A STOCHASTIC ANALOGUE OF THE VINOGRADOV MEAN VALUE THEOREM

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Abstract. We prove a stochastic analogue of the well-known Vinogradov mean value theorem. In the special case of random variables taking values $1, 2, \dots, P$ with equal probabilities our result yields an estimate of the same order in P as the original Vinogradov estimate.

1. Introduction and main results

Let $\mathcal{J}_m(P)$ denote the number of solutions of the simultaneous diophantine equations

$$\sum_{i=1}^{m}(x_i^j - y_i^j) = 0,\ j = 1, \dots, k,$$

where $1 \le x_i,\ y_i \le P$ for $i = 1, \dots, m$.
Since for any integer x we have

$$\int_0^1 \exp\{2\pi i\alpha x\}\, d\alpha = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0, \end{cases}$$

we get

$$\mathcal{J}_m(P) = \int_0^1 \dots \int_0^1 |\sum_{x=1}^{P} \exp\{2\pi i f(x)\}|^{2m} d\alpha_k \dots d\alpha_1,$$

where $f(x) \triangleq \alpha_k x^k + \dots + \alpha_1 x$.
Estimates of the order of magnitude of the growth of $\mathcal{J}_m(P)$ with increasing P have important applications in analytic number theory. The mean value $\mathcal{J}_m(P)$ was investigated by I.M.Vinogradov (1934).

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Theorem (I.M.Vinogradov). *Let $\tau > 0,\ k > 2$ be integers with $m \geq k\tau$ and let $P \geq 1$. Then*

$$\mathcal{J}_m(P) \leq C_\tau \cdot P^{2m-\triangle(\tau)},$$

where

$$\triangle(\tau) = 0.5k(k+1)(1-(1-1/k)^\tau),$$

$$C_\tau = (k\tau)^{6k\tau}\,(2k)^{4k(k+1)\tau}.$$

The bounds for $\mathcal{J}_m(P)$ provide information about the size of the trigonometric Weyl sums

$$F \triangleq \sum_{x=1}^{P} \exp\{2\pi i f(x)\}$$

and trigonometric sums involving functions to which there are reasonable polynomial approximations. Since there are numerous applications for such bounds Vinogradov was able to use his method with great success in many different problems of number theory. For later refinements of the Vinogradov mean value theorem see e.g. Stechkin (1975) and Wooley (1992), and the discussion and references therein.

It is obvious that if a random variable S takes values $1,\ 2,\ \ldots,\ P$ with the equal probability $1/P$ then

$$\mathbb{E}\exp\{2\pi i f(S)\} = F/P.$$

Moreover, using estimates for $|F|$ known in number theory it is possible to get bounds for $|\,\mathbb{E}\exp\{2\pi i f(S_n)\}|$, where S_n is a normed sum of independent identically distributed random variables $X_1,\ X_2, \ldots,\ X_n$ provided that a distribution of X_1 has a nondegenerate discrete component (see Götze et al. (1994)).

The aim of the present paper is to extend the Vinogradov mean value theorem to the case of general random variables. Although the stochastic extension is of independent interest, we intend to use these results to construct bounds for $|\,\mathbb{E}\exp\{2\pi i f(S_n)\}|$ without any assumption on the type of distribution of X_1. Our main result is the following

Theorem 1. *Let S denote a random variable, let $P \geq 1,\ k > 2,\ \tau \geq 1,\ m$ be positive integers. Put $f(x) = \alpha_k x^k + \cdots + \alpha_1 x$ and*

$$\mathcal{I}_m(P) \triangleq P^{-(k-1)k/2} \times \int_{-P^{k-1}}^{P^{k-1}} \int_{-P^{k-2}}^{P^{k-2}} \cdots \int_{-1}^{1} |\,\mathbb{E}\exp\{2\pi i f(S)\}\cdot \mathbf{1}_{\{-P<S\leq P\}}|^{2m}\, d\alpha_k \ldots d\alpha_1, \tag{0-1}$$

where $\mathbf{1}_{\{A\}}$ denotes an indicator function of an event A.

Then for all $m = k\tau$ there exists a constant $D_\tau = D(k,\tau)$ depending on k and τ only such that

$$\mathcal{I}_m(P) \leq D_\tau \cdot P^{2m-\triangle(\tau)} \left(\sup_I \mathbb{P}\,(S \in I)\right)^{2m}, \tag{0-2}$$

where

$$\triangle(\tau) = 0.5k(k+1)(1-(1-1/k)^{\tau})$$

and the supremum is taken over all intervals $\{x : a < x \le a+1\}$ *with any real* a.

Remark 1. If S takes values $1, 2, \ldots, P$ with equal probabilities then $\mathbb{E}\exp\{2\pi i f(S)\}$ is a periodic function with respect to each coefficient $\alpha_1, \ldots, \alpha_k$ with period 1. Thus

$$\mathcal{I}_m(P) = 2^k\, P^{-2m}\, \mathcal{J}_m(P)$$

and our result coincides with the original Vinogradov estimate with respect to P.

Remark 2. If S has a uniform distribution on $[-P, P]$ Theorem 1 yields bounds less precise than Lemma 9 below. Compare Lemmas 8 and 10 as well.

Remark. 3 Our proof of Theorem 1 is based on the original proof of the Vinogradov mean value theorem (see Vinogradov (1985)). It seems that the original proof is more suitable for extension to the case of general random variables with non-lattice support than the p-adic method of proof which is due to Linnik and Karatsuba (see e.g. Linnik (1943) and Arkhipov et al. (1987)). In particular, we prove Theorem 1 by induction with respect to both parameters P and τ simultaneously. the following Theorem 2 justifies the first step of induction where τ is equal to 1 and P is arbitrary. In Vinogradov's mean value theorem for lattice points this case is obvious.

Theorem 2. *Assume that the conditions of Theorem 1 are fulfilled. Then we have for all*
m, $1 \le m \le k$,

$$\mathcal{I}_m(P) \le M_m \cdot P^{(3m-1)/2} \left(\sup_I \mathbb{P}(S \in I)\right)^{2m}, \tag{0-3}$$

where M_m *is a constant depending only on* k *and* m.

The proofs of the Theorems are given in Section 3. They are based on the Lemmas in Section 2.

2. LEMMAS

In the following the notion "an interval of length l" means that we consider a set of the form $\{x \in \mathbb{R}^1 : a \le x \le a+l\}$ for an *arbitrary* $a \in \mathbb{R}^1$.

Lemma 1. *Let* $k > 2$, H *be a constant depending on* k *and* $H > (2k)^4$, $P > H^k$ *and let* R *be the smallest integer such that* $HR \ge P$. *Assume that for some* ω *with* $0 \le \omega < P$ *there are points* $X_i, Y_i \triangleq X_i + Rl$, $i = 1, 2, \ldots, k$, *satisfying for some* $l \ge 1$

$$-\omega l < X_1,\ Y_1 + Rl \le X_2, \ldots, Y_k \le (P-\omega)l.$$

Cover every interval $(X_i, Y_i]$ *by* $\bigcup\limits_{j=1}^{N}(a_{ij}, b_{ij}]$, *where*

$$b_{ij} \triangleq a_{ij+1},\ b_{ij} - a_{ij} = l,\ a_{i1} \leq X_i,\ b_{iN} \geq Y_i,\ N \leq R+1.$$

We say that the vectors $v = (v_1, \dots, v_k)$ *and* $v' = (v_1', \dots, v_k')$ *with* $X_i < v_i \leq Y_i$, $X_i < v_i' \leq Y_i$, $i = 1, 2, \dots, k$, *belong to the same class* V *if for every* $i = 1, 2, \dots, k$ *there exists an index* $j_0 = j_0(i)$ *such that*

$$v_i \in (a_{ij_0}, b_{ij_0}] \ \text{ and } \ v_i' \in (a_{ij_0}, b_{ij_0}].$$

Then the number E_1 *of classes* V *for which there exists a vector* $v \in V$ *such that the sums*

$$V_1 = v_1 + \dots + v_k, \dots, V_k = v_1^k + \dots + v_k^k$$

lie in some intervals of lengths, respectively

$$lP^0, \dots, l^k P^{k-1}, \tag{1-1}$$

satisfies the inequality

$$E_1 \leq c_1(k), \tag{1-2}$$

where $c_1(k)$ *is a constant depending only on* k.
Let

$$V_1' = v_1' + \dots + v_k', \dots, V_k' = (v_1')^k + \dots + (v_k')^k$$

where the vector $(v_1', \dots, v_k')$ *belongs to some class* V'. *Then the number* E *of classes* V *and* V' *such that for some vectors* $v \in V$ *and* $v' \in V'$ *the differences*

$$V_1 - V_1', \dots, V_k - V_k'$$

lie in some intervals of lengths

$$lP^{1-1/k}, \dots, (lP^{1-1/k})^k, \tag{1-3}$$

respectively satisfies the inequality

$$E \leq c_2(k) P^{(3k-1)/2} \tag{1-4}$$

with some constant $c_2(k)$ *depending only on* k.

Remark. Lemma 1 is a generalization of the corresponding Vinogradov Lemma (see Lemma 3, ch. 4 in Vinogradov (1985)). In the proof of Lemma 1 we use the same arguments as in Vinogradov's proof together with some additional arguments (see also Lemma 6, §1, ch. 3 in Arkhipov et al. (1987)).

Proof. In the proof we denote by c (with or without index) constants which may depend on k only. The same symbol may be used for different constants.

First we shall evaluate E_1. Let s denote an integer such that $1 < s \le k$. If for given $v_{s+1}, \dots, v_n$ the sums $V_1, \dots, V_n$ lie in certain intervals of lengths (1-1) then the sums

$$v_1 + \cdots + v_s, \dots, v_1^s + \cdots + v_s^s$$

lie again in certain intervals of lengths

$$c_1 l P^0, \dots, c_s l^s P^{s-1}.$$

Let $(\eta_1, \dots, \eta_s)$ and $(\eta_1 + \xi_1, \dots, \eta_s + \xi_s)$ denote two vectors with this property and such that

$$\eta_s \in (a_{sj_0}, b_{sj_o}], \qquad \eta_s + \xi_s \in (a_{sj_1}, b_{sj_1}],$$

where $j_1 > j_0$, i.e. $\xi_s > 0$, and the vectors η and $\eta + \xi$ belong to different classes of vectors.

According to our assumptions we have

$$\begin{aligned} \frac{(\eta_1+\xi_1)-\eta_1}{\xi_1}\cdot\xi_1 \quad &+\cdots+ \quad \frac{(\eta_s+\xi_s)-\eta_s}{\xi_s}\cdot\xi_s = c_1 l \\ &\cdots \\ \frac{(\eta_1+\xi_1)^s-\eta_1^s}{s\xi_1}\cdot\xi_1 \quad &+\cdots+ \quad \frac{(\eta_s+\xi_s)^s-\eta_s^s}{s\xi_s}\cdot\xi_s = c_s l^s P^{s-1}, \end{aligned}$$

Hence we find

$$\triangle\xi_s - \triangle' = 0, \tag{1-5}$$

where

$$\triangle = \begin{vmatrix} \frac{(\eta_1+\xi_1)-\eta_1}{\xi_1} & \cdots & \frac{(\eta_s+\xi_s)-\eta_s}{\xi_s} \\ & \cdots & \\ \frac{(\eta_1+\xi_1)^s-\eta_1^s}{s\xi_1} & \cdots & \frac{(\eta_s+\xi_s)^s-\eta_s^s}{s\xi_s} \end{vmatrix}$$

and

$$\triangle' = \begin{vmatrix} \frac{(\eta_1+\xi_1)-\eta_1}{\xi_1} & \cdots & \frac{(\eta_{s-1}+\xi_{s-1})-\eta_{s-1}}{\xi_{s-1}} & c_1 l \\ & \cdots & & \\ \frac{(\eta_1+\xi_1)^s-\eta_1^s}{s\xi_1} & \cdots & \frac{(\eta_{s-1}+\xi_{s-1})^s-\eta_{s-1}^s}{s\xi_{s-1}} & c_s l^s P^{s-1} \end{vmatrix}.$$

Now we apply the following transformation to (1-5). Expand both determinants in terms of the elements of the first column using Lagrange's formula und multiply the equation by ξ_1. The result can be written as a difference of a certain function of v_1 evaluated at the points $v_1 = \eta_1 + \xi_1$ and $v_1 = \eta_1$. By the mean value theorem we obtain a new equality for the derivative of that function at some intermediate point $X_1 < x_1 < Y_1$. This leads to a new equation like (1-5) where the elements of the first column of each determinant are replaced by $1, \dots, x_1^{s-1}$. Further transformations with respect to subsequent columns up to the $(s-1)$-th column and finally the last column where only the first determinant is transformed, we obtain

$$\triangle_s \xi_s - \triangle_s' = 0,$$

where

$$\triangle_s = \begin{vmatrix} 1 & \dots & 1 \\ & \dots & \\ x_1^{s-1} & \dots & x_s^{s-1} \end{vmatrix},$$

$$\triangle_s' = \begin{vmatrix} 1 & \dots & 1 & c_1 l \\ & \dots & & \\ x_1^{s-1} & \dots & x_{s-1}^{s-1} & c_s l^s P^{s-1} \end{vmatrix}$$

and $X_i < x_i < Y_i, \quad i = 1, 2, \dots, s.$

Thus we obtain by the wellknown formula for Vandermonde's determinant

$$\triangle_s = \prod_{1 \le i < j \le s} (x_i - x_j)$$

the equality

$$\triangle_s' = c \sum_{r=0}^{s-1} l^{r+1} P^r U_r,$$

where U_r is the coefficient of x_s^r in the expansion of

$$\triangle_s = (x_s - x_1) \dots (x_s - x_{s-1}) \triangle_{s-1}$$

in powers of x_s and hence equals the product of $\triangle_{s-1}$ and the sum of all products of $(s-1-r)$ different numbers chosen from $-x_1, \dots, -x_{s-1}$.

Our assumptions about the the location of the numbers x_j imply

$$U_r \le c \triangle_{s-1} (lP)^{s-1-r},$$

$$\xi_s \le c \sum_{r=0}^{s-1} \frac{l^s P^{s-1}}{(x_s - x_1) \dots (x_s - x_{s-1})},$$

where $x_i \in (X_i, Y_i]$. Since for $t \ge 1$ we have

$$x_{j+t} - x_j \ge (2t-1) Rl,$$

we get

$$\xi_s < cl$$

Thus, we found that v_s (for $s > 1$ and given $v_{s+1}, \dots, v_k$) cannot be an element of more than c different classes of vectors. Since v_1 for given $v_2, \dots, v_k$ lies in an interval of length l (recall that we consider the intervals of type $[a, b], \ a < b$), v_1 cannot be element of more than 2 classes of vectors V. Thus we get (1-2)
Let E' denote the number of classes X such that the sums $V_1, \dots, V_k$ lie in some intervals of lengths (1-3) respectively. In order to derive a bound for E' from (1-2) it is sufficient to note that

$$\left(\frac{lP^{1-1/k}}{l} + 1 \right) \cdot \dots \cdot \left(\frac{(lP^{1-1/k})^{k-1}}{l^{k-1} P^{k-2}} + 1 \right) < eP^{(k-1)/2}.$$

Thus,

$$E' < cP^{(k-1)/2}. \tag{1-6}$$

Finally, note that the number of all classes V is not larger than

$$(R+1)^k < (P/H+2)^k < cP^k$$

This together with (1-6) yields the desired inequality (1-4) which completes the proof of Lemma 1.

Let μ be a nonnegative finite Borel measure on $\mathbb{R}^k$. Define for $\bar{l} = (l_1, \dots, l_k)$ with $l_i > 0, i = 1, \dots, k,$

$$Q_\mu(\bar{l}) = \sup \mu(I(\bar{l})),$$

where supremum is taken over all rectangles of the form

$$I(\bar{l}) = I(\bar{l}, \bar{a}) = \{x \in \mathbb{R}^k : a_i \le x_i \le a_i + l_i, i = 1, 2, \dots, k\}$$

with any real a_i.

Lemma 2. *Define $(t, x) \triangleq \sum_{j=1}^k t_j x_j$. Assume that the function*

$$\Phi_\mu(t) = \int_{\mathbb{R}^k} \exp\{i(t, x)\} \mu(dx)$$

is nonnegative. Then for any $\bar{l} = (l_1, \dots, l_k)$ with $l_i > 0,\ i = 1, 2, \dots, k$, there exist constants $c_3(k)$ and $c_4(k)$ depending only on k such that

$$c_3(k) \prod_{j=1}^k l_j \int_{|t| \le \bar{l}^{-1}} \Phi_\mu(t) dt \le Q_\mu(\bar{l}) \le c_4(k) \prod_{j=1}^k l_j \int_{|t| \le \bar{l}^{-1}} \Phi_\mu(t) dt,$$

where $\int_{|t| \le \bar{l}^{-1}}$ stands for an integral over the region $\{t \in \mathbb{R}^k : |t_j| \le l_j^{-1}, j = \overline{1, k}\}$.

The Lemma follows from Lemmas 2.5 and 2.6 in Zaitsev (1988a), see also Lemma 6.1 in Esseen (1968), Hengartner and Theodorescu (1973) and Zaitsev (1988b).

Lemma 3. *Under assumptions of Lemma 2 and in its notation for any $\bar{b} = (b_1, \dots, b_k)$ and for any $\bar{l} = (l_1, \dots, l_k)$ with $l_i > 0,\ i = 1, 2, \dots, k$, there exists a constant $c_5(k)$ depending only on k such that*

$$c_5(k) \int_{W(\bar{b}, \bar{l})} \Phi_\mu(t) dt \le \int_{|t| \le \bar{l}^{-1}} \Phi_\mu(t) dt,$$

where $W(\bar{b}, \bar{l}) = \{t \in \mathbb{R}^k : |t_j - b_j| \le l_j^{-1},\ j = \overline{1, k}\}$.

The Lemma follows from Lemmas 2 above and from Lemma 5.1 in Zaitsev (1988b).

Lemma 4. *Let X and Y be independent random vectors taking values in $\mathbb{R}^k$. Then for any real $a_i,\ l_i \ge 0,\ d_i,\ D_i, d_i \le D_i,\ i=\overline{1,k}$ we have*

$$\begin{aligned}&\mathbb{P}(a_i \le X_i + Y_i \le a_i + l_i,\ d_i \le Y_i \le D_i,\ i=\overline{1,k})\\ &\le \sup_1 \mathbb{P}(b_i \le Y_i \le b_i + l_i,\ i=\overline{1,k})\\ &\quad\times \mathbb{P}(a_i - D_i \le X_i \le a_i + l_i - d_i,\ i=\overline{1,k})\end{aligned} \tag{4-1}$$

where $\sup_1$ denotes the supremum over all numbers $b_i,\ i=\overline{1,k}$.

Proof. Put

$$A \triangleq \{x : a_i - D_i \le x_i \le a_i + l_i - d_i,\ i=\overline{1,k}\}.$$

We have

$$\begin{aligned}&\mathbb{P}(a_i \le X_i + Y_i \le a_i + l_i, d_i \le Y_i \le D_i,\ i=\overline{1,k})\\ &= \int \mathbb{P}(a_i \le x_i + Y_i \le a_i + l_i, d_i \le Y_i \le D_i,\ i=\overline{1,k})\ \mathbb{P}_X(dx)\\ &\le \sup_1 \mathbb{P}(b_i \le Y_i \le b_i + l_i,\ i=\overline{1,k}) \cdot \int \mathbf{1}_A(x)\ \mathbb{P}_X(dx)\end{aligned} \tag{4-2}$$

where $\mathbb{P}_X$ is a distribution of X and $\mathbf{1}_A(x)$ denotes the indicator function of an event A.

Thus (4-1) follows from (4-2).

Lemma 5. *Let k and m be natural integers. Denote by $I_i(b_i)$ an interval of length $b_i > 0,\ i=\overline{1,k}$. The same symbol may be used in the following for different intervals of the same length. Assume that there exist real numbers $a,\ x_i,\ \bar{x}_i,\ y_i,\ \bar{y}_i,\ i=\overline{1,k}$, such that*

$$\begin{aligned}&x_1 - \bar{x}_1 + \cdots + x_k - \bar{x}_k\\ &\quad + (y_1 + a) - (\bar{y}_1 + a) + \cdots + (y_m + a) - (\bar{y}_m + a) \in I_1(b_1)\end{aligned} \tag{5-1}$$

$$\cdots$$

$$\begin{aligned}&x_1^k - \bar{x}_1^k + \cdots + x_k^k - \bar{x}_k^k\\ &\quad + (y_1 + a)^k - (\bar{y}_1 + a)^k + \cdots + (y_m + a)^k - (\bar{y}_m + a)^k \in I_k(b_k)\end{aligned} \tag{5-k}$$

and for some $P \ge 1$

$$|a| \le P \ \ \text{and}\ \ b_{i+1} \ge P b_i,\ \ i=\overline{1,k-1}.$$

Then

$$\begin{aligned}&(x_1 - a) - (\bar{x}_1 - a) + \cdots + (x_k - a) - (\bar{x}_k - a)\\ &\quad + y_1 - \bar{y}_1 + \cdots + y_m - \bar{y}_m \in I_1(c b_1)\end{aligned} \tag{5-(k+1)}$$

$$\cdots$$

$$(x_1-a)^k-(\bar{x}_1-a)^k+\cdots+(x_k-a)^k-(\bar{x}_k-a)^k$$
$$+y_1^k-\bar{y}_1^k+\cdots+y_m^k-\bar{y}_m^k\in I_k(cb_k), \tag{5-2k}$$

where c is a constant depending only on k and m.

Proof. The relation (5-(k+1)) obviously holds with $c=1$.
We take now $j: 1<j\le k$, and consider (5-1)-(5-j). Note that for any $i\le j$ we have

$$\sum_{l=i}^{j}\binom{j}{l}\binom{l}{l-i}(-1)^{k-l}=\binom{j}{i}\sum_{l=i}^{j}\binom{j-i}{l-i}(-1)^{j-l}=0. \tag{5-(2k+1)}$$

We multiply the relation (5-l) by $\binom{j}{l}(-a)^{j-l}$ for all $l=1,\dots,j$. Then we add all multiplied in this way rows (5-1)-(5-j) and add also $(-a)^j-(-a)^j$ k times. Thus by (5-(2k+1)) we get on the left-hand side the expression

$$(x_1-a)^j-(\bar{x}_1-a)^j+\cdots+(x_k-a)^j-(\bar{x}_k-a)^j$$
$$+y_1^j-\bar{y}_1^j+\cdots+y_m^j-\bar{y}_m^j.$$

It follows from this and the conditions on a and b_i the statement of Lemma 5.

Let $f(x)=\alpha_k x^k+\cdots+\alpha_1 x$, where $\alpha_k,\dots,\alpha_1$ are real numbers and k is a natural number.

Lemma 6 (see Lemma 4 in Arkhipov et al. (1987), §3, ch. 1). *We have*

$$\left|\int_0^1 \exp\{2\pi i f(x)\}dx\right|\le\min(1;32\alpha^{-1/k}), \tag{6-1}$$

where $\alpha=\max_{1\le j\le k}|\alpha_j|$.

Lemma 7. *Let Y be a random variable with uniform distribution on $[-P;P]$, where $P>0$. Then*

$$|\mathbb{E}\exp\{itf(Y)\}|\le\min_{1\le j\le k}\left(1;32\cdot\left(\frac{|t|}{2\pi}\cdot|\alpha_j|\cdot P^j\right)^{-1/k}\right). \tag{7-1}$$

Proof. We have

$$\mathbb{E}\exp\{itf(Y)\}=\frac{1}{2P}\int_{-P}^{P}\exp\{itf(x)\}dx$$
$$=\frac{1}{2}\left(\int_0^1\exp\{itf(xP)\}dx+\int_0^1\exp\{itf(-xP)\}dx\right). \tag{7-2}$$

Combining (6-1) and (7-2) we obtain (7-1).

Lemma 8. *Let k and b be positive integers and $b \geq (k+1)/2$. Let P be a real number, $P \geq 32^k$, and Y be a random variable with uniform distribution on $[-P; P]$. Then for $m = bk$ we have*

$$
\begin{aligned}
I_m &\triangleq \int_{-1}^{1} \cdots \int_{-1}^{1} |\mathbb{E}\exp\{2\pi i f(Y)\}|^{2m} d\alpha_k \ldots d\alpha_1 \\
&\leq 2^k \cdot 32^{k^2} \cdot P^{-k(k+1)/2} \cdot (1 + k/(2b-k)). \qquad (8\text{-}1)
\end{aligned}
$$

Proof. By Lemma 7 we have

$$
\begin{aligned}
I_m &\leq \int_{-1}^{1} \cdots \int_{-1}^{1} \min_{1 \leq j \leq k} (1;\ 32^{2m} (|\alpha_j| \cdot P^j)^{-2m/k})\, d\alpha_k \ldots d\alpha_1 \\
&= 2^k \int_{0}^{1} \cdots \int_{0}^{1} \min_{1 \leq j \leq k} (1;\ 32^{2m} (\alpha_j P^j)^{-2b})\, d\alpha_k \ldots d\alpha_1 \\
&= 2^k \sum_{j=0}^{k} I_{mj}, \qquad (8\text{-}2)
\end{aligned}
$$

where $I_{m0} = Vol(\Omega_0 \cap K_1)$;

$$
I_{mj} = 32^{2m} \int \cdots \int_{\Omega_j} (\alpha_j P^j)^{-2b} d\alpha_k \ldots d\alpha_1,\ j = \overline{1,k};
$$

$$
K_1 = \{\bar{\alpha} = (\alpha_k, \ldots, \alpha_1) : 0 \leq \alpha_j \leq 1,\ j = \overline{1,k}\};
$$

$$
\Omega_0 = \{\bar{\alpha} : 1 \leq 32^{2m} \min_{1 \leq j \leq k} (\alpha_j P^j)^{-2b}\};
$$

$$
\Omega_j = K_1 \cap \{\bar{\alpha} : 32^{2m} (\alpha_j P^j)^{-2b} \leq \min_{\substack{1 \leq l \leq k \\ l \neq j}} (1; 32^{2m} (\alpha_l P^l)^{-2b})\}.
$$

Note that

$$
\Omega_0 = \{\bar{\alpha} : \alpha_j \leq 32^k \cdot P^{-j}, \quad j = \overline{1,k}\}.
$$

Therefore,

$$
I_{mo} = \prod_{j=1}^{k} 32^k \cdot P^{-j} = 32^{k^2} \cdot P^{-k(k+1)/2}. \qquad (8\text{-}3\text{-}0)
$$

Moreover, we have for any $j : 1 \leq j \leq k$,

$$
\Omega_j = K_1 \cap \{\bar{\alpha} : \alpha_j \geq 32^k \cdot P^{-j},\ \alpha_l \leq \alpha_j P^{j-l},\ l = \overline{1,k},\ l \neq j\}.
$$

Hence, for all $j : 1 \le j \le k$,

$$
\begin{aligned}
I_{mj} &\le 32^{2m} \int_{32^k \cdot P^{-j}}^{1} \prod_{\substack{l=1 \\ l \ne j}}^{k} P^{j-l} \cdot \frac{d\alpha_j}{\alpha_j^{2b-k+1} P^{2bj}} \\
&= 32^{2m} \cdot P^{k(j-1+j-k)/2} \cdot P^{-2bj} \int_{32^k \cdot P^{-j}}^{1} \frac{d\alpha_j}{\alpha_j^{2b-k+1}} \\
&= 32^{2m} \cdot P^{k(2j-1-k)/2} \cdot P^{-2bj} \cdot (2b-k)^{-1} \\
&\quad \times \Big((P^j / 32^k)^{(2b-k)} - 1 \Big) \\
&\le 32^{k^2} \cdot (2b-k)^{-1} \cdot P^{-k(k+1)/2}. \qquad \text{(8-3-j)}
\end{aligned}
$$

Combining (8-2), (8-3-0)-(8-3-k) we obtain (8-1).

Lemma 9. *Let k and b be positive integers and $b \ge (k+1)/2$. Let P be a real number, $P \ge 32$, and Y be a random variable with uniform distribution on $[-P, P]$. Then for $m = bk$ we have*

$$
\begin{aligned}
I_m(P) &\triangleq P^{-(k-1)k/2} \int_{-P^{k-1}}^{P^{k-1}} \int_{-P^{k-2}}^{P^{k-2}} \dots \int_{-1}^{1} |\mathbb{E} \exp\{2\pi i f(Y)\}|^{2m} d\alpha_k \dots d\alpha_1 \qquad \text{(9-1)} \\
&\le 2^{5k^2+k} \cdot P^{-k^2} \cdot (1 - k/(2b))^{-1}.
\end{aligned}
$$

Proof. We change variables and integrate over the transformed set $\{\bar{\gamma} \in \mathbb{R}^k : -1 \le \gamma_i \le 1,\, i = \overline{1,k}\}$. Similarly to (8-2) we get

$$
\begin{aligned}
I_m(P) &\le 2^k \int_0^1 \cdots \int_0^1 \min_{1 \le j \le k} (1; 32^{2m} (\gamma_j P^k)^{-2b}) d\gamma_k \dots d\gamma_1 \\
&= 2^k \sum_{j=0}^{k} I'_{mj}, \qquad \text{(9-2)}
\end{aligned}
$$

where $I'_{mo} = Vol(\Omega'_0 \cap K_1)$;

$$
I'_{mj} = 32^{2m} \cdot P^{-2m} \int \dots \int_{\Omega'_j} \gamma_j^{-2b} d\gamma_k \dots d\gamma_1, \quad j = \overline{1,k};
$$

$$
\Omega'_0 = \left\{ \bar{\gamma} : 1 \le (32/P)^k \min_{1 \le j \le k} \gamma_j^{-1} \right\};
$$

$$
\Omega'_j = K_1 \cap \{ \bar{\gamma} : (32/P)^k \gamma_j^{-1} \le \min_{\substack{1 \le l \le k \\ l \ne j}} \big(1; (32/P)^k \gamma_l^{-1}\big) \}
$$

and K_1 is defined similarly as in Lemma 8. Repeating the arguments of Lemma 8 the inequality (9-1) follows from (9-2).

Lemma 10. *Let k and m denote positive integers and $2m > 0.5(k^2+k)+1$. Let P and $M_1, \dots, M_k$ denote any positive real numbers and let Y denote a random variable with uniform distribution on $[-P;P]$. Then*

$$J_m(M) \triangleq \left(\prod_1^k M_i\right)^{-1} \int_{-M_1}^{M_1} \cdots \int_{-M_k}^{M_k} |\mathbb{E}\exp\{2\pi i f(Y)\}|^{2m} d\alpha_k \dots d\alpha_1$$
$$\leq c(k,m)P^{-k(k+1)/2} \left(\prod_1^k M_i\right)^{-1}, \tag{10-1}$$

where $c(k,m) = 2^{k^2+k} + (12ek^3)^{2m} \cdot 2^{2k^2+4} \sum_{j=0}^{\infty} 2^{j(0.5(k^2+k)+1-2m)}$.

Proof. In view of (7-2) we get

$$J_m(M) \leq \left(\prod_1^k M_i\right)^{-1} P^{-k(k+1)/2}$$
$$\times \int_{-M_1P}^{M_1P} \cdots \int_{-M_kP^k}^{M_kP^k} |\int_0^1 \exp\{2\pi i f(x)\}dx|^{2m} d\alpha_k \dots d\alpha_1$$
$$\leq \left(\prod_1^k M_i\right)^{-1} P^{-k(k+1)/2}$$
$$\times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\int_0^1 \exp\{2\pi i f(x)\}|^{2m} d\alpha_k \dots d\alpha_1. \tag{10-2}$$

The last integral is the so-called special integral Θ_0 in the Tarry problem. Then (10-2) and the upper bound for Θ_0 in the proof of Theorem 3 (see Arkhipov et al. (1987), ch. 1 together) imply (10-1).

Remark 1. It is clear that Lemma 10 extends the results of Lemmas 8 and 9. At the same time the conditions on m become weaker. However the constants depending on k in Lemmas 8–9 are smaller than $c(k,m)$ in (10-1).

Remark 2. It follows from (10-1) that the bound for $I_m(M)$ does not depend on the individual values of $M_i,\ i=\overline{1,k}$, but on the product $\prod_1^k M_i$.

3. Proofs of Theorems

Notation Let $F(k;x,y;l,m,A)$ denote a function with values in $\mathbb{R}^k$ such that its $i-th$ component equals

$$(x_l+A)^i - (y_l+A)^i + (x_{l+1}+A)^i - (y_{l+1}+A)^i + \ldots + (x_m+A)^i - (y_m+A)^i$$

where k, l, m are positive integers and x_i, y_i, A are real numbers.

Proof of Theorem 1. By the definition (0-1) of $\mathcal{I}_m(P)$ we obtain

$$\mathcal{I}_m(P) \le 2^k \, (2P)^{2m} \, Q^{2m}(S,1), \tag{11-1}$$

where $Q(S,1) \triangleq \sup_I \mathbb{P}(S \in I)$. Thus, (0-2) obviously holds for all $\tau \ge 1$ if

$$D_\tau \, P^{-\triangle(\tau)} \ge 2^{k+2m},$$

that is

$$P \le \left(D_\tau \cdot 2^{-k-2m}\right)^{1/\triangle(\tau)}. \tag{11-2}$$

Moreover, for $m = k$ and any $P \ge 1$ the inequality (0-2) holds too. It follows from Theorem 2. Hence in view of (11-1) and (11-2) it suffices to consider the case $\tau > 1$ and $P > (2^{-k-2m} \cdot D)^{1/\triangle(r)}$ only.

Let τ_0 and P_0 denote natural numbers and assume that the theorem holds for $\tau \le \tau_0$ and $P \le P_0$ as well as for $\tau \le \tau_0 + 1$ and $P < P_0$. Under these assumptions we shall prove that the theorem will hold for $\tau \le \tau_0 + 1$ and $P = P_0$ as well. Thus by induction the inequality (0-4) will hold for all τ and P.

Let $S_1, S_1', S_2, S_2', \ldots$ denote independent copies of S.
Note that Lemma 2 implies

$$c_4^{-1}(k) \, Q_{P,m} \le \mathcal{I}_m(P) \le c_3^{-1}(k) Q_{P,m}, \tag{11-3}$$

where the constants $c_3(k)$ and $c_4(k)$ are the same as in Lemma 2,

$$Q_{P,m} = \sup \mathbb{P} \begin{pmatrix} F(k; S, S'; 1, m, 0) \in (I_1(P^{-k+1}), \ldots, I_k(1)) \\ S_j \in I_0(P), \; S_j' \in I_0(P), \; j = \overline{1,m} \end{pmatrix}, \tag{11-4}$$

$I_0(P) \triangleq \{x \in \mathbb{R}^1 : -P < x \le P\}$ and the supremum extends over all intervals $I_1(P^{-k+1}), I_2(P^{-k+2}), \ldots, I_k(1)$ of lengths $P^{-k+1}, P^{-k+2}, \ldots, 1$ respectively.
Firstly note that for $P_1 > P$ we have

$$\begin{aligned} Q_{P,m} &\le \sup \mathbb{P} \begin{pmatrix} F(k; S, S'; 1, m, 0) \in (I_1(P^{-k+1}), \ldots, I_k(1)) \\ S_j \in I_0(P_1), \; S_j' \in I_0(P_1), \; j = \overline{1,m} \end{pmatrix} \\ &\le ((P_1/P)^{k-1} + 1) \\ &\quad \times \sup \mathbb{P} \begin{pmatrix} F(k; S, S', 1, m, 0) \in (I_1(P_1^{-k+1}), I_2(P^{-k+2}), \ldots, I_k(1)) \\ S_j \in I_0(P_1), \; S_j' \in I_0(P_1), \; j = \overline{1,m} \end{pmatrix} \\ &\le Q_{P_1,m} \prod_{j=1}^{k-1} ((P_1/P)^{k-j} + 1). \end{aligned}$$

Therefore,

$$\mathcal{I}_m(P) \le c_4(k)\, c_3^{-1}(k)\, \mathcal{I}_m(P_1) \prod_{j=1}^{k-1}((P_1/P)^{k-j}+1). \tag{11-5}$$

In the proof we shall use the following notation

$$b = k(\tau_0+1), \;\; H = (2k)^4\, 2^k\, (c_3^{-1}(k)\, c_4(k))^{1/2}, \;\; R = [PH^{-1}+1].$$

Thus, we have $P \le RH$ and by (11-5) we get

$$\mathcal{I}_b(P) \le c_5\, \mathcal{I}_b(RH). \tag{11-6}$$

In the following we denote by c (with or without index) constants which may depend on k and τ_0, only. In particular, since $P > H^k$, c_5 in (11-6) satisfies the inequality

$$c_5 < c_3^{-1}(k)\, c_4(k)\, 4^k.$$

By definition (0-1) the expectation in $\mathcal{I}_b(RH)$ can be decomposed into $2RH$ intervals of length one for the values of S. Define the function $K(z)$ on $(-RH, RH]$ with positive integer values by the formula

$$\begin{aligned}
&K(z) = 1, && \text{if } z \in (-RH; -RH+1];\\
&K(z) = 2, && \text{if } z \in (-RH+1; -RH+2];\\
&\quad\cdots\cdots\\
&K(z) = 2RH, && \text{if } z \in (RH-1; RH].
\end{aligned}$$

Write

$$\begin{aligned}
G &\triangleq \mathbb{E}\exp\{2\pi i f(S)\}\cdot \mathbf{1}_{\{-RH<S\le RH\}}\\
&= \sum_{x=1}^{2RH} \mathbb{E}\exp\{2\pi i f(S)\}\cdot \mathbf{1}_{\{S\in K^{-1}(x)\}}\\
&= \sum_{y=0}^{H-1} \mathbb{E}G(y,S),
\end{aligned}$$

where $G(y,S) \triangleq \sum\limits_{x=1}^{2R} \exp\{2\pi i f(S)\}\cdot \mathbf{1}_{\{S\in K^{-1}(x+2Ry)\}}$ and K^{-1} is the mapping inverse to K.

Thus,

$$G^b = \sum_{y_1=0}^{H-1}\cdots\sum_{y_b=0}^{H-1} \mathbb{E}G(y_1,S_1)\cdot\ldots\cdot G(y_b,S_b).$$

A sequence of type $y_1,\ldots,y_b$ and its associated product $G(y_1,S_1)\cdot\ldots\cdot G(y_b,S_b)$ are called *regular* if there exists an increasing subsequence consisting of k of the numbers $y_1, \ldots, y_b$ such that the difference between any two consecutive terms is larger than 1. A sequence and its associated product are called *irregular*, if such a sequence does not exist.

We now decompose $G^b = W_1 + W_2$, where W_1 contains regular products $G(y_1, S_1) \cdot \ldots \cdot G(y_b, S_b)$ only while W_2 contains the irregular products. Obviously, we have

$$\mathcal{I}_b(RH) \leq 2\,\mathcal{I}_1 + 2\,\mathcal{I}_2, \tag{11-7}$$

where

$$\mathcal{I}_l = (RH)^{-(k-1)k/2} \int\limits_{-(RH)^{k-1}}^{(RH)^{k-1}} \int\limits_{-(RH)^{k-2}}^{(RH)^{k-2}} \ldots \int\limits_{-1}^{1} |W_l|^2 d\alpha_k \ldots d\alpha_1, \;\; l = 1, 2.$$

We shall first evaluate $\mathcal{I}_1$. It is clear that

$$\begin{aligned} \mathcal{I}_1 \leq{} & H^{2b}\,(RH)^{-(k-1)k/2} \\ & \times \max_{y_1, \ldots, y_b} \int\limits_{-(RH)^{k-1}}^{(RH)^{k-1}} \ldots \int\limits_{-1}^{1} |\,\mathbb{E}\,G(y_1, S_1) \cdot \ldots \cdot G(y_b, S_b)|^2 \overline{d\alpha}, \end{aligned}$$

where $\overline{d\alpha} \triangleq d\alpha_k \ldots d\alpha_1$

Without loss of generality we may assume that $y_1 < \cdots < y_k$ and $y_{\nu+1} - y_\nu > 1$, $\nu = \overline{1, k-1}$. Each sum $G(y_\nu, S_\nu)$ with $\nu \geq k+1$ will be divided into at most t smaller sums, where $t = [RP^{-1+1/k} + 1]$, and these small sums consist of $2P^{1-1/k}$ summands or possibly less for the last sum. Thus $G(y_{k+1}, S_{k+1}) \cdot \ldots \cdot G(y_b, S_b)$ can be written as the sum of at most t^{b-k} terms of the type $G'(y_{k+1}, S_{k+1}) \cdot \ldots \cdot G'(y_b, S_b)$, where $G'(y_j, S_j)$ denotes the sums obtained in the subdivision of $G(y_j, S_j)$. By the geometric-arithmetic mean inequality for nonnegative numbers we obtain

$$\begin{aligned} & |\,\mathbb{E}\,G'(y_{k+1}, S_{k+1})|^2 \cdot \ldots \cdot |\,\mathbb{E}\,G'(y_b, S_b)|^2 \\ & \quad \leq \frac{|\,\mathbb{E}\,G'(y_{k+1}, S_{k+1})|^{2(b-k)} + \cdots + |\,\mathbb{E}\,G'(y_b, S_b)|^{2(b-k)}}{b-k}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{I}_1 \leq{} & t^{2(b-k)}\,H^{2b}\,(RH)^{-(k-1)k/2} \\ & \times \int\limits_{-(RH)^{k-1}}^{(RH)^{k-1}} \ldots \int\limits_{-1}^{1} |\,\mathbb{E}\,G(y_1, S_1) \cdot \ldots \cdot G(y_k, S_k)|^2 \cdot |\,\mathbb{E}\,G'(y, S_{k+1})|^{2(b-k)} \overline{d\alpha}, \end{aligned}$$

where y is one of the numbers $y_{k+1}, \ldots, y_b$.

By Lemmas 2–5 and (11-5) we get for $|a| < P$

$c\, t^{-2(b-k)}\, H^{-2b}\, \mathcal{I}_1$

$$\leq \sup_1 \mathbb{P} \left(\begin{array}{c} F(k;S,S';1,k,0) + F(k;S,S';k+1,b,a) \in ((I_1(P^{-k+1}),\dots,I_k(1)) \\ S_l \in \cup_{x=1}^{2R} K^{-1}(x+2Ry_l),\ S'_l \in \cup_{x=1}^{2R} K^{-1}(x+2Ry_l),\ l=\overline{1,k} \\ S_j \in I_0(P^{1-1/k}),\ S'_j \in I_0(P^{1-1/k}),\ j=\overline{k+1,b} \end{array} \right)$$

$$\leq \sup_2 \mathbb{P} \left(\begin{array}{c} F(k;S,S';1,k,-a) + F(k;S,S';k+1,b,0) \in ((I_1(c_1P^{-k+1}),\dots,I_k(c_k)) \\ S_l \in \cup_{x=1}^{2R} K^{-1}(x+2Ry_l),\ S'_l \in \cup_{x=1}^{2R} K^{-1}(x+2Ry_l),\ l=\overline{1,k} \\ S_j \in I_0(P^{1-1/k}),\ S'_j \in I_0(P^{1-1/k}),\ j=\overline{k+1,b} \end{array} \right)$$

$$\begin{aligned} &\leq \sup_2 \mathbb{P} \left(\begin{array}{c} F(k;S,S';k+1,b,0) \in ((I_1(c_1P^{-k+1}),\dots,I_k(c_k)) \\ S_j \in I_0(P^{1-1/k}),\ S'_j \in I_0(P^{1-1/k}),\ j=\overline{k+1,b} \end{array} \right) \\ &\quad \times \sup_3 \mathbb{P} \left(\begin{array}{c} F(k;S,S';1,k,-a) \in ((I^{(1)}(c_{k+1}P^{1-1/k}),\dots,I^{(k)}(c_{2k}P^{k(1-1/k)}) \\ S_l \in \cup_{x=1}^{2R} K^{-1}(x+2Ry_l),\ S'_l \in \cup_{x=1}^{2R} K^{-1}(x+2Ry_l),\ l=\overline{1,k} \end{array} \right) \\ &\leq c\ \sup_4 \mathbb{P} \left(\begin{array}{c} F(k;S,S';k+1,b,0) \in ((I_1(P^{(-k+1)(1-1/k)}),\dots,I_k(1)) \\ S_j \in I_0(P^{1-1/k}),\ S'_j \in I_0(P^{1-1/k}),\ j=\overline{k+1,b} \end{array} \right) \\ &\quad \times \sup_5 \mathbb{P} \left(\begin{array}{c} F(k;S,S';1,k,-a) \in ((I^{(1)}(P^{1-1/k}),\dots,I^{(k)}(P^{k(1-1/k)}) \\ S_l \in \cup_{x\in 1}^{2R} K^{-1}(x+2Ry_l),\ S'_l \in \cup_{x-1}^{2R} K^{-1}(x+2Ry_l),\ l=\overline{1,k} \end{array} \right), \end{aligned}$$

where $I_0(\cdot)$ is defined after (11-4) and $\sup_1$ denotes the supremum over all intervals $I_1(P^{-k+1}),\dots,I_k(1)$ of lengths $P^{-k+1},\dots,1$, respectively. The other suprema like $\sup_i,\ i=\overline{2,5}$, are defined similarly. Applying Lemma 2 to $\sup_4 \mathbb{P}(\dots)$ and Lemma 1 to $\sup_5 \mathbb{P}(\dots)$ we obtain

$$\mathcal{I}_1 \leq c\, t^{2(b-k)}\, \mathcal{I}_{b-k}(P^{1-1/k})\, P^{(3k-1)}\, Q^{2k}(S,1),$$

Finally, using the induction hypothesis we have

$$\begin{aligned} \mathcal{I}_1 &\leq c\, D_{\tau_0}\, t^{2(b-k)} P^{(2(b-k)-\triangle(\tau_0))(1-1/k)+(3k-1)/2}\, Q^{2b}(S,1) \\ &\leq c\, D_{\tau_0}\, (RP^{-1+1/k}+2)^{2(b-k)} \\ &\quad \times P^{(2b-k)-\triangle(\tau_0))(1-1/k)+(3k-1)/2}\, Q^{2b}(S,1) \\ &\leq c\, D_{\tau_0}\, Q^{2b}(S,1)\, P^{2(b-k)/k+(2(b-k)-\triangle(\tau_0))(1-1/k)+(3k-1)/2}. \end{aligned}$$

Note that

$$\begin{aligned} &2(b-k)/k + (2(b-k)-\triangle(\tau_0))(1-1/k) + (3k-1)/2 \\ &\quad = 2b - \triangle(\tau_0+1). \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{I}_1 &\leq c\, D_{\tau_0}\, P^{2b-\triangle(\tau_0+1)}\, Q^{2b}(S,1) \\ &\leq 0.25\, c_5^{-1}\, D_{\tau_0+1}\, P^{2b-\triangle(\tau_0+1)}\, Q^{2b}(S,1), \end{aligned} \tag{11-8}$$

provided that D_τ satisfies the following recursion

$$D_{\tau+1} \geq 4\, c_5\, c\, D_\tau, \tag{11-9}$$

where c_5 and c are the constants from (11-6) and (11-8) respectively.
The estimation of $\mathcal{I}_2$ is completely similar to the proof of the original Vinogradov mean value theorem (see Theorem 1, ch. 4 in Vinogradov (1985) and Theorem 1, §1, ch. 3 in Arkhipov et al. (1987)). That is, the total number of irregular sequences $y_1, \ldots, y_b$ is not larger than

$$B \triangleq \frac{H^{k-1}}{(k-1)!}\,(2k-2)^b.$$

Therefore, the integral $\mathcal{I}_2$ does not exceed

$$B^2\,(RH)^{-(k-1)k/2} \int\limits_{-(RH)^{k-1}}^{(RH)^{k-1}} \cdots \int\limits_{-1}^{1} \Big| \mathbb{E}\, G(y_1, S_1) \cdot \ldots \cdot G(y_b, S_b) \Big|^2 \overline{d\alpha},$$

where $y_1, \ldots, y_b$ is the sequence for which the last integral achieves its maximal value. Again using the geometric-arithmetic mean inequality and our choice of R and H, we obtain by induction

$$\begin{aligned}
\mathcal{I}_2 &\leq B^2\,(RH)^{-(k-1)k/2} \int\limits_{-(RH)^{k-1}}^{(RH)^{k-1}} \cdots \int\limits_{-1}^{1} \big| \mathbb{E}\, G(y, S) \big|^{2b} \overline{d\alpha} \\
&\leq B^2\, \mathcal{I}_b(R) \\
&\leq B^2\, D_{\tau_0+1}\, R^{2b-\triangle(\tau_0+1)}\, Q^{2b}(S,1) \\
&\leq 0.25\, c_5^{-1}\, D_{\tau_0+1}\, P^{2b-\triangle(\tau_0+1)}\, Q^{2b}(S,1).
\end{aligned} \tag{11-10}$$

Thus, (11-6)–(11-8) and (11-10) together imply (0-2), if we choose the function D_τ such that the inequalities (11-9) and

$$D_\tau \geq 2^{k+2k\tau}\, H^{k^2(k+1)/2}$$

hold. Obviously this choice of D_τ is possible.

This completes the proof of Theorem 1.

Proof of Theorem 2. We use the same notation and the similar arguments as in the proof of Theorem 1. Let $S_1, S_1', S_2, S_2', \ldots$ denote independent copies of S.

Similar to the proof of Theorem 1 we can show that (0-3) holds for all m, $1 \leq m \leq k$, when

$$P \leq (2^{-3k} \cdot M)^{2/(k+1)}. \tag{12-1}$$

Moreover by Lemma 2 we have

$$\begin{aligned}\mathcal{I}_1(P) &\le c\ \sup \mathbb{P}\begin{pmatrix} S_1 - S_1' \in I_1(P^{-k+1}) \\ S_1 \in I_0(P);\ S_1' \in I_0(P)\end{pmatrix} \\ &\le c\ \sup \mathbb{P}\begin{pmatrix} S_1 - S_1' \in I_1(1) \\ S_1 \in I_0(P);\ S_1' \in I_0(P)\end{pmatrix} \\ &\le 2\, c\, P\, Q^2(S,1). \end{aligned} \tag{12-2}$$

Thus we get (0-3) for $m = 1$ and any integer $P \ge 1$.
Furthermore, by geometric arguments on the plane $\mathbb{R}^2$ (see e.g. Prudnikov et al. (1988)) we obtain

$$\begin{aligned}\mathcal{I}_2(P) &\le c\, P^2\, Q^2(S,1) \\ &\qquad \times \sup \mathbb{P}\begin{pmatrix} S_1 + S_2 \in I_1(1) \\ S_1^2 + S_2^2 \in I_2(1) \\ S_j \in I_0(P),\ j = 1,2 \end{pmatrix} \\ &\le c\, P^{5/2}\, Q^4(S,1). \end{aligned} \tag{12-3}$$

Thus (12-2) and (12-3) together show that (0-3) holds for $m = 1$ and $m = 2$ with arbitrary integer $P \ge 1$. We already remarked that (0-3) holds for all $m, 1 \le m \le k$, when P satisfies (12-1).

Let m_0 and P_0 denote natural numbers and assume that Theorem 2 holds for $m \le m_0$ and $P \le P_0$ as well as for $m \le m_0 + 1$ and $P < P_0$. Under these assumptions we shall prove that the Theorem will hold for $m \le m_0 + 1$ and $P = P_0$ as well. Thus by induction the inequality (0-3) will hold for all m and P.

From here on we can apply the induction arguments with respect to m and P almost in the same way as in the proof of Theorem 1. The difference is that now we call sequences either *regular* or *irregular* depending on the existence of a special increasing subsequence consisting of

$$T = [(m+1)/2] + 1 \tag{12-4}$$

numbers, where we put $m = m_0$ for simplicity.

In other words we call a sequence $y_1, \dots, y_{m+1}$ *regular* if there exists an increasing subsequence consisting of T of the numbers $y_1, \dots, y_{m+1}$ such that the difference between any two consecutive terms is larger than 1. A sequence is called *irregular*, if such a sequence does not exist.

In the same way as in the proof of Theorem 1 we show the inequalities

$$\mathcal{I}_{m+1}(P) \le c_6\, \mathcal{I}_{m+1}(RH) \tag{12-5}$$

and

$$\mathcal{I}_{m+1}(RH) \le 2\, (\mathcal{I}_1 + \mathcal{I}_2), \tag{12-6}$$

where

$$H = (2k)^4\, c_6^{1/2},\ R = [PH^{-1} + 1], \tag{12-7}$$

$\mathcal{I}_1$ and $\mathcal{I}_2$ correspond to regular and irregular sequences, respectively. We have for $\mathcal{I}_1$ and $|A| < P$ similarly as in the proof of Theorem 1

$$\begin{aligned}
&t^{-2(m_0+1-T)}\, H^{-2(m+1)}\, \mathcal{I}_1 \\
&\le c\, \sup_1 \mathbb{P}\begin{pmatrix} F(k; S, S'; T+1, m+1, 0) \in (I_1(P^{-k+1}), \dots, I_k(1)) \\ S_j \in I_0(P^{1-1/T}),\ S'_j \in I_0(P^{1-1/T}),\ j = \overline{T+1, m+1} \end{pmatrix} \\
&\quad \times \sup_2 \mathbb{P}\begin{pmatrix} F(T; S, S'; 1, T, -A) \in (I^{(1)}(P^{1-1/T}), \dots, I^{(T)}(P^{T-1})) \\ S_j \in \cup_{x=1}^{2R} K^{-1}(x + 2Ry_j);\ S'_j \in \cup_{x=1}^{2R} K^{-1}(x + 2Ry_j),\ j = \overline{1, T} \end{pmatrix} \\
&\le c\, \sup_1 \mathbb{P}\begin{pmatrix} F(k; S, S'; T+1, m+1, 0) \in (I_1(P^{(1-1/T)(-k+1)}), \dots, I_k(1)) \\ S_j \in I_0(P^{1-1/T}),\ S'_j \in I_0(P^{1-1/T}),\ j = \overline{T+1, m+1} \end{pmatrix} \\
&\quad \times P^{(3T-1)/2}\, Q^{2T}(S, 1).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\mathcal{I}_1 &\le c\, M_m\, Q^{2(m+1)}(S, 1) \\
&\quad \times P^{2(m+1-T)/T + \frac{1}{2}(3m+3-3T-1)(1-1/T) + \frac{1}{2}(3T-1)} \\
&\le 0.25\, c_6^{-1}\, M_{m+1}\, P^{(3(m+1)-1)/2}\, Q^{2(m+1)}(S, 1),
\end{aligned} \tag{12-8}$$

provided that M_m is chosen such that

$$M_{m+1} \ge 4\, c\, c_6\, M_m,$$

where c_6 and c are the constants from (12-7) and (12-8) respectively, and

$$2(m+1-T)/T - 1/2 - (3(m+1-T) - 1)/(2T) \le 0. \tag{12-9}$$

By choice of T we have that (12-9) holds.

To estimate $\mathcal{I}_2$ we have to repeat the arguments of Theorem 1 and by choice of H and T (see (12-7) and (12-4) respectively) we get

$$\mathcal{I}_2 \le 0.25\, c_6^{-1}\, M_{m+1}\, P^{(3(m+1)-1)/2}\, Q^{2(m+1)}(S, 1). \tag{12-10}$$

The inequalities (12-5), (12-6), (12-8), (12-10) together imply Theorem 2 which completes the proof.

References

Arkhipov, G.I., Karatsuba, A.A., and Chubarikov, V.N., *The theory of multiple trigonometric sums*, Nauka, Moscow (Russian) (1987).

Esseen, C.-G., *On the concentration function of a sum of independent random variables*, Z. Wahrscheinlichkeitstheor. verw. Geb. **9** (1968), 290–308.

Götze, F., *Asymptotic expansions for bivariate von Mises functionals*, Z. Wahrscheinlichkeitstheor. verw. Geb. **50** (1979), 333–355.

Götze, F., Prohorov, Yu.V., and Ulyanov, V.V., *Bounds for characteristic functions of polynomials in asymptotically normal variables*, SFB 343 "Discrete Strukturen in der Mathematik", Preprint 94-006, Universität Bielefeld (1994 (to appear in Russian Math. Surveys)).

Hengartner, W., and Theodorescu, R,, *Concentration functions*, Academic Press N.-Y., London, (1973).

Linnik Yu., V., *On Weyl's sums*, Mat Sbornik (Rec.Math.) **12**, 23-39.

Prudnikov, A.P., Brychkov, Yu. A, and Marichev, O.I., *Integrals and series*, Gordon and Breach, N.-Y. **1-2,** (1988).

Stechkin, S.B., *On mean values of the modulus of a trigonometric sum*, Proc. Steklov Inst. Math. **134** (1975), 321–350.

Vinogradov, I.M, *Selected Works*, Springer–Verlag, Berlin, (1985).

Wooley, T.D., *On Vinogradov's mean value theorem*, Mathematika **39** (1992), 379–399.

Zaitsev, A.Yu., *Estimates for the closeness of successive convolutions of multidimensional symmetric distributions*, Probab. Th. Rel. Fields **79** (1988), 175–200.

Zaitsev, A.Yu., *Multidimensional generalized method of triangular functions*, J. Soviet Math. **43** (1988), 2797–2810.