_F} O_{F'})^\times\big) = 1$. For the orbital and twisted orbital integrals, we also need measures on the centralizer $D_\gamma^\times$ of $\gamma \in D^\times$ and the twisted centralizer $D_{\delta\sigma}^\times = \{\, d \in D^\times(\mathcal{K}_r) \mid \delta\sigma(d) = d\delta \,\}$ of $\delta \in D^\times(\mathcal{K}_r)$. These measures are normalized such that the volume of an Iwahori subgroup is 1. It will be useful to have a characterization of the function f in terms of its Satake transform. This is provided by the following easy lemma.

Lemma 9.1 *For every irreducible admissible representation π of $D^\times$, let*

$$\theta(r',\pi) = \begin{cases} (1+q')\mu(p)^{-r'} & \textit{if } \pi = \mu\circ \mathrm{Nm}_{D|F} \textit{ for an unramified character } \mu \textit{ of } F^\times\,; \\ q'^{1/2}(\mu_1(p)^{-r'}+\mu_2(p)^{-r'}) & \textit{if } D \cong \mathrm{M}_2(F)\,,\ \pi = \mathrm{PS}(\mu_1,\mu_2) \textit{ for unramified characters } \mu_i \textit{ of } F^\times \textit{ such that } \mu_1\mu_2^{-1} \neq |\ |_F^{\pm 1}\,; \\ 0 & \textit{otherwise}\,; \end{cases}$$

where PS *denotes the principal series representation corresponding to a given pair of characters. Then*

$$\mathrm{Tr}\big(\pi(f)\big) = \theta(r',\pi)^{a_0}\,.$$

Proof. Since f is $O_D^\times$–bi-invariant, $\mathrm{Tr}\big(\pi(f)\big) = 0$ if π does not admit a non-zero $O_D^\times$–invariant vector. If D is a division algebra, this means that $\pi = \mu\circ\mathrm{Nm}_{D|F}$ for an unramified character μ of $F^\times$, and the lemma follows immediately from the explicit definition of f. If $D \cong \mathrm{M}_2(F)$, we have to treat the first two cases in the definition of $\theta(r',\pi)$. If we are in the first case, we put $\mu_1 = \mu|\ |_F^{-1/2}$ and $\mu_2 = \mu|\ |_F^{1/2}$. Then $\theta(r',\pi)$ can be expressed in terms of these characters by the formula given for the second case. Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{H}\big((D\otimes_F F')^\times//(O_D\otimes_{O_F} O_{F'})^\times\big) \arrow[r, "S"] \arrow[d, "b"'] & \mathbb{C}[\,x_1^{\pm1}, x_2^{\pm1}\,] \arrow[d] & x_i \arrow[d, mapsto] \\
\mathcal{H}(D^\times//O_D^\times) \arrow[r, "S"] \arrow[dr, "\mathrm{Tr}\,\pi"'] & \mathbb{C}[\,x_1^{\pm1}, x_2^{\pm1}\,] \arrow[d] & \substack{x_i^{r'} \\ x_i} \arrow[d, mapsto] \\
 & \mathbb{C} & \mu_i(p)
\end{tikzcd}$$

where S denotes the Satake isomorphism onto $\mathbb{C}[\,x_1+x_2, (x_1x_2)^{\pm1}\,]$. In particular,

$$S(\Phi') = q'^{1/2}(x_1^{-1}+x_2^{-1})$$

(cf. [5], Theorem 2.5 and Theorem 3.12). Since $f = b(\Phi')^{a_0}$, this implies our claim. ∎

Remark 9.2 We could, of course, give an explicit definition of f' in the matrix algebra case, as well. We leave it as an exercise to the reader to check that in this case $f' = f_1' - q_F f_2'$ where q_F denotes the number of elements in the residue class field of F, and f_1' and f_2' are the characteristic functions of the subsets of elements $d \in D^\times$ such that $\mathrm{ord}_p(\mathrm{Nm}_{D|F}(d)) = -r'$ and $d^{-1} \in O_D$ or P_D, respectively.

Definition 9.3 We say that an element $\delta \in D^\times(\mathcal{K}_r)$ is σ*-semi-simple* if its norm $\mathrm{Nm}_\sigma(\delta) = \delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ is semi-simple. Similarly, $\delta' \in (D \otimes_F F')^\times$ is called τ*-semi-simple* if $\mathrm{Nm}_\tau(\delta') = \delta'\tau(\delta')\cdots\tau^{r'-1}(\delta')$ is semi-simple. Here $\tau \in \mathrm{Gal}(F'|F)$ is induced by $\sigma^{r_0} \in \mathrm{Gal}(\mathcal{K}_r|F_0)$ where $r_0 = [\,F_0 : \mathbb{Q}_p\,]$. An element $\gamma \in D^\times$ and δ (resp. δ') are *associated* if γ is conjugate to $\mathrm{Nm}_\sigma(\delta)$ (resp. $\mathrm{Nm}_\tau(\delta')$).

Proposition 9.4 *For all σ-semi-simple elements $\delta \in D^\times(\mathcal{K}_r)$ and all semi-simple elements $\gamma \in D^\times$, the following assertions hold.*

(i) *If δ is not associated to an element of $D^\times$, then $\mathrm{TO}_{\delta\sigma}(\Phi) = 0$.*

(ii) *If γ is not associated to an element of $D^\times(\mathcal{K}_r)$, then $\mathrm{O}_\gamma(f) = 0$.*

(iii) *If δ and γ are associated, then $\mathrm{TO}_{\delta\sigma}(\Phi) = \xi(\gamma)\,\mathrm{O}_\gamma(f)$ where $\xi(\gamma) = (-1)^{\#\mathcal{Z}}$ if $\gamma \in F^\times$, and $\xi(\gamma) = 1$ otherwise.*

Proof. The natural $\mathbb{Q}_p$-algebra homomorphism $\beta\colon D \otimes_{\mathbb{Q}_p} \mathcal{K}_r \to D \otimes_F F'$ induces an isomorphism

$$D^\times_{\delta\sigma} \longrightarrow D^\times_{\delta'\tau} = \{\, d \in (D \otimes_F F')^\times \mid \delta'\tau(d) = d\delta' \,\}$$

where $\delta' = \beta\big(\delta\sigma(\delta)\cdots\sigma^{r_0-1}(\delta)\big)$, and a bijection

$$\begin{aligned} D^\times(\mathcal{K}_r) &\longrightarrow \prod_{i=0}^{r_0-1} (D \otimes_F F')^\times \\ d &\longmapsto (d_0, \ldots, d_{r_0-1}) \end{aligned}$$

where $d_0 = \beta(d)$ and $d_i = \beta\big(\sigma^i(d)^{-1}\sigma^i(\delta)\cdots\sigma^{r_0-1}(\delta)\sigma^{r_0}(d)\big)$ for $i \in \{1, \ldots, r_0 - 1\}$. These two maps are compatible if we consider the action of $D^\times_{\delta\sigma}$ by left multiplication on the left side and the action of $D^\times_{\delta'\tau}$ by left multiplication in the factor indexed by 0 on the right side. Moreover,

$$\Phi(d^{-1}\delta\sigma(d)) = \Phi_0(d_0^{-1}\delta'\tau(d_0)\,d_1^{-1})\Phi_1(d_1d_2^{-1})\cdots\Phi_{r_0-2}(d_{r_0-2}d_{r_0-1}^{-1})\Phi_{r_0-1}(d_{r_0-1})$$

where $\Phi_i = \Phi'$ if $\sigma^{-i}v_0$ can be extended to an element in $\mathcal{Z}$, and Φ_i is the unit element in $\mathcal{H}\big((D \otimes_F F')^\times /\!/ (O_D \otimes_{O_F} O_{F'})^\times\big)$ otherwise. This implies

$$\mathrm{TO}_{\delta\sigma}(\Phi) = \mathrm{TO}_{\delta'\tau}(\Phi'^{(a_0)}) \tag{9.5}$$

where $\Phi'^{(a_0)}$ denotes the a_0-th power (with respect to the convolution product) of Φ', and where the Haar measure on $D^\times_{\delta'\tau}$ is normalized such that the volume of an Iwahori subgroup is 1. Obviously, $\beta\big(\mathrm{Nm}_\sigma(\delta)\big) = \mathrm{Nm}_\tau(\delta')$, and γ and δ are associated if and only if γ and δ' are associated.

Now we consider the three cases in the definition of Φ'. The first two are easily dealt with. If $D \cong \mathrm{M}_2(F)$, f is the image of $\Phi'^{(a_0)}$ under the base change map. Hence we can apply the "Fundamental Lemma" for Hecke functions (cf. [1], Chapter 1, Theorem 4.5 and Lemma 3.6). Of course, in this case, $\mathrm{Nm}_\tau(\delta')$ is always conjugate to an element in $D^\times$, so that we only have to prove (ii) and (iii). For $\gamma \in F^\times$, the factor $e(\delta')$ in Lemma 3.6 (loc. cit.) equals $(-1)^{\mathrm{ord}_p(\mathrm{Nm}_{D|F}(\delta'))}$. But $\mathrm{TO}_{\delta'\tau}(\Phi'^{(a_0)}) = 0$ unless $\mathrm{ord}_p\big(\mathrm{Nm}_{D|F}(\delta')\big) = -a_0$, and $\#\mathcal{Z} = a_0[\,F : F_0\,]$. Thus $e(\delta') = \xi(\gamma)$ if $[\,F : F_0\,]$ is odd. On the other hand, if $\gamma \in F^\times$ and δ' are associated, $r' \,\mathrm{ord}_p\big(\mathrm{Nm}_{D|F}(\delta')\big) = 2\,\mathrm{ord}_p(\gamma)$ is even. Since $[\,F : F_0\,]$ and r' are relatively prime, $e(\delta') = 1 = \xi(\gamma)$ if $[\,F : F_0\,]$ is even.

Consider now the case where $D \otimes_F F'$ is a division algebra, i.e. D is a division algebra and r' is odd. Then, again, $\mathrm{Nm}_\tau(\delta')$ is always conjugate to an element in $D^\times$, so that we only have to prove (ii) and (iii). Moreover, γ is associated to an element in $D^\times(\mathcal{K}_r)$ if and only if its order in $D^\times$ is divisible by r', $\Phi'^{(a_0)}$ is $(1+q')^{a_0}$ times the characteristic function of the set of elements of order $-a_0$ in $(D \otimes_F F')^\times$, and f is $(1+q')^{a_0}$ times the characteristic function of the set of elements of order $-r'a_0$ in $D^\times$. Thus (ii) is obvious, and (iii) is equivalent to the equation

$$\mathrm{vol}\big(D^\times_{\delta'\tau}\backslash (D \otimes_F F')^\times\big) = \xi(\gamma)\,\mathrm{vol}\big(D^\times_\gamma \backslash D^\times\big)$$

if δ' has order $-a_0$ and γ is associated to δ'. But since r' is odd, it follows as in the previous case that $\#\mathcal{Z}$ is even if $\mathrm{Nm}_\tau(\delta') \in F^\times$. Therefore, $\xi(\gamma)$ is always 1. Finally, our choice of measures implies that both volumes above are 2 if $F(\gamma)$ is an unramified quadratic extension of F, and that they are 1 otherwise.

It remains to consider the case where D is a division algebra and r' is even. This case is much more difficult than the other two. We treat it in a way similar to [33] (§5). But it should be mentioned at this point that loc. cit. is incorrect in several places, so that we cannot refer to the details. To start with, we fix an F'–algebra isomorphism $\alpha\colon D \otimes_F F' \to \mathrm{M}_2(F')$ such that, for all $d \in D \otimes_F F'$,

$$\alpha(\tau(d)) = \eta\tau(\alpha(d))\eta^{-1} \quad \text{with} \quad \eta = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}. \tag{9.6}$$

Such an isomorphism is unique up to an F–algebra automorphism of D, and it satisfies

$$\alpha\big((O_D \otimes_{O_F} O_{F'})^\times\big) = G_0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(O_{F'}) \;\middle|\; c \in pO_{F'} \right\}.$$

We normalize the Haar measure on $\mathrm{Gl}_2(F')$ such that $\mathrm{vol}(G_0) = 1$ and consider the function

$$\widetilde{\Phi} = H^{-1}W + (1-q')H^{-1} + WH^{-1} \in \mathcal{H}(\mathrm{Gl}_2(F')/\!/G_0)$$

where H and W are the generators of the Hecke algebra introduced in [5] (§2), i.e. the characteristic functions of $G_0\eta$ and $G_0 w G_0$ where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, respectively. A straightforward computation shows that $\Phi' = \widetilde{\Phi} \circ \alpha$, Using (9.6) this yields

$$\mathrm{TO}_{\delta'\tau}(\Phi'^{(a_0)}) = \mathrm{TO}_{\widetilde{\delta}\tau}(\widetilde{\Psi}) \quad \text{with} \quad \begin{array}{l} \widetilde{\delta} = \alpha(\delta')\eta \\ \widetilde{\Psi} = \widetilde{\Phi}^{(a_0)} H\,. \end{array} \tag{9.7}$$

We choose a smooth compactly supported function f_1 on $\mathrm{Gl}_2(F)$ which is associated to $\widetilde{\Psi}$ in the sense of [1] (Chapter 1, §3), i.e. for every semi-simple element $\widetilde{\gamma} \in \mathrm{Gl}_2(F)$,

$$\mathrm{O}_{\widetilde{\gamma}}(f_1) = \begin{cases} 0 & \text{if } \widetilde{\gamma} \notin \mathrm{Nm}_\tau(\mathrm{Gl}_2(F'))\,; \\ e(\widetilde{\delta})\,\mathrm{TO}_{\widetilde{\delta}\tau}(\widetilde{\Psi}) & \text{if } \widetilde{\gamma} \text{ is conjugate to } \mathrm{Nm}_\tau(\widetilde{\delta})\,; \end{cases} \tag{9.8}$$

where $e(\widetilde{\delta}) = -1$ if $\mathrm{Nm}_\tau(\widetilde{\delta}) \in F^\times$ and $\mathrm{ord}_p\big(\det(\widetilde{\delta})\big)$ is odd, and $e(\widetilde{\delta}) = 1$ otherwise. On the other hand, $\widetilde{\delta} = \alpha(\delta')\eta$ satisfies $\mathrm{Nm}_\tau(\widetilde{\delta}) = p^{r'/2}\alpha(\mathrm{Nm}_\tau(\delta'))$. Therefore, we consider the central shift of f which maps $d \in D^\times$ to $f(p^{-r'/2}d)$ and choose a smooth compactly supported function f_2 on $\mathrm{Gl}_2(F)$ which has matching orbital integrals (cf. [14], §8), i.e. for every semi-simple element $\widetilde{\gamma} \in \mathrm{Gl}_2(F)$,

$$\mathrm{O}_{\widetilde{\gamma}}(f_2) = \begin{cases} 0 & \text{if } F(\widetilde{\gamma}) \cong F \times F\,; \\ e(\gamma)\,\mathrm{O}_\gamma(f) & \text{if } \gamma \text{ is the image of } p^{-r'/2}\widetilde{\gamma} \text{ under an} \\ & F\text{-algebra homomorphism } F(\widetilde{\gamma}) \to D\,; \end{cases} \tag{9.9}$$

where $e(\gamma) = -1$ if $\gamma \in F^\times$ and $e(\gamma) = 1$ otherwise. Note that formula (8.8) of loc. cit. is incorrect. There should be a minus sign on one side of the equation. The proposition will follow if we show that, for every semi-simple element $\widetilde{\gamma} \in \mathrm{Gl}_2(F)$,

$$\mathrm{O}_{\widetilde{\gamma}}(f_2) = \mathrm{O}_{\widetilde{\gamma}}(f_1)\,.$$

In fact, if δ is not associated to an element in $D^\times$ and $\widetilde{\gamma} \in \mathrm{Gl}_2(F)$ is conjugate to $p^{r'/2}\alpha(\mathrm{Nm}_\tau(\delta'))$, $F(\widetilde{\gamma})$ is isomorphic to $F \times F$, hence (9.9) implies that $\mathrm{O}_{\widetilde{\gamma}}(f_2) = 0$. On the other hand, (9.5), (9.7) and (9.8) imply $\mathrm{TO}_{\delta\sigma}(\Phi) = \mathrm{O}_{\widetilde{\gamma}}(f_1)$. This proves (i).

Consider now a $\gamma \in D^\times$ which is not associated to an element in $D^\times(\mathcal{K}_r)$, and fix a $\widetilde{\gamma} \in \mathrm{Gl}_2(F)$ which is conjugate to $p^{r'/2}\alpha(\gamma)$. Then there is an F-algebra homomorphism $F(\widetilde{\gamma}) \to D$ which maps $p^{-r'/2}\widetilde{\gamma}$ to γ, hence by (9.9), $\mathrm{O}_\gamma(f) = e(\gamma)\,\mathrm{O}_{\widetilde{\gamma}}(f_2)$, and by (9.8), $\mathrm{O}_{\widetilde{\gamma}}(f_1) = 0$ since $\widetilde{\gamma}$ cannot be a norm. This proves (ii).

Assume, finally, that γ and δ are associated, and choose a $\widetilde{\gamma} \in \mathrm{Gl}_2(F)$ which is conjugate to $\mathrm{Nm}_\tau(\widetilde{\delta})$ with $\widetilde{\delta} = \alpha(\delta')\eta$. Then (9.5), (9.7) and (9.8) imply that $\mathrm{TO}_{\delta\sigma}(\Phi) = e(\widetilde{\delta})\,\mathrm{O}_{\widetilde{\gamma}}(f_1)$, and (9.9) implies that $\mathrm{O}_\gamma(f) = e(\gamma)\,\mathrm{O}_{\widetilde{\gamma}}(f_2)$. Thus, in order to get (iii), we only have to check that the sign factors agree if the integrals are non-zero. If $\gamma \notin F^\times$, these are all 1, and if the order of $\gamma \in D^\times$ is not equal to $-r'a_0$, the orbital integral, obviously, vanishes. In the remaining case, we have to show that $\operatorname{ord}_p\bigl(\det(\widetilde{\delta})\bigr)$ is even if and only if $\#\mathcal{Z}$ is odd. Since $[\,F : F_0\,]$ and r' are relatively prime, $[\,F : F_0\,]$ is odd, hence $\#\mathcal{Z} \equiv a_0 \bmod 2$. On the other hand, since $\mathrm{Nm}_\tau(\delta')$ is conjugate to γ, $\mathrm{Nm}_{F'|F}\bigl(\mathrm{Nm}_{D|F}(\delta')\bigr) = \mathrm{Nm}_{D|F}(\gamma)$, hence

$$\operatorname{ord}_p\bigl(\det(\widetilde{\delta})\bigr) = 1 + \operatorname{ord}_p\bigl(\mathrm{Nm}_{D|F}(\delta')\bigr) = 1 + r'^{-1}\operatorname{ord}_p\bigl(\mathrm{Nm}_{D|F}(\gamma)\bigr) = 1 - a_0\,.$$

This completes the proof of (iii).

By a theorem of Kazhdan (cf. [11], page 45), the orbital integrals of f_1 and f_2 coincide if, for every irreducible tempered (hence infinite-dimensional) representation π of $\mathrm{Gl}_2(F)$,

$$\mathrm{Tr}\bigl(\pi(f_1)\bigr) = \mathrm{Tr}\bigl(\pi(f_2)\bigr)\,. \tag{9.10}$$

The right side of this equation is easily calculated using (8.5) and (8.9) of [14] and Lemma 9.1. We obtain

$$\mathrm{Tr}\bigl(\pi(f_2)\bigr) = \begin{cases} -\pi'(p)^{r'/2}\,\mathrm{Tr}\bigl(\pi'(f)\bigr) & \text{if } \pi \text{ corresponds to the representation } \pi' \text{ of } D^\times; \\ 0 & \text{if } \pi \text{ is not square-integrable}; \end{cases} \tag{9.11}$$

$$= \begin{cases} -(1+q')^{a_0}\mu(p)^{r'(1-a_0)} & \text{if } \pi = \mathrm{Sp}(\mu) \text{ for an unramified character } \mu \text{ of } F^\times; \\ 0 & \text{otherwise}; \end{cases}$$

where $\mathrm{Sp}(\mu)$ denotes the special representation corresponding to μ, i.e. the (essentially) square-integrable (modulo the centre) representation of $\mathrm{Gl}_2(F)$ corresponding to the 1–dimensional representation $\mu\circ\mathrm{Nm}_{D|F}$ of $D^\times$. On the other hand, we can consider the base change lift Π of π (cf. [1], Chapter 1, §6). This is an irreducible tempered representation of $\mathrm{Gl}_2(F')$ which admits a canonical intertwining operator $I_\tau\colon \Pi \to \Pi^\tau$, and, for $g \in \mathrm{Gl}_2(F')$ such that $\mathrm{Nm}_\tau(g)$ is conjugate to a regular element $h \in \mathrm{Gl}_2(F)$,

$$\mathrm{Tr}\,(\Pi(g)I_\tau) = \mathrm{Tr}\,(\pi(h))\,.$$

Together with (9.8) and the (twisted) Weyl integration formula (cf. loc. cit. page 36) applied to the functions $g \mapsto \widetilde{\Psi}(g)\,\mathrm{Tr}\,(\Pi(g)I_\tau)$ on $\mathrm{Gl}_2(F')$ and $h \mapsto f_1(h)\,\mathrm{Tr}\,(\pi(h))$ on $\mathrm{Gl}_2(F)$, this implies

$$\mathrm{Tr}\,(\Pi(\widetilde{\Psi})I_\tau) = \mathrm{Tr}\,(\pi(f_1))\,. \tag{9.12}$$

Since $\widetilde{\Psi}$ is a G_0–bi-invariant function, the left side of this equation vanishes if Π does not admit a non-zero G_0–fixed vector. Thus we can restrict our attention to the cases where either $\pi = \mathrm{Sp}(\mu)$ and $\Pi = \mathrm{Sp}(\mu')$ for an unramified character μ of $F^\times$ and $\mu' = \mu \circ \mathrm{Nm}_{F'|F}$, or $\pi = \mathrm{PS}(\mu_1, \mu_2)$ and $\Pi = \mathrm{PS}(\mu_1', \mu_2')$ for unramified characters μ_i of $F^\times$ and $\mu_i' = \mu_i \circ \mathrm{Nm}_{F'|F}$ of $F'^\times$ such that $\mu_1' {\mu_2'}^{-1} \neq |\ |_{F'}^{\pm 1}$. To treat these cases, we need the following simple lemma.

Lemma 9.13 *Let I_τ denote the canonical intertwining operator of the representation $\varrho = \mathrm{PS}(\nu_1, \nu_2)$ of $\mathrm{Gl}_2(F')$ where ν_i are arbitrary τ–invariant unramified characters of $F'^\times$. Then $\mathrm{Tr}\,(\varrho(\widetilde{\Psi}) I_\tau) = 0$.*

Proof. By definition, ϱ acts as the right regular representation on the space V of locally constant $\mathbb{C}$–valued functions φ on $\mathrm{Gl}_2(F')$ which satisfy

$$\varphi\left(\begin{pmatrix} a & * \\ 0 & b \end{pmatrix} g\right) = \nu_1(a)\nu_2(b) \left|\frac{a}{b}\right|_{F'}^{1/2} \varphi(g)\,,$$

and I_τ maps φ to $\varphi \circ \tau^{-1}$. Since $\widetilde{\Psi}$ is G_0–bi-invariant, we can compute the trace on the subspace V^{G_0} of G_0–invariants in V. According to [5] (Theorem 3.13), this space is 2–dimensional and admits a basis with respect to which

$$\varrho(W) = \begin{pmatrix} 0 & q' \\ 1 & q'-1 \end{pmatrix} \quad \text{and} \quad \varrho(H) = \begin{pmatrix} 0 & \nu_2(p) {q'}^{1/2} \\ \nu_1(p) {q'}^{-1/2} & 0 \end{pmatrix}.$$

Moreover, it is immediate from the definition on page 35 of loc. cit. that I_τ acts trivially on the elements of this basis. Since $\widetilde{\Psi}$ is explicitly given in terms of W and H, a straightforward computation yields

$$\varrho(\widetilde{\Psi}) I_\tau = \big(\nu_1(p)^{-1} + \nu_2(p)^{-1}\big)^{a_0} {q'}^{a_0/2} \begin{pmatrix} 0 & \nu_2(p) {q'}^{1/2} \\ \nu_1(p) {q'}^{-1/2} & 0 \end{pmatrix}$$

with respect to the chosen basis of V^{G_0}. Obviously, this has trace 0, and the lemma is proved. ∎

We can now complete the proof of the proposition. In the case of an unramified special representation, we consider μ and μ' as above and denote the 1–dimensional representation $\mu' \circ \det$ of $\mathrm{Gl}_2(F')$ by $\widetilde{\mu}$. The preceding lemma and Theorem 3.5 of loc. cit. imply

$$\mathrm{Tr}\,(\Pi(\widetilde{\Psi}) I_\tau) + \widetilde{\mu}(\widetilde{\Psi}) = 0\,.$$

Using the explicit definition of $\widetilde{\Psi}$ in terms of W and H and the fact that r' is even, we compute

$$\begin{aligned} \widetilde{\mu}(W) &= \mathrm{vol}(G_0 w G_0) = q'\,; \\ \widetilde{\mu}(H) &= \widetilde{\mu}(\eta) = \mu'(-p) = \mu(p)^{r'}\,; \\ \widetilde{\mu}(\widetilde{\Psi}) &= (2\widetilde{\mu}(W) + 1 - q')^{a_0} \widetilde{\mu}(H)^{1-a_0} = (1+q')^{a_0} \mu(p)^{r'(1-a_0)}\,. \end{aligned}$$

In the second case to be considered, Lemma 9.13 simply asserts that the left side of (9.12) vanishes. Thus the preceding calculations together with (9.11) and (9.12) imply (9.10) and the proof of the proposition is complete. ▪

§ 10. The trace formula

In this section, we shall identify the semi-simple trace (8.7) as the trace of a function $f^{(j)} \in C_c^\infty(D^\times(\mathbb{A})/F^\times(\mathbb{R}))$ acting on the space $\mathcal{A}$ of automorphic forms on $D^\times F^\times(\mathbb{R})\backslash D^\times(\mathbb{A})$. This function is defined as follows. Recall from § 9 that there is a function $f_p^{(j)} \in \mathcal{H}(D^\times(\mathbb{Q}_p)//C_p)$ such that, for every irreducible admissible representation $\pi = \bigotimes_{i=1}^m \pi_i$ of $D^\times(\mathbb{Q}_p)$,

$$\operatorname{Tr}\left(\pi(f_p^{(j)})\right) = \prod_{i=1}^m \theta(r_i,\pi_i)^{a_i} \quad \text{with} \quad r_i = \frac{r}{(r,f_i)} \quad \text{and} \quad a_i = \frac{(r,f_i)d_i}{f_i} \tag{10.1}$$

where the measure on $D^\times(\mathbb{Q}_p)$ is such that $\operatorname{vol}(C_p) = 1$ (cf. Lemma 9.1). Moreover, for every $\gamma \in D^\times$,

$$\xi(\gamma)\,\mathrm{O}_\gamma(f_p^{(j)}) = \begin{cases} \mathrm{TO}_{\delta\sigma}(\Phi) & \text{if } \gamma \text{ is associated to } \delta \in D^\times(\mathcal{K}_r)\,; \\ 0 & \text{if there is no such } \delta\,; \end{cases} \tag{10.2}$$

where $\xi(\gamma) = (-1)^{\#\mathcal{Z}}$ if $\gamma \in F^\times$ and $\xi(\gamma) = 1$ otherwise, and where the measure on $D_\gamma^\times(\mathbb{Q}_p)$ is normalized such that the volume of an Iwahori subgroup is 1 (cf. Proposition 9.4). On the other hand, if we fix a Haar measure on $D^\times(\mathbb{R})/F^\times(\mathbb{R})$, there is a function $f_\infty \in C_c^\infty(D^\times(\mathbb{R})/F^\times(\mathbb{R}))$ such that, for every irreducible admissible representation $\pi = \bigotimes_{v\colon F\to\mathbb{R}} \pi_v$ of $D^\times(\mathbb{R})/F^\times(\mathbb{R})$,

$$\operatorname{Tr}\left(\pi(f_\infty)\right) = m(\pi) = \prod_v m(\pi_v) \quad \text{with} \quad m(\pi_v) = \begin{cases} 1 & \text{if } \dim\pi_v = 1\,; \\ -1 & \text{if } v \in \mathcal{Z},\ \pi_v = \pi_0\,; \\ 0 & \text{otherwise}\,; \end{cases}$$

where π_0 denotes the unique discrete series representation of $\mathrm{PGl}_2(\mathbb{R})$ with trivial infinitesimal character (cf. [6], § 3). Moreover, it is known (cf. loc. cit. § 5) that, for every $\gamma \in D^\times$,

$$\xi(\gamma)\,\mathrm{O}_\gamma(f_\infty) = \begin{cases} \operatorname{vol}(D'^\times(\mathbb{R})/F^\times(\mathbb{R}))^{-1} & \text{if } \gamma \in F^\times\,; \\ \operatorname{vol}(D_\gamma^\times(\mathbb{R})/F^\times(\mathbb{R}))^{-1} & \text{if } F(\gamma) \text{ is totally imaginary}\,; \\ 0 & \text{otherwise}\,; \end{cases} \tag{10.3}$$

where we have fixed arbitrary Haar measures on the centralizer of γ in $D^\times(\mathbb{R})/F^\times(\mathbb{R})$ while, for $\gamma \in F^\times$, we take the related measure on $D'^\times(\mathbb{R})/F^\times(\mathbb{R})$ in the sense of

Jacquet-Langlands (cf. [17], §15). Finally, we put $f^{(j)} = f_\infty \cdot f^p \cdot f_p^{(j)}$ where, as in §8, f^p denotes the characteristic function of C^p. Then, for every irreducible admissible representation $\pi = \pi_\infty \otimes \pi_f$ of $D^\times(\mathbb{A})/F^\times(\mathbb{R})$ with $\pi_f = \pi^p \otimes \pi_p$,

$$(10.4) \qquad \operatorname{Tr}\big(\pi(f^{(j)})\big) = m(\pi, C)\operatorname{Tr}\big(\pi_p(f_p^{(j)})\big) \quad \text{where} \quad m(\pi, C) = m(\pi_\infty)\dim(\pi_f^C).$$

Remark 10.5 Let k be any local field, ψ a non-trivial additive character of k and, for every finite-dimensional semi-simple algebra H over k, $d_H h$ the unique Haar measure on H which is self-dual with respect to the Fourier transform determined by the character $\psi \circ \operatorname{Tr}_{H|k}$ where $\operatorname{Tr}_{H|k}$ denotes the reduced trace. If H_1 and H_2 are two such algebras, and C is a closed subgroup of $k^\times$, a measure d_1x on $H_1^\times/C$ is *related* to a measure d_2x on $H_2^\times/C$ if there is a Haar measure dc on C such that

$$d_1x = \frac{|h|_{H_1}^{-1} d_{H_1}h}{dc} \quad \text{and} \quad d_2x = \frac{|h|_{H_2}^{-1} d_{H_2}h}{dc}.$$

This, obviously, does not depend on the choice of ψ. In particular, measures on $\mathrm{Gl}_2(k)$ and on the unique quaternion division algebra over a non-archimedean local field k are related if and only if the volumes of their Iwahori subgroups are equal (cf. [6], Lemma 5.2.1).

Similarly, one can consider finite-dimensional semi-simple algebras H_1 and H_2 over a global field k, and a closed subgroup C in $\mathbb{A}_k^\times$. A measure d_1x on $H_1^\times(\mathbb{A}_k)/C$ is *related* to a measure d_2x on $H_2^\times(\mathbb{A}_k)/C$ if there is a Haar measure dc on C such that

$$d_1x = \frac{\prod_v d_1h_v}{dc} \quad \text{and} \quad d_2x = \frac{\prod_v d_2h_v}{dc}$$

where v runs over all places of k and d_1h_v and d_2h_v are measures on $H_1^\times(k_v)$ and $H_2^\times(k_v)$, respectively, which are related as above and such that the product measures are defined. Let, in particular, H_i $(i = 1, 2)$ be quaternion algebras over k, and choose related measures on $\mathcal{G}_i = H_i^\times(\mathbb{A}_k)/C$ and normalized measures on the discrete subgroups $\Gamma_i = H_i^\times/(C \cap H_i^\times) \subset \mathcal{G}_i$. Then, with respect to the quotient measures,

$$\operatorname{vol}(\Gamma_1\backslash\mathcal{G}_1) = \operatorname{vol}(\Gamma_2\backslash\mathcal{G}_2)$$

(cf. [17], (16.1.8)).

Let $\mathfrak{R} \subset D^\times$ be a set of representatives for the conjugacy classes in $D^\times/(F^\times \cap C)$. The Selberg trace formula expresses the trace of $f^{(j)}$ on $\mathcal{A}$ as

$$(10.6) \qquad \operatorname{Tr}(f^{(j)}|\mathcal{A}) = \sum_{\gamma\in\mathfrak{R}} \operatorname{vol}\big(D_\gamma^\times(F^\times(\mathbb{R}) \times C_F)\backslash D_\gamma^\times(\mathbb{A})\big)\, \mathrm{O}_\gamma(f^{(j)}).$$

For the comparison of this expression with (8.7) we will need the following lemma.

Lemma 10.7 *Let r be a positive integer and $\gamma \in D^\times$ an element which is contained in a totally imaginary quadratic extension of F. Assume that there is a $\delta \in D^\times(\mathcal{K}_r)$ which is contained in the σ–conjugacy class of an element in $Y_D \subset D^\times(\mathcal{K})$ and such that γ and $\delta\sigma(\delta)\cdots\sigma^{r-1}(\delta)$ are conjugate. If, for every prime $\mathfrak{p}$ of $F(\gamma)$ above p, $\mathrm{ord}_\mathfrak{p}(\gamma) = \mathrm{ord}_\mathfrak{p}(\gamma^*)$, then there is an F–algebra embedding $F(\gamma) \subset D'$. In this case let $\mathfrak{t} = D'$. Otherwise let $\mathfrak{t} = (F(\gamma), \varepsilon)$ with*

$$\varepsilon(\mathfrak{p}) = \begin{cases} d_i & \text{if } \overline{\mathfrak{p}} = \mathfrak{p} \,|\, \mathfrak{p}_i\,; \\ \dfrac{-f_i \,\mathrm{ord}_\mathfrak{p}(\gamma)}{r} & \text{if } \overline{\mathfrak{p}} \neq \mathfrak{p} \,|\, \mathfrak{p}_i\,. \end{cases}$$

Then $\mathfrak{t} \in \mathfrak{I}_D$. In both cases the set $\{\, d \in D^\times(\mathcal{K}) \mid \mathcal{F}_\mathfrak{t}^r(d) = j_\mathfrak{t}(\gamma) d \,\}$ is not empty.

Proof. We choose $d_1 \in D^\times(\mathcal{K}_r)$ such that $\delta\sigma(\delta)\cdots\sigma^{r-1}(\delta) = d_1 \gamma d_1^{-1}$ and replace δ by $d_1^{-1}\delta\sigma(d_1)$. Then, in the isocrystal $(N, \mathcal{F})$ with $N = D \otimes_\mathbb{Q} \mathcal{K}$ and $\mathcal{F}(d) = \sigma(d)\delta^{-1}$, we have

$$\mathcal{F}^r(d) = \sigma^r(d)\gamma^{-1}\,. \tag{10.8}$$

The natural action of D induces a decomposition

$$N = \bigoplus_{\varphi \in \mathrm{Hom}_\mathbb{Q}(L\,, \mathcal{K})} N_\varphi$$

into 2–dimensional $\mathcal{K}$–vector spaces. Since δ is σ–conjugate to an element in Y_D, there are lattices $M_\varphi \subset N_\varphi$ which satisfy the conditions (4.2) and (4.3). As in §4 we fix, for each $i \in \{1, \ldots, m\}$, a $\varphi_i \in \mathrm{Hom}_\mathbb{Q}(L\,, \mathcal{K})$ such that $\varphi_i|_F \in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}\,, \mathcal{K})$ and denote the restriction of the σ^{f_i}–linear automorphism $\Pi\mathcal{F}^{f_i}$ to N_{φ_i} by $\mathcal{F}_i$. Since

$$\delta\sigma(\gamma) = \delta\sigma(\delta)\cdots\sigma^r(\delta) = \gamma\delta\,,$$

we have an F–algebra embedding

$$\begin{aligned} F(\gamma) &\longrightarrow \mathrm{End}(N_{\varphi_i}, \mathcal{F}_i)\,. \\ \alpha &\longmapsto (d \mapsto d\alpha^*) \end{aligned}$$

Applying (10.8) and Lemma 4.4 we derive the following facts. For a prime $\mathfrak{p} \,|\, \mathfrak{p}_i$ of $F(\gamma)$,

$$f_i\big(\mathrm{ord}_\mathfrak{p}(\gamma) + \mathrm{ord}_\mathfrak{p}(\gamma^*)\big) = -r d_i \,\mathrm{ord}_\mathfrak{p}(p)$$

and, if $\mathrm{ord}_\mathfrak{p}(\gamma) = \mathrm{ord}_\mathfrak{p}(\gamma^*)$, then $(N_{\varphi_i}, \mathcal{F}_i)$ is isotypic and the endomorphism ring is isomorphic to $D'_{\mathfrak{p}_i}$. In particular, we have an embedding $F(\gamma) \subset D'_{\mathfrak{p}_i}$. Assume, on the other hand, that $\mathrm{ord}_\mathfrak{p}(\gamma) \neq \mathrm{ord}_\mathfrak{p}(\gamma^*)$. Then, since $F(\gamma) \subset D$, $\delta_i = 0$ and the slopes of $(N_{\varphi_i}, \mathcal{F}_i)$ are the numbers $\varepsilon(\mathfrak{p})$ and $\varepsilon(\overline{\mathfrak{p}})$ occuring in the assertion of the lemma. In particular, these are non-negative integers. Since D' and D have the same invariants at all places of F not dividing p and $F(\gamma) \subset D$ is either equal to F or a totally imaginary quadratic extension of F, these facts imply the first part of our claim.

For the second part, we may replace r by ar and γ by γ^a, for any positive integer a, since in the exact sequence of pointed sets

$$\mathrm{H}^1(\langle\sigma^r\rangle, D^\times(\mathcal{K}_{ar})) \longrightarrow \mathrm{H}^1(\langle\sigma^r\rangle, D^\times(\mathcal{K})) \longrightarrow \mathrm{H}^1(\langle\sigma^{ar}\rangle, D^\times(\mathcal{K}))$$

the first term vanishes by Hilbert's Theorem 90, and the restriction map sends the σ^r–conjugacy class of $j_{\mathfrak{t}}(\gamma)^{-1} d_{\mathfrak{t}} \sigma(d_{\mathfrak{t}}) \cdots \sigma^{r-1}(d_{\mathfrak{t}})$ to the σ^{ar}–conjugacy class of $j_{\mathfrak{t}}(\gamma)^{-a} d_{\mathfrak{t}} \sigma(d_{\mathfrak{t}}) \cdots \sigma^{ar-1}(d_{\mathfrak{t}})$. In fact, this is an immediate consequence of the choice of $d_{\mathfrak{t}}$ (cf. Lemma 5.2). Hence we may assume that $r = bn$ for a positive integer b. Again by the choice of $d_{\mathfrak{t}}$ this implies

$$j_{\mathfrak{t}}(\gamma)^{-1} d_{\mathfrak{t}} \sigma(d_{\mathfrak{t}}) \cdots \sigma^{r-1}(d_{\mathfrak{t}}) = j_{\mathfrak{t}}(\gamma^{-1} z_{\mathfrak{t}}^b).$$

From the definition of $z_{\mathfrak{t}}$ in §5 we see that, for any prime $\mathfrak{p} | \mathfrak{p}_i$ of $F(\gamma)$,

$$\mathrm{ord}_{\mathfrak{p}}(z_{\mathfrak{t}}) = \begin{cases} \dfrac{-n\varepsilon(\mathfrak{p})}{f_i} & \text{if } \mathfrak{t} = (F(\gamma), \varepsilon) \text{ and } \varepsilon(\overline{\mathfrak{p}}) \neq \varepsilon(\mathfrak{p}); \\[2ex] \dfrac{-n d_i \, \mathrm{ord}_{\mathfrak{p}}(p)}{2 f_i} & \text{otherwise}. \end{cases}$$

Thus $\gamma^{-1} z_{\mathfrak{t}}^b$ is a unit in $F(\gamma) \otimes_{\mathbb{Q}} \mathbb{Q}_p$. But then it is easy to see that this element is contained in the σ^r–conjugacy class of 1 in $F(\gamma)^\times(\mathcal{K})$. In fact, this is a trivial case of a general result of Greenberg (cf. [15], §3, Proposition 3). Since $j_{\mathfrak{t}}$ commutes with σ^r, this completes the proof. ∎

Proposition 10.9 *For every $j > 0$, $\mathrm{Tr}^{\mathrm{ss}}_{\wp}(Sh_{D,C}, j) = \mathrm{Tr}(f^{(j)} | \mathcal{A})$.*

Proof. We will prove that there is a bijective correspondence between the non-vanishing terms in the sums (8.7) and (10.6), and that the corresponding terms agree. Let us start with an element $h \in \mathfrak{R}'(\mathfrak{t})$ for some $\mathfrak{t} \in \mathfrak{T}_D$. Consider a $\delta \in D^\times(\mathcal{K}_r)$ as in Lemma 7.4 and assume that $\mathrm{TO}_{\delta\sigma}(\Phi) \neq 0$. Then, by Proposition 9.4, δ is associated to an element in $D^\times(\mathbb{Q}_p)$, hence by Lemma 7.4 there is an embedding of $F(h)$ into D. The conjugacy class of the image of h in $D^\times/(F^\times \cap C)$ is uniquely determined. Let $\gamma \in \mathfrak{R}$ be its representative. Since $F(\gamma) \cong F(h)$ is either equal to F or a totally imaginary quadratic extension, and $\gamma \in D^\times$ is associated to $\delta \in D^\times(\mathcal{K}_r)$, (10.2), (10.3), and Remark 10.5 imply that the term corresponding to γ in (10.6) equals

$$\mathrm{vol}\big(D'^\times C_F \backslash D'^\times(\mathbb{A}_f)\big)\, \mathrm{O}_\gamma(f^p)\, \mathrm{TO}_{\delta\sigma}(\Phi)$$

if $h \in F^\times$, and

$$\mathrm{vol}\big(D_\gamma^\times C_F \backslash D_\gamma^\times(\mathbb{A}_f)\big)\, \mathrm{O}_\gamma(f^p)\, \mathrm{TO}_{\delta\sigma}(\Phi)$$

otherwise, which in turn is equal to the contribution of h to (8.7). Moreover, since by Lemma 7.4 $\mathfrak{t}$ is uniquely determined by $\mathrm{ord}_\mathfrak{p}(\gamma) = \mathrm{ord}_\mathfrak{p}(h)$ for all places $\mathfrak{p}$ of $F(\gamma) \cong F(h)$ above p, the map $(\mathfrak{t}, h) \mapsto \gamma$ just constructed is injective.

Consider now any element $\gamma \in \mathfrak{R}$ such that $\mathrm{O}_\gamma(f_\infty) \neq 0$ and $\mathrm{O}_\gamma(f_p^{(j)}) \neq 0$. Then (10.3) implies that either $\gamma \in F^\times$ or $F(\gamma)$ is totally imaginary, and (10.2) implies that γ is associated to an element $\delta \in D^\times(\mathcal{K}_r)$ with $\mathrm{TO}_{\delta\sigma}(\Phi) \neq 0$. Using Proposition 7.7 and (7.5) this in turn implies that δ is contained in the σ–conjugacy class of an element in $Y_D \subset D^\times(\mathcal{K})$, hence we can apply Lemma 10.7 to construct $\mathfrak{t}$ together with an embedding $F(\gamma)^\times \subset H_\mathfrak{t}(\mathbb{Q})$. Since either $\mathfrak{t} = D'$ or $\mathfrak{t} = (L', \varepsilon)$ such that L' splits D, $\mathfrak{t}$ is contained in $\mathfrak{T}_D$ (cf. Lemma 4.8). Let $h \in \mathfrak{R}(\mathfrak{t})$ denote the representative of the conjugacy class of the image of γ in $H_\mathfrak{t}(\mathbb{Q})/(F^\times \cap C)$. By Lemma 10.7 this is contained in $\mathfrak{R}'(\mathfrak{t})$, and the map $\gamma \mapsto (\mathfrak{t}, h)$ is inverse to the map constructed above. ∎

§11. Semi-simple automorphic L–functions

Similar to the semi-simple variant of a local factor of the zeta function of a projective variety one can introduce a semi-simple variant of a local factor of the L–function of an automorphic representation. We will do this only in the case relevant for our purposes. To this end, we consider the L–group ${}^L D^\times = \widehat{D^\times} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ where $\gamma \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ acts on

$$\widehat{D^\times} = \prod_{v\colon F \to \overline{\mathbb{Q}}} \mathrm{Gl}_2(\mathbb{C})$$

as $(\, g_v \,) \mapsto (\, g_{\gamma^{-1}v} \,)$. Let (ϱ, V) denote the representation induced from the representation (ϱ°, V°) of the subgroup ${}^L D^\times_{E(D)} = \widehat{D^\times} \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}|E(D)) \subset {}^L D^\times$ where

$$V^\circ = \bigotimes_{v \in \mathcal{Z}} \mathbb{C}^2 \qquad \text{and} \qquad \varrho^\circ\big((\, g_v \,), \gamma\big)\colon \textstyle\bigotimes_v x_v \longmapsto \bigotimes_v (\det g_v)^{-1} g_v x_{\gamma^{-1}v}\,.$$

This representation can be characterized in terms of h_D as in [19] (§9) and is contragredient to the representation considered by Langlands [21]. The difference is explained by the fact that we use Deligne's normalization of the reciprocity law for local fields while Langlands (loc. cit.) uses the inverse. Let us fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. The restriction ϱ_p of ϱ to the subgroup ${}^L D^\times_{\mathbb{Q}_p}$ is, up to isomorphism, independent of this embedding.

Consider now any irreducible unramified representation $\pi = \bigotimes_{i=1}^{m} \pi_i$ of $D^\times(\mathbb{Q}_p)$ (i.e. $\pi^{C_p} \neq 0$). It corresponds to a $\widehat{D^\times}$–conjugacy class of admissible homomorphisms

$$\varphi(\pi) \colon W'_{\mathbb{Q}_p} \longrightarrow {}^L D^\times_{\mathbb{Q}_p}$$

where $W'_{\mathbb{Q}_p} = \mathbb{C} \rtimes W_{\mathbb{Q}_p}$ denotes the Weil-Deligne group of $\mathbb{Q}_p$ (cf. [2], §8). A representative of this conjugacy class can be obtained as follows. For every $(x,w) \in W'_{\mathbb{Q}_p}$ and $i \in \{1,\ldots,m\}$, let

$$g_i(x,w) = \begin{pmatrix} \chi_{i,1}(w) & 0 \\ 0 & \chi_{i,2}(w) \end{pmatrix}$$

if D splits at $\mathfrak{p}_i$ and $\chi_{i,1}, \chi_{i,2}$ are unramified characters of $\mathbb{Q}_p^\times$ such that π_i is the unramified principal series representation corresponding to the pair of characters $\chi_{i,j} \circ \mathrm{Nm}_{F_{\mathfrak{p}_i}|\mathbb{Q}_p}$ $(j=1,2)$ of $F_{\mathfrak{p}_i}^\times$, and

$$g_i(x,w) = \chi_i(w) \begin{pmatrix} |w|^{1/2} & x\,|w|^{1/2} \\ 0 & |w|^{-1/2} \end{pmatrix}$$

if D is ramified at $\mathfrak{p}_i$ and χ_i is an unramified character of $\mathbb{Q}_p^\times$ such that π_i is the 1–dimensional representation $\chi_i \circ \mathrm{Nm}_{D_{\mathfrak{p}_i}|\mathbb{Q}_p}$. Now let

$$\varphi(\pi)(x,w) = \big((\, g_v \,)\, , w \big) \quad \text{where} \quad g_v = g_i(x,w) \text{ if } v \in \mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \overline{\mathbb{Q}}_p)\,.$$

Assume that, for every prime $\wp$ of $E(D)$ dividing p, we have fixed a diagram (2.4). ϱ_p decomposes into a direct sum over all $\wp$ of representations $\varrho_\wp$ which are obtained by induction from the restrictions of ϱ° to ${}^L D_E^\times$ where $E = E(D)_\wp$. This implies that the representation $\varrho_p \circ \varphi(\pi)$ of $W'_{\mathbb{Q}_p}$ decomposes into a direct sum of the representations $\varrho_\wp \circ \varphi(\pi)$, and these are obtained by induction from the representations $\varrho^\circ \circ \varphi(\pi)|_{W'_E}$ of W'_E on V°. Let $\{e_1, e_2\}$ denote the natural basis of $\mathbb{C}^2$ and consider a fixed $\wp$. For every integer j, let $F_\wp^j V^\circ$ be the subspace of V° generated by

$$\Big\{\, x_J = \big(\textstyle\bigotimes_{v\in J} e_1\big) \otimes \big(\textstyle\bigotimes_{v\in \mathcal{Z}\smallsetminus J} e_2\big) \;\Big|\; J \subseteq \mathcal{Z} \text{ with } \nu(J) \leqslant j \,\Big\}.$$

where $\nu(J) = \#(\mathcal{Z} \smallsetminus \mathcal{Z}_0) - 2\#(J \smallsetminus \mathcal{Z}_0)$. This depends on $\wp$ since $\mathcal{Z}_0$ does. It defines a filtration on V° which is stable with respect to the action of all $g = \big((\, g_v\,)\, , \gamma\big) \in {}^L D_E^\times$ where g_v is upper triangular if $v \in \mathcal{Z} \smallsetminus \mathcal{Z}_0$. Moreover, if g_v is unipotent upper triangular for all $v \in \mathcal{Z}$, $g_v = 1$ for all $v \in \mathcal{Z}_0$, and γ acts trivially on $\mathcal{Z}$, then g acts trivially on $\mathrm{gr}^{F_\wp^\bullet} V^\circ$. In particular, $F_\wp^\bullet$ is stable with respect to the action of W'_E, and $I' = \mathbb{C} \times I$ acts trivially on the associated graded module. Here $I \subset W_{\mathbb{Q}_p}$ denotes the inertia subgroup which is contained in W_E since $E|\mathbb{Q}_p$ is unramified.

Remark 11.1 Consider the nilpotent endomorphism N of V° given by

$$Nx_J = \sum_{v \in \mathcal{Z} \smallsetminus (J \cup \mathcal{Z}_0)} x_{J \cup \{v\}} \,.$$

Then, for every irreducible unramified representation π of $D^\times(\mathbb{Q}_p)$ and $(x,\gamma) \in I'$,

$$\varrho^\circ\big(\varphi(\pi)(x,\gamma)\big) = \exp(xN)\,.$$

The corresponding Schmid filtration $F_N^\bullet$ of V° is determined by the conditions that, for every integer j, N maps $F_N^j V^\circ$ to $F_N^{j-2} V^\circ$ and, for every positive j, N^j induces an isomorphism

$$\mathrm{gr}_j^{F_N^\bullet} V^\circ \longrightarrow \mathrm{gr}_{-j}^{F_N^\bullet} V^\circ\,.$$

By induction on $\#(\mathcal{Z} \smallsetminus \mathcal{Z}_0)$ one can show that $F_\wp^\bullet$ satisfies these conditions. Thus $F_\wp^\bullet$ is the common monodromy filtration of the representations $\varrho^\circ \circ \varphi(\pi)|_{W_E'}$.

By induction, this filtration provides a filtration on the representation space of $\varrho_\wp \circ \varphi(\pi)$. Taking the direct sum over all $\wp$ dividing p, we obtain a $W'_{\mathbb{Q}_p}$–invariant filtration $F_p^\bullet$ on V such that I' acts trivially on $\mathrm{gr}^{F_p^\bullet} V$. Hence the following definition makes sense.

Definition 11.2 For every irreducible admissible representation π of $D^\times(\mathbb{A})/F^\times(\mathbb{R})$ with unramified component π_p at p, let

$$L_p^{\mathrm{ss}}(\pi,\varrho,s) = \det\big(1 - (\varrho \circ \varphi(\pi_p))(\sigma^{-1})p^{-s} \;\big|\; \mathrm{gr}^{F^\bullet} V\,\big)^{-1}$$

where $\sigma \in W'_{\mathbb{Q}_p}/I' \subset \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}\,|\,\mathbb{Q}_p)$ is the Frobenius automorphism over $\mathbb{Q}_p$ and $F^\bullet$ is any admissible Filtration of V with respect to $I' \subset W'_{\mathbb{Q}_p}$ in the sense of Definition 8.1.

For every irreducible unramified representation π of $D^\times(\mathbb{Q}_p)$, every place $\wp$ of $E(D)$ above p, and every integer $j > 0$, let

$$\begin{aligned} L_\wp^{\mathrm{ss}}(\pi,\varrho^\circ,s) &= \det\big(1 - (\varrho^\circ \circ \varphi(\pi))(\mathrm{Fr}^{-1})q^{-s} \;\big|\; \mathrm{gr}^{F^\bullet} V^\circ\,\big)^{-1} \\ \mathrm{Tr}_\wp^{\mathrm{ss}}(\pi,\varrho^\circ,j) &= \mathrm{Tr}\,\big((\varrho^\circ \circ \varphi(\pi))(\mathrm{Fr}^{-j}) \;\big|\; \mathrm{gr}^{F^\bullet} V^\circ\,\big) \end{aligned}$$

where $\mathrm{Fr} \in W'_E/I' \subset \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}\,|\,E)$ is the Frobenius automorphism over $E = E(D)_\wp$, q is the number of elements in the residue class field of E, and $F^\bullet$ is any filtration of V° which is admissible with respect to $I' \subset W'_E$.

Using the same argument as in the proof of Proposition 3.8 in [9] we obtain

$$L_p^{\mathrm{ss}}(\pi,\varrho,s) = \prod_{\wp \mid p} L_\wp^{\mathrm{ss}}(\pi_p,\varrho^\circ,s)\,.$$

Now we return to the notation of the previous sections and consider again the fixed place $\wp$ of $E(D)$. As in §8 we have the equation

$$\log L_\wp^{\mathrm{ss}}(\pi, \varrho^\circ, s) = \sum_{j=1}^{\infty} \mathrm{Tr}_\wp^{\mathrm{ss}}(\pi, \varrho^\circ, j) \frac{q^{-js}}{j} . \tag{11.3}$$

Lemma 11.4 *For every irreducible unramified representation π of $D^\times(\mathbb{Q}_p)$,*

$$\mathrm{Tr}_\wp^{\mathrm{ss}}(\pi, \varrho^\circ, j) = q^{-j\#\mathcal{Z}/2}\, \mathrm{Tr}\big(\pi(f_p^{(j)})\big) .$$

Proof. For every $i \in \{1, \ldots, m\}$, let

$$\begin{array}{llll} \alpha_i = \chi_{i,1}(p)^{-1} & \text{and} & \beta_i = \chi_{i,2}(p)^{-1} & \text{if } \delta_i = 0 ; \\ \alpha_i = p^{1/2}\chi_i(p)^{-1} & \text{and} & \beta_i = p^{-1/2}\chi_i(p)^{-1} & \text{if } \delta_i = 1 ; \end{array}$$

where $\chi_{i,j}$ or χ_i, respectively, are fixed as in the construction of $\varphi(\pi)$ above, and for every subset $J \subseteq \mathcal{Z}$, let

$$\nu_i(J) = \#\big(\mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathbb{Q}_p^{\mathrm{nr}}) \cap J\big) .$$

The basis $\{\, x_J \mid J \subseteq \mathcal{Z} \,\}$ of V° induces a basis $\{\, y_J \mid J \subseteq \mathcal{Z} \,\}$ of $\mathrm{gr}^{F_\wp^\bullet} V^\circ$ such that

$$(\varrho^\circ \circ \varphi(\pi))(\mathrm{Fr}^{-1})(y_J) = c_J^e\, y_{\mathrm{Fr}^{-1} J} \quad \text{with} \quad c_J = \prod_{i=1}^{m} \alpha_i^{\nu_i(\mathcal{Z} \smallsetminus J)} \beta_i^{\nu_i(J)} .$$

Hence, for every $j > 0$,

$$\mathrm{Tr}_\wp^{\mathrm{ss}}(\pi, \varrho^\circ, j) = \sum_{\substack{J \subseteq \mathcal{Z} \\ \mathrm{Fr}^j J = J}} c_J^r = \prod_{i=1}^{m} c_{i,j}$$

where $r = ej$ and

$$c_{i,j} = \sum_{k=0}^{a_i} \binom{a_i}{k} \alpha_i^{(a_i - k) f_i r_i} \beta_i^{k f_i r_i} = \big(\alpha_i^{f_i r_i} + \beta_i^{f_i r_i}\big)^{a_i}$$

with a_i and r_i as in (10.1). In fact, a_i is the number of orbits of the action of Fr^j on $\mathrm{Hom}_{\mathbb{Q}_p}(F_{\mathfrak{p}_i}, \mathbb{Q}_p^{\mathrm{nr}}) \cap \mathcal{Z}$ and $f_i r_i$ is r times the number of elements in such an orbit. On the other hand,

$$\alpha_i^{f_i r_i} + \beta_i^{f_i r_i} = p^{-f_i r_i/2} \theta(r_i, \pi_i)$$

if $\pi = \bigotimes_{i=1}^m \pi_i$ (cf. Lemma 9.1). The lemma follows by comparing the resulting expression for the left side of our claim with (10.1). ∎

Remark 11.5 If π is an irreducible unramified representation which is tempered, i.e. the characters χ_i and $\chi_{i,j}$, respectively, considered above are unitary, then for every subset $J \subseteq \mathcal{Z}$, the absolute value of the complex number c_J occuring in the proof of Lemma 11.4 is given by $|c_J| = p^{\nu(J)/2}$. By Remark 11.1 this means that the monodromy filtration of $\varrho^\circ \circ \varphi(\pi)|_{W'_E}$ is pure of weight 0, i.e. the absolute value of every eigenvalue of Fr^{-1} acting on the j–th component of the associated graded module equals $q^{j/2}$.

Theorem 11.6 *The semi-simple local zeta function of $Sh_{D,C}$ can be expressed as*

$$Z^{\mathrm{ss}}_\wp(Sh_{D,C}, s) = \prod_\pi L^{\mathrm{ss}}_\wp(\pi_p, \varrho^\circ, s - \#\mathcal{Z}/2)^{m(\pi,C)}$$

where the product is extended over all irreducible constituents of $\mathcal{A}$ for which π_p is unramified.

Proof. Combine (8.4), (10.4), Proposition 10.9, (11.3), and Lemma 11.4. ∎

Remark 11.7 Under the following equivalent assumptions $Z^{\mathrm{ss}}_\wp$ and $L^{\mathrm{ss}}_\wp$ in Theorem 11.6 can be replaced by $Z_\wp$ and $L_\wp$, respectively.

(i) The map from the set of archimedean places of F to the set of primes of F dividing p which is induced by embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$ such that diagram (2.4) is commutative respects the subsets of places at which D splits.

(ii) For one (hence every) open subgroup $C \subset D^\times(\mathbb{A}_f)$ which satisfies (6.1), $\widetilde{Sh}_{D,C}$ is smooth.

(iii) For one (hence every) irreducible unramified representation π of $D^\times(\mathbb{Q}_p)$, the representation $\varrho^\circ \circ \varphi(\pi)|_{W'_E}$ of W'_E is unramified, i.e. I' acts trivially.

(iv) The filtration $F^\bullet_\wp$ of V° is trivial, i.e. the associated graded module equals V°.

(cf. Remark 6.7 and Remark 11.1) If, in particular, D is unramified at all places above p, the filtration $F^\bullet_p$ of V considered above is trivial, and in the corollary below Z^{ss}_p and L^{ss}_p can be replaced by Z_p and L_p, respectively.

Corollary 11.8 *For every rational prime p which is unramified in F and every open subgroup $C \subset D^\times(\mathbb{A}_f)$ which satisfies* (6.1),

$$Z^{\mathrm{ss}}_p(Sh_{D,C}, s) = \prod_\pi L^{\mathrm{ss}}_p(\pi, \varrho, s - d/2)^{m(\pi,C)}$$

where d denotes the dimension of $Sh_{D,C}$ and the product is extended over the same set as in Theorem 11.6.

Appendix A

§ A1. A mysterious invariant of a CM–type

This section is slightly more general than necessary for the application in the present paper. We consider a CM–field K with maximal totally real subfield F and two CM–algebras Q and Q' of degree 2 with non-trivial automorphisms * and $\tilde{}$ over K. Let $\bar{}$ denote complex conjugation on the CM–algebra $R = Q \otimes_K Q'$ and Φ a CM–type of R. We fix a prime p, embeddings $\overline{\mathbb{Q}} \to \mathbb{C}$ and $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, and an abelian variety A_0 of CM–type (R, Φ) over $\mathbb{C}$. Then there is a number field $L \subset \overline{\mathbb{Q}}$ such that A_0 and all of its endomorphisms are defined over L and A_0 has good reduction at the place defined by the embedding into $\overline{\mathbb{Q}}_p$ (cf. [38], Theorem 6). Let $I \subset \mathrm{Gal}(\overline{\mathbb{Q}}|L)$ denote the corresponding inertia subgroup and A the abelian variety over the residue class field $\mathbb{F}$ of $\overline{\mathbb{Q}}_p$ obtained this way. Up to isogeny, A is determined by the isogeny class of A_0 and the embeddings above. Let $H = \mathrm{H}_1(A_0, \mathbb{Q})$ and, for every prime $\mathfrak{r}$ of R above p, $\Phi_{\mathfrak{r}} = \Phi \cap \mathrm{Hom}_{\mathbb{Q}_p}(R_{\mathfrak{r}}, \overline{\mathbb{Q}}_p)$ and $n(\mathfrak{r}) = \#\Phi_{\mathfrak{r}}$. Here we have identified $\mathrm{Hom}_{\mathbb{Q}_p}(R_{\mathfrak{r}}, \overline{\mathbb{Q}}_p)$ with a subset of $\mathrm{Hom}_{\mathbb{Q}}(R, \mathbb{C})$ using the fixed embeddings of $\overline{\mathbb{Q}}$. The automorphisms of R induced by * or $\tilde{}$, respectively, and the identity on the other factor of the tensor product are again denoted by the same symbol. We impose the following conditions on the CM–type Φ:

(A1.1) for every prime $\mathfrak{r}$ of R above p, $n(\mathfrak{r}) = n(\mathfrak{r}^*)$;

(A1.2) $\{ \varphi|_Q \mid \varphi \in \Phi \}$ is invariant with respect to the automorphism * of Q.

Lemma A1.3 *There is a quasi-isogeny Π of A such that, for every $r \in R$, $\Pi r = r^* \Pi$ and $\Pi^2 \in K$.*

Proof. From (A1.1) we infer, applying the formula of Shimura and Taniyama and Tate's theorem (cf. [41], Lemme 5 and Théorème 1), that $\mathrm{End}^\circ_{Q'} A$ is a quaternion algebra over Q' which has invariant 0 at all places not dividing p. At a prime $\mathfrak{q}$ above p the invariant is 0 if $\mathfrak{q}$ splits in R, and it equals $n(\mathfrak{r})/2$ for the prime $\mathfrak{r}$ of R above $\mathfrak{q}$ otherwise.

Fix now a prime $\mathfrak{r}$ of R above p which is invariant under *. We claim that either $n(\mathfrak{r})$ is even, or $\mathfrak{r}$, $\bar{\mathfrak{r}}$, $\tilde{\mathfrak{r}}$, and $\tilde{\bar{\mathfrak{r}}}$ are pairwise distinct and $n(\mathfrak{r})$, $n(\bar{\mathfrak{r}})$, $n(\tilde{\mathfrak{r}})$, and $n(\tilde{\bar{\mathfrak{r}}})$ are all odd. In fact, (A1.2) implies that $\Phi_{\mathfrak{r}} \cap \widetilde{\Phi}_{\tilde{\mathfrak{r}}}$ and $\Phi_{\mathfrak{r}} \cup \widetilde{\Phi}_{\tilde{\mathfrak{r}}}$ are stable under * hence have an even number of elements. If, in addition, $\mathfrak{r}$ is invariant under $\tilde{}$, these two sets are stable under $\tilde{}$, hence have a cardinality which is divisible by 4. Now the identity

$$n(\mathfrak{r}) + n(\tilde{\mathfrak{r}}) = \#(\Phi_{\mathfrak{r}} \cup \widetilde{\Phi}_{\tilde{\mathfrak{r}}}) + \#(\Phi_{\mathfrak{r}} \cap \widetilde{\Phi}_{\tilde{\mathfrak{r}}})$$

implies $n(\mathfrak{r}) \equiv n(\tilde{\mathfrak{r}}) \pmod 2$ and, if $\mathfrak{r}$ is invariant under $\tilde{}$, $n(\mathfrak{r}) \equiv 0 \pmod 2$. Moreover, we have $n(\mathfrak{r})+n(\bar{\mathfrak{r}}) = [\, R_{\mathfrak{r}} : \mathbb{Q}_p \,]$ which is even, hence $n(\mathfrak{r}) \equiv n(\bar{\mathfrak{r}}) \pmod 2$, and, if $\mathfrak{r}$ is invariant under $\bar{}$, $n(\mathfrak{r}) = [\, R_{\mathfrak{r}} : \mathbb{Q}_p \,]/2 \equiv 0 \pmod 2$. If, on the other hand, $\bar{\mathfrak{r}} = \tilde{\mathfrak{r}}$, $\Phi_{\mathfrak{r}} \cap \widetilde{\overline{\Phi}}_{\mathfrak{r}}$ is invariant under $\tilde{\bar{}}$ hence has an even number of elements. But then

$$n(\mathfrak{r}) = \#(\Phi_{\mathfrak{r}} \cap \widetilde{\Phi}_{\tilde{\mathfrak{r}}}) + \#(\Phi_{\mathfrak{r}} \cap \widetilde{\overline{\Phi}}_{\mathfrak{r}}) \equiv 0 \pmod 2$$

which proves our claim.

What we have just shown implies that the number of places of K dividing p which split in Q' but not in Q and above which there is a prime $\mathfrak{r}$ of R such that $n(\mathfrak{r}) \equiv 1 \pmod 2$ is even. Hence there is a quaternion algebra B over K which is ramified exactly at these places. Obviously, Q is a splitting field of B and $\mathrm{End}^\circ_{Q'} A$ is isomorphic to $Q' \otimes_K B$. By the Theorem of Skolem and Noether, we find an isomorphism of Q'–algebras which maps Q into B. But then any element $\Pi \in B^\times$ with $\Pi q = q^*\Pi$ for all $q \in Q$ gives rise to a quasi-isogeny of A with the desired properties. ∎

The action of $\mathrm{Gal}(\overline{\mathbb{Q}}|L)$ on the rational Tate module $\widehat{\mathrm{V}}(A_0)$ which is canonically isomorphic to $H \otimes_{\mathbb{Q}} \mathbb{A}_f$ is given by a homomorphism

$$r_\Phi\colon\ \mathrm{Gal}(\overline{\mathbb{Q}}|L) \longrightarrow (R \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$$

such that $r_\Phi(I) \subset (R \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ (cf. [38], Theorem 1). On the other hand, we can consider the reflex field $E \subseteq L$ and the reciprocity law

$$\rho\colon E^\times \longrightarrow R^\times$$

of the CM–type Φ. This is a morphism of $\mathbb{Q}$–tori which is characterized by the following property. For every homomorphism $\varphi\colon R \to \overline{\mathbb{Q}} \subset \mathbb{C}$ and every $e \in E^\times$,

$$\varphi(\rho(e)) = \prod_{\tau \in T} \tau(e) \quad \text{where} \quad T = \{\, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \mid \tau^{-1}\varphi \in \Phi \,\} / \,\mathrm{Gal}(\overline{\mathbb{Q}}|E)\,.$$

r_Φ and ρ are related by a commutative diagram (cf. [39], Theorem 4.3)

$$\begin{tikzcd}
\mathrm{Gal}(\overline{\mathbb{Q}}|L) \arrow[r] \arrow[d, "r_\Phi"'] & \mathrm{Gal}(E^{\mathrm{ab}}|E) \arrow[d] & (E \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times \arrow[l] \arrow[d, "\rho"] \\
(R \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times \arrow[r] & (R \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times / R^\times & (R \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times \arrow[l]
\end{tikzcd}$$

where the upper left horizontal map is defined by the embedding $E^{\mathrm{ab}} \subset \overline{\mathbb{Q}}$ and the upper right horizontal map by global class field theory (using Deligne's normalization which is inverse to that in loc. cit.). (A1.2) implies that $\rho\tilde{\rho}$ is invariant under *. Hence $(r_\Phi \tilde{r}_\Phi)|_I$ is invariant, too.

Consider now the canonical isomorphisms

$$\widehat{\mathrm{V}}^p(A_0) \cong \widehat{\mathrm{V}}^p(A)$$

and

$$\mathrm{V}_p(A_0) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong M \otimes_{\mathrm{W}(\mathbb{F})} B_{\mathrm{cris}}$$

where M denotes the covariant Dieudonné module of A, and B_{cris} is Fontaine's topological $\mathrm{W}(\mathbb{F}) \otimes_{\mathbb{Z}} \mathbb{Q}$–algebra which is equipped with a Frobenius endomorphism σ and an action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ (cf. [13]). If V denotes the Verschiebung on M, the second isomorphism carries the action of $\mathrm{id} \otimes \sigma$ on the left into $V^{-1} \otimes \sigma$ on the right and, for every $\tau \in I \subset \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$, the action of $\tau \otimes \tau$ on the left into $\mathrm{id} \otimes \tau$ on the right. The choice of an R–module isomorphism $H \cong R$ identifies $\widehat{\mathrm{V}}^p(A) \times (M \otimes_{\mathrm{W}(\mathbb{F})} B_{\mathrm{cris}})$ with $R \otimes_{\mathbb{Q}} \widetilde{\mathbb{A}}_f$ where $\widetilde{\mathbb{A}}_f = \mathbb{A}_f^p \times B_{\mathrm{cris}}$. Hence, if we fix a quasi-isogeny Π of A as in Lemma A1.3, it defines an element $r_\Pi \in (R \otimes_{\mathbb{Q}} \widetilde{\mathbb{A}}_f)^\times$ such that the action of Π on $R \otimes_{\mathbb{Q}} \widetilde{\mathbb{A}}_f$ is given by

$$\Pi \colon r \longmapsto r_\Pi r^* .$$

Up to a factor in $R^\times$, r_Π is independent of the choices of Π and the isomorphism $H \cong R$.

Lemma A1.4 *The element r_Π has the following properties:*

$$\begin{aligned} r_\Pi r_\Pi^* &= \Pi^2 \in K^\times ; \\ \frac{\tau(r_\Pi)}{r_\Pi} &= \frac{r_\Phi(\tau)^*}{r_\Phi(\tau)} \qquad \text{for every } \tau \in I ; \\ \sigma(r_\Pi) &= r_\Pi ; \\ r_\Pi \overline{r}_\Pi &\in F^\times R_+^\times ; \end{aligned}$$

where the actions of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ and σ on $R \otimes_{\mathbb{Q}} \widetilde{\mathbb{A}}_f$ are defined by the corresponding actions on B_{cris}, and $R_+^\times$ denotes the subgroup of totally positive real elements in $R^\times$.

Proof. The first assertion is trivial and the next two simply express the obvious fact that the action of Π on $M \otimes_{\mathrm{W}(\mathbb{F})} B_{\mathrm{cris}}$ commutes with the actions of I and $V^{-1} \otimes \sigma$. The last property follows from (A1.2). To see this, we choose a polarization λ_0 of A_0 whose Rosati involution leaves R invariant. The Riemann form on H induced by λ_0 satisfies the Riemann period relations, hence under the isomorphism $H \cong R$ fixed above, it corresponds to a map

$$(r_1, r_2) \longmapsto \mathrm{Tr}_{R|\mathbb{Q}}(r_\lambda r_1 \overline{r}_2)$$

for an element $r_\lambda \in R^\times$ such that, for each $\varphi \in \Phi$, $\varphi(r_\lambda) \in \mathbb{R}_+ i$. In particular, r_λ^*/r_λ is totally real, and, since $\Phi \cup \widetilde{\Phi}$ and $\Phi \cap \widetilde{\Phi}$ are invariant under *, $(r_\lambda \widetilde{r}_\lambda)^*/(r_\lambda \widetilde{r}_\lambda)$ is totally positive. This shows that the set of infinite places of the maximal totally real subalgebra of R at which r_λ^*/r_λ is negative is invariant with respect to * and $\widetilde{\ }$, i.e. $r_\lambda^*/r_\lambda \in F^\times R_+^\times$. Now we consider the polarization λ on A induced by λ_0 and the element $r_0 = \Pi'\Pi \in R_+^\times$ where $'$ denotes the Rosati involution defined by λ. Since the canonical isomorphisms above are compatible with the Riemann forms for λ_0 and λ, we obtain the equation

$$\mathrm{Tr}_{R|\mathbb{Q}}(r_\lambda r_\Pi r_1^* \overline{r}_\Pi \overline{r}_2^*) = \mathrm{Tr}_{R|\mathbb{Q}}(r_\lambda r_0 r_1 \overline{r}_2)$$

for every pair of elements $r_1, r_2 \in R \otimes_{\mathbb{Q}} \widetilde{\mathbb{A}}_f$. But this implies that $r_\Pi \overline{r}_\Pi = (r_0 r_\lambda)^*/r_\lambda$ is contained in $F^\times R_+^\times$. ∎

Consider now $x = \widetilde{r}_\Pi / r_\Pi^*$. Since $(r_\Phi \widetilde{r}_\Phi)|_I$ is fixed by *, and the set of invariants of I in B_{cris} equals $\mathrm{W}(\mathbb{F}) \otimes_{\mathbb{Z}} \mathbb{Q}$, the first three assertions of Lemma A1.4 imply that x is contained in $(Q \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$ and $xx^* = 1$. By Hilbert's Theorem 90, we can choose an element $q \in (Q \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$ such that $x = q/q^*$, i.e.

$$r_\Pi \in q(S \otimes_{\mathbb{Q}} \widetilde{\mathbb{A}}_f)^\times \quad \text{where} \quad S = \{\, r \in R \mid \widetilde{r} = r^* \,\}.$$

Up to a factor in $Q^\times (K \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$, q is independent of all choices. The last assertion of the lemma implies that $z = x\overline{x}$ is contained in $R_+^\times \cap (Q \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times = Q_+^\times$, the subgroup of totally positive real elements in $Q^\times$. Since $zz^* = 1$, we conclude as above that $z = y/y^*$ for an element $y \in Q_+^\times$, hence $q\overline{q}$ is contained in $Q_+^\times (F \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times$. Finally, we note that

$$Q_+^\times (F \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times / Q_+^\times \, \mathrm{Nm}_{K|F}(K \otimes_{\mathbb{Q}} \mathbb{A}_f)^\times \cong \{+1, -1\}$$

which leads to the following definition.

Definition A1.5 $\varrho(Q|K, Q'|K, \Phi)$ denotes the image of $q\overline{q}$ in $\{+1, -1\}$.

The significance of this invariant (which is independent of the choices of A_0, Π, the isomorphism $H \cong R$, and q, but may depend on the embeddings of $\overline{\mathbb{Q}}$) is explained in the following two sections. In §A2 we introduce an obstruction for the existence of a certain PEL structure on a given abelian variety over $\mathbb{F}$, and in §A3 we show that this obstruction is closely related to the invariant defined above. If the following question has an affirmative answer, it can be replaced by an explicit condition which in every particular case should be easy to check.

Question A1.6 Is $\varrho(Q|K, Q'|K, \Phi)$ always equal to $+1$ (at least, if all primes of F dividing p split in K)?

§ A2. Polarized abelian varieties over $\mathbb{F}$

As in § A1 we fix a CM–field K with maximal totally real subfield F and a rational prime p. Let B denote a quaternion algebra over K and A an abelian variety of dimension $2[K : \mathbb{Q}]$ over $\mathbb{F}$ together with an embedding $\iota_K\colon K \to \mathrm{End}^\circ A$. We assume that the following conditions are satisfied:

(A2.1) all primes $\wp$ of F above p split in K;

(A2.2) there is a K–algebra homomorphism $\iota\colon B \to \mathrm{End}^\circ A$ and a polarization λ of A such that $\iota(B)$ is invariant under the Rosati involution defined by λ.

In this situation we can again define an invariant. To this end, we fix

a K–algebra homomorphism ι and a polarization λ as above;

an isomorphism $\eta\colon B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \widehat{\mathrm{V}}^p(A)$ of $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–modules;

an isomorphism $\tau\colon \mathbb{A}_f^p(1) \to \mathbb{A}_f^p$ over $\mathrm{Spec}(\mathbb{F})$.

Let E_λ denote the alternating form on $\widehat{\mathrm{V}}^p(A)$ induced by λ and $'$ the positive involution on B induced by the Rosati involution on $\mathrm{End}^\circ A$. Then there is a unique $b \in (B \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ such that

$$\tau E_\lambda(\eta(b_1), \eta(b_2)) = \mathrm{Tr}_{B|\mathbb{Q}}(b_1 b b_2') \qquad \text{for all } b_1, b_2 \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p\,.$$

Obviously, $b' = -b$, hence $\mathrm{Nm}_{B|K}(b) \in (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$. Again, we have

$$(F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times / F_+^\times\, \mathrm{Nm}_{K|F}(K \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times \cong \{+1, -1\}\,,$$

where $F_+^\times$ denotes the subgroup of totally positive elements in $F^\times$.

Definition A2.3 $\beta(A, \iota_K, B|K) \in \{+1, -1\}$ denotes the image of $\mathrm{Nm}_{B|K}(b)$.

Lemma A2.4 *$\beta(A, \iota_K, B|K)$ does not depend on the choices of ι, λ, η, and τ.*

Proof. Let ι_1, λ_1, η_1, and τ_1 be another choice of those data, and let b_1 be the corresponding element of $(B \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$. Then, by the Theorem of Skolem and Noether, there is an isogeny γ of A such that $\iota(b)\gamma = \gamma\iota_1(b)$ for every $b \in B$. Put now $\lambda_2 = \widehat{\gamma}\lambda\gamma$, $\eta_2 = \widehat{\mathrm{V}}^p(\gamma)^{-1}\eta$, $\iota_2 = \iota_1$, and $\tau_2 = \tau$. Then ι_2, λ_2, η_2, and τ_2 is also a set of data of the type considered above, and the corresponding element of $(B \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ is again b. Hence we may assume that $\iota_1 = \iota$. Moreover, there are elements $a \in \mathbb{A}_f^{p\,\times}$ and $c \in (B \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ such that $\eta_1(b) = \eta(bc)$ for all $b \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ and $\tau_1 = a\tau$. But then, if we replace η by η_1 and τ by τ_1, b is replaced by $acbc'$, and this means that

$\mathrm{Nm}_{B|K}(b)$ has to be multiplied by $\mathrm{Nm}_{K|F}(a\,\mathrm{Nm}_{B|K}(c))$. Therefore we may further assume that $\eta_1 = \eta$ and $\tau_1 = \tau$.

The Rosati involutions associated to λ and λ_1 induce positive involutions on B. By a lemma of Zink (cf. [44], Satz 1.1), they differ by an inner automorphism of B defined by an element $c \in B^\times$ such that $\mathbb{Q}[\,c\,]$ is a product of totally real number fields and c is totally positive and invariant under both involutions. We may choose it such that $\lambda\iota(c)$ is a polarization of A which induces the same involution on B as λ_1. If we replace λ with this involution, b is replaced with bc, and $\mathrm{Nm}_{B|K}(b)$ is multiplied by $\mathrm{Nm}_{B|K}(c) \in F_+^\times$.

Assuming now that both involutions on B coincide, we may apply the same lemma to the commutant B' of B in $\mathrm{End}^\circ A$. This gives us a $\gamma \in B'$ which is totally positive real and invariant under both involutions, and such that $\lambda\gamma$ is a polarization which induces the same involution as λ_1 on the commutant $\mathrm{End}_K^\circ A$ of K in $\mathrm{End}^\circ A$. It differs from λ_1 by a factor in $K^\times$. But this factor then has to be totally positive real, i.e. is contained in $F_+^\times$. Hence we may choose γ such that $\lambda\gamma = \lambda_1$. The isomorphism η carries the action of γ on $\widehat{\mathrm{V}}^p(A)$ into right multiplication by a certain $c \in (B \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ and we get $b_1 = bc'$. Since $\mathrm{Nm}_{B|K}(c) = \mathrm{Nm}_{B'|K}(\gamma) \in F_+^\times$ the lemma follows. ∎

The invariant $\beta(A, \iota_K, B|K)$ determines all possible structures of a symplectic $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–module on $\widehat{\mathrm{V}}^p(A)$ which are obtained by choosing an embedding ι and a polarization λ as above. To make this more precise, we fix a non-degenerate, alternating $\mathbb{Q}$–bilinear form

$$\Psi\colon B \times B \longrightarrow \mathbb{Q}$$

which induces a positive involution $'$ on B, i.e.

$$\Psi(bx, y) = \Psi(x, b'y) \qquad \text{for all } b, x, y \in B\,.$$

This form defines a sub–F–algebra

$$D = \{\, b \in B \mid \Psi(xb, y) = \Psi(x, yb^*) \quad \text{for all } x, y \in B \,\}$$

in B where * denotes the main involution on B.

Lemma A2.5 *D is a quaternion algebra over F such that $B \cong D \otimes_F K$.*

Proof. Consider the F–linear automorphism γ of B defined by the condition

$$\Psi(xb, y) = \Psi(x, y\gamma(b)^*) \qquad \text{for all } b, x, y \in B\,.$$

Since Ψ is skew-symmetric and γ is a homomorphism of simple F–algebras, hence commutes with the main involution, γ^2 is the identity. Therefore, B decomposes into the direct sum of eigenspaces for the eigenvalues $+1$ and -1 of γ. Since D is the eigenspace for the eigenvalue $+1$ and multiplication with any element $k \in K^\times$ such that $\mathrm{Tr}_{K|F}(k) = 0$ defines an isomorphism between these two eigenspaces, this proves our claim. ▪

We attach to Ψ the invariant

$$\beta(\Psi) = (-1)^{\#\mathcal{V}} \in \{+1, -1\}\,, \tag{A2.6}$$

where $\mathcal{V}$ denotes the set of infinite places of F at which D is ramified, and we denote the centralizer of B in $\mathrm{End}^\circ A$, for any fixed K-algebra embedding $\iota\colon B \to \mathrm{End}^\circ A$, by B'. Up to K-algebra isomorphism, B' is independent of the choice of ι.

Proposition A2.7 *There are*

a K-algebra homomorphism $\iota\colon B \to \mathrm{End}^\circ A$;

a polarization λ of A;

an isomorphism $\eta\colon B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \widehat{\mathrm{V}}^p(A)$ of $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$-modules;

an isomorphism $\tau\colon \mathbb{A}_f^p(1) \to \mathbb{A}_f^p$ over $\mathrm{Spec}(\mathbb{F})$;

an element $f \in (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$;

such that

$$\tau E_\lambda(\eta(b_1), \eta(b_2)) = \Psi(f b_1, b_2) \qquad \text{for all } b_1, b_2 \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$$

if and only if $\beta(A, \iota_K, B|K) = \beta(\Psi)$ and one of the following two conditions is satisfied:

(i) *$B' \cong L' \otimes_F K$ for a totally imaginary quadratic extension $L'|F$ which splits D at all finite places not dividing p;*

(ii) *$B' \cong D' \otimes_F K$ for a quaternion algebra $D'|F$ which is ramified at all infinite places and has the same invariants as D at all finite places not dividing p.*

Proof. Suppose first that we are given ι, λ, η, τ, and f as above, and let $b_\Psi \in B^\times$ be the element that satisfies

$$\Psi(b_1, b_2) = \mathrm{Tr}_{B|\mathbb{Q}}(b_1 b_\Psi b_2') \qquad \text{for all } b_1, b_2 \in B\,.$$

Then, by definition, $\beta(A, \iota_K, B|K)$ is the image of $\mathrm{Nm}_{B|K}(f b_\Psi) \in (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ in the quotient

$$(F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times / F_+^\times \, \mathrm{Nm}_{K|F}(K \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times \cong \{+1, -1\}\,,$$

and this is, obviously, the same as the image of $\mathrm{Nm}_{B|K}(b_\Psi) \in F^\times$ which can easily be computed as follows. Let $\widehat{\ }$ denote the involution on $B = D \otimes_F K$ defined by the main involution on D and complex conjugation on K. Then $b' = b_\Psi^{-1} \widehat{b} b_\Psi$ for all $b \in B$, and $\widehat{b}_\Psi = -b_\Psi$. The positivity of the involution $'$ is now equivalent to the fact that the set of infinite places of F at which $\mathrm{Nm}_{B|K}(b_\Psi)$ is negative equals $\mathcal{V}$, and this implies

$$\beta(A, \iota_K, B|K) = \beta(\Psi)\,.$$

To deduce the assertion concerning the structure of B' we consider the embedding $\varepsilon\colon B' \to B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ defined by the condition

$$\eta(b\varepsilon(x)^*) = \widehat{\mathrm{V}}^p(x)\eta(b) \qquad \text{for all } x \in B' \text{ and } b \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p\,.$$

It follows easily from the equation connecting E_λ and Ψ that ε is compatible with the involutions on both sides given by restriction of the Rosati involution to B' and $\widehat{\ }$ on $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. Now B' is either a quaternion algebra or a CM–algebra of degree 2 over K. Let * be the main involution or the non-trivial automorphism of B' over K, respectively, and denote the set of elements in B' on which the Rosati involution coincides with * by D' or L', respectively. The positivity of the Rosati involution implies that D' is ramified everywhere at infinity or that L' is a totally imaginary quadratic extension of F. Moreover, ε is compatible with the involution * on B' and the main involution on $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$, hence maps D' or L', respectively, into $D \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. This proves the assertion concerning the finite places of F not dividing p.

Now we come to the proof of the opposite implication. To this end, we fix τ, ι, and λ such that $\iota(B)$ is invariant under the Rosati involution defined by λ. Using the lemma of Zink [44] as above, we may assume that the involution induced on B is $'$. Consider first the case where B' is a quaternion algebra over K, and choose an isomorphism $D' \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong D \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ of $F \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–algebras. It induces an isomorphism

$$\varepsilon\colon B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p \longrightarrow B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$$

of $K \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–algebras. Since $B \otimes_K B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong \mathrm{M}_4(K \otimes_{\mathbb{Q}} \mathbb{A}_f^p)$ and $\widehat{\mathrm{V}}^p(A)$ and $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ are both free of rank 4 as $K \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–modules, we find an isomorphism

$$\eta\colon B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \longrightarrow \widehat{\mathrm{V}}^p(A)$$

of $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–modules such that

$$\eta(b\varepsilon(x)^*) = \widehat{\mathrm{V}}^p(x)\eta(b) \qquad \text{for all } x \in B' \text{ and } b \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p\,.$$

Using Zink's lemma again, we may assume that the restriction of the Rosati involution to D' is the main involution, hence ε is compatible with the restriction of the Rosati involution to B' and the involution $\widehat{\ }$ on $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. But then there is an element $f \in (K \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ such that

$$\tau E_\lambda(\eta(b_1), \eta(b_2)) = \Psi(f b_1, b_2) \qquad \text{for all } b_1, b_2 \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p .$$

Since both forms are skew-symmetric, this actually lies in $(F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$, and the proof is complete in this case.

Consider now the case where B' is of degree 2 over K, and choose an embedding $L' \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to D \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ of $F \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–algebras. It induces an embedding

$$\varepsilon\colon B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p \longrightarrow B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$$

of $K \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–algebras which is compatible with the restriction of the Rosati involution to B' and the involution $\widehat{\ }$ on $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. Since $B \otimes_K B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p \cong \mathrm{M}_2(B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p)$ and $\widehat{\mathrm{V}}^p(A)$ and $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ are both free of rank 2 as $B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–modules, we find an isomorphism

$$\eta\colon B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \longrightarrow \widehat{\mathrm{V}}^p(A)$$

of $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–modules such that

$$\eta(b\varepsilon(x)^*) = \widehat{\mathrm{V}}^p(x)\eta(b) \qquad \text{for all } x \in B' \text{ and } b \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p .$$

As above, this implies that there is an element $f \in (B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ such that

$$\tau E_\lambda(\eta(b_1), \eta(b_2)) = \Psi(b_1\varepsilon(f), b_2) \qquad \text{for all } b_1, b_2 \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p .$$

Let F' denote the maximal totally real subalgebra in B'. Since both forms are skew-symmetric, f is contained in $(F' \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$. By definition, $\beta(A, \iota_K, B|K)$ is the image of $\mathrm{Nm}_{F'|F}(f)\,\mathrm{Nm}_{B|K}(b_\Psi) \in (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ in the quotient $\{+1, -1\}$, and from the calculation at the beginning of the current proof we know that the image of $\mathrm{Nm}_{B|K}(b_\Psi)$ in $\{+1, -1\}$ equals $\beta(\Psi)$. Using the assumption of the lemma concerning the β's, this implies that $\mathrm{Nm}_{F'|F}(f)$ is contained in $F_+^\times\,\mathrm{Nm}_{K|F}(K \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$. Now we observe that the map

$$\begin{array}{c} (F' \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times / F_+'^\times (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times\,\mathrm{Nm}_{B'|F'}(B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times \\ \Big\downarrow{\scriptstyle \mathrm{Nm}_{F'|F}} \\ (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times / F_+^\times\,\mathrm{Nm}_{K|F}(K \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times \end{array}$$

is a bijection. Indeed, this is obvious if $B' = K \times K$ and follows via global class field theory from the fact that complex conjugation on B' restricts to complex conjugation on K if B' is a field. This yields

$$f \in F_+'^\times (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times\,\mathrm{Nm}_{B'|F'}(B' \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times .$$

Therefore, we can change λ by a factor in $F_+^{\prime\times}$ and η by a factor in $(B'\otimes_{\mathbb{Q}}\mathbb{A}_f^p)^\times$ such that f is replaced by an element of $(F\otimes_{\mathbb{Q}}\mathbb{A}_f^p)^\times$. This completes the proof of the lemma in the second case. ∎

Remark A2.8 The last part of the previous proof shows, in addition, that, if B' is a field, ι, λ, η, and τ may be chosen such that

$$\tau E_\lambda(\eta(b_1),\eta(b_2)) = \Psi(b_1,b_2) \qquad \text{for all } b_1,b_2\in B\otimes_{\mathbb{Q}}\mathbb{A}_f^p\,.$$

If, on the other hand, ι and λ are fixed, the K–algebra isomorphism in (i) or (ii), respectively, may be chosen such that the Rosati involution defined by λ induces complex conjugation on L' or the main involution on D', respectively.

§ A3. The comparison of invariants

We consider $B|K$, A, and ι_K as in § A2 and explain the relation between the invariants defined in § A1 and § A2. To this end, we choose a finite field $\mathbb{F}_q\subset\mathbb{F}$ which is a field of definition for A and all of its endomorphisms, and let π_A denote the Frobenius endomorphism of A over $\mathbb{F}_q$. Then $[\,K(\pi_A):K\,]\leqslant 2$, and we put

$$k_A = \begin{cases}\mathrm{Nm}_{K(\pi_A)|K}(\pi_A) & \text{if } \pi_A\notin K\,;\\ \pi_A^2 & \text{if } \pi_A\in K\,.\end{cases}$$

For every place $\wp$ of F above p, we choose a place $\mathfrak{p}$ of K above $\wp$ and define

$$a(\wp) = \frac{[\,F_\wp:\mathbb{Q}_p\,]\,\mathrm{ord}_{\mathfrak{p}}(k_A)}{\mathrm{ord}_\wp(q)}\,.$$

Lemma A3.1 *$a(\wp)$ is an integer whose class modulo 2 is independent of the choices of $\mathfrak{p}$ and $\mathbb{F}_q$.*

Proof. The independence of $\mathbb{F}_q$ is obvious. Let Q' denote a maximal semi-simple commutative subalgebra in the centralizer of B in $\mathrm{End}^\circ A$, and let $\mathfrak{q}$ be a prime of Q' above $\mathfrak{p}$. Then Q' has degree 2 over K, and ι induces an isomorphism between $B\otimes_K Q'$ and the centralizer of Q' in $\mathrm{End}^\circ A$. Therefore, Tate's Theorem [41] implies

$$\frac{[\,Q'_{\mathfrak{q}}:\mathbb{Q}_p\,]\,\mathrm{ord}_{\mathfrak{q}}(\pi_A)}{\mathrm{ord}_{\mathfrak{q}}(q)} \equiv [\,Q'_{\mathfrak{q}}:F_\wp\,]\,\mathrm{inv}_{\mathfrak{p}}(B) \pmod{\mathbb{Z}}\,,$$

hence, if $\tilde{\mathfrak{q}}$ denotes the image of $\mathfrak{q}$ with respect to the conjugation of $Q'|K$,

$$\frac{[\,F_\wp:\mathbb{Q}_p\,]\,\mathrm{ord}_{\mathfrak{p}}(k_A)}{\mathrm{ord}_\wp(q)} \equiv [\,F_\wp:\mathbb{Q}_p\,]\left(\frac{\mathrm{ord}_{\mathfrak{q}}(\pi_A)}{\mathrm{ord}_{\mathfrak{q}}(q)}+\frac{\mathrm{ord}_{\tilde{\mathfrak{q}}}(\pi_A)}{\mathrm{ord}_{\tilde{\mathfrak{q}}}(q)}\right) \equiv 2\,\mathrm{inv}_{\mathfrak{p}}(B)\equiv 0 \pmod{\mathbb{Z}}\,.$$

On the other hand, the equation $k_A\overline{k}_A = q^2$ implies

$$\frac{[\,F_\wp : \mathbb{Q}_p\,]\,(\mathrm{ord}_{\mathfrak{p}}(k_A) + \mathrm{ord}_{\overline{\mathfrak{p}}}(k_A))}{\mathrm{ord}_\wp(q)} = 2[\,F_\wp : \mathbb{Q}_p\,]$$

which completes the proof. ∎

Definition A3.2 Let $\alpha(A,\iota_K) = (-1)^{[F:\mathbb{Q}]}\prod_{\wp|p}(-1)^{a(\wp)} \in \{+1,-1\}$.

From now on, we fix embeddings $\overline{\mathbb{Q}} \to \mathbb{C}$ and $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$, and a CM–algebra $Q \subset B$ of degree 2 over K in which all primes of K above p stay prime.

Lemma A3.3 *There is a K–algebra homomorphism $\iota\colon B \to \mathrm{End}^\circ A$ and a polarization λ of A such that $\iota(B)$ and $\iota(Q)$ are invariant under the Rosati involution defined by λ.*

Proof. Since there are λ and ι such that $\iota(B)$ is invariant, and since any two positive involutions on B differ by an inner automorphism defined by a totally positive real element in B, any such involution is induced by a polarization of A. Hence it suffices to show that there is a positive involution on B fixing Q.

Since Q is a field, there is a prime in the maximal totally real subfield that does not split in Q. This implies that there is a prime in F which does not split neither in K, nor in the CM–field

$$L = \{\, q \in Q \mid q^* = \overline{q}\,\}\,.$$

Here * denotes the non-trivial automorphism of $Q|K$ and $^-$ complex conjugation on Q. Obviously, $Q = L\otimes_F K$. On the other hand, we know that there is an involution on B which induces complex conjugation on K. Clearly, this implies that B splits at all primes of K which are invariant under complex conjugation. Hence there is a quaternion algebra D over F containing L which is ramified everywhere at infinity and such that $B \cong D\otimes_F K$. In fact, these conditions can obviously be satisfied locally, and there is only a finite set of places where D has to be ramified. The prime mentioned above can then be used to make sure that the sum of the invariants is zero. We may choose this isomorphism such that it respects the embeddings of Q. But then, the involution on B induced by the main involution on D and complex conjugation on K is positive and induces complex conjugation on Q. ∎

Fix λ and ι as in the preceding lemma. The commutant of B in $\mathrm{End}^\circ A$ is either an algebra Q' of degree 2 over K, or a quaternion algebra over K. In the second case we choose a K–algebra Q' of degree 2 in this commutant which is invariant under the Rosati involution. As in §A1, we denote the non-trivial automorphisms of Q and Q' over K by * and $^\sim$, respectively.

Lemma A3.4 *There is a CM–type Φ of $R = Q \otimes_K Q'$ with the following properties:*

(i) $\Phi \cap \widetilde{\Phi}$ *is invariant under* * *;*

(ii) *A together with the given embedding $R \to \mathrm{End}^\circ A$ is isogenous to the reduction of an abelian variety of CM–type (R, Φ) over $\mathbb{C}$ defined by the embeddings of $\overline{\mathbb{Q}}$ fixed above.*

Proof. Choose a field of definition $\mathbb{F}_q \subset \mathbb{F}$ for A and all of its endomorphisms, and let $\pi_A \in Q'$ denote the Frobenius endomorphism of A over $\mathbb{F}_q$. Since ι induces an isomorphism between $B \otimes_K Q'$ and the centralizer of Q' in $\mathrm{End}^\circ A$, Tate's theorem implies that, for every prime $\mathfrak{q}$ of Q' above p,

$$\mathrm{inv}(B \otimes_K Q'_\mathfrak{q}) = \frac{[\, Q'_\mathfrak{q} : \mathbb{Q}_p \,]\, \mathrm{ord}_\mathfrak{q}(\pi_A)}{\mathrm{ord}_\mathfrak{q}(q)}\,.$$

In particular, since R splits $B \otimes_K Q'$, for every prime $\mathfrak{r}$ of R above p,

$$n(\mathfrak{r}) = \frac{[\, R_\mathfrak{r} : \mathbb{Q}_p \,]\, \mathrm{ord}_\mathfrak{r}(\pi_A)}{\mathrm{ord}_\mathfrak{r}(q)}$$

is an integer such that $n(\mathfrak{r}^*) = n(\mathfrak{r})$. Moreover, if $\mathfrak{r}^* = \mathfrak{r}$ and $\mathfrak{q} = \mathfrak{r} \cap Q'$, we have $\mathrm{inv}(B \otimes_K Q'_\mathfrak{q}) = n(\mathfrak{r})/2$, hence $n(\widetilde{\mathfrak{r}}) \equiv n(\mathfrak{r}) \pmod 2$, and if, in addition, $\widetilde{\mathfrak{r}} = \mathfrak{r}$, $n(\mathfrak{r}) \equiv 0 \pmod 2$.

We come to the construction of the CM–type now. By the formula of Shimura and Taniyama (cf. [41], Lemme 5), condition (ii) is equivalent to $\#\Phi_\mathfrak{r} = n(\mathfrak{r})$ for every prime $\mathfrak{r}$ of R above p where $\Phi_\mathfrak{r}$ denotes the intersection of Φ with $\mathrm{Hom}_{\mathbb{Q}_p}(R_\mathfrak{r}, \overline{\mathbb{Q}}_p)$ in the usual sense. For every prime $\wp$ of F above p, we fix a prime $\mathfrak{q}$ of K dividing $\wp$. Recall that according to our assumption (A2.1) $\overline{\mathfrak{q}} \neq \mathfrak{q}$. We shall construct $\Phi_\mathfrak{r}$ for all primes $\mathfrak{r}$ of R dividing $\mathfrak{q}$, and put

$$\Phi_{\overline{\mathfrak{r}}} = \mathrm{Hom}_{\mathbb{Q}_p}(R_{\overline{\mathfrak{r}}}, \overline{\mathbb{Q}}_p) \smallsetminus \overline{\Phi}_\mathfrak{r}\,.$$

It will be obvious from the construction that the union of these sets $\Phi_\mathfrak{r}$, for all $\mathfrak{r}$ above p, is a CM–type with the desired properties. We have to distinguish three cases.

If $\mathfrak{q}$ splits in Q', we choose one of the two primes $\mathfrak{r}$ of R above $\mathfrak{q}$ such that $n(\widetilde{\mathfrak{r}}) \leqslant n(\mathfrak{r})$. In this case, $\mathfrak{r}^* = \mathfrak{r} \neq \widetilde{\mathfrak{r}} = \widetilde{\mathfrak{r}}^*$, and we choose a set

$$\Phi_{1,\mathfrak{r}} \subset \mathrm{Hom}_{\mathbb{Q}_p}(R_\mathfrak{r}, \overline{\mathbb{Q}}_p)$$

which is invariant under * and has cardinality $n(\mathfrak{r}) - n(\widetilde{\mathfrak{r}})$. Since this number is even, such a subset, obviously, exists. Then we choose an arbitrary subset

$$\Phi_{2,\mathfrak{r}} \subset \mathrm{Hom}_{\mathbb{Q}_p}(R_\mathfrak{r}, \overline{\mathbb{Q}}_p) \smallsetminus \Phi_{1,\mathfrak{r}}$$

with $n(\widetilde{\mathfrak{r}})$ elements and put $\Phi_{\mathfrak{r}} = \Phi_{1,\mathfrak{r}} \cup \Phi_{2,\mathfrak{r}}$ and $\Phi_{\tilde{\mathfrak{r}}} = \widetilde{\Phi}^*_{2,\mathfrak{r}}$.

If $\mathfrak{q}$ is prime in Q', we choose any of the two primes $\mathfrak{r}$ of R above $\mathfrak{q}$. Then $\mathfrak{r}^* = \widetilde{\mathfrak{r}} \neq \mathfrak{r} = \widetilde{\mathfrak{r}}^*$. We take an arbitrary subset $\Phi_{\mathfrak{r}} \subset \mathrm{Hom}_{\mathbb{Q}_p}(R_{\mathfrak{r}}, \overline{\mathbb{Q}}_p)$ with $n(\mathfrak{r})$ elements and put $\Phi_{\mathfrak{r}^*} = {\Phi_{\mathfrak{r}}}^*$.

If $\mathfrak{q}$ is ramified in Q', there is only one prime $\mathfrak{r}$ of R above $\mathfrak{q}$, and $n(\mathfrak{r})$ is even. Hence we can choose a subset $\Phi_{\mathfrak{r}} \subset \mathrm{Hom}_{\mathbb{Q}_p}(R_{\mathfrak{r}}, \overline{\mathbb{Q}}_p)$ with $n(\mathfrak{r})$ elements which is invariant under *. ∎

We fix a CM–type Φ of R as in Lemma A3.4. Using this CM–type we can relate the considerations of § A1 and § A2. Note that (i) in Lemma A3.4 is equivalent to condition (A1.2), and (ii) implies that Φ satisfies condition (A1.1), hence the construction in § A1 can be applied. The assumption (A2.1) simplifies the situation considerably since it ensures that the component in $(Q \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ of the element q occuring in § A1 does not affect the invariant in Definition A1.5. Let us start with an easy lemma which gives an expression for $\alpha(A, \iota_K)$ in terms of Φ. We consider the set $(\Phi \cap \widetilde{\Phi})_F = \{ \varphi|_F \mid \varphi \in \Phi \cap \widetilde{\Phi} \}$.

Lemma A3.5 $\alpha(A, \iota_K) = (-1)^{\#(\Phi \cap \widetilde{\Phi})_F}$.

Proof. Fix a prime $\wp$ of F above p and a prime $\mathfrak{r}$ of R dividing it. Let $\Phi_{\mathfrak{r}}$ and $n(\mathfrak{r})$ have the usual meaning. The formula of Shimura and Taniyama gives an expression for the integer $a(\wp)$ in the definition of $\alpha(A, \iota_K)$. We obtain

$$a(\wp) \equiv \frac{n(\mathfrak{r}) + n(\widetilde{\mathfrak{r}})}{[\, R_{\mathfrak{r}} : F_{\wp} \,]} \pmod 2 .$$

On the other hand, if we denote the intersection of $(\Phi \cap \widetilde{\Phi})_F$ with $\mathrm{Hom}_{\mathbb{Q}_p}(F_{\wp}, \overline{\mathbb{Q}}_p)$ by $(\Phi \cap \widetilde{\Phi})_{\wp}$, the invariance of $\Phi \cap \widetilde{\Phi}$ with respect to $\mathrm{Aut}(R|K)$

$$\begin{aligned}
\#(\Phi \cap \widetilde{\Phi})_{\wp} &= \frac{\#(\Phi_{\mathfrak{r}} \cap \widetilde{\Phi}_{\tilde{\mathfrak{r}}})}{[\, R_{\mathfrak{r}} : F_{\wp} \,]} + \frac{\#(\Phi_{\tilde{\mathfrak{r}}} \cap \widetilde{\Phi}_{\tilde{\tilde{\mathfrak{r}}}})}{[\, R_{\tilde{\mathfrak{r}}} : F_{\wp} \,]} \\
&= [\, F_{\wp} : \mathbb{Q}_p \,] + \frac{\#(\Phi_{\mathfrak{r}} \cap \widetilde{\Phi}_{\tilde{\mathfrak{r}}}) - \#(\Phi_{\mathfrak{r}} \cup \widetilde{\Phi}_{\tilde{\mathfrak{r}}})}{[\, R_{\mathfrak{r}} : F_{\wp} \,]} \\
&= [\, F_{\wp} : \mathbb{Q}_p \,] + \frac{n(\mathfrak{r}) + n(\widetilde{\mathfrak{r}}) - 2\#(\Phi_{\mathfrak{r}} \cup \widetilde{\Phi}_{\tilde{\mathfrak{r}}})}{[\, R_{\mathfrak{r}} : F_{\wp} \,]} \\
&\equiv [\, F_{\wp} : \mathbb{Q}_p \,] + a(\wp) \pmod 2
\end{aligned}$$

which proves our assertion. ∎

Proposition A3.6 *For every CM–type Φ of $R = Q \otimes_K Q'$ which satisfies the conditions in Lemma A3.4,*

$$\beta(A, \iota_K, B\,|\,K) = \alpha(A, \iota_K)\, \varrho(Q\,|\,K, Q'\,|\,K, \Phi) .$$

Proof. Since we are free to change A within its isogeny class, we may assume that A is a reduction of an abelian variety A_0 of CM–type (R,Φ) over $\mathbb{C}$ with respect to the fixed embeddings of $\overline{\mathbb{Q}}$. Moreover, since the Rosati involution defined by λ induces complex conjugation on R, we can lift λ to a polarization λ_0 of A_0. We fix an isomorphism $H \cong R$ of R–modules where H denotes the first homology group of A_0 as usual. This defines an element $r_\lambda \in R^\times$ such that the Riemann form associated to λ_0 corresponds to the map

$$(r_1, r_2) \longmapsto \mathrm{Tr}_{R|\mathbb{Q}}(r_\lambda r_1 \overline{r}_2)\,.$$

Moreover, since $\Phi \cap \widetilde{\Phi}$ is invariant under * and $\varphi(r_\lambda) \in \mathbb{R}_+ i$ for all $\varphi \in \Phi$, $r_\lambda \widetilde{r}_\lambda$ is contained in $F^\times R_+^\times$, and Lemma A3.5 implies that its image in

$$R_+^\times (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times / R_+^\times \,\mathrm{Nm}_{K|F}(K \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times \cong \{+1,-1\}$$

equals $\alpha(A, \iota_K)$. We fix an isomorphism

$$\eta\colon B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \longrightarrow \widehat{\mathrm{V}}^p(A)$$

of $B \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–modules. Via the canonical isomorphisms $\widehat{\mathrm{V}}^p(A) \cong \widehat{\mathrm{V}}^p(A_0) \cong H \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ this defines an isomorphism

$$\mu\colon B \otimes_{\mathbb{Q}} \mathbb{A}_f^p \longrightarrow R \otimes_{\mathbb{Q}} \mathbb{A}_f^p$$

of $Q \otimes_{\mathbb{Q}} \mathbb{A}_f^p$–modules. Moreover, we can choose the isomorphism $\tau\colon \mathbb{A}_f^p(1) \to \mathbb{A}_f^p$ over $\mathrm{Spec}(\mathbb{F})$ such that, under the canonical isomorphisms above, the Riemann form of λ_0 on H induces the form τE_λ on $\widehat{\mathrm{V}}^p(A)$ where E_λ denotes the alternating form defined by λ.

Consider now any element $\Pi \in B^\times$ such that $\Pi q = q^*\Pi$ for all $q \in Q$. Obviously, $\Pi^* = -\Pi$ where * denotes the main involution on B, and $\Pi^2 \in K$. Since the positive involution ${}'$ on B induces complex conjugation on Q, $q_0 = \Pi'\Pi$ is contained in $Q_+^\times$. Let $r_\Pi \in (R \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ be defined by the condition

$$\mu(\Pi b) = r_\Pi \mu(b)^* \quad \text{for all } b \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p\,.$$

The identity $E_\lambda \circ (\mathrm{id} \times \Pi) = E_\lambda \circ (\Pi' \times \mathrm{id})$ can be expressed as

$$\mathrm{Tr}_{R|\mathbb{Q}}(r_\lambda r_1 \overline{r}_\Pi \overline{r}_2^*) = \mathrm{Tr}_{R|\mathbb{Q}}(r_\lambda q_0 (r_\Pi^{-1} r_1)^* \overline{r}_2) \qquad \text{for all } r_1, r_2 \in R \otimes_{\mathbb{Q}} \mathbb{A}_f^p$$

which in turn is equivalent to the equation

$$r_\lambda r_\Pi \overline{r}_\Pi = (r_\lambda q_0)^*\,.$$

Let, finally, $b \in (B \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ be determined by the condition

$$\mathrm{Tr}_{R|\mathbb{Q}}(r_\lambda \mu(b_1)\overline{\mu(b_2)}) = \mathrm{Tr}_{B|\mathbb{Q}}(b_1 b b_2') \qquad \text{for all } b_1, b_2 \in B \otimes_{\mathbb{Q}} \mathbb{A}_f^p\,,$$

and $r = \mu(1) \in R \otimes_{\mathbb{Q}} \mathbb{A}_f^p$. It is easy to see that $b = x + \Pi y$ where

$$x = \mathrm{Tr}_{R|Q}(r_\lambda r \overline{r})\,;$$
$$y = \mathrm{Tr}_{R|Q}(r_\lambda (r_\Pi^{-1} r)^* \overline{r})\,.$$

By definition, $\beta(A, \iota_K, B|K)$ is the image of $\mathrm{Nm}_{B|K}(b) = xx^* - \Pi^2 yy^* \in (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ in

$$R_+^\times (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times / R_+^\times \mathrm{Nm}_{K|F}(K \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times \cong \{+1, -1\}\,,$$

and a straightforward computation yields

$$q_0^* \mathrm{Nm}_{B|K}(b) = r_\lambda \widetilde{r}_\lambda c\overline{c} \qquad \text{where } c = r_\Pi \widetilde{r} r^* - \widetilde{r}_\Pi r \widetilde{r}^* \in R \otimes_{\mathbb{Q}} \mathbb{A}_f^p\,.$$

In particular, since q_0 and b are invertible, c has to be invertible, too. Moreover, $\widetilde{c} = -c$ and $r_\Pi^* c + \widetilde{r}_\Pi c^* = 0$. Now we choose an element $s \in S^\times$, where

$$S = \{\, s \in R \mid \widetilde{s} = s^* \,\}\,,$$

such that $\mathrm{Tr}_{S|K}(s) = 0$ and put $q = sc \in (Q \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$ and $r_0 = q_0^* s\overline{s} \in R_+^\times$. Then $r_\Pi \in q(S \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times$, and

$$r_0 \,\mathrm{Nm}_{B|K}(b) = r_\lambda \widetilde{r}_\lambda q\overline{q}\,.$$

Since the image of $q\overline{q}$ in $R_+^\times (F \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times / R_+^\times \mathrm{Nm}_{K|F}(K \otimes_{\mathbb{Q}} \mathbb{A}_f^p)^\times \cong \{+1, -1\}$ equals $\varrho(Q|K, Q'|K, \Phi)$, this equation proves our claim. ∎

Remark A3.7 An affirmative answer to Question A1.6 combined with the previous proposition would imply that the condition on $\beta(A, \iota_K, B|K)$ in Proposition A2.7 could be replaced by the much easier condition $\alpha(A, \iota_K) = \beta(\Psi)$, i.e.

$$\sum_{\wp | p} a(\wp) \equiv \#\mathcal{Z} \pmod 2$$

where $\mathcal{Z}$ denotes the set of infinite places of F at which D splits.

Appendix B

§ B1. Construction of algebraic groupoids

Langlands and Rapoport [24] gave a conjectural description for the set of points of a suitable reduction of a general Shimura variety which applies not only to the reduction at good places but also at certain bad places. Their conjecture is expressed in terms of *admissible homomorphisms* from a certain Galois gerb which they call the *quasi-motivic gerb* into the neutral Galois gerb corresponding to the reductive group which is part of the data defining the Shimura variety. The notion of an admissible homomorphism depends on a functor which assigns to each pair (T,μ) consisting of a $\mathbb{Q}$–torus T and a cocharacter $\mu \in X_*(T)$ a homomorphism from the quasi-motivic gerb into the neutral Galois gerb defined by T. Unfortunately, there is a mistake in the construction of this functor on page 144 (loc. cit.). The element $\lambda(\gamma)$ which in order for the construction to work has to be a so called *quasi Weil number* in general fails to satisfy the defining property at infinity (loc. cit. Definition 3.3(a)). There seems to be no way to overcome this difficulty without modifying the definition of the quasi-motivic gerb. Moreover, Milne has suggested to state the conjecture in terms of groupoids rather than Galois gerbs (cf. [29]). Following this suggestion, we give in § B2 a definition for a *quasi-motivic groupoid* which is used in § B3 to restate the conjecture of Langlands and Rapoport. The present section contains some technical preparations.

For a survey of groupoids in schemes we refer to [29] (Appendix A). The groupoids that we have to consider are projective limits of affine algebraic $k\,|\,k_0$–groupoids $\mathfrak{G}$ which admit a special section, i.e. a section u over $S \times_{S_0} S$ such that the induced map

$$\mathrm{ad}(u)\colon\ \mathrm{pr}_2^*(\mathfrak{G}^\Delta) \longrightarrow \mathrm{pr}_1^*(\mathfrak{G}^\Delta)$$

satisfies the obvious cocycle condition. Here $S_0 = \mathrm{Spec}(k_0)$ and $S = \mathrm{Spec}(k)$ where $k\,|\,k_0$ is a normal extension of fields of characteristic 0, and $\mathfrak{G}^\Delta$ denotes the kernel of the groupoid, i.e. the affine group scheme over S which is the pull back of $\mathfrak{G}$ with respect to the diagonal embedding.

Let Γ denote the Galois group of the extension $k\,|\,k_0$, and consider the category of all pairs (G,g) where G is an affine group scheme of finite type over S_0, Z_G denotes its centre, and $g = (\,g_{\varrho,\tau} \mid \varrho,\tau \in \Gamma\,)$ is a continuous 2-cocycle with values in $Z_G(k)$ such that $g_{1,1} = 1$. Note that this implies $g_{\varrho,1} = g_{1,\varrho} = 1$ for all $\varrho \in \Gamma$. g will be considered as the element of $Z_G(S \times_{S_0} S \times_{S_0} S)$ whose composite with $(\mathrm{id},\varrho,\varrho\tau)\colon S \to S \times_{S_0} S \times_{S_0} S$, for any $\varrho,\tau \in \Gamma$, is $g_{\varrho,\tau}$. A Morphism $(G',g') \to (G,g)$

in this category is a pair (φ_0, f) where $\varphi_0\colon G'_S \to G_S$ is a homomorphism of group schemes over S and $f = (\, f_\varrho \mid \varrho \in \Gamma\,)$ is a continuous 1-cochain with values in $G(k)$ such that, for all $\varrho, \tau \in \Gamma$,

$$\begin{aligned} &g_{\varrho,\tau} f_\varrho \varrho(f_\tau) f_{\varrho\tau}^{-1} = \varphi_0(g'_{\varrho,\tau})\,; \\ &\mathrm{ad}(f_\varrho) \circ \varrho^*(\varphi_0) = \varphi_0\,. \end{aligned}$$

In particular,

$$\varphi_0(\varrho(x)) = f_\varrho \varrho(\varphi_0(x)) f_\varrho^{-1} \in G(k) \quad \text{for all } x \in G'(k),\ \varrho \in \Gamma\,.$$

Again, f will be considered as an element of $G(S \times_{S_0} S)$. Composition of morphisms is given by the rule

$$(\varphi_0, f) \circ (\varphi'_0, f') = (\varphi_0 \circ \varphi'_0, f'') \quad \text{where } f''_\varrho = \varphi_0(f'_\varrho) f_\varrho \quad \text{for all } \varrho \in \Gamma\,.$$

Every object (G, g) defines a groupoid $\mathfrak{G}$ whose underlying $S \times_{S_0} S$–scheme is $S \times_{S_0} G \times_{S_0} S$. The law of composition is the map

$$\mathfrak{G} \times_S \mathfrak{G} = S \times_{S_0} G \times_{S_0} S \times_{S_0} G \times_{S_0} S \longrightarrow S \times_{S_0} G \times_{S_0} Z_G \times_{S_0} G \times_{S_0} S \longrightarrow \mathfrak{G}$$

where the first arrow is obtained from $g\colon S \times_{S_0} S \times_{S_0} S \to Z_G$ and the obvious projection maps, and the second arrow is induced by multiplication in G. The unit section of G defines a section of this groupoid. If we apply this construction in the case where all $g_{\varrho,\tau} = 1$, we get the neutral groupoid $\mathfrak{G}_G$ together with its natural section. A morphism (φ_0, f) as above induces a morphism of groupoids

$$\mathfrak{G}' \longrightarrow G'_S \times_{S_0} G \times_{S_0} S \longrightarrow G_S \times_{S_0} G \times_{S_0} S \longrightarrow \mathfrak{G}$$

where the first arrow is obtained from $f\colon S \times_{S_0} S \to G$ and the obvious projection maps, the second arrow is induced by φ_0, and the third arrow comes from multiplication in G. This morphism is an isomorphism if and only if φ_0 is an isomorphism, and it respects the sections if and only if, for all $\varrho \in \Gamma$, $f_\varrho = 1$. This construction defines a functor into the category of affine algebraic $k\,|\,k_0$–groupoids which admit a special section. It is easy to see that it is, in fact, an equivalence of categories.

If φ^i $(i = 1, 2)$ are two morphisms from $\mathfrak{G}'$ into $\mathfrak{G}$ corresponding to (φ_0^i, f^i), $\mathrm{Isom}(\varphi^1, \varphi^2)$ is an affine S_0–scheme. In particular, $\mathrm{Aut}(\varphi^1)$ is a group scheme over S_0 such that $\mathrm{Aut}(\varphi^1)(k)$ can be identified with the centralizer of φ_0^1 in $G(k)$ on which $\varrho \in \Gamma$ acts as $x \mapsto f_\varrho^1 \varrho(x) (f_\varrho^1)^{-1}$, and $\mathrm{Isom}(\varphi^1, \varphi^2)$ is either empty or a right $\mathrm{Aut}(\varphi^1)$–torsor. For every S_0–scheme T_0 and $\pi\colon T = T_0 \times_{S_0} S \to S$,

$$\mathrm{Isom}(\varphi^1, \varphi^2)(T_0) = \left\{ x \in G(T) \;\middle|\; \begin{array}{l} \mathrm{ad}(x) \circ \pi^*(\varphi_0^1) = \pi^*(\varphi_0^2)\colon G'_T \longrightarrow G_T \\ x\pi^*(f_\varrho^1)\varrho(x)^{-1} = \pi^*(f_\varrho^2) \text{ for all } \varrho \in \Gamma \end{array} \right\}.$$

In particular, φ^1 and φ^2 are equivalent (or isomorphic) if and only if, for $T_0 = S_0$, this set is not empty.

As mentioned above, we will consider projective limits of groupoids of the type just described which are, in general, not algebraic. For this it will be convenient to have a weaker notion of equivalence of morphisms. Note that in [29] Milne does not prove his Theorem 3.28. Our proof of Theorem B2.8 below indicates how this can be done if in the uniqueness assertion this modified equivalence of morphisms is used.

Definition B1.1 Two morphisms φ^1 and $\varphi^2 \colon \mathfrak{G}' \to \mathfrak{G}$ of affine $k\,|\,k_0$–groupoids are called algebraically equivalent if, for every morphism $\psi \colon \mathfrak{G} \to \mathfrak{H}$ into an affine algebraic $k\,|\,k_0$–groupoid, $\psi \circ \varphi^1$ and $\psi \circ \varphi^2$ are equivalent.

Remark B1.2 It is not hard to see that the notions of equivalence and algebraic equivalence are really different. As an example, we construct an automorphism of a neutral $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoid $\mathfrak{G}_R$ which is algebraically equivalent but not equivalent to the identity. To this end, we fix a rational prime p and, for every positive integer n, a number field $L_n \subset \overline{\mathbb{Q}}$ such that $L_n \subset L_{n+1}$ and $\bigcup_n L_n = \overline{\mathbb{Q}}$. This determines a sequence of positive integers d_n such that $\operatorname{ord}_p(\operatorname{Nm}_{L_n|\mathbb{Q}}(L_n^\times)) = d_n\mathbb{Z}$. Obviously, $\varprojlim_n \mathbb{Z}/d_n\mathbb{Z} = \widehat{\mathbb{Z}}$. We choose integers $c_n \in d_n\mathbb{Z}$ such that

$$c = \sum_n c_n \in \widehat{\mathbb{Z}} \smallsetminus \mathbb{Z}$$

and, for all $n \geqslant m > 0$, elements $x_{n,m} \in L_n^\times(\overline{\mathbb{Q}})$ such that $x_{n,n} \in L_n^\times$ and

$$\begin{aligned} \operatorname{Nm}_{L_{n+1}|L_n}(x_{n+1,m}) &= x_{n,m}\,; \\ \operatorname{ord}_p(\operatorname{Nm}_{L_n|\mathbb{Q}}(x_{n,n})) &= c_n\,. \end{aligned}$$

We define continuous 1-cochains $f_n = (\, f_{n,\varrho} \mid \varrho \in \operatorname{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\,)$ with values in $L_n^\times(\overline{\mathbb{Q}})$ such that

$$f_{n,\varrho} = \varrho(x_n)x_n^{-1} \quad \text{with} \quad x_n = \prod_{m=0}^{n-1} x_{n,m}\,.$$

Let R be the projective limit of the $\mathbb{Q}$–tori $L_n^\times$ with respect to the norm maps. The pairs $(\mathrm{id}_{L_n^\times}, f_n)$ correspond to a projective system of automorphisms of the neutral groupoids $\mathfrak{G}_{L_n^\times}$ which are all equivalent to the identity. The limit of these morphisms is an automorphism of $\mathfrak{G}_R$ which is not equivalent to the identity since, otherwise, we would find elements $y_n \in L_n^\times(\overline{\mathbb{Q}})$ such that, for all $n > 0$ and all $\varrho \in \operatorname{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$,

$$\begin{aligned} y_n &= \operatorname{Nm}_{L_{n+1}|L_n}(y_{n+1})\,; \\ f_{n,\varrho} &= \varrho(y_n)y_n^{-1}\,. \end{aligned}$$

But then $z_n = x_n^{-1} y_n \in L_n^\times$ and $z_n = \mathrm{Nm}_{L_{n+1}|L_n}(z_{n+1}) x_{n,n}$ which gives a contradiction since it implies

$$\mathrm{ord}_p(\mathrm{Nm}_{L_n|\mathbb{Q}}(z_n)) = c_n + \mathrm{ord}_p(\mathrm{Nm}_{L_{n+1}|\mathbb{Q}}(z_{n+1}))\,,$$

and, in particular, $c = \mathrm{ord}_p(\mathrm{Nm}_{L_1|\mathbb{Q}}(z_1)) \in \mathbb{Z}$.

As an application of the description given above, we define the local groupoids needed in the sequel. To this end, we fix a rational prime p. Let $\mathfrak{G}_\infty$ be the $\mathbb{C}|\mathbb{R}$–groupoid corresponding to $(\mathbb{G}_{m,\mathbb{R}}\,,\, g_{\iota,\iota} = -1)$ where ι denotes complex conjugation. For every prime $\ell \neq p$, $\mathfrak{G}_\ell$ is the trivial $\overline{\mathbb{Q}}_\ell|\mathbb{Q}_\ell$–groupoid, and $\mathfrak{G}_p$ is the $\overline{\mathbb{Q}}_p|\mathbb{Q}_p$–groupoid obtained as pull-back of the Dieudonné groupoid $\mathfrak{D}$ which is defined as the projective limit of affine $\mathbb{Q}_p^{\mathrm{nr}}|\mathbb{Q}_p$–groupoids $\mathfrak{D}_n$ with respect to morphisms $\delta_{n,n'}$ described below. Let $\sigma \in \Gamma = \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}|\mathbb{Q}_p)$ denote the Frobenius automorphism, and, for any rational number x, let $[x]$ denote the greatest integer such that $[x] \leqslant x$. For every positive integer n, there is a unique continuous map $\kappa_n \colon \Gamma \times \Gamma \to \mathbb{Z}$ such that, for all integers i, j,

$$\kappa_n(\sigma^i, \sigma^j) = \left[\frac{i}{n}\right] + \left[\frac{j}{n}\right] - \left[\frac{i+j}{n}\right].$$

The groupoid $\mathfrak{D}_n$ corresponds to $(\mathbb{G}_{m,\mathbb{Q}_p}\,,\, g_{\varrho,\tau} = p^{\kappa_n(\varrho,\tau)})$. Moreover, if n' is another positive integer which is divisible by n, there is a unique continuous map $\lambda_{n,n'} \colon \Gamma \to \mathbb{Z}$ such that, for all integers i,

$$\lambda_{n,n'}(\sigma^i) = \left[\frac{i}{n'}\right]\frac{n'}{n} - \left[\frac{i}{n}\right].$$

The morphism $\delta_{n,n'} \colon \mathfrak{D}_{n'} \to \mathfrak{D}_n$ is induced by $(x \mapsto x^{n'/n}\,,\, f_\varrho = p^{\lambda_{n,n'}(\varrho)})$. It is obvious that these morphisms define a projective system. If we consider $\mathbb{Q}_p^{\mathrm{nr}}$ as a $\mathbb{Q}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}}$–algebra via the map $x \otimes y \mapsto x\sigma^i(y)$ for some integer i, there is a unique element $d_{\sigma^i} \in \mathfrak{D}(\mathbb{Q}_p^{\mathrm{nr}})$ which, for every n, maps to $p^{-[i/n]} \in \mathbb{G}_m(\mathbb{Q}_p^{\mathrm{nr}}) = \mathfrak{D}_n(\mathbb{Q}_p^{\mathrm{nr}})$. Note that, if we replace σ^i with an element of Γ which is not an integral power of σ, $\mathfrak{D}(\mathbb{Q}_p^{\mathrm{nr}})$ becomes empty, hence $\mathfrak{D}$ does not have a section over $\mathrm{Spec}(\mathbb{Q}_p^{\mathrm{nr}} \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}})$.

Remark B1.3 For any affine group scheme G of finite type over $\mathbb{Q}_p$, there is a canonical map

$$\mathrm{cls}\colon\ \mathrm{Hom}_{\overline{\mathbb{Q}}_p|\mathbb{Q}_p}(\mathfrak{G}_p\,,\, \mathfrak{G}_G) \longrightarrow \mathbb{B}(G)$$

where $\mathbb{B}(G)$ denotes the set of σ–conjugacy classes of elements in $G(\mathcal{K})$. It can be described as follows. If $\theta\colon \mathfrak{G}_p \to \mathfrak{G}_G$ is a morphism of $\overline{\mathbb{Q}}_p|\mathbb{Q}_p$–groupoids, there is a morphism $\theta^{\mathrm{nr}}\colon \mathfrak{D} \to \mathfrak{G}_G$ of $\mathbb{Q}_p^{\mathrm{nr}}|\mathbb{Q}_p$–groupoids whose pull-back to $\overline{\mathbb{Q}}_p|\mathbb{Q}_p$ is equivalent to θ and which is uniquely determined up to equivalence (cf. [29], Proposition 3.35). The image of $\theta^{\mathrm{nr}}(d_\sigma) \in G(\mathbb{Q}_p^{\mathrm{nr}})$ in $\mathbb{B}(G)$ is, therefore, uniquely determined by θ. It is denoted by $\mathrm{cls}(\theta)$.

We finish this section with a classification of all equivalence classes of morphisms from $\mathfrak{G}_p$ into the neutral $\overline{\mathbb{Q}}_p|\mathbb{Q}_p$–groupoid $\mathfrak{G}_{B^\times}$ where B denotes a finite-dimensional simple $\mathbb{Q}_p$–algebra with centre K. We will use the following lemma.

Lemma B1.4 *Let B be a finite-dimensional simple algebra over a field k_0 with centre $K \supseteq k_0$, and let A be a finite-dimensional semi-simple algebra over k_0 with centre $Z \supseteq K$ such that the dimensions of any maximal semi-simple commutative subalgebras of A and B over k_0 are equal. Then, for every normal separable extension $k\,|\,k_0$ which splits $A \otimes_K B^{\mathrm{opp}}$, there is an embedding of $K \otimes_{k_0} k$–algebras*

$$j\colon A \otimes_{k_0} k \longrightarrow B \otimes_{k_0} k$$

which is unique up to conjugation with an element of $(B \otimes_{k_0} k)^\times$. For every such embedding and every continuous 2–cocycle $(\, z_{\varrho,\tau} \mid \varrho, \tau \in \mathrm{Gal}(k\,|\,k_0)\,)$ with values in $(Z \otimes_{k_0} k)^\times$ which represents the class of $A \otimes_K B^{\mathrm{opp}}$ in $\mathrm{H}^2(k\,|\,k_0\,, (Z \otimes_{k_0} k)^\times)$, there is a continuous 1–cochain $(\, b_\varrho \mid \varrho \in \mathrm{Gal}(k\,|\,k_0)\,)$ with values in $(B \otimes_{k_0} k)^\times$ such that

$$\begin{aligned} j(\varrho(a)) &= b_\varrho \varrho(j(a)) b_\varrho^{-1} && \textit{for all } a \in A \otimes_{k_0} k\,,\ \varrho \in \mathrm{Gal}(k\,|\,k_0)\,; \\ j(z_{\varrho,\tau}) &= b_\varrho \varrho(b_\tau) b_{\varrho\tau}^{-1} && \textit{for all } \varrho, \tau \in \mathrm{Gal}(k\,|\,k_0)\,. \end{aligned}$$

Proof. Let $Z_1, \ldots, Z_r$ denote the simple factors of Z. The assumption concerning $k\,|\,k_0$ implies that there are positive integers n_i such that

$$A \otimes_K B^{\mathrm{opp}} \otimes_{k_0} k \cong \prod_{i=1}^{r} \mathrm{M}_{n_i}(Z_i \otimes_{k_0} k)\,,$$

and the assumption about the commutative subalgebras of A and B implies that the dimension of $V_0 = \prod_{i=1}^r Z_i^{n_i}$ over k_0 is the same as that of B. Via the above isomorphism, $V_0 \otimes_{k_0} k$ has the structure of an $A \otimes_K B^{\mathrm{opp}} \otimes_{k_0} k$–left module. Thus it may be considered as a $B \otimes_{k_0} k$–right module with

$$A \otimes_{k_0} k \subseteq \mathrm{End}_{B \otimes_{k_0} k}(V_0 \otimes_{k_0} k) \cong B \otimes_{k_0} k\,.$$

This proves the first part of the lemma. The second part follows immediately from the comparison of the classical and the cohomological definitions of the Brauer group (cf. [37], Chap. X, §5). ■

Definition B1.5 $\mathcal{H}(B)$ denotes the set of maps $\gamma\colon \mathbb{Q} \to \mathbb{Z}$ such that

(i) for all $u \in \mathbb{Q}$, $\gamma(u) \geqslant 0$ and $\gamma(u)\big(\mathrm{inv}(B) + [\,K : \mathbb{Q}_p\,]\,u\big) \in \mathbb{Z}\,$;

(ii) $\Big(\sum_{u \in \mathbb{Q}} \gamma(u)\Big)^2 = [\,B : K\,]$.

To any fixed $\gamma \in \mathcal{H}(B)$, we associate a semi-simple K–algebra

$$A = \prod_{\substack{u\in\mathbb{Q} \\ \gamma(u)>0}} A_u \tag{B1.6}$$

where A_u denotes the simple algebra with centre K determined by

$$[\, A_u : K\,] = \gamma(u)^2\,; \tag{B1.7}$$

$$\operatorname{inv}(A_u) = \operatorname{inv}(B) + [\, K : \mathbb{Q}_p\,]\, u\,. \tag{B1.8}$$

We fix an integer $n > 1$ such that $\{\, u \in \mathbb{Q} \mid \gamma(u) > 0 \,\} \subset n^{-1}\mathbb{Z}$. In particular, this implies that the unramified extension $\mathcal{K}_n|\mathbb{Q}_p$ of degree n in $\mathbb{Q}_p^{\mathrm{nr}}$ splits $A \otimes_K B^{\mathrm{opp}}$. Let Z denote the centre of A, and consider the homomorphism of $\mathbb{Q}_p$–tori

$$\begin{aligned} \theta_1^n\colon \mathbb{G}_{m,\mathbb{Q}_p} &\longrightarrow Z^\times = \prod_{\substack{u\in\mathbb{Q} \\ \gamma(u)>0}} K^\times \\ x &\longmapsto (\, x^{nu} \mid u \in \mathbb{Q}, \gamma(u) > 0\,)\,. \end{aligned} \tag{B1.9}$$

θ_1^n maps the cohomology class of $\mathfrak{D}_n$ to the class of $A \otimes_K B^{\mathrm{opp}}$ in $\mathrm{H}^2(\Gamma\,, Z^\times)$. Using condition (ii) in Definition B1.5 and Lemma B1.4, we see that there is an embedding of $K \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}}$–algebras

$$j\colon A \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}} \longrightarrow B \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}}$$

which commutes with σ^n and a continuous 1–cochain $b = (\, b_\varrho \mid \varrho \in \Gamma\,)$ with values in $B^\times(\mathcal{K}_n)$ such that $b_{\sigma^n} = 1$ and

$$j(\varrho(a)) = b_\varrho \varrho(j(a)) b_\varrho^{-1} \qquad \text{for all } a \in A \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}},\ \varrho \in \Gamma\,; \tag{B1.10}$$

$$b_\varrho \varrho(b_\tau) b_{\varrho\tau}^{-1} = j \circ \theta_1^n(p^{\kappa_n(\varrho,\tau)}) \quad \text{for all } \varrho, \tau \in \Gamma\,. \tag{B1.11}$$

Let

$$\theta_0^n\colon \mathbb{G}_{m,\mathbb{Q}_p^{\mathrm{nr}}} \longrightarrow (Z \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}})^\times \subset (A \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}})^\times \longrightarrow (B \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}})^\times$$

be the morphism of algebraic groups over $\mathbb{Q}_p^{\mathrm{nr}}$ induced by θ_1^n and j. The pair (θ_0^n, b) corresponds to a morphism of $\mathbb{Q}_p^{\mathrm{nr}}|\mathbb{Q}_p$–groupoids $\theta^n\colon \mathfrak{D}_n \to \mathfrak{G}_{B^\times}$ which defines a morphism of $\overline{\mathbb{Q}}_p|\mathbb{Q}_p$–groupoids $\theta\colon \mathfrak{G}_p \to \mathfrak{G}_{B^\times}$ such that $\operatorname{Aut}(\theta) = A^\times$. Let $z_\gamma = (\, p^{-nu} \mid u \in \mathbb{Q}, \gamma(u) > 0\,) \in Z^\times$.

Proposition B1.12 *The preceding construction provides a bijection between $\mathcal{H}(B)$ and the set of equivalence classes of morphisms $\theta\colon \mathfrak{G}_p \to \mathfrak{G}_{B^\times}$. If θ corresponds to $\gamma \in \mathcal{H}(B)$, L is any maximal semi-simple commutative subalgebra of the K–algebra A associated to γ, and*

$$j_0\colon L \otimes_{\mathbb{Q}_p} \mathcal{K} \longrightarrow B \otimes_{\mathbb{Q}_p} \mathcal{K}$$

is an embedding of $K \otimes_{\mathbb{Q}_p} \mathcal{K}$–algebras which commutes with σ^n, then $\operatorname{cls}(\theta)$ *contains every element $b_0 \in B^\times(\mathcal{K})$ which satisfies*

$$j_0(\sigma(l)) = b_0\,\sigma(j_0(l))\,b_0^{-1} \quad \textit{for all } l \in L \otimes_{\mathbb{Q}_p} \mathcal{K}$$

and

$$b_0\,\sigma(b_0)\cdots\sigma^{n-1}(b_0) = j_0(z_\gamma)\,.$$

Proof. If $\theta\colon \mathfrak{G}_p \to \mathfrak{G}_{B^\times}$ is any morphism of $\overline{\mathbb{Q}}_p|\mathbb{Q}_p$–groupoids, we fix θ^{nr} as in Remark B1.3 and an $n > 1$ such that θ^{nr} factors through a morphism $\theta^n\colon \mathfrak{D}_n \to \mathfrak{G}_{B^\times}$. It corresponds to a pair (θ_0^n, b), and $\operatorname{Aut}(\theta)(\mathbb{Q}_p^{\mathrm{nr}})$ is the centralizer of θ_0^n in $B^\times(\mathbb{Q}_p^{\mathrm{nr}})$. The action of $\varrho \in \Gamma$ is given by the map $x \mapsto b_\varrho \varrho(x) b_\varrho^{-1}$. By descent, we get a semi-simple algebra A with centre Z together with an embedding

$$j\colon A \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}} \longrightarrow B \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{\mathrm{nr}}$$

which identifies $A^\times$ with $\operatorname{Aut}(\theta)$, and a homomorphism of $\mathbb{Q}_p$–tori

$$\theta_1^n\colon \mathbb{G}_{m,\mathbb{Q}_p} \longrightarrow Z^\times$$

such that θ_0^n is induced by j and θ_1^n, and such that (B1.10) and (B1.11) hold. In particular, $(\, b_\varrho \mid \varrho \in \operatorname{Gal}(\mathbb{Q}_p^{\mathrm{nr}}|\mathcal{K}_n)\,)$ is a cocycle. Since $\mathrm{H}^1(\mathbb{Q}_p^{\mathrm{nr}}|\mathcal{K}_n\,, B^\times)$ vanishes by Hilbert's Theorem 90, this implies that we may replace θ^{nr} with an equivalent morphism such that, for the resulting 1–cochain b, $b_{\sigma^n} = 1$. But then j commutes with σ^n.

Using Lemma B1.4 together with (B1.10) and (B1.11), we see that the cocycle $(\, \theta_1^n(p^{\kappa_n(\varrho,\tau)}) \mid \varrho, \tau \in \Gamma\,)$ represents the cohomology class of $A \otimes_K B^{\mathrm{opp}}$ in $\mathrm{H}^2(\Gamma\,, Z^\times)$ which, therefore, is the image of the fundamental class of $\mathcal{K}_n|\mathbb{Q}_p$ with respect to θ_1^n. Moreover, by definition of A, θ_1^n does not factor through a subtorus of $Z^\times$ corresponding to a proper sub–K–algebra of Z, i.e. Z is a product of copies of K, and we obtain a unique map γ from $\mathbb{Q}$ into the set of non-negative integers such that A is given by (B1.6) where each A_u is a simple algebra with centre K which satisfies (B1.7), and θ_1^n is defined by (B1.9). In particular, (B1.8) holds, hence γ has property (i) in Definition B1.5. Since, obviously, $A^\times$ and $B^\times$ have the same rank, γ satisfies condition (ii) in Definition B1.5, and since $\mathrm{H}^1(\Gamma\,, A^\times) = 0$, Proposition A.18 in [29] implies that the equivalence class of θ^{nr} is uniquely determined by θ_1^n. This proves the first part of the proposition.

By the Theorem of Skolem and Noether, we find an element $x \in B^\times(\mathcal{K}_n)$ with

$$j|_{L \otimes_{\mathbb{Q}_p} \mathcal{K}} = \operatorname{ad}(x) \circ j_0\,.$$

Hence, replacing j_0 with $\mathrm{ad}(x) \circ j_0$ and b_0 with $x b_0 \sigma(x)^{-1}$, we may assume that j_0 is the restriction of j. But then (B1.10) and (B1.11) imply that b_σ satisfies the assumptions on b_0, i.e. there is an element $l \in L^\times(\mathcal{K})$ such that $b_0 = j(l)\, b_\sigma$ and

$$l\sigma(l) \cdots \sigma^{n-1}(l) = 1\,.$$

Since $\mathrm{H}^1(\mathcal{K}_n \,|\, \mathbb{Q}_p\,, L^\times) = 0$ by Hilbert's Theorem 90, the restriction map

$$\mathrm{H}^1(\langle\sigma\rangle\,, L^\times(\mathcal{K})) \longrightarrow \mathrm{H}^1(\langle\sigma^n\rangle\,, L^\times(\mathcal{K}))$$

is injective. This implies that there is an $l_0 \in L^\times(\mathcal{K})$ such that $l = l_0\sigma(l_0)^{-1}$, hence $b_0 = j(l_0)\, b_\sigma\, \sigma(j(l_0))^{-1}$. Since, by definition, $\mathrm{cls}(\theta)$ is the σ–conjugacy class of b_σ, this completes the proof of the second part. ∎

§B2. The quasi-motivic groupoid

In this section we construct the quasi-motivic groupoid $\mathfrak{Q}$. For every place ℓ of $\mathbb{Q}$, we fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell$. The pull-back of a $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoid $\mathfrak{G}$ to $\overline{\mathbb{Q}}_\ell|\mathbb{Q}_\ell$ is denoted by $\mathfrak{G}(\ell)$. For every Galois extension $L|\mathbb{Q}$ in $\overline{\mathbb{Q}}$, we denote by $L(\ell)$ the intersection and by L_ℓ the composite of L and $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$. Then $\mathrm{Gal}(L_\ell|\mathbb{Q}_\ell)$ is identified with $\mathrm{Gal}(L|L(\ell))$, hence $[\,L : L(\ell)\,] = [\,L_\ell : \mathbb{Q}_\ell\,]$. Let Q^L be the $\mathbb{Q}$–torus defined by the exact sequence

$$\text{(B2.1)} \qquad 1 \longrightarrow \mathbb{G}_m \xrightarrow{\;\Delta\;} L(\infty)^\times \times L(p)^\times \longrightarrow Q^L \longrightarrow 1$$

where Δ is induced by the diagonal embedding of $\mathbb{Q}$ into $L(\infty) \times L(p)$, and let

$$\psi^L \colon Q^L \longrightarrow L^\times$$

be the homomorphism of $\mathbb{Q}$–tori induced by the natural inclusion of $L(p)^\times$ and the inverse of the natural inclusion of $L(\infty)^\times$ into $L^\times$. The embedding of $L(\ell)$ into $\mathbb{Q}_\ell$ determines a place of $L(\ell)$ and identifies the corresponding factor of $L(\ell) \otimes_\mathbb{Q} \mathbb{Q}_\ell$ with $\mathbb{Q}_\ell$. This induces a cocharacter of $L(\ell)^\times$ which is defined over $\mathbb{Q}_\ell$. For $\ell \in \{\infty, p\}$, let $\nu(\ell)^L$ denote the resulting cocharacter of Q^L and $q(\ell)^L \in \mathrm{H}^2(\mathbb{Q}_\ell\,, Q^L)$ the image of the fundamental class of $L_\ell|\mathbb{Q}_\ell$ with respect to the map induced by $\nu(\ell)^L$.

Lemma B2.2 *$(Q^L, \nu(\infty)^L, \nu(p)^L)$ is an initial object in the category of all triples (T, ν_∞, ν_p) where T is a $\mathbb{Q}$–torus which splits over L, and, for $\ell \in \{\infty, p\}$, ν_ℓ is a cocharacter of T which is defined over $\mathbb{Q}_\ell$, and such that*

$$\sum_{\ell \in \{\infty, p\}} [\,L_\ell : \mathbb{Q}_\ell\,]^{-1}\, \mathrm{Tr}_{L|\mathbb{Q}}(\nu_\ell) = 0\,.$$

Q^L satisfies the Hasse principle for H^1, i.e. the natural map

$$\mathrm{H}^1(\mathbb{Q}, Q^L) \longrightarrow \bigoplus_{\text{all } \ell} \mathrm{H}^1(\mathbb{Q}_\ell, Q^L)$$

is injective. For $\ell \in \{\infty, p\}$, $q(\ell)^L$ is contained in the kernel of the map

$$\mathrm{H}^2(\mathbb{Q}_\ell, Q^L) \longrightarrow \mathrm{H}^2(\mathbb{Q}_\ell, L^\times)$$

induced by ψ^L.

Proof. For $\ell \in \{\infty, p\}$, we may consider $\nu(\ell)^L$ as a cocharacter of $L(\ell)^\times$. For any L–split $\mathbb{Q}$–torus T and any cocharacter ν_ℓ of T which is defined over $\mathbb{Q}_\ell$, there is a unique homomorphism

$$f_\ell \colon L(\ell)^\times \longrightarrow T$$

such that $f_\ell \circ \nu(\ell)^L = \nu_\ell$. For a triple (T, ν_∞, ν_p) as above, this gives a unique homomorphism

$$f \colon L(\infty)^\times \times L(p)^\times \longrightarrow T\,,$$

and since

$$\Delta = [\,L_\infty : \mathbb{R}\,]^{-1}\,\mathrm{Tr}_{L|\mathbb{Q}}\,\nu(\infty)^L \times [\,L_p : \mathbb{Q}_p\,]^{-1}\,\mathrm{Tr}_{L|\mathbb{Q}}\,\nu(p)^L\,,$$

it satisfies $f \circ \Delta = 0$, i.e. it factors through Q^L. This proves the first claim.

The cocharacter $\psi^L \circ \nu(\ell)^L$ of $L^\times$ if $\ell = p$, or its negative if $\ell = \infty$, is induced by the inclusion $\mathbb{Q}_\ell \subseteq L_\ell \subseteq L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. The fundamental class of the extension $L_\ell|\mathbb{Q}_\ell$ is contained in the kernel of the corresponding homomorphism of the second cohomology groups which implies the last assertion. The top row of the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathbb{Q}, Q^L) \arrow[r] \arrow[d] & \mathrm{H}^2(\mathbb{Q}, \mathbb{G}_m) \arrow[d] \\
\bigoplus_{\text{all } \ell} \mathrm{H}^1(\mathbb{Q}_\ell, Q^L) \arrow[r] & \bigoplus_{\text{all } \ell} \mathrm{H}^2(\mathbb{Q}_\ell, \mathbb{G}_m)
\end{tikzcd}$$

obtained from (B2.1) is injective since the first cohomology groups $\mathrm{H}^1(\mathbb{Q}, L(\infty)^\times)$ and $\mathrm{H}^1(\mathbb{Q}, L(p)^\times)$ vanish. Therefore, the injectivity of the left column follows from the injectivity of the right column. ∎

Our construction of Q^L is inspired by the definition of the Weil number torus P^L (cf. [24], §3). To make this clear, we characterize P^L by a universal property similar to that in Lemma B2.2. For this, we may assume that L is a CM–field (loc. cit. Lemma 3.4). Let L_0 denote its maximal totally real subfield. Recall that P^L is equipped with two cocharacters ν_1^L and ν_2^L.

Lemma B2.3 *(P^L, ν_1^L, ν_2^L) is an initial object in the category of all triples (T, ν_∞, ν_p) where T is a $\mathbb{Q}$–torus which splits over L, and ν_∞ and ν_p are cocharacters of T defined over $\mathbb{Q}$ and $\mathbb{Q}_p$, respectively, and such that*

$$\mathrm{Tr}_{L|L_0}(\nu_p) + [\,L_p : \mathbb{Q}_p\,]\,\nu_\infty = 0\,.$$

Proof. Let $X(L) = X^*(P^L)$. The map

$$(T, \nu_\infty, \nu_p) \longmapsto (X^*(T), \nu_p)$$

gives an anti-equivalence between the category considered in the claim and the category of all pairs (X, ν) where X is a finitely generated free $\mathrm{Gal}(L|\mathbb{Q})$–module and $\nu\colon X \to \mathbb{Z}$ is a homomorphism which is $\mathrm{Gal}(L_p|\mathbb{Q}_p)$–invariant and such that $\mathrm{Tr}_{L|L_0}(\nu)$ is $\mathrm{Gal}(L|\mathbb{Q})$–invariant and maps X into $[\,L_p : \mathbb{Q}_p\,]\mathbb{Z}$. We have to show that $(X(L), \nu_2^L)$ is a final object in this category. The finiteness of the ideal class group of L implies that, for every $x \in X$, there are a positive integer m and a $\pi \in O_L[\,p^{-1}\,]^\times$ such that, for all $\varrho \in \mathrm{Gal}(L|\mathbb{Q})$,

$$\mathrm{ord}_p(\mathrm{Nm}_{L_p|\mathbb{Q}_p}(\varrho(\pi))) = m\nu(\varrho(x))\,.$$

Let ι denote complex conjugation on L. Then $p^{-m[\,L_p:\mathbb{Q}_p\,]^{-1}\mathrm{Tr}_{L|L_0}(\nu)(x)}\iota(\pi)\pi^{-1}$ is a Weil number attached to L and $2m$ (cf. loc. cit. 3.1). Its image $\alpha(x)$ in $X(L)$ does not depend on the choice of m and π. Obviously, this construction provides a homomorphism $\alpha\colon (X,\nu) \to (X(L), \nu_2^L)$. It is unique since there is no non-trivial $\mathrm{Gal}(L|\mathbb{Q})$–invariant submodule in the kernel of ν_2^L. ∎

If $L' \subset \overline{\mathbb{Q}}$ is another Galois extension of $\mathbb{Q}$ containing L, and ℓ is any place of $\mathbb{Q}$, $[\,L'(\ell) : L(\ell)\,][\,L'_\ell : L_\ell\,] = [\,L' : L\,]$. This implies that the maps

$$\begin{aligned} L'(\ell)^\times &\longrightarrow L(\ell)^\times \\ x &\longmapsto \mathrm{Nm}_{L'(\ell)|L(\ell)}(x)^{[L'_\ell:L_\ell]}\,, \end{aligned}$$

for $\ell \in \{\infty, p\}$, define a surjective homomorphism $\omega_{L'|L}$ of $\mathbb{Q}$–tori such that the following diagram commutes.

$$\begin{tikzcd} Q^{L'} \arrow[r, "\psi^{L'}"] \arrow[d, "\omega_{L'|L}"'] & L'^\times \arrow[d, "\mathrm{Nm}_{L'|L}"] \\ Q^L \arrow[r, "\psi^L"'] & L^\times \end{tikzcd}$$

Thus, we obtain a projective system whose limit is denoted by Q. The maps ψ^L define a homomorphism

$$\psi^\Delta\colon Q \longrightarrow R = \varprojlim_L L^\times$$

from Q into the pro-torus R whose character group is the set of all continuous maps $f\colon \operatorname{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to \mathbb{Z}$ with $\varrho(f)(\tau) = f(\varrho^{-1}\tau)$ for all $\varrho, \tau \in \operatorname{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$. Moreover,

$$(\omega_{L'|L})_*(\nu(\ell)^{L'}) = [\, L'_\ell : L_\ell\,]\,\nu(\ell)^L \in X_*(Q^L)\,. \tag{B2.4}$$

For $\ell = \infty$, this shows that there is a homomorphism $\nu(\infty)\colon \mathbb{G}_{m,\mathbb{R}} \to Q_{\mathbb{R}}$ whose composite with $Q \to Q^L$, for totally imaginary L, is $\nu(\infty)^L$. For $\ell = p$, we consider the pro-torus $\mathbb{D}$ over $\mathbb{Q}_p$ with character group $\mathbb{Q}$ which is the projective limit of $D_n = \mathbb{G}_{m,\mathbb{Q}_p}$ over all positive integers n with respect to the maps

$$\begin{aligned} D_{n'} &\longrightarrow D_n \\ x &\longmapsto x^{n'/n} \end{aligned}$$

if n' is divisible by n. The maps $\nu(p)^L\colon D_n \to Q^L_{\mathbb{Q}_p}$, for all L and $n = [\, L_p : \mathbb{Q}_p\,]$, define a homomorphism $\nu(p)\colon \mathbb{D} \to Q_{\mathbb{Q}_p}$.

Remark B2.5 We may as well describe Q together with the homomorphisms ψ^Δ, $\nu(\infty)$, and $\nu(p)$ via its character group. For $\ell \in \{\infty, p\}$, let X_ℓ denote the set of all functions on $\operatorname{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})/\operatorname{Gal}(\overline{\mathbb{Q}}_\ell|\mathbb{Q}_\ell)$ which take values in $\mathbb{Z}$ if $\ell = \infty$, or in $\mathbb{Q}$ if $\ell = p$, and which are invariant under an open subgroup of $\operatorname{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ with respect to the obvious action. If $\{f_1, \ldots, f_r\}$ is the $\operatorname{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$–orbit of an element $f \in X_\ell$, let

$$\operatorname{Tr}(f) = r^{-1} \sum_{i=1}^r f_i(\mathrm{id}) \in \mathbb{Q}.$$

Moreover, if $\{f_1, \ldots, f_r\}$ is the orbit of a continuous function $f\colon \operatorname{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to \mathbb{Z}$ with respect to the action of $\operatorname{Gal}(\overline{\mathbb{Q}}_\ell|\mathbb{Q}_\ell)$ given by $\varrho(f)(\tau) = f(\tau\varrho)$, we define

$$\operatorname{Tr}_\ell(f) = r^{-1} \sum_{i=1}^r f_i\,.$$

Then, immediately from our explicit constructions, we obtain an identification

$$X^*(Q) = \{\, (f_\infty, f_p) \in X_\infty \times X_p \mid \operatorname{Tr}(f_\infty) + 2\operatorname{Tr}(f_p) = 0\,\}$$

such that $\nu(\infty)$ and $\nu(p)$ induce the map

$$\begin{aligned} X^*(Q) &\longrightarrow \mathbb{Z} \times \mathbb{Q} \\ (f_\infty, f_p) &\longmapsto (f_\infty(\mathrm{id}), f_p(\mathrm{id}))\,, \end{aligned}$$

and ψ^Δ corresponds to the map

$$\begin{aligned} X^*(R) &\longrightarrow X^*(Q) \\ f &\longmapsto (-2\operatorname{Tr}_\infty(f), \operatorname{Tr}_p(f))\,. \end{aligned}$$

Lemma B2.6 *Let $L \subseteq L' \subset \overline{\mathbb{Q}}$ be two Galois extensions of $\mathbb{Q}$ such that $[\,L'_p : \mathbb{Q}_p\,]$ is divisible by $[\,L : \mathbb{Q}\,]$.*

(i) *For every place ℓ of $\mathbb{Q}$, $\omega_{L'|L}$ induces the zero map $\mathrm{H}^1(\mathbb{Q}_\ell\,, Q^{L'}) \to \mathrm{H}^1(\mathbb{Q}_\ell\,, Q^{L})$.*

(ii) *$\omega_{L'|L}$ induces the zero map $\mathrm{H}^1(\mathbb{Q}, Q^{L'}) \to \mathrm{H}^1(\mathbb{Q}, Q^{L})$.*

(iii) *The kernel of the natural map $\mathrm{H}^2(\mathbb{Q}, Q^{L'}) \to \bigoplus_{\text{all } \ell} \mathrm{H}^2(\mathbb{Q}_\ell\,, Q^{L'})$ is contained in the kernel of the map $\mathrm{H}^2(\mathbb{Q}, Q^{L'}) \to \mathrm{H}^2(\mathbb{Q}, Q^{L})$ induced by $\omega_{L'|L}$.*

Proof. Consider the homomorphism

$$\begin{aligned} \xi\colon L'(\infty)^\times \times L'(p)^\times &\longrightarrow \mathbb{G}_m \\ (x\,,\,y) &\longmapsto \mathrm{Nm}_{L'(p)|\mathbb{Q}}(y)^{[\,L:\mathbb{Q}\,]^{-1}[\,L'_p:\mathbb{Q}_p\,]}\,. \end{aligned}$$

The exact sequences (B2.1) for L and L', respectively, induce a commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathbb{Q}_\ell\,, Q^{L'}) \arrow[r] \arrow[d, "\omega_{L'|L}"'] & \mathrm{H}^2(\mathbb{Q}_\ell\,, \mathbb{G}_m) \arrow[r] \arrow[d, dotted] & \mathrm{H}^2(\mathbb{Q}_\ell\,, L'(\infty)^\times \times L'(p)^\times) \arrow[dl, "\xi"] \\ \mathrm{H}^1(\mathbb{Q}_\ell\,, Q^{L}) \arrow[r] & \mathrm{H}^2(\mathbb{Q}_\ell\,, \mathbb{G}_m) & \end{tikzcd}$$

whose bottom row is injective since $\mathrm{H}^1(\mathbb{Q}_\ell\,, L(\infty)^\times \times L(p)^\times) = 0$, whose top row is exact, and where the dotted arrow is multiplication by $[\,L' : L\,]$. This proves (i). (ii) follows from (i) and Lemma B2.2. Let now $C = (\mathbb{A} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times / \overline{\mathbb{Q}}^\times$ denote the idèle class group of $\overline{\mathbb{Q}}$. Since

$$\mathrm{H}^1(\mathbb{Q}, C \otimes_{\mathbb{Z}} X_*(L'(\infty)^\times \times L'(p)^\times)) = \mathrm{H}^1(L'(\infty)\,, C) \oplus \mathrm{H}^1(L'(p)\,, C) = 0\,,$$

we have again a commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathbb{Q}, C \otimes_{\mathbb{Z}} X_*(Q^{L'})) \arrow[r] \arrow[d, "\omega_{L'|L}"'] & \mathrm{H}^2(\mathbb{Q}, C) \arrow[r] \arrow[d, dotted] & \mathrm{H}^2(\mathbb{Q}, C \otimes_{\mathbb{Z}} X_*(L'(\infty)^\times \times L'(p)^\times)) \arrow[dl, "\xi"] \\ \mathrm{H}^1(\mathbb{Q}, C \otimes_{\mathbb{Z}} X_*(Q^{L})) \arrow[r] & \mathrm{H}^2(\mathbb{Q}, C) & \end{tikzcd}$$

with the same properties as the preceding one. As above, we conclude that the left vertical map is zero. But this map appears in another commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathbb{Q}, C \otimes_{\mathbb{Z}} X_*(Q^{L'})) \arrow[r] \arrow[d, "\omega_{L'|L}"'] & \mathrm{H}^2(\mathbb{Q}, Q^{L'}) \arrow[r] \arrow[d, "\omega_{L'|L}"] & \bigoplus_{\text{all } \ell} \mathrm{H}^2(\mathbb{Q}_\ell\,, Q^{L'}) \\ \mathrm{H}^1(\mathbb{Q}, C \otimes_{\mathbb{Z}} X_*(Q^{L})) \arrow[r] & \mathrm{H}^2(\mathbb{Q}, Q^{L}) & \end{tikzcd}$$

whose top row is exact. This proves (iii). ∎

Definition B2.7 A *quasi-motivic groupoid* is a system $(\mathfrak{Q}, \zeta_\ell \mid \text{all } \ell)$ consisting of an affine $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoid $\mathfrak{Q}$ which has a section over $\mathrm{Spec}(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ and morphisms $\zeta_\ell\colon \mathfrak{G}_\ell \to \mathfrak{Q}(\ell)$ for all places ℓ of $\mathbb{Q}$ such that

(i) $(\mathfrak{Q}^\Delta, \zeta_\infty^\Delta, \zeta_p^\Delta) = (Q, \nu(\infty), \nu(p))$;

(ii) the morphisms ζ_ℓ, for all $\ell \neq \infty, p$, are induced by a section of $\mathfrak{Q}$ over $\mathrm{Spec}(\overline{\mathbb{A}_f^p} \otimes_{\mathbb{A}_f^p} \overline{\mathbb{A}_f^p})$;

where $\overline{\mathbb{A}_f^p}$ denotes the image of the map $\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{A}_f^p \to \prod\limits_{\ell \neq \infty, p} \overline{\mathbb{Q}}_\ell$.

Note that an analogue of condition (ii) is missing in Milne's definition of a pseudo-motivic groupoid (cf. [29], 3.27). It should, however, be added since, otherwise, the restricted product $X^p(\varphi)$ considered on page 187 (loc. cit.) is not well defined.

Theorem B2.8 *There exists a quasi-motivic groupoid $(\mathfrak{Q}, \zeta_\ell)$. If $(\widetilde{\mathfrak{Q}}, \widetilde{\zeta}_\ell)$ is a second quasi-motivic groupoid, then there is an isomorphism $\alpha\colon \mathfrak{Q} \to \widetilde{\mathfrak{Q}}$ such that, for all places ℓ of $\mathbb{Q}$, $\widetilde{\zeta}_\ell$ is algebraically equivalent to $\alpha \circ \zeta_\ell$. If $(\mathfrak{P}, \zeta_\ell^P)$ is a pseudo-motivic groupoid, there is a surjective morphism $\pi\colon \mathfrak{Q} \to \mathfrak{P}$ such that, for all ℓ, ζ_ℓ^P is algebraically equivalent to $\pi \circ \zeta_\ell$. Moreover, there is a morphism $\psi\colon \mathfrak{Q} \to \mathfrak{G}_R$ whose restriction to the kernel is the homomorphism ψ^Δ defined above. α, π, and ψ are uniquely determined up to algebraic equivalence.*

Proof. For every Galois extension $L|\mathbb{Q}$ in $\overline{\mathbb{Q}}$, let H^L denote the set of all elements of $\mathrm{H}^2(\mathbb{Q}, Q^L)$ whose image in $\mathrm{H}^2(\mathbb{Q}_\ell, Q^L)$ is $q(\ell)^L$ for $\ell \in \{\infty, p\}$, and zero for the remaining places ℓ of $\mathbb{Q}$. As a consequence of (B2.4), we see that, for $\ell \in \{\infty, p\}$, the classes $q(\ell)^L$ form a projective system with respect to the maps induced by $\omega_{L'|L}$. This implies that the sets H^L form a projective system, too. Let H denote its limit. For every $\mathbb{Q}$–torus T which splits over L, there is a natural commutative diagram

$$\begin{tikzcd}
 & \bigoplus\limits_{\text{all } \ell} X_*(T)_\ell \arrow[r, "t^L"] \arrow[d, "\oplus u_\ell^L"] & X_*(T)_0 \arrow[d, "u_0^L"] \\
\mathrm{H}^2(\mathbb{Q}, T) \arrow[r] & \bigoplus\limits_{\text{all } \ell} \mathrm{H}^2(\mathbb{Q}_\ell, T) \arrow[r] & \mathrm{H}^2(\mathbb{Q}, C \otimes_{\mathbb{Z}} X_*(T))
\end{tikzcd}$$

whose bottom row is exact, and which can easily be derived from [40] (1_M). Here $X_*(T)_\ell$ denotes the set of cocharacters of T that are defined over $\mathbb{Q}_\ell$ if ℓ is a place of $\mathbb{Q}$, or over $\mathbb{Q}$ if $\ell = 0$. Moreover, u_ℓ^L is the map which sends a $\nu \in X_*(T)_\ell$ to the image of the fundamental class of the extension $L_\ell|\mathbb{Q}_\ell$ if ℓ is a place of $\mathbb{Q}$, or to the image of the fundamental class of $L|\mathbb{Q}$ if $\ell = 0$, respectively, under the map induced by ν, and t^L is given by the formula

$$t^L(\,\nu_\ell \mid \text{all } \ell\,) = \sum_\ell [\,L_\ell : \mathbb{Q}_\ell\,]^{-1}\,\mathrm{Tr}_{L|\mathbb{Q}}(\nu_\ell)\,.$$

If we apply this to the torus Q^L, and consider the collection (ν_ℓ) where $\nu_\ell = \nu(\ell)^L$ if $\ell \in \{\infty, p\}$, and $\nu_\ell = 0$ for the remaining places ℓ of $\mathbb{Q}$, which is contained in the kernel of t^L, it follows that H^L is not empty. Using Lemma B2.6(iii) it follows that H has precisely one element. Since the maps $\omega_{L'|L}$ are surjective, this element is represented by a projective system of continuous 2-cocycles $q^L = (\,q^L_{\varrho,\tau} \mid \varrho, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\,)$ with values in $Q^L(\overline{\mathbb{Q}})$. From the construction in §B1 we get a projective system of commutative algebraic $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoids $\mathfrak{Q}^L$. Let $\mathfrak{Q}$ denote its limit.

Similarly, using Lemma B2.6(i) and the surjectivity of the maps $\omega_{L'|L}$, we construct local morphisms ζ_ℓ satisfying the conditions in Definition B2.7. To start with, we consider an increasing sequence of Galois extensions $(\,L^i|\mathbb{Q} \mid i \geqslant 0\,)$ such that $\bigcup_{i\geqslant 0} L^i = \overline{\mathbb{Q}}$ and such that, for all $i \geqslant 0$ and all places ℓ of $\mathbb{Q}$, $\omega^{i+1} = \omega_{L^{i+1}|L^i}$ induces the zero map $\mathrm{H}^1(\mathbb{Q}_\ell\,, Q^{i+1}) \to \mathrm{H}^1(\mathbb{Q}_\ell\,, Q^i)$ where $Q^i = Q^{L^i}$. For every $i > 0$, we fix a continuous 1–cochain $f^i = (\,f^i_\varrho \in Q^i(\mathbb{A}^p_f \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \mid \varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\,)$ with coboundary q^{L^i} and, for $\ell \in \{\infty, p\}$, a continuous 1–cochain $f^i_\ell = (\,f^i_{\ell,\varrho} \in Q^i(\overline{\mathbb{Q}}_\ell) \mid \varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell|\mathbb{Q}_\ell)\,)$ such that $(\nu(\ell), f^i_\ell)$ corresponds to a morphism $\mathfrak{G}_\ell \to \mathfrak{Q}^{L^i}(\ell)$. Then, for all $i > 0$, we choose continuous 1–cocycles $c^i = (\,c^i_\varrho \in Q^i(\mathbb{A}^p_f \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \mid \varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\,)$ and put $g^{i-1} = \omega^i(c^i f^i)$. This is done inductively starting with the trivial cocycle for $i = 1$ and such that $\omega^i(g^i) = g^{i-1}$. If, for some $i > 0$, c^i is given, we consider the cocycle c which satisfies $c^i f^i = c\,\omega^{i+1}(f^{i+1})$. The choice of the fields L^i implies that $\omega^i(c)$ is a coboundary. Using the surjectivity of the morphisms ω^i we find an element in $Q^{i+1}(\mathbb{A}^p_f \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$ whose coboundary c^{i+1} satisfies $\omega^i\omega^{i+1}(c^{i+1}) = \omega^i(c)$. For every place $\ell \neq \infty, p$, the g^i induce a projective system of continuous 1–cochains $g^i_\ell = (\,g^i_{\ell,\varrho} \in Q^i(\overline{\mathbb{Q}}_\ell) \mid \varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}_\ell|\mathbb{Q}_\ell)\,)$ and the corresponding local morphisms ζ_ℓ satisfy condition (ii) in Definition B2.7. For $\ell \in \{\infty, p\}$, the same construction applied to the cochains f^i_ℓ, provides local morphisms ζ_ℓ which satisfy condition (i) in Definition B2.7. This proves the existence of a quasi-motivic groupoid.

If $\widetilde{q}^L$ is another projective system of cocycles representing the element in H, we can use Lemma B2.6(ii) and the surjectivity of the maps $\omega_{L'|L}$ to find a projective system of continuous 1-cochains $f^L = (\,f^L_\varrho \mid \varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\,)$ with values in $Q^L(\overline{\mathbb{Q}})$ such that $\widetilde{q}^L_{\varrho,\tau} f^L_\varrho\, \varrho(f^L_\tau)(f^L_{\varrho\tau})^{-1} = q^L_{\varrho,\tau}$. Together with Lemma B2.6(i) this shows that, for every other quasi-motivic groupoid, there is a morphism α as claimed. That it is unique up to algebraic equivalence follows from Lemma B2.6(ii).

The assertion concerning the neutral groupoid $\mathfrak{G}_R$ is an obvious consequence of the last statement in Lemma B2.2, because, for every L, $\mathrm{H}^2(\mathbb{Q}, L^\times)$ satisfies the Hasse principle and $\mathrm{H}^1(\mathbb{Q}, L^\times) = 0$. Since the pro-torus P has the same properties as Q concerning its first and second cohomology groups (cf. [24], 3.5, 3.10, 3.12), the existence and uniqueness of π is completely analogous to that of α as soon as the following observation has been made. For a CM–field L, the category considered in Lemma B2.3 is a full subcategory of the category considered in Lemma B2.2. Thus there is a unique homomorphism $Q^L \to P^L$ which respects the distinguished cocharacters. The universal property for P^L implies that it is surjective, and that, for varying L, these maps form a projective system. ▪

Remark B2.9 Using condition (ii) in Definition B2.7 it is not hard to see that any isomorphism $\alpha\colon \mathfrak{Q} \to \widetilde{\mathfrak{Q}}$ in Theorem B2.8 has the following property. For every morphism $\varphi\colon \widetilde{\mathfrak{Q}} \to \mathfrak{G}$ into an affine algebraic $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoid, there is an element $g \in \mathfrak{G}^{\Delta}(\overline{\mathbb{A}_f^p})$ which, for every place $\ell \neq \infty, p$ of $\mathbb{Q}$, induces an element $g_\ell \in \mathrm{Isom}(\varphi(\ell) \circ \widetilde{\zeta}_\ell, \varphi(\ell) \circ \alpha \circ \zeta_\ell)(\mathbb{Q}_\ell) \subset \mathfrak{G}^{\Delta}(\overline{\mathbb{Q}}_\ell)$. If the analogue of property (ii) is added to Milne's definition of a pseudo-motivic groupoid, a similar statement is true for the morphism π.

From now on we fix a quasi-motivic groupoid together with morphisms ψ and π as in the theorem.

Definition B2.10 For any $\mathbb{Q}$–torus T and any cocharacter μ of T, there is a unique homomorphism $R \to T$ for which μ is the image of the cocharacter of R defined by evaluating a continuous function $f\colon \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q}) \to \mathbb{Z}$ at the identity. The composite of the corresponding morphism of neutral groupoids with ψ is a morphism $\mathfrak{Q} \to \mathfrak{G}_T$ which is denoted by ψ_μ.

Remark B2.11 Let $\mathfrak{G}$ be any affine algebraic $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoid which admits a special section. Using the surjectivity of the morphism π, we see that a morphism $\varphi\colon \mathfrak{Q} \to \mathfrak{G}$ is induced by a morphism $\mathfrak{P} \to \mathfrak{G}$ if and only if its restriction to the kernel is induced by a homomorphism $P_{\overline{\mathbb{Q}}} \to \mathfrak{G}^{\Delta}$. Let T denote the unique algebraic quotient of Q such that φ^{Δ} is given by an embedding $T_{\overline{\mathbb{Q}}} \subseteq \mathfrak{G}^{\Delta}$. The characterizations of Q and P in Lemma B2.2 and Lemma B2.3, respectively, imply that φ factors through π if and only if the following conditions are satisfied:

(i) the image of $\nu(\infty)$ in $X_*(T)$ is defined over $\mathbb{Q}$;

(ii) the image of $\nu(p)$ in $X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfies the Serre condition, i.e. it is defined over a CM–field and its weight is defined over $\mathbb{Q}$.

In particular, if μ is a cocharacter of a $\mathbb{Q}$–torus T which satisfies the Serre condition, ψ_μ factors through $\mathfrak{P}$.

Remark B2.12 The existence and uniqueness of the quasi-motivic groupoid and the morphisms ψ_μ together with the universal property of the tori Q^L, obviously, imply the results of [24] Satz 2.2 and Satz 2.3 over the base field $F = \mathbb{Q}$.

Remark B2.13 In his thesis [32] Pfau gives another definition of a quasi-motivic groupoid. It is easy to see that the morphisms π and ψ^{-1} in Theorem B2.8 induce a morphism from $\mathfrak{Q}$ into this groupoid which respects the local morphisms ζ_ℓ up to algebraic equivalence. This morphism is neither injective nor surjective, but its existence implies that in [32] Pfau's groupoid can be replaced by our $\mathfrak{Q}$.

In the remainder of this section we give a down to earth description of the set of equivalence classes of morphisms from $\mathfrak{Q}$ into the neutral $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoid $\mathfrak{G}_{B^\times}$, where B denotes an arbitrary finite-dimensional simple $\mathbb{Q}$–algebra. For a finite-dimensional semi-simple commutative $\mathbb{Q}$–algebra Z and $\ell \in \{\infty, p\}$, $\mathcal{P}_\ell(Z)$ denotes the set of all places of Z above ℓ. Let K be the centre of B.

Definition B2.14 $\mathcal{H}(B)$ is the set of isomorphism classes of triples $(A, \varphi_\infty, \varphi_p)$ where A is a finite-dimensional semi-simple K–algebra with centre $Z \supseteq K$ such that the maximal semi-simple commutative subalgebras in A and B have equal dimensions over K, and $\varphi_\infty\colon \mathcal{P}_\infty(Z) \to \mathbb{Z}$ and $\varphi_p\colon \mathcal{P}_p(Z) \to \mathbb{Q}$ are maps such that

(i) there is no proper sub–K–algebra $Z_0 \subset Z$ for which φ_∞ and φ_p are induced by maps $\mathcal{P}_\infty(Z_0) \to \mathbb{Z}$ and $\mathcal{P}_p(Z_0) \to \mathbb{Q}$, respectively;

(ii) for all simple factors Z_0 of Z, $\sum\limits_{v \in \mathcal{P}_\infty(Z_0)} [\, Z_v : \mathbb{R}\,] \varphi_\infty(v) = \sum\limits_{v \in \mathcal{P}_p(Z_0)} -2 [\, Z_v : \mathbb{Q}_p\,] \varphi_p(v)$;

(iii) $$\mathrm{inv}_v(A) = \mathrm{inv}_v(B \otimes_K Z) + \begin{cases} [\, Z_v : \mathbb{R}\,] \varphi_\infty(v)/2 & \text{if } v \in \mathcal{P}_\infty(Z)\,; \\ [\, Z_v : \mathbb{Q}_p\,] \varphi_p(v) & \text{if } v \in \mathcal{P}_p(Z)\,; \\ 0 & \text{for any other place } v \text{ of } Z\,. \end{cases}$$

Fix A, φ_∞, and φ_p as in Definition B2.14. For all $\ell \in \{\infty, p\}$, $\alpha \in \mathrm{Hom}(Z, \overline{\mathbb{Q}})$, and $\varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$, there is a unique $v \in \mathcal{P}_\ell(Z)$ such that $\varrho^{-1}\alpha\colon Z \to \overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell$ factors through Z_v. Put $\varphi_{\ell,\alpha}(\varrho) = \varphi_\ell(v)$. This defines an element $\varphi_{\ell,\alpha}$ of the set X_ℓ introduced in Remark B2.5 with

$$\mathrm{Tr}\,(\varphi_{\ell,\alpha}) = [\, Z_i : \mathbb{Q}]^{-1} \sum_{v \in \mathcal{P}_\ell(Z_i)} [\, Z_v : \mathbb{Q}_\ell\,] \varphi_\ell(v)$$

where Z_i is the simple component of Z which α factors through. Condition (ii) in Definition B2.14 implies that we obtain a map

$$\begin{aligned} X^*(Z^\times) = \bigoplus_{\alpha\colon Z \to \overline{\mathbb{Q}}} \mathbb{Z} e_\alpha &\longrightarrow X^*(Q) \\ e_\alpha &\longmapsto (\varphi_{\infty,\alpha}\,,\, \varphi_{p,\alpha})\,, \end{aligned}$$

and condition (i) ensures that the induced homomorphism $\varphi_1 \colon Q \to Z^\times$ of group schemes over $\mathbb{Q}$ does not factor through a subtorus of $Z^\times$ corresponding to a proper sub–K–algebra of Z. We choose a totally imaginary Galois extension $L \subset \overline{\mathbb{Q}}$ of $\mathbb{Q}$ such that, for all $v \in \mathcal{P}_p(Z)$, $n(v) = [\, L_p : \mathbb{Q}_p \,]\varphi_p(v)$ is an integer, and put $n(v) = \varphi_\infty(v)$ if $v \in \mathcal{P}_\infty(Z)$. Then φ_1 factors through a homomorphism $\varphi_1^L \colon Q^L \to Z^\times$ such that, for $\ell \in \{\infty, p\}$, the diagram

$$\begin{array}{ccc} Q^L_{\mathbb{Q}_\ell} & \xrightarrow{\varphi_1^L} & (Z \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \\ {\scriptstyle \nu(\ell)^L}\uparrow & & \Vert \\ \mathbb{G}_{m,\mathbb{Q}_\ell} & \longrightarrow & \prod\limits_{v \in \mathcal{P}_\ell(Z)} Z_v^\times \\ x & \longmapsto & (\, x^{n(v)} \mid v \in \mathcal{P}_\ell(Z)\,) \end{array}$$

commutes. In particular, condition (iii) in Definition B2.14 implies that φ_1^L maps the cohomology class of $\mathfrak{Q}^L$ to the class of $A \otimes_K B^{\mathrm{opp}}$ in $\mathrm{H}^2(\mathbb{Q}, Z^\times)$. Via the construction in §B1, $\mathfrak{Q}^L$ corresponds to a pair (Q^L, q^L). We fix an embedding of $K \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$–algebras

$$j \colon A \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \longrightarrow B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}\,.$$

Up to conjugation with an element in $B^\times(\overline{\mathbb{Q}})$, there is precisely one such embedding. Let $\varphi_0^L \colon Q^L_{\overline{\mathbb{Q}}} \to (B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})^\times$ be the homomorphism of algebraic groups over $\overline{\mathbb{Q}}$ induced by φ_1^L and j. Using Lemma B1.4, we see that there is a continuous 1–cochain b with values in $B^\times(\overline{\mathbb{Q}})$ such that

$$\begin{aligned} j(\varrho(a)) &= b_\varrho \varrho(j(a)) b_\varrho^{-1} && \text{for all } a \in A \otimes_{\mathbb{Q}} \overline{\mathbb{Q}},\ \varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\,; \\ \varphi_0^L(q^L_{\varrho,\tau}) &= b_\varrho \varrho(b_\tau) b_{\varrho\tau}^{-1} && \text{for all } \varrho, \tau \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\,. \end{aligned}$$

But then (φ_0^L, b) corresponds to a morphism $\varphi^L \colon \mathfrak{Q}^L \to \mathfrak{G}_{B^\times}$ which defines a morphism $\varphi \colon \mathfrak{Q} \to \mathfrak{G}_{B^\times}$ such that $\mathrm{Aut}(\varphi) = A^\times$.

Proposition B2.15 *The preceding construction provides a bijection between $\mathcal{H}(B)$ and the set of equivalence classes of morphisms $\varphi \colon \mathfrak{Q} \to \mathfrak{G}_{B^\times}$.*

Proof. If $\varphi \colon \mathfrak{Q} \to \mathfrak{G}_{B^\times}$ is any morphism of $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoids, there is an L such that φ factors through $\mathfrak{Q}^L$. It corresponds to a pair (φ_0^L, b), and $\mathrm{Aut}(\varphi)(\overline{\mathbb{Q}})$ is the centralizer of φ_0^L in $B^\times(\overline{\mathbb{Q}})$. The action of $\varrho \in \mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})$ is given by the map $x \mapsto b_\varrho \varrho(x) b_\varrho^{-1}$. By descent, we get a semi-simple algebra A with centre Z together with an embedding $j \colon A \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \to B \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ which identifies $A^\times$ with $\mathrm{Aut}(\varphi)$, and a homomorphism of $\mathbb{Q}$–tori $\varphi_1^L \colon Q^L \to Z^\times$ which does not factor through a subtorus of $Z^\times$ corresponding to a proper sub–K–algebra of Z. Obviously, $A^\times$ and $B^\times$ have the same rank. Since $\mathrm{H}^1(\mathbb{Q}, A^\times) = 0$, Proposition A.18 in [29] implies that the equivalence class of φ is uniquely determined by φ_1^L. Using Remark B2.5 and Lemma B1.4, it is not hard to see that φ_1^L corresponds to a pair of maps $(\varphi_\infty, \varphi_p)$ as in Definition B2.14. ∎

Remark B2.16 The same construction may be applied with B replaced by K. In particular, for every cocharacter $\mu\colon X^*(K^\times) \to \mathbb{Z}$, we have a triple $(K, \psi_{\mu,\infty}, \psi_{\mu,p})$ which corresponds to the equivalence class of the morphism ψ_μ. A straightforward computation yields

$$\begin{aligned} \psi_{\mu,\infty}(w) &= -2[\,K_w : \mathbb{R}]^{-1}\mu(e_w) \quad \text{for all } w \in \mathcal{P}_\infty(K)\,; \\ \psi_{\mu,p}(w) &= [\,K_w : \mathbb{Q}_p\,]^{-1}\mu(e_w) \qquad \text{for all } w \in \mathcal{P}_p(K)\,; \end{aligned}$$

where, for $\ell \in \{\infty, p\}$ and every $w \in \mathcal{P}_\ell(K)$, $e_w \in X^*(K^\times)$ is the homomorphism of $\mathbb{Q}_\ell$–tori

$$(K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \longrightarrow K_w^\times \xrightarrow{\mathrm{Nm}_{K_w|\mathbb{Q}_\ell}} \mathbb{G}_{m,\mathbb{Q}_\ell}\,.$$

Moreover, the reduced norm of B over K induces a map $\varphi \mapsto \varphi^{\mathrm{ab}}$ from the set of morphisms from $\mathfrak{Q}$ into $\mathfrak{G}_{B^\times}$ to the set of morphisms from $\mathfrak{Q}$ into $\mathfrak{G}_{K^\times}$. This map, obviously, respects equivalence of morphisms, hence corresponds to a map $\mathcal{H}(B) \to \mathcal{H}(K)$. A simple computation shows that it sends a triple $(A, \varphi_\infty, \varphi_p)$ to $(K, \varphi_\infty^{\mathrm{ab}}, \varphi_p^{\mathrm{ab}})$ where, for $\ell \in \{\infty, p\}$ and $w \in \mathcal{P}_\ell(K)$,

$$\varphi_\ell^{\mathrm{ab}}(w) = \sum_{v \in \mathcal{P}_w(Z)} a_v\, \varphi_\ell(v)\,.$$

Here $\mathcal{P}_w(Z)$ denotes the set of all places v of Z above w, and, for every such v, a_v is the dimension of a maximal semi-simple commutative subalgebra of A_v over K_w.

Remark B2.17 Localization at p induces a map $\varphi \mapsto \varphi(p) \circ \zeta_p$ from the set of morphisms from $\mathfrak{Q}$ into $\mathfrak{G}_{B^\times}$ to the set of morphisms from $\mathfrak{G}_p$ into $\mathfrak{G}_{(B\otimes_{\mathbb{Q}}\mathbb{Q}_p)^\times}$ which respects equivalence of morphisms, hence corresponds to a map

$$\begin{aligned} \mathcal{H}(B) &\longrightarrow \prod_{w \in \mathcal{P}_p(K)} \mathcal{H}(B_w) \\ (A, \varphi_\infty, \varphi_p) &\longmapsto (\,\gamma_w \mid w \in \mathcal{P}_p(K)\,) \end{aligned}$$

where $\mathcal{H}(B_w)$ are the sets introduced in Definition B1.5. A simple computation shows that, for all $u \in \mathbb{Q}$,

$$\gamma_w(u) = \sum_{\substack{v \in \mathcal{P}_w(Z) \\ \varphi_p(v) = u}} a_v$$

where $\mathcal{P}_w(Z)$ and a_v are defined in Remark B2.16.

§B3. The conjecture of Langlands and Rapoport

In this section we review the conjecture of Langlands and Rapoport [24] (§5) concerning the reduction of a Shimura variety Sh_G attached to a connected reductive group G over $\mathbb{Q}$ and a $G(\mathbb{R})$–conjugacy class of homomorphisms $h\colon \mathbb{S} \to G_{\mathbb{R}}$ satisfying the usual conditions (cf. [10], 2.1.1.1–3). Taking the image of the canonical cocharacter of $\mathbb{S}$ with respect to h, we obtain a $G(\mathbb{C})$–conjugacy class of homomorphisms $\mu_h\colon \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$. Moreover, restriction of h^{-1} to $\mathbb{G}_{m,\mathbb{R}} \subset \mathbb{S}$ determines a unique cocharacter w_h of the maximal $\mathbb{R}$–split torus in the center of G. The pair $(w_h, f_\iota = \mu_h(-1))$ corresponds to a morphism of $\mathbb{C}|\mathbb{R}$–groupoids

$$\xi_\infty\colon \mathfrak{G}_\infty \longrightarrow \mathfrak{G}_G(\infty)$$

which is uniquely determined up to equivalence. Note that this is the inverse of the morphism considered by Langlands and Rapoport (cf. [24], page 165) and agrees with the one defined by Milne (cf. [29], page 191). This difference is the reason for our choice of ψ in place of ψ^{-1}. Apparently, Milne forgot to make the corresponding change (cf. loc. cit. page 184) and, therefore, condition (a) of his Theorem 4.6 (taking the composition of morphisms in the order which makes sense) is not correct.

For a prime ℓ different from the fixed prime p, let

$$\xi_\ell\colon \mathfrak{G}_\ell \longrightarrow \mathfrak{G}_G(\ell)$$

be the canonical section. Let G^{der} denote the derived group, $G^{\mathrm{ad}} = G/Z_G$ the adjoint group, and $G^{\mathrm{ab}} = G/G^{\mathrm{der}}$ the maximal abelian quotient of G. The cocharacter of G^{ab} induced by μ_h defines a morphism of $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoids

$$\psi_h\colon \mathfrak{Q} \longrightarrow \mathfrak{G}_{G^{\mathrm{ab}}}\,.$$

The following observation concerning the stabilizer of a point in the building of G (cf. [42]) will be useful when we extract the information concerning the place p which is not as easily obtained as for the other places.

Lemma B3.1 *Let K be a local field, i.e. a field which is complete with respect to a non-trivial discrete valuation and has a perfect residue class field. Let G be a connected reductive group over K which splits over an unramified extension of K, and let x be a point in the building $\mathcal{B}(G,K)$.*

(i) *If x' denotes the image of x under the projection map $\mathcal{B}(G,K) \to \mathcal{B}(G^{\mathrm{der}},K)$, and $\mathcal{A}$ is any apartment containing x, then*

$$\mathrm{Stab}_{G(K)}(x) = \mathrm{Stab}_{G^{\mathrm{der}}(K)}(x') \cdot \bigcap_{y\in\mathcal{A}} \mathrm{Stab}_{G(K)}(y)\,.$$

(ii) *If G^{der} is simply connected and y is a point in the closure of the facet of $\mathcal{B}(G,K)$ containing x, then*

$$\mathrm{Stab}_{G(K)}(x) \subseteq \mathrm{Stab}_{G(K)}(y)\,.$$

Proof. To start with, we choose a maximal K–split torus $S \subseteq G$ such that x is contained in the corresponding apartment $\mathcal{A}_S$. Let N_S denote the normalizer of S in G. Then $\mathrm{Stab}_{G(K)}(x)$ is generated by the subgroup $\mathrm{Stab}_{N_S(K)}(x)$ and the groups X_α which correspond to the affine roots α of G relative to S and K such that $\alpha(x) \geqslant 0$ (cf. [42], 3.1). Moreover, the group $N_S(K)$ acts on the apartment $\mathcal{A}_S$ through the modified Weyl group $\widetilde{W}_S$. Since the X_α consist of unipotent elements, they must be contained in G^{der}. Therefore, for the proof of (i), it is sufficient to show that we have an inclusion

$$\mathrm{Stab}_{\widetilde{W}_S}(x) \subseteq \widetilde{W}_{S'}$$

where S' denotes the maximal K–split torus in G^{der} which is contained in S.

Recall that the modified Weyl group is equipped with two canonical homomorphisms, an injection $X_*(S) \to \widetilde{W}_S$ and a surjection $\widetilde{W}_S \to W_S$ where W_S denotes the usual Weyl group of G with respect to S. Since we have assumed that G splits over an unramified extension of K, these homomorphisms give rise to an exact sequence

$$0 \longrightarrow X_*(S) \longrightarrow \widetilde{W}_S \longrightarrow W_S \longrightarrow 1$$

(cf. [42], 1.3). The apartment $\mathcal{A}_S$ is a product $\mathcal{A}_{S'} \times \mathcal{A}_0$ where $\mathcal{A}_{S'}$ denotes the apartment in the building of G^{der} corresponding to S' and $\mathcal{A}_0$ is an affine space with translation vector space $X_*(S_0) \otimes_{\mathbb{Z}} \mathbb{R}$ for the maximal K–split torus S_0 in G^{ab}. The action of $\widetilde{W}_S$ is given by separate actions on both factors. On the second factor it acts via a canonical homomorphism $\widetilde{W}_S \to X_*(S_0)$. Since the argument applied above to the group G may as well be applied to G^{der}, we get a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 1 \arrow[d] & \\
0 \arrow[r] & X_*(S') \arrow[r] \arrow[d] & \widetilde{W}_{S'} \arrow[r] \arrow[d] & W_{S'} \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & X_*(S) \arrow[r] \arrow[d] & \widetilde{W}_S \arrow[r] & W_S \arrow[r] \arrow[d] & 1 \\
 & X_*(S_0) & & 1 &
\end{tikzcd}$$

with exact rows and columns. In particular, this shows that the sequence

$$1 \longrightarrow \widetilde{W}_{S'} \longrightarrow \widetilde{W}_S \longrightarrow X_*(S_0)$$

is exact which completes the proof of (i). Applying it to the points x, y and an apartment containing both of them reduces (ii) to the case of a semi-simple group G which is well known (cf. [42], 3.1.1). ▪

Now we fix a maximal compact subgroup $C_p \subset G(\mathbb{Q}_p)$ and assume that G^{der} is simply connected, $G_{\mathbb{Q}_p} = G_1 \times G_2$ splits over an unramified extension of $\mathbb{Q}_p$, and $C_p = C_1 \times C_2$ where $C_1 \subset G_1(\mathbb{Q}_p)$ is hyperspecial, $C_2 \subset G_2(\mathbb{Q}_p)$ and G_2^{ad} is anisotropic. Note that this is slightly weaker than (but in the spirit of) the condition in [24]. The product of the hyperspecial facet in the building $\mathcal{B}(G_1, \mathbb{Q}_p)$ corresponding to C_1 and the whole building $\mathcal{B}(G_2, \mathbb{Q}_p)$ of G_2 is contained in a unique facet $\mathfrak{F} \subset \mathcal{B}(G, \mathcal{K})$ which is invariant under the action of the Galois group $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}} | \mathbb{Q}_p)$. Let $\mathfrak{F}_1, \ldots, \mathfrak{F}_r$ be all facets of minimal dimension in the closure of $\mathfrak{F}$ in $\mathcal{B}(G, \mathcal{K})$, and put

$$C_p(\mathcal{K}) = \bigcap_{i=1}^{r} \mathrm{Stab}_{G(\mathcal{K})}(x_i) \tag{B3.2}$$

where $x_i \in \mathfrak{F}_i$ are any points. Lemma B3.1 implies that this is uniquely determined by the group C_p, hence invariant under $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}} | \mathbb{Q}_p)$, and that

$$C_p = C_p(\mathcal{K}) \cap G(\mathbb{Q}_p)\,.$$

Consider now any two points $x, y \in \mathcal{B}(G, \mathcal{K})$. For every maximal split torus $S \subseteq G_{\mathcal{K}}$ and every $g \in G(\mathcal{K})$ such that x and y are contained in the apartment corresponding to gSg^{-1}, there is a unique element $v \in X_*(S) \otimes_{\mathbb{Z}} \mathbb{R}$ such that $x = y + g(v)$. If S' is another maximal split torus in $G_{\mathcal{K}}$, there is an element $g' \in G(\mathcal{K})$ such that $S' = g'Sg'^{-1}$. The bijection

$$(X_*(S) \otimes_{\mathbb{Z}} \mathbb{R})/W_S \longrightarrow (X_*(S') \otimes_{\mathbb{Z}} \mathbb{R})/W_{S'}$$

induced by $\mathrm{ad}(g')$ does not depend on the choice of g', and the image of v in

$$\widetilde{X}(G, \mathcal{K}) = \varprojlim_{S} (X_*(S) \otimes_{\mathbb{Z}} \mathbb{R})/W_S \tag{B3.3}$$

is uniquely determined by the $G(\mathcal{K})$–orbit of the pair (x, y). It is denoted by $\mathrm{inv}(x, y)$. Moreover, for every field K of characteristic 0, $\mathcal{C}(G, K)$ denotes the set of $G(K)$–conjugacy classes of homomorphisms of algebraic groups $\mathbb{G}_{m,K} \to G_K$. Since G splits over $\mathcal{K}$, we have natural bijections

$$\varprojlim_{S} X_*(S)/W_S \longrightarrow \mathcal{C}(G, \mathcal{K}) \longrightarrow \mathcal{C}(G, \overline{\mathcal{K}}) \longleftarrow \mathcal{C}(G, \overline{\mathbb{Q}}) \longrightarrow \mathcal{C}(G, \mathbb{C})$$

(cf. [29], 1.7), hence an injection $\mathcal{C}(G, \mathbb{C}) \subset \widetilde{X}(G, \mathcal{K})$. Using this, we may define the set

$$Y_h = \{\, g \in G(\mathcal{K}) \mid \mathrm{inv}(gx_i, x_i) = [\mu_h] \text{ for all } i \in \{1, \ldots, r\} \,\} \tag{B3.4}$$

where $x_i \in \mathfrak{F}_i$ are any points and $[\mu_h]$ denotes the $G(\mathbb{C})$–conjugacy class of μ_h considered as an element of $\widetilde{X}(G,\mathcal{K})$. Since the action of $G(\mathcal{K})$ on the building $\mathcal{B}(G,\mathcal{K})$ commutes with translation by an element of $X_*(S_0)\otimes_{\mathbb{Z}}\mathbb{R}$, this does not depend on the choice of the points x_i. It is uniquely determined by the $G(\mathbb{R})$–conjugacy class of h and the compact subgroup C_p. Obviously, Y_h is a union of $C_p(\mathcal{K})$–double cosets in $G(\mathcal{K})$. Let E denote the completion of the reflex field of the Shimura variety Sh_G at the place above p which is determined by the fixed embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. Since G splits over an unramified extension of $\mathbb{Q}_p$, E is unramified and $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}\,|\,E)$ is the stabilizer of $[\mu_h]$ for the natural action of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}\,|\,\mathbb{Q}_p)$ on $\widetilde{X}(G,\mathcal{K})$. In particular, $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}\,|\,E)$ acts on Y_h.

For every open compact subgroup $C^p\subset G(\mathbb{A}_f^p)$, we consider the Shimura variety Sh_{G,C_pC^p} as a variety over E and put

$$Sh_G^p = \varprojlim_{C^p} Sh_{G,C_pC^p}\,.$$

This is a scheme over E with a continuous right action of $G(\mathbb{A}_f^p)$ and an action of $Z_G(\mathbb{Q}_p)$. We are now prepared to introduce the notion of an admissible morphism which is the essential ingredient in the conjectural description of a suitable model of Sh_G^p over O_E.

Definition B3.5 An *admissible morphism* for Sh_G^p is a morphism $\varphi\colon \mathfrak{Q}\to\mathfrak{G}_G$ of $\overline{\mathbb{Q}}|\mathbb{Q}$–groupoids which satisfies the following conditions:

(i) the morphism $\varphi^{\mathrm{ab}}\colon \mathfrak{Q}\to\mathfrak{G}_{G^{\mathrm{ab}}}$ induced by φ is equivalent to ψ_h;

(ii) for every place $\ell\neq p$ of $\mathbb{Q}$, $\varphi(\ell)\circ\zeta_\ell$ is equivalent to ξ_ℓ;

(iii) the σ–conjugacy class $\mathrm{cls}(\varphi(p)\circ\zeta_p)$ contains an element of Y_h.

To every morphism $\varphi\colon \mathfrak{Q}\to\mathfrak{G}_G$ we can associate a set $S(\varphi)$ with a right action of $Z_G(\mathbb{Q}_p)\times G(\mathbb{A}_f^p)$ and an automorphism Φ as follows. Let $I_\varphi=\mathrm{Aut}(\varphi)$ and

$$X^p(\varphi)=\{\,g\in G(\overline{\mathbb{A}_f^p}) \mid \text{for all } \ell\neq\infty,p,\ g_\ell\in\mathrm{Isom}(\xi_\ell,\varphi(\ell)\circ\zeta_\ell)(\mathbb{Q}_\ell)\,\}$$

where, for every $g\in G(\overline{\mathbb{A}_f^p})$ and every place $\ell\neq\infty,p$ of $\mathbb{Q}$, g_ℓ denotes the image of g in $G(\overline{\mathbb{Q}}_\ell)$. I_φ is an algebraic group over $\mathbb{Q}$, and $X^p(\varphi)$ is a subset of $G(\overline{\mathbb{A}_f^p})$ on which $I_\varphi(\mathbb{Q})\subset G(\overline{\mathbb{Q}})$ acts by left multiplication and $G(\mathbb{A}_f^p)$ acts transitively by right multiplication. Remark B2.9 implies that, up to isomorphism, this construction is independent of the choice of a quasi-motivic groupoid.

Lemma B3.6 *The set $X^p(\varphi)$ is not empty if and only if φ satisfies condition* (ii) *in Definition B3.5 for all finite places $\ell\neq p$ of $\mathbb{Q}$.*

Proof. We fix a $\mathbb{Z}$–structure $G_{\mathbb{Z}}$ on G and, using condition (ii) in Definition B2.7, a Galois extension $L|\mathbb{Q}$ in $\overline{\mathbb{Q}}$ and an element $g \in G(L_f^p \otimes_{\mathbb{A}_f^p} L_f^p)$, where L_f^p denotes the image of $L \otimes_{\mathbb{Q}} \mathbb{A}_f^p$ in $\prod_{\ell \neq \infty, p} \overline{\mathbb{Q}}_\ell$, such that, for all finite places $\ell \neq p$ of $\mathbb{Q}$, $\varphi(\ell) \circ \zeta_\ell$ corresponds to the image g_ℓ of g in $G(\overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell)$. But then, for almost all ℓ, g_ℓ can be considered as a 1–cocycle $\mathrm{Gal}(L_\ell|\mathbb{Q}_\ell) \to G_{\mathbb{Z}}(O_{L_\ell})$. On the other hand, for almost all ℓ, the pull-back of $G_{\mathbb{Z}}$ to $\mathrm{Spec}(\mathbb{Z}_\ell)$ is a smooth group scheme with connected fibres and the extension $L_\ell|\mathbb{Q}_\ell$ is unramified. This implies that $\mathrm{H}^1(L_\ell|\mathbb{Q}_\ell, G_{\mathbb{Z}}(O_{L_\ell}))$ vanishes (cf. [4], §3, Lemme 2 and [20], §2, Proposition 3). Hence, for almost all ℓ, $\mathrm{Isom}(\xi_\ell, \varphi(\ell) \circ \zeta_\ell)(\mathbb{Q}_\ell)$ contains an element of $G_{\mathbb{Z}}(O_{L_\ell})$. The assertion of the lemma is an obvious consequence of this fact. ▪

For the component at p, we choose a morphism $\theta\colon \mathfrak{D} \to \mathfrak{G}_G$ of $\mathbb{Q}_p^{\mathrm{nr}}|\mathbb{Q}_p$–groupoids such that the pull-back $\bar\theta$ to $\overline{\mathbb{Q}}_p|\mathbb{Q}_p$ is equivalent to $\varphi(p) \circ \zeta_p$ and an element g_θ in $\mathrm{Isom}(\varphi(p) \circ \zeta_p, \bar\theta)(\mathbb{Q}_p)$. We define a map $\mathcal{F}\colon G(\mathcal{K}) \to G(\mathcal{K})$ such that $\mathcal{F}(g) = b\sigma(g)$, where $b = \theta(d_\sigma) \in \mathrm{cls}(\varphi(p) \circ \zeta_p)$, and consider the set

$$X_p(\varphi) = \{\, g \in G(\mathcal{K}) \mid \mathcal{F}(g) \in gY_h \,\}/C_p(\mathcal{K})$$

on which $J_\theta = \mathrm{Aut}(\theta)(\mathbb{Q}_p) \subset G(\mathbb{Q}_p^{\mathrm{nr}}) \subset G(\mathcal{K})$ acts by left multiplication since, for every $g \in J_\theta$, $b\sigma(g) = gb$. Via the embedding

$$\begin{aligned} I_\varphi(\mathbb{Q}) &\longrightarrow \mathrm{Aut}(\bar\theta)(\mathbb{Q}_p) = J_\theta \\ \alpha &\longmapsto g_\theta \alpha g_\theta^{-1} \end{aligned}$$

this induces a left action of $I_\varphi(\mathbb{Q})$. Moreover, since Y_h is invariant with respect to σ^e where $e = [\,E : \mathbb{Q}_p\,]$, $\mathcal{F}^e$ induces an automorphism Φ of $X_p(\varphi)$ which commutes with the action of $I_\varphi(\mathbb{Q})$ and the obvious action of $Z_G(\mathbb{Q}_p) \subset G(\mathcal{K})$. $X_p(\varphi)$ can be embedded into the product of r copies of the building $\mathcal{B}(G, \mathcal{K})$ by choosing points $x_i \in \mathfrak{F}_i$ and sending $gC_p(\mathcal{K}) \in X_p(\varphi)$ to $(gx_1, \ldots, gx_r)$. It is obvious that $X_p(\varphi)$ is not empty if and only if φ satisfies condition (iii) in Definition B3.5, and up to unique isomorphism, it is independent of the choice of θ and g_θ. In fact, if $X_p'(\varphi)$ is obtained using θ' and $g_{\theta'}$, left multiplication by $g_{\theta'} g_\theta^{-1} \in \mathrm{Isom}(\bar\theta, \overline{\theta'})(\mathbb{Q}_p) = \mathrm{Isom}(\theta, \theta')(\mathbb{Q}_p) \subset G(\mathcal{K})$ provides a bijection $X_p(\varphi) \to X_p'(\varphi)$ which respects the actions of $I_\varphi(\mathbb{Q})$, $Z_G(\mathbb{Q}_p)$ and Φ.

We define

$$S(\varphi) = \varprojlim_{C^p} I_\varphi(\mathbb{Q}) \backslash \big(X_p(\varphi) \times (X^p(\varphi)/C^p) \big),$$

where the limit is over all open compact subgroups $C^p \subset G(\mathbb{A}_f^p)$. The construction above provides a right action of $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$ on $S(\varphi)$ and an automorphism Φ which commutes with it. It is easy to see (cf. [29], lemma 4.1) that, up to isomorphism, $S(\varphi)$ together with these actions is determined by the equivalence class of

φ. It is not empty if φ is admissible. Let now $\mathbb{F}$ denote an algebraic closure of the residue class field of O_E and Fr the corresponding Frobenius automorphism. The conjecture of Langlands and Rapoport (cf. [24], page 169) is expressed as follows.

Conjecture B3.7 *For every C^p which is sufficiently small, there exists a model $\widetilde{Sh}_{G,C_pC^p}$ of Sh_{G,C_pC^p} over O_E with the following properties:*

(i) *there is a bijection*

$$\widetilde{Sh}_{G,C_pC^p}(\mathbb{F}) \cong \bigsqcup_{\varphi} S(\varphi)/C^p\,,$$

where the disjoint union is taken over a set of representatives for the equivalence classes of admissible morphisms $\varphi\colon \mathfrak{Q} \to \mathfrak{G}_G$, which carries the action of Fr *on the left to the action of Φ on the right;*

(ii) *if Sh_{G,C_pC^p} is proper over E, the model is proper over O_E;*

(iii) *if C_p is hyperspecial, the model is smooth over O_E.*

Remark B3.8 Langlands and Rapoport forgot to include the compatibility with the actions of $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$. Moreover, they do not specify the meaning of the word "model". If it just means any O_E–scheme whose general fibre is Sh_{G,C_pC^p}, the conjecture says almost nothing if the Shimura variety is not proper and C_p is not hyperspecial since then the model may be a disjoint union of its two fibres. For the case where the subgroup $C_p \subset G(\mathbb{Q}_p)$ is hyperspecial, the torus G^{ab} splits over a CM–field, and the weight w_h is defined over $\mathbb{Q}$, Milne has given a more precise version of the conjecture (cf. [29], 2.7, 4.8). It is expressed in terms of the pseudo-motivic groupoid. That this is possible is a consequence of the following lemma.

Lemma B3.9 *If G^{ab} splits over a CM–field and w_h is defined over $\mathbb{Q}$, every admissible morphism $\varphi\colon \mathfrak{Q} \to \mathfrak{G}_G$ factors through $\mathfrak{P}$.*

Proof. Let T denote the algebraic quotient of Q such that φ^{Δ} induces an embedding $T_{\overline{\mathbb{Q}}} \subseteq G_{\overline{\mathbb{Q}}}$. Since w_h is defined over $\mathbb{Q}$, condition (ii) for $\ell = \infty$ in Definition B3.5 implies that condition (i) in Remark B2.11 is satisfied. Moreover, using Lemma 5.1 in [24], we obtain an $\mathbb{R}$–form G' of $G_{\mathbb{R}}$ such that $G'^{\,\mathrm{ad}}(\mathbb{R})$ is compact and $T_{\mathbb{R}} \subseteq G'_{\mathbb{R}}$. If T^{ad} denotes the quotient of T such that $T^{\mathrm{ad}}_{\overline{\mathbb{Q}}} \subseteq G^{\mathrm{ad}}_{\overline{\mathbb{Q}}}$, this implies that $T^{\mathrm{ad}}_{\mathbb{R}}$ is anisotropic. In particular, T^{ad} splits over a CM–field and every cocharacter has weight zero. Therefore, in order to check condition (ii) in Remark B2.11, it is sufficient to show that the image of $\nu(p)$ in $X_*(G^{\mathrm{ab}}) \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfies the Serre condition. By condition (i) in Definition B3.5, this image is the element $\mu = r^{-1}\sum_{i=1}^{r}\mu_i$ if $\{\mu_1,\ldots,\mu_r\}$ is the $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$–orbit of the cocharacter μ_h^{ab}. Since G^{ab} splits over a CM–field and w_h is defined over $\mathbb{Q}$, μ_h^{ab} satisfies the Serre condition, hence μ satisfies it, too. ∎

We finish this section with the statement of a somewhat more precise variant of the conjecture of Langlands and Rapoport. Inspired by a discussion with Blasius, we include a condition for the existence of a smooth model. For this we consider the following general situation.

Definition B3.10 Let G be any reductive group over $\mathbb{Q}_p$, c a conjugacy class in $\mathcal{C}(G, \overline{\mathbb{Q}}_p)$, and $G^{\mathrm{ad}} = G_1 \times \cdots \times G_n$ the decomposition into $\mathbb{Q}_p$-simple groups. For every $i \in \{1, \ldots, n\}$, let c_i denote the image of c in $\mathcal{C}(G_i, \overline{\mathbb{Q}}_p)$, $E_i \subset \overline{\mathbb{Q}}_p$ its field of definition, and H_i the smallest closed, normal subgroup of G_i over E_i such that c_i is contained in the image of $\mathcal{C}(H_i, \overline{\mathbb{Q}}_p)$. The class c is called *unramified* if all H_i are unramified (i.e. quasi-split over E_i and split over an unramified extension).

Now we return to our original situation. Since there is so far no good notion of an *integral canonical model* for a general Shimura variety, we will use the following provisional definition.

Definition B3.11 An *admissible model* for Sh^p_G over O_E is a projective system of O_E-schemes $\widetilde{Sh}_{G,C_pC^p}$ indexed by the set of all open compact subgroups $C^p \subset G(\mathbb{A}^p_f)$ and equipped with a continuous action of $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}^p_f)$ such that the following conditions hold:

(i) for $\widetilde{Sh}^p_G = \varprojlim_{C^p} \widetilde{Sh}_{G,C_pC^p}$, there is a $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}^p_f)$-equivariant isomorphism of E-schemes

$$\widetilde{Sh}^p_G \otimes_{O_E} E \cong Sh^p_G \, ;$$

(ii) there is a $Z_G(\mathbb{Q}_p) \times G(\mathbb{A}^p_f)$-equivariant bijection

$$\widetilde{Sh}^p_G(\mathbb{F}) \cong \bigsqcup_{\varphi} S(\varphi) \, ,$$

where the disjoint union is taken over a set of representatives for the equivalence classes of admissible morphisms $\varphi \colon \mathfrak{Q} \to \mathfrak{G}_G$, which carries the action of Fr on the left to the action of Φ on the right;

(iii) for every sufficiently small C^p, $\widetilde{Sh}_{G,C_pC^p}$ can be embedded as a dense open subscheme into a normal projective flat O_E-scheme such that the complement (with its reduced structure) is flat over O_E.

Conjecture B3.12 *In the situation of the present section, there exists an admissible model for Sh^p_G over O_E. There exists a smooth admissible model (i.e. for every sufficiently small C^p, $\widetilde{Sh}_{G,C_pC^p}$ is smooth over O_E), if and only if the class $c_h \in \mathcal{C}(G, \overline{\mathbb{Q}}_p)$ determined by μ_h is unramified.*

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