OPTIMAL RATES OF CONVERGENCE IN FUNCTIONAL LIMIT THEOREMS FOR QUADRATIC FORMS

V. Bentkus[1] F. Götze[1]

University of Bielefeld

July 1995

Abstract. Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random vectors taking values in a k-dimensional real linear space $\mathbf{R}^k$. Assume that $\mathbf{E}\, X = 0$ and that X is not concentrated in a proper subspace of $\mathbf{R}^k$. Let G denote a mean zero Gaussian random variable with the same covariance operator as that of X. We investigate the distributions of non-degenerated quadratic forms $Q[S_N]$ of the normalized sums $S_N = N^{-1/2}\,(X_1 + \dots + X_N)$ and show that

$$\sup_x \Big|\, \mathbf{P}\,\{\, Q[S_N] \le x \,\} - \,\mathbf{P}\,\{\, Q[G] \le x \,\}\,\Big| = \mathcal{O}\big(N^{-1}\big),$$

provided that $k \ge 9$ and the fourth moment of X exists. The bound $\mathcal{O}\big(N^{-1}\big)$ is optimal and improves, e.g., the well-known bound $\mathcal{O}\big(N^{-k/(k+1)}\big)$ due to Esseen (1945). The result extends to the case of random vectors taking values in a Hilbert space.

1. Introduction

Let $\mathbf{R}^k$ denote the k-dimensional space of real vectors $x = (x_1, \dots, x_k)$ with scalar product $\langle \cdot, \cdot \rangle$ and norm $|x|^2 = \langle x, x \rangle = x_1^2 + \dots + x_k^2$. Since our results are independent of the dimension (provided $k \ge 9$), it will be convenient to denote by $\mathbf{R}^\infty$ a real separable Hilbert space. Thus, $\mathbf{R}^\infty$ consists of all real sequences $x = (x_1, x_2, \dots)$ such that $|x|^2 = x_1^2 + x_2^2 + \dots < \infty$.

Let $X, X_1, X_2, \dots$ be a sequence of i.i.d. random variables taking values in $\mathbf{R}^k$. If $\mathbf{E}\, X = 0$ and $\mathbf{E}\,|X|^2 < \infty$, then the sums

$$S_N = N^{-1/2}\,(X_1 + \dots + X_N)$$

1991 *Mathematics Subject Classification.* Primary 62E20; secondary 60F05.

Key words and phrases. Central Limit Theorem, concentration inequalities, convergence rates, Berry–Esseen bounds, Edgeworth expansions, multidimensional spaces, Hilbert spaces, quadratic forms, ellipsoids, hyperboloids, lattice point problem.

[1]Research supported by the SFB 343 in Bielefeld.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

converge weakly to a mean zero Gaussian random variable, say G, such that its covariance is equal to the covariance of X. We shall assume that G (or, equivalently, X) is not concentrated on a subspace of $\mathbf{R}^k$, i.e., that $\mathbf{P}\{G \in \mathbf{L}\} < 1$ whenever $\mathbf{L} \subset \mathbf{R}^k$ is a proper closed linear subspace of $\mathbf{R}^k$.

For a linear symmetric and bounded operator $Q : x \mapsto Qx$ mapping $\mathbf{R}^k$ into $\mathbf{R}^k$, define the quadratic form $Q[x] = \langle Qx, x\rangle$. Assume that the operator Q is non-degenerated, i.e., that $\ker Q = \{0\}$.

Notice that the distribution of the quadratic form $Q[G]$ may be represented up to a shift as the distribution of a finite (resp., infinite) weighted sum of squares of i.i.d. standard Gaussian variables, for $k < \infty$ (resp., $k = \infty$).

Theorem 1.1. *Let* $\mathbf{E}\,X = 0$ *and* $\beta_4 = \mathbf{E}\,|X|^4 < \infty$. *Assume that* $k \geq 9$ *or* $k = \infty$. *Then*

$$\sup_x \Big|\, \mathbf{P}\{Q[S_N] \leq x\} - \mathbf{P}\{Q[G] \leq x\}\,\Big| = \mathcal{O}\big(N^{-1}\big).$$

The constant in this bound depends on β_4, *the eigenvalues of* Q *and the covariance operator of* G *only.*

Remark. A rather straightforward inspection of the proofs shows that Theorem 1.1 holds for $k = 8$ with $\mathcal{O}(N^{-1}\ln^{\delta} N)$ instead of $\mathcal{O}(N^{-1})$, for some $\delta > 0$.

For an orthonormal system of vectors $e_1, \ldots, e_s \in \mathbf{R}^k$ and a random variable $X \in \mathbf{R}^k$, consider the condition

$$\mathbf{P}\Big\{\,|\widetilde{X} - e_j| \leq (100\,s)^{-1}\Big\} \geq p, \qquad \mathbf{P}\Big\{\,|\widetilde{X} - Qe_j| \leq (100\,s)^{-1}\Big\} \geq p, \tag{1.1}$$

for all $1 \leq j \leq s$, where $\widetilde{X}$ denotes the symmetrization of X, i.e., $\widetilde{X}$ has the distribution $\mathcal{L}(\widetilde{X}) = \mathcal{L}(X_1 - X_2)$.

Introduce the concentration function

$$Q(X;\,\lambda) = Q(X;\,\lambda;\,Q) = \sup_{a,\,x} \mathbf{P}\big\{\,x \leq Q[X + a] \leq x + \lambda\,\big\}, \quad \text{for} \quad \lambda \geq 0.$$

Theorem 1.2. *Assume that the operator* Q *is isometric. If* $s \geq 9$ *and* X *satisfies* (1.1) *with some* $p > 0$, *then, for sums* $Z_N = X_1 + \cdots + X_N$,

$$Q(Z_N;\,\lambda) \leq \frac{c_s}{p\,N}\,\max\big\{1;\,\lambda\,\big\}. \tag{1.2}$$

The bound (1.2) may be useful for investigations of infinitely divisible approximations. Notice that the bound (1.2) is an analogue for $Q[Z_N]$ of the classical bound $\mathcal{O}\big(N^{-1/2}\max\{1;\,\lambda\}\big)$ for the concentration of sums (see, e.g., Th. 9 of Ch. 3 in Petrov (1975)). Furthermore, the assumption that Q is isometric, i.e., that $Q^2 = I$, does not restrict the generality. Indeed, any symmetric operator Q

may be decomposed as $Q = Q_+ Q_0 Q_+$, where Q_0 is isometric and Q_+ is non-negative. Thus, $Q(X;\, \lambda;\, Q) = Q(Q_+ X;\, \lambda;\, Q_0)$, and we may apply Theorem 1.2 provided $Q_+ X$ satisfies (1.1) (with Q_0 instead of Q).

Remark 1.3. Theorem 1.2 is applicable to any $X \in \mathbf{R}^k$, $k \geq 9$, which is not concentrated in an affine subspace of $\mathbf{R}^k$. For such X, it implies

$$Q(Z_N;\, \lambda) \leq \frac{c_s}{p\, N}\ \max\bigl\{ m;\, \lambda \bigr\}, \tag{1.3}$$

where the natural number m is chosen such that $m^{-1/2} Z_m$ satisfies (1.1). Such an m exists (see the proof of (1.3) in Section 2).

The bound of Theorem 1.1 is optimal since the distribution function of $|S_N|^2$ (for bounded $X \in \mathbf{R}^k$) may have jumps of precise order $\mathcal{O}\bigl(N^{-1}\bigr)$, for all $1 \leq k \leq \infty$. See, for example, Bentkus and Götze (1994a). In that paper a similar bound $\mathcal{O}(N^{-1})$ as in Theorem 1.1 is proved even for $k \geq 5$ assuming that Q is diagonal on the subspace spanned by five coordinates of X, which have to be stochastically independent. Both results are based on discretization (i.e., a reduction to lattice valued real random variables) and symmetrization techniques. The independence assumption allowed to apply an adaption of the Hardy–Littlewood circle method. For the general case described in Theorem 1.1, we had to develop a new tool—a multiplicative inequality for characteristic functions.

Theorem 1.1 and the method of its proof is related to the well known lattice point problem for conic sections in additive number theory. Assume that $\mathbf{R}^k$ is finite dimensional and that $\bigl\langle Qx, x \bigr\rangle > 0$, for $x \neq 0$. Define the ellipsoid

$$E_s = \bigl\{\, x \in \mathbf{R}^k :\ Q[x] \leq s \,\bigr\}, \qquad\qquad \text{for} \quad s \geq 0.$$

Let $\mathbb{Z}^k$ be the standard lattice of points with integer coordinates in $\mathbf{R}^k$. For a (measurable) set $E \subset \mathbf{R}^k$ let $\operatorname{vol} E$ denote the Lebesgue measure of E. By $\operatorname{vol}_{\mathbb{Z}} E$ we shall denote the lattice volume of E, that is the number of points in $E \cap \mathbb{Z}^k$.

The following result corresponds to Theorem 1.1.

Theorem 1.4 (Bentkus and Götze (1994b, 1995)). *For* $k \geq 9$,

$$\sup_{a \in \mathbf{R}^k} \left| \frac{\operatorname{vol}_{\mathbb{Z}} (E_s + a) - \operatorname{vol} E_s}{\operatorname{vol} E_s} \right| = \mathcal{O}\left(\frac{1}{s} \right), \qquad\qquad \textit{for} \quad s \geq 1.$$

The constant in $\mathcal{O}(s^{-1})$ *depends on the dimension* k *and on the lengths of axes of the ellipsoid* E_1 *only.*

Theorem 1.4 solves the lattice point problem for $k \geq 9$, and it improves the classical estimate $\mathcal{O}(s^{-k/(k+1)})$ due to Landau (1915), just as Theorem 1.1 improves the bound $\mathcal{O}(N^{-k/(k+1)})$ by Esseen (1945) for the probabilistic problem.

There exists an extensive literature about convergence rates in the CLT in multi-dimensional and Hilbert spaces and on the lattice point problem. See Bentkus and Götze (1994a,b, 1995) for a discussion and references.

We conclude the Introduction by a brief description of basic elements of the proof—a discretization procedure, a double large sieve and multiplicative inequalities.

Let $\varepsilon_1, \varepsilon_2, \dots$ denote i.i.d. symmetric Rademacher random variables. Let $\delta > 0$ and $e_1, \dots, e_s \in \mathbf{R}^k$. We say that a discrete random variable $Z \in \mathbf{R}^k$ belongs to the class

$$\Gamma = \Gamma(\delta;\, e_1, \dots, e_s)$$

if Z is distributed as $\varepsilon_1 z_1 + \dots + \varepsilon_s z_s$, with some (non-random) $z_1, \dots, z_s \in \mathbf{R}^k$ such that $|z_j - e_j| \le \delta$, for all $1 \le j \le s$.

For a bounded and measurable function $H : \mathbf{R}^k \to \mathbf{B}$ taking values in a Banach space $\big(\mathbf{B}, |\cdot|_{\mathbf{B}}\big)$, define the norm $|H|_\infty = \sup_x \big|H(x)\big|_{\mathbf{B}}$.

Discretization. Assume that a random variable $W \in \mathbf{R}^k$ is independent of the sum $V = X_1 + \dots + X_N$. Let $\widetilde{X}$ denote the symmetrization of X, i.e., $\widetilde{X}$ has the distribution of $X_1 - X_2$. Assume that

$$\mathbf{P}\left\{|\widetilde{X} - e_j| \le \delta\right\} \ge p > 0, \qquad \text{for all } 1 \le j \le s.$$

Then, for any $d \ge 0$ and natural $0 \le K \le pN/2$,

$$\big|\,\mathbf{E}\,H(2V + W)\,\big| \le c_d\,(pN)^{-d}|H|_\infty + \sup_{Z_1,\dots,Z_K}\ \sup_b \big|\,\mathbf{E}\,H(Z_1 + \dots + Z_K + W + b)\,\big|$$

(cf. Lemmas 5.1 and 5.2 of Section 5) where the first sup (respectively, the second) is taken over all sequences $Z_1, \dots, Z_K$ of independent random variables of the class Γ (respectively, over all $b \in \mathbf{R}^k$).

This discretization allows to reduce the estimation of the characteristic function

$$f(t) = \mathbf{E}\,\exp\big\{\,it\,Q[S_N - a]\,\big\}, \qquad \text{for} \quad t \in \mathbf{R},$$

to the estimation of functions like

$$\varphi(t) = \mathbf{E}\,\exp\big\{\,it\,Q[S'_N]\,\big\},$$

where $S'_N = N^{-1/2}\,(Z_1 + \dots + Z_N)$ is a sum of independent (non-identically distributed!) random variables $Z_1, \dots, Z_N$ of class Γ. Notice that norms of the random variables $Z_1, \dots, Z_K$ are bounded from above by a constant independent of N.

The symmetrization inequality (see Lemma 4.1) reduces the estimation of the function φ to the estimation of the characteristic function

$$\mathbf{E}\,\exp\big\{\,it\,\langle T_N, T'_N\rangle\,\big\}, \tag{1.4}$$

where T_N and T'_N denote independent random variables, which are normalized sums of i.i.d. random variables of class Γ, taking values in a s-dimensional subspace of $\mathbf{R}^k$.

Let us choose $\delta = (100\,s)^{-1}$ and $e_1, \ldots, e_s \in \mathbf{R}^k$ to be an orthornomal system. The double large sieve (see Lemma 3.2) allows to estimate (1.4) and to obtain:

$$\big|\,\varphi(t)\,\big| \ll |t|^{-s/2}, \text{ for } |t| \le \sqrt{N}, \qquad \big|\,\varphi(t)\,\big| \ll \Big|\,\frac{t}{N}\,\Big|^{s/2}, \text{ for } |t| \ge \sqrt{N}.$$

These inequalities imply that

$$\big|\,f(t)\,\big| \ll |t|^{-s/2}, \text{ for } |t| \le \sqrt{N}, \qquad \big|\,f(t)\,\big| \ll \Big|\,\frac{t}{N}\,\Big|^{s/2}, \text{ for } |t| \ge \sqrt{N}. \tag{1.5}$$

Since for Gaussian X,

$$\big|\,f(t)\,\big| \sim |t|^{-s/2}, \qquad\qquad \text{for } |t| \ge 1,$$

the first bound in(1.5) is precise (up to a constant).

The inequalities (1.5) allow to prove in Theorem 1.1 only error bounds $\mathcal{O}(N^{-\alpha})$, for some $\alpha < 1$. This is due to possible oscillations of $\big|\,f(t)\,\big|$ between zero and 1, as $|t| \sim N$. The oscillations are restricted by the following *multiplicative inequality* (cf. Lemma 7.1): for any $t \in \mathbf{R}$,

$$\big|\,\varphi(t)\,\varphi(t+\delta)\,\big| \ll \delta^{-s/2}, \qquad\qquad \text{for } 0 < \delta \le \sqrt{N}, \tag{1.6}$$

and

$$\big|\,\varphi(t)\,\varphi(t+\delta)\,\big| \ll \Big(\,\frac{\delta}{N}\,\Big)^{s/2}, \qquad\qquad \text{for } \delta \ge \sqrt{N}. \tag{1.7}$$

Notice that the right-hand sides of (1.6) and (1.7) are independent of t. In other words, (1.6) and (1.7) reflect a certain stationarity in the behavior of the characteristic function φ (and f as well): if $\big|\,\varphi(t_0)\,\big|$ is sufficiently large, then $\big|\,\varphi(t)\,\big|$ is bounded from above (near t_0) similarly as it is bounded near $t = 0$. The inequalities (1.6) and (1.7) guarantee that the distance between maxima of $\big|\,\varphi(t)\,\big|$ is sufficiently large, and that the integral of $\big|\,\varphi(t)/t\,\big|$ around its maxima is $\mathcal{O}(N^{-1})$ provided that $s \ge 9$.

Section 2 contains the formulation and an outline of the proof of the main result which extends Theorem 1.1—instead of $Q[S_N]$ we consider $Q[S_N - a]$, for $a \in \mathbf{R}^k$. In Section 3 we extend the well-known double large sieve used in number theory to arbitrary (unbounded) probability distributions. Section 4 contains some simple but useful auxiliary Lemmas. Section 5 contains a description of discretization of expectations. In Section 6 we prove estimates for characteristic functions. The proofs of these estimates are based on conditioning, discretization, as well as on the double large sieve. Finally, Section 7 contains the crucial multiplicative inequality for the characteristic functions. This is an extension of an inequality for trigonometric sums introduced in Bentkus and Götze (1994b, 1995).

Notation. By $c, c_1, \dots$ we shall denote generic absolute positive constants. If a constant depends on, say s, then we shall point out the dependence as c_s or $c(s)$.

By $[\alpha]$ we shall denote the integer part of a number α.

By $\varepsilon_1, \varepsilon_2, \dots$ we shall denote i.i.d. symmetric Rademacher random variables.

We assume that all random variables are independent in aggregate if the contrary is not clear from the context. By $\overline{X}$ and $X_1, X_2, \dots$ we shall denote independent copies of a random variable X. By $\widetilde{X}$ we shall denote a symmetrization of X, for example $\widetilde{X} = X - \overline{X}$. By $\mathbf{I}\{A\}$ we denote the indicator of an event A. The expectation $\mathbf{E}_Y$ with respect to a random variable Y we define as the conditional expectation

$$\mathbf{E}_Y f(X, Y, Z, \dots) = \mathbf{E}\left(f(X, Y, Z \dots) \,\middle|\, X, Z, \dots\right)$$

given all random variables but Y.

Acknowledgment. We would like to thank R. Ahlswede, Yu. Prohorov and A. Zaitsev for helpful discussions.

2. Statement and proof of the result

In order to simplify the notation we shall assume throughout that the symmetric operator Q is isometric, i.e., that $Q^2 = I$ is the identity operator I. This will not restrict the generality, since any symmetric operator Q can be represented as $Q = Q_+ Q_0 Q_+$, where Q_+ is a non-negative operator and $Q_0^2 = I$, and we may replace the random vector X by $Q_+ X$, the Gaussian random vector G by $Q_+ G$, etc.

Let us introduce the Edgeworth correction $\varkappa = \varkappa(x) = \varkappa_a(x)$, where $x \in \mathbf{R}$ and $a \in \mathbf{R}^k$. If $\mathbf{E}\,|X|^3 < \infty$ and $k \geq 9$, it is a function of bounded variation in x, which has the characteristic function

$$\widehat{\varkappa}(t) = 4\,(it)^2\,\mathbf{E}\,\exp\{Q[G-a]\}\big(3\,\langle G-a, QX\rangle\,\langle QX, X\rangle + 2it\,\langle G-a, QX\rangle^3\big).$$

An application of the symmetrization Lemma 4.1 and Lemma 3.2 shows that this characteristic function is integrable for $k \geq 9$, and thus $\varkappa$ is well defined. See Bentkus and Götze (1994a) for representations of $\varkappa$ by an integral over $\mathbf{R}^k$ whenever $k < \infty$ (as in Bhattacharya and Rao (1986)). Furthermore, notice that $\varkappa = 0$ if $a = 0$ or if $\mathbf{E}\,\langle X, y\rangle^3 = 0$, for all $y \in \mathbf{R}^k$. In particular, $\varkappa = 0$ if X is symmetric.

We shall estimate

$$\Delta_N = \sup_x \left| \mathbf{P}\left\{Q[S_N - a] < x\right\} - \mathbf{P}\left\{Q[G-a] < x\right\} - \frac{\varkappa}{6\sqrt{N}} \right|,$$

where $a \in \mathbf{R}^k$.

In the case $a = 0$, the next Theorem 2.1 is equivalent to Theorem 1.1.

Theorem 2.1. *Let* $\mathbf{E}\, X = 0$ *and* $\beta_4 = \mathbf{E}\,|X|^4 < \infty$. *If* $k = \infty$, *then*

$$\Delta_N = \mathcal{O}\Big(\big(1 + |a|^6\big)\, N^{-1}\Big).$$

If the dimension $k \geq 9$ *is finite, then*

$$\Delta_N = \mathcal{O}\Big(\big(1 + |a|^3\big)\, N^{-1}\Big).$$

Remark. It is likely that the result of Theorem 2.1 remains valid for $k \geq 5$. We need the assumption $k \geq 9$ for estimation of integrals over the characteristic function of the quadratic form for Fourier frequencies $N^{3/5} \ll |t| \ll N$. All other estimates are valid assuming that $k \geq 5$.

Remark. In cases where X and Q are degenerated, Theorem 2.1 may be applied as well. Any symmetric operator Q may be decomposed as $Q = Q_+ Q_0 Q_+$, where Q_+ is a non-negative symmetric operator, and where Q_0 is a partial isometry, i.e., Q_0 is isometric on a subspace, say $\mathbf{L} \subset \mathbf{R}^k$, and $Q_0 = 0$ on the orthogonal complement to $\mathbf{L}$. Let P be the orthogonal projector of $\mathbf{R}^k$ onto $\mathbf{L}$. Then Theorem 2.1 is still applicable provided the covariance operator of the random variable PQ_+X has 9 positive eigenvalues.

Let us define the following quantity which will appear in our bounds. Let $s \leq k$ be a natural number. For an orthonormal system of vectors $e_1, \ldots, e_s \in \mathbf{R}^k$, define $p = \min\{p_1,\, p_2\}$, where

$$p_1 = \min_{1 \leq j \leq N} \mathbf{P}\left\{\, |G - e_j| \leq (100\, s)^{-1} \right\}, \qquad p_2 = \min_{1 \leq j \leq N} \mathbf{P}\left\{\, |G - Qe_j| \leq (100\, s)^{-1} \right\}. \tag{2.1}$$

The number p is positive since the Gaussian vector G is not concentrated in any proper closed linear subspace of $\mathbf{R}^k$ and therefore $\mathbf{P}\left\{ G \in B \right\} > 0$, for any ball $B \subset \mathbf{R}^k$ with positive radius.

Introduce the distribution function $F(x) = \mathbf{P}\left\{ Q[S_N - a] < x \right\}$ and its characteristic function

$$f(t) = \mathbf{E}\, \exp\big\{\, it\, Q[S_N - a] \,\big\}, \qquad \text{for} \quad t \in \mathbf{R}.$$

Write $t_0 = c_0(s)\, p^{-1} (p^2\, N/\beta_4)^{2/s}$, where $c_0(s)$ is a sufficiently large positive constant. Introduce the integral

$$I = \frac{i}{2\pi}\ \mathrm{V.P.} \int_{|t| \leq t_0} \exp\big\{ -ixt \big\} f(t)\ \frac{dt}{t},$$

where Cauchy's Principal Value is defined as

$$\mathrm{V.P.} \int u(t)\, dt = \lim_{h \downarrow 0} \int_{|t| \geq h} u(t)\, dt.$$

Theorem 2.2. *Let* $\mathbf{E}\,X = 0$. *Assume that* $k \geq 9$ *is finite or infinite. Then*

$$\Big| F(x) - \frac{1}{2} - I \Big| \leq \frac{c_s\,\beta_4}{p^2\,N}\,,$$

where p *is defined by* (2.1), $9 \leq s \leq k$, *and* $\beta_4 = \mathbf{E}\,|X|^4$.

Proof of Theorem 2.1. It is known (see Bentkus and Götze (1994a), Lemmas 2.3–2.5) that

$$\delta_N = \sup_x \Big| \frac{1}{2} + I - \mathbf{P}\left\{ Q[G-a] < x \right\} - \frac{\varkappa}{6\sqrt{N}} \Big|$$

satisfies

$$\delta_N = \mathcal{O}\Big(\big(1 + |a|^6\big)\, N^{-1} \Big), \qquad \text{for } k = \infty,$$

and

$$\delta_N = \mathcal{O}\Big(\big(1 + |a|^3\big)\, N^{-1} \Big), \qquad \text{for } 9 \leq k < \infty,$$

provided that $\beta_4 < \infty$. This result together with Theorem 2.2 implies Theorem 2.1.

For $s > 0$, introduce the function

$$M_s(t;N) = \begin{cases} |t|^{-s/2}, & \text{for} \quad |t| \leq \sqrt{N}, \\ \big(|t|/N\big)^{s/2}, & \text{for} \quad |t| \geq \sqrt{N}. \end{cases} \tag{2.2}$$

Notice that

$$c_1(s)\,\Big(|t|^{-s/2} + \big(|t|/N\big)^{s/2} \Big) \leq M_s(t;N) \leq c_2(s)\,\Big(|t|^{-s/2} + \big(|t|/N\big)^{s/2} \Big). \tag{2.3}$$

Proof of Theorem 2.2. For a sufficiently large constant c_s, let m be the minimal natural number such that $m \geq c_s\,\beta_4/p$. Write

$$K = c_1(s)\,N/m \qquad \text{and} \qquad N_0 = p\,K = c_1(s)\,p\,N/m.$$

The constant $c_1(s)$ will be specified later. In this notation we have to prove that

$$\Big| F(x) - \frac{1}{2} - I \Big| \leq \frac{c_s}{N_0}\,.$$

Indeed, $c_2(s)\,p^2\,N/\beta_4 \leq N_0 \leq c_3(s)\,p^2\,N/\beta_4$.

We shall now express F in terms of its Fourier transform. A smoothing inequality of Prawitz (1972) implies (see Bentkus and Götze (1994a), Section 4) that

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\ \mathrm{V.P.} \int_{-K}^{K} \exp\big\{ -ixt \big\} f(t)\ \frac{dt}{t} + R, \tag{2.4}$$

for any $K > 0$ and any distribution function F with characteristic function f, where

$$|R| \le \frac{1}{K} \int_{-K}^{K} |f(t)|\, dt.$$

Write

$$t_0 = N_0^{2/s}/p, \qquad t_1 = \sqrt{N_0}/p, \qquad t_2 = c_4(s)\, N_0^{1-2/s}/p.$$

Notice that $0 < t_0 < t_1 < t_2 < K$, for $N_0 \ge c_5(s)$. The function $M_s\big(pt; N_0\big)$ assumes its minimal value at $t = t_1$. Splitting the integrals over t and using $1/K \le 1/|t|$, for $|t| \le K$, we derive from (2.4) that

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\ \mathrm{V.P.} \int_{|t|\le t_0} \exp\big\{-ixt\big\} f(t)\ \frac{dt}{t} + R_1,$$

with $|R_1| \le I_1 + I_2 + 2\, I_3 + 2\, I_4$, where we denote

$$I_1 = \frac{1}{K} \int_{|t|\le t_1} \big|\, f(t)\,\big|\, dt, \qquad I_2 = \int_{t_0 \le |t| \le t_1} \big|\, f(t)\,\big|\ \frac{dt}{|t|}\,,$$

and

$$I_3 = \int_{t_1 \le |t| \le t_2} |f(t)|\ \frac{dt}{|t|}\,, \qquad I_4 = \int_{t_2 \le |t| \le K} |f(t)|\ \frac{dt}{|t|}\,.$$

To prove the Theorem it is sufficient to show that each of the integrals I_j is bounded from above by c_s/N_0.

To bound the integrals I_1, I_2 and I_3 we shall use the bound of Lemma 6.1, which can be written as

$$\big|\, f(t)\,\big| \le c_s\, M_s\big(pt; N_0\big).$$

We have $\big|\, f(t)\,\big| \le c_s\, \min\Big\{1;\, M_s\big(pt; N_0\big)\Big\}$, and

$$I_1 \le \frac{c_s}{K} \int_0^{1/p} dt + \frac{c_s}{K} \int_{1/p}^{\infty} \frac{dt}{(p\,t)^{s/2}} = \frac{c_s}{p\,K} = \frac{c_s}{N_0}\,.$$

Similarly

$$I_2 \le c_s \int_{t_0}^{t_1} \frac{1}{(p\,t)^{s/2}}\ \frac{dt}{t} \le c_s \int_{p\,t_0}^{\infty} \frac{dt}{t^{1+s/2}} = \frac{c_s}{N_0}\,,$$

and

$$I_3 \le c_s \int_0^{t_2} \Big(\frac{p\,t}{N_0}\Big)^{s/2}\ \frac{dt}{t} \le c_s \Big(\frac{p\,t_2}{N_0}\Big)^{s/2} = \frac{c_s}{N_0}\,.$$

In order to bound I_4 we shall apply Corollary 5.5. Combined with Remark 6.2, it implies

$$I_4 \le c_d(s)\, N_0^{-d} \log N_0 + \sup_{\Lambda,\, b} \int_{t_2 \le |t| \le K} \sqrt{\varphi(c_s\, p\, t\, r\, N_0^{-1})}\, \frac{dt}{|t|} \tag{2.5}$$

with

$$\varphi(\tau) = \Big| \mathbf{E}\, \exp\Big\{ \tau\, r^{-1}\, Q[\Lambda_r + Q\Lambda_r' + b] \Big\} \Big|^2,$$

where the sums Λ_r and Λ_r' are defined by (7.1), and $r \le c_s\, N_0$. Changing variables and choosing appropriate constants $c_1(s)$, $c_4(s)$ in the definitions of t_2 and K, we have

$$\int_{t_2 \le |t| \le K} \sqrt{\varphi(c_s\, t\, p\, r\, N_0^{-1})}\, \frac{dt}{|t|} \le \int_{N_0^{(s-2)/s} \le |\tau| \le N_0} \sqrt{\varphi(\tau)}\, \frac{d\tau}{|\tau|}.$$

By Lemma 7.1,

$$\varphi(\tau+\delta)\, \varphi(\tau-\delta) \le c_s\, M_s^2(\delta;\, r) \le c_s\, M_s^2(\delta;\, N_0), \qquad \text{for any} \quad \delta > 0.$$

The following result (see Lemma 6.1 in Bentkus and Götze (1994b)) now implies the desired bound for I_4. Assume that the function $\psi \ge 0$ is continuous, for $t \ge 0$, and $\psi(0) = 1$. Furthermore, assume that there exists a number $\lambda \ge 1$ such that, for some $s \ge 9$,

$$\psi(t)\, \psi(t+\delta) \le \lambda^2 M_s^2(\delta;\, N), \qquad \text{for all} \quad t \ge 0 \quad \text{and} \quad \delta \ge 0.$$

Then

$$\int_{N^{(s-2)/s}}^{N} \sqrt{\psi(t)}\, \frac{dt}{t} \le \frac{c_s\, \lambda}{N}.$$

Therefore, choosing $d = 2$ in (2.5), we obtain $I_4 \le c_s/N_0$, that concludes the proof of Theorem 2.2.

Proof of Remark 1.3. Let us prove (1.3). Assume that $U = m^{-1/2} Z_m$ satisfies (1.1). Let $mr \le N$. Denote $V_r = U_1 + \cdots + U_r$, where U_j are independent copies of U. Then we can write $Z_N = m^{1/2} V_r + b$, where the (random) shift b is independent of V_r. An application of Theorem 1.2 to the sums V_r concludes the proof of (1.3). The existence of the desired $m = m\big(\mathcal{L}(X)\big)$ can be proved as follows. Let $\tau > 0$. Split the distribution $\mu = \mathcal{L}(\widetilde{X}) = u\,\nu + v\,\vartheta$, with the conditional distributions

$$\nu(A) = \mathbf{P}\left\{ \widetilde{X} \in A \,\middle|\, |\widetilde{X}| \le \tau \right\}, \qquad \vartheta(A) = \mathbf{P}\left\{ \widetilde{X} \in A \,\middle|\, |\widetilde{X}| > \tau \right\},$$

and parameters $u = \mathbf{P}\left\{ |\widetilde{X}| \le \tau \right\}$ and $v = 1 - u$. If the dimension $k < \infty$, then the distribution ν is not concentrated in a linear subspace provided τ is sufficiently

large. If $k = \infty$, then denote the minimal closed linear subspace supporting ν by $\mathbf{L}_\tau$. In this case $\dim \mathbf{L}_\tau \to \infty$, as $\tau \to \infty$. In both cases we may choose (and fix) τ such that $\dim \mathbf{L}_\tau \geq 9$. Notice that the choice of τ implies that $u > 0$. Writing μ^m for the m-fold convolution of the measure μ, we have

$$\mathcal{L}(m^{-1/2}\,\widetilde{S}_m)(\,\cdot\,) = (u\,\nu + v\,\vartheta)^m(m^{1/2}\,\cdot\,) \geq u^m\,\nu^m(m^{1/2}\,\cdot\,).$$

The measure ν is the distribution of a bounded and symmetric random variable, which is not concentrated in a closed subspace of $\mathbf{L}_\tau$. Hence, it satisfies the Central Limit Theorem, and the limiting Gaussian distribution, say γ, is not concentrated in a subspace of $\mathbf{L}_\tau$. Therefore, for sufficiently large $m = m(\nu, B)$, we have

$$\mathcal{L}(m^{-1/2}\,\widetilde{S}_m)(B) \geq u^m\,\nu^m(m^{1/2}B) \geq 2^{-1}\,u^m\,\gamma(B) > 0,$$

for any ball $B \subset \mathbf{L}_\tau$ since the Gaussian measure of balls with positive radius is positive. Thus, in order to prove (1.1), it suffices to choose an arbitrary orthonormal system in $\mathbf{L}_\tau$, say $e_1, \ldots, e_s$.

Proof of Theorem 1.2. Rescaling and using a well-known inequality (see, for example Petrov (1975), Lemma 3 of Ch. 3), we have

$$Q(Z_N;\,\lambda) \leq 2\,\sup_a \max\Big\{\,\frac{\lambda}{N}\,;\;\frac{1}{T}\,\Big\}\int_{-T}^{T}\big|\,f(t)\,\big|\,dt,$$

for any $T > 0$. Let $r = \big\lceil\, pN/(4\,s)\,\big\rceil$. Write

$$t_1 = \sqrt{N/p}, \qquad\qquad t_2 = r^{-2/s}N/4,$$

and choose $T = N/4$. Using $1 \leq T/|t|$, for $|t| \leq T$, we have

$$\int_{|t|\leq T}\big|\,f(t)\,\big|\,dt \leq I_1 + I_3 + I_4,$$

where

$$I_1 = \int_{-t_1}^{t_1}\big|\,f(t)\,\big|\,dt, \qquad I_3 = T\int_{t_1\leq|t|\leq t_2}\big|\,f(t)\,\big|\,\frac{dt}{|t|} \qquad I_4 = T\int_{t_2\leq|t|\leq T}\big|\,f(t)\,\big|\,\frac{dt}{|t|}\,.$$

To prove the Theorem, it is sufficient to show that each of the integrals I_1, I_3 and I_4 is bounded from above by c_s/p.

By Theorem 6.3, $\big|\,f(t)\,\big| \leq c_s\,M_s\big(pt;pN\big)$. Therefore, as in the proof of Theorem 2.2, we derive $I_1 \leq c_s/p$ and $I_3 \leq c_s/p$.

Thus, it remains to estimate I_4. By Corollary 5.5, choosing $r = \lceil pN/(4\,s)\rceil$ (and replacing K by N and t by tN^{-1}), we have

$$\int\limits_{t_1\le|t|\le T} \big|\,f(t)\,\big|\,\frac{dt}{|t|} \le I + c_d(s)\,(p\,N)^{-d}\,\log\frac{T}{t_2}\,,$$

where

$$I = \sup_{\Lambda}\sup_{b}\int\limits_{t_2\le|t|\le T}\sqrt{\varphi(4\,t\,N^{-1}\,r)}\,\frac{dt}{|t|} = \sup_{\Lambda}\sup_{b}\int\limits_{4\,t_1\,N^{-1}\,r\le|t|\le 4\,T\,N^{-1}\,r}\sqrt{\varphi(t)}\,\frac{dt}{|t|}$$

with

$$\varphi(t) = \Big|\,\mathbf{E}\,\exp\big\{\,itr^{-1}\,Q[\Lambda_r + Q\Lambda_r' + b]\,\big\}\,\Big|^2.$$

Choose $d = 2$. Then it remains to show that $I \le c_s/r$. Notice that $4\,t_1\,N^{-1}\,r = r^{\,1-2/s}$ and $4\,T\,N^{-1}\,r = r$. By Lemma 7.1,

$$\varphi(t+\gamma)\,\varphi(t-\gamma) \le c_s\,M_s^2(\gamma;\,r), \qquad \text{for any} \quad \gamma > 0.$$

Hence we obtain

$$I = \int\limits_{r^{\,1-2/s}\le|t|\le r}\sqrt{\varphi(t)}\,\frac{dt}{|t|} \le \frac{c_s}{r}\,,$$

by an application of Lemma 6.1 in Bentkus and Götze (1994b), similarly as in the proof of Theorem 2.2. Consequently, $I_4 \le c_s/p$, that concludes the proof of the Theorem.

3. An extension of the double large sieve for unbounded distributions

The large sieves of Linnik (1941) and Rényi (1962) (see also Halberstam and Roth (1966), Bombieri and Iwaniec (1986), Graham and Kolesnik (1991)) play a key role in some problems of analytic number theory. The following Lemma 3.1 presents a special probabilistic version of this bound for trigonometric sums.

In this section we shall assume that the space $\mathbf{R}^k$ is finite dimensional, i. e., $k < \infty$.

Let $|x|_\infty$ denote the max-norm $|x|_\infty = \max\limits_{1\le j\le k}|x_j\,|$.

Lemma 3.1. *Let $U, V \in \mathbf{R}^k$ be independent random variables with non-negative characteristic functions. Assume that the random variables $|U|_\infty \le T$ and $|V|_\infty \le R$ are bounded by some positive constants T and R such that $TR > 1$. Then*

$$\mathbf{E}\,\exp\big\{\,i\,\langle U, V\rangle\,\big\} \le c(k)\big(1 + (RT)^k\big)\,\mathbf{P}\,\big\{\,|TV|_\infty \le 1\,\big\}\,\mathbf{P}\,\big\{\,|RU|_\infty \le 1\,\big\}.$$

The next Lemma extends the double large sieve for unbounded random variables.

Lemma 3.2. *Let m denote an integer $m > k/2$. Assume that the independent random variables $U, V \in \mathbf{R}^k$ are sums of independent random variables U_j and V_j with non-negative characteristic functions, such that*

$$U = U_1 + \cdots + U_{2m+2} \qquad \text{and} \qquad V = V_1 + \cdots + V_{2m+2}.$$

Let $R > 0$ and $T > 0$ denote some positive real numbers. Write

$$A = 1 + \mathbf{E}\,\big|\,U/T\,\big|^{2m}, \qquad B = 1 + \mathbf{E}\,\big|\,V/R\,\big|^{2m},$$

and

$$C = 1 + (T\,R)^{-2m}, \qquad D = (T\,R)^k.$$

Then there exists a constant $c = c(m,k)$ such that

$$\mathbf{E}\,\exp\big\{\,i\,\big\langle U, V\big\rangle\,\big\} \le c\,A\,B\,C\,D \max_{1\le a\le 2m+1} \mathbf{P}\,\big\{\,|R\,U_a| \le 1\,\big\} \max_{1\le b\le 2m+1} \mathbf{P}\,\big\{\,|T\,V_b| \le 1\,\big\}.$$

Corollary 3.3. *For arbitrary positive numbers R_0 and T_0 write*

$$\overline{R}^2 = \min\big\{\,R^2;\,R_0^2\,\big\} \qquad \text{and} \qquad \overline{T}^2 = \min\big\{\,T^2;\,T_0^2\,\big\}.$$

Then, under the assumptions of Lemma 3.2, we have

$$\mathbf{E}\,\exp\big\{\,i\,\big\langle U, V\big\rangle\,\big\} \le c(m,k)\,A\,B\,C\,D \max_{1\le a\le 2m+1} I_a \max_{1\le b\le 2m+1} \bar{I}_b,$$

where we denote

$$I_a = \int_{|s|\le 1} \exp\big\{\,i\,\big\langle \overline{R}^2 U_a/R, s\big\rangle\,\big\}\,ds, \qquad \bar{I}_b = \int_{|s|\le 1} \exp\big\{\,i\,\big\langle \overline{T}^2 V_b/T, s\big\rangle\,\big\}\,ds.$$

Remark. Lemma 3.2. can be extended to vectors

$$T = (T_1, \ldots, T_k) \qquad \text{and} \qquad R = (R_1, \ldots, R_k)$$

with positive coordinates. Define

$$T^{-1} = \big(T_1^{-1}, \ldots, T_k^{-1}\big) \qquad \text{and} \qquad Ts = (T_1 s_1, \ldots, T_k s_k),$$

for $s = (s_1, \ldots, s_k) \in \mathbf{R}^k$. Then Lemma 3.2 still holds with

$$A = 1 + \mathbf{E}\,\big|\,T^{-1}U\,\big|^{2m}, \qquad B = 1 + \mathbf{E}\,\big|\,R^{-1}V\,\big|^{2m},$$

and

$$C = 1 + \sum_{j=1}^{k} (T_j\,R_j)^{-2m}, \qquad D = \prod_{j=1}^{k} T_j\,R_j.$$

Similar extensions hold for other results of the section.

To prove Lemma 3.2 and its Corollary 3.3 we need several lemmas which may be of separate interest.

Lemma 3.4. *Let U and V be independent random variables taking values in $\mathbf{R}^k$. Let $R > 0$ and $T > 0$. Let g and h denote positive functions. Assume that g as well as its inverse Fourier transform, say $\widehat{g}$, are Lebesgue integrable and $\mathbf{E}\, 1/g(V/R) < \infty$. Then*

$$\Big|\, \mathbf{E}\, \exp\big\{\, i\, \big\langle U, V\big\rangle\, \big\}\, \Big| \le c(k) J_1 J_2,$$

where

$$J_1 = \int_{\mathbf{R}^k} \left| h(s/T)\ \mathbf{E}\ \frac{\exp\{\, i\, \langle V, s\rangle\, \}}{g(V/R)} \right| ds,$$

$$J_2 = \sup_{s\in\mathbf{R}^k} \left| \int_{\mathbf{R}^k} \frac{\exp\{\, -i\, \langle \tau, s\rangle\, \}}{h(s/T)}\ \mathbf{E} \exp\big\{\, i\, \big\langle U, \tau\big\rangle\, \big\}\, g(\tau/R)\, d\tau \right|.$$

Proof of Lemma 3.4. Since g is the Fourier transform of $\widehat{g}$ we get

$$\begin{aligned}
\mathbf{E}\, \exp\big\{\, i\, \big\langle U, V\big\rangle\, \big\} &= \mathbf{E}\, \exp\big\{\, i\, \big\langle U, V\big\rangle\, \big\}\, \frac{g(V/R)}{g(V/R)} \\
&= c(k) \int_{\mathbf{R}^k} \mathbf{E}\ \frac{\exp\{\, i\, \langle V, U + \tau/R\rangle\, \}}{g(V/R)}\, \widehat{g}(\tau)\, d\tau \\
&= c(k)\, \mathbf{E} \int_{\mathbf{R}^k} \frac{\exp\{\, i\, \langle V, s\rangle\, \}}{g(V/R)}\, \widehat{g}(Rs - RU)\, R^k\, ds,
\end{aligned}$$

and

$$\Big|\, \mathbf{E}\, \exp\big\{\, i\big\langle U, V\big\rangle\, \big\}\, \Big| \le c(k) \int_{\mathbf{R}^k} \left| \mathbf{E}\ \frac{h(s/T) \exp\{\, i\, \langle V, s\rangle\, \}}{g(V/R)} \right| \left| \frac{\mathbf{E}\, \widehat{g}(Rs - RU)\, R^k}{h(s/T)} \right| ds.$$

Consequently,

$$\Big|\, \mathbf{E}\, \exp\big\{\, i\big\langle U, V\big\rangle\, \big\}\, \Big| \le c(k)\, J_1\, J_3, \qquad \text{where} \quad J_3 = \sup_{s\in\mathbf{R}^k} \left| \frac{\mathbf{E}\, \widehat{g}(Rs - RU)}{h(s/T)}\, R^k \right|.$$

Representing $\widehat{g}$ as the inverse Fourier transform of g, we obtain $J_3 \le c(k) J_2$, and the result of the Lemma follows.

Choosing $g(s) = h(s) = \big(1 + |s|^2\big)^{-m}$ in Lemma 3.4 and integrating by parts we obtain the following

Corollary 3.5. *Assume that $g(s) = \big(1 + |s|^2\big)^{-m}$. Furthermore, let $\mathbf{E}\, |U|^{2m} < \infty$ and $\mathbf{E}\, |V|^{2m} < \infty$, for some $m > k/2$. Then*

$$\Big|\, \mathbf{E}\, \exp\big\{\, i\, \big\langle U, V\big\rangle\, \big\}\, \Big| \le c(k)\, J_1\, \overline{J}_2,$$

where we denote

$$J_1 = \int_{\mathbf{R}^k} g(s/T) \Big| \mathbf{E} \left(1 + \big| V/R \big|^2\right)^m \exp\big\{ i \big\langle V, s \big\rangle \big\} \Big| \, ds,$$
$$\overline{J}_2 = \int_{\mathbf{R}^k} \Big| \big(1 - T^{-2} \Delta_\tau\big)^m \Big(\mathbf{E} \exp\big\{ i \big\langle U, \tau \big\rangle \big\} \, g(\tau/R) \Big) \Big| \, d\tau,$$

and where the Laplace operator $\Delta_\tau = \partial_1^2 + \dots + \partial_k^2$ *acts on the variables* τ.

Proof. Indeed,

$$\begin{aligned} J_2 &= \sup_{s \in \mathbf{R}^k} \left| \int_{\mathbf{R}^k} \left(1 + \big| s/T \big|^2\right)^m \exp\big\{ -i \big\langle \tau, s \big\rangle \big\} \, \mathbf{E} \exp\big\{ i \big\langle U, \tau \big\rangle \big\} \, g(\tau/R) \, d\tau \right| \\ &= \sup_{s \in \mathbf{R}^k} \left| \int_{\mathbf{R}^k} \Big(\big(1 - T^{-2} \Delta_\tau\big)^m \exp\big\{ -i(\tau, s) \big\} \Big) \, \mathbf{E} \exp\big\{ i \big\langle U, \tau \big\rangle \big\} \, g(\tau/R) \, d\tau \right| \\ &\le \overline{J}_2. \end{aligned}$$

Lemma 3.6. *Assume all conditions of Lemma* 3.2 *except positivity of the characteristic functions which is replaced by the assumption that the random variables have mean zero. Denote*

$$J_a = \int_{\mathbf{R}^k} g(s/R) \Big| \mathbf{E} \exp\big\{ i \big\langle U_a, s \big\rangle \big\} \Big| \, ds, \qquad \overline{J}_b = \int_{\mathbf{R}^k} g(s/T) \Big| \mathbf{E} \exp\big\{ i \big\langle V_b, s \big\rangle \big\} \Big| ds.$$

Then

$$\Big| \mathbf{E} \, \exp\big\{ i \big\langle U, V \big\rangle \big\} \Big| \le c(m,k) \, A \, B \, C \max_{1 \le a \le 2m+1} J_a \max_{1 \le b \le 2m+1} \overline{J}_b.$$

Proof. We shall derive the result from Corollary 3.5. It is sufficient to show that there exists a constant $c = c(m,k)$ such that

$$\Big| \mathbf{E} \, (1 + |V/R|^2)^m \exp\big\{ i \big\langle V, s \big\rangle \big\} \Big| \le c \, B \sum_{1 \le b \le 2m+1} \Big| \mathbf{E} \, \exp\big\{ i \big\langle V_b, s \big\rangle \big\} \Big|, \tag{3.1}$$

$$\Big| (1 - T^{-2} \Delta_\tau)^m \Big(u(\tau) \, g(\tau/R) \Big) \Big| \le c \, A \, C \sum_{1 \le a \le 2m+1} \Big| \mathbf{E} \, \exp\big\{ i \big\langle U_a, \tau \big\rangle \big\} \Big| \, g(\tau/R), \tag{3.2}$$

where we denote $u(\tau) = \mathbf{E} \, \exp\big\{ i \big\langle U, \tau \big\rangle \big\}$.

Let us prove (3.1). We have

$$\Big| \mathbf{E} \left(1 + |V/R|^2\right)^m \exp\big\{ i \big\langle V, s \big\rangle \big\} \Big| \le c \sum_{s=0}^{m} R^{-2\,s} \, I_s,$$

where

$$I_s = \Big| \mathbf{E}\,|V|^{2\,s}\ \exp\big\{\, i\,\langle V, s\rangle\,\big\}\Big|.$$

In order to estimate I_s recall that $V = V_1 + \cdots + V_{2m+2}$. Thus

$$|V|^{2\,s} = \left(\sum_{l=1}^{2m+2}\sum_{r=1}^{2m+2}\langle V_l, V_r\rangle\right)^s = \sum_{1\le l_1,\ldots,l_s\le 2m+2}\ \sum_{1\le r_1,\ldots,r_s\le 2m+2}\ \prod_{q=1}^{s}\langle V_{l_q}, V_{r_q}\rangle,$$

and instead of I_s we have to bound

$$J \stackrel{\rm def}{=} \left| \mathbf{E}\prod_{q=1}^{s}\langle V_{l_q}, V_{m_q}\rangle\,\exp\big\{\, i\,\langle V, s\rangle\,\big\}\right|.$$

The number of (different) V_j in the product $\prod_{q=1}^{s}\langle V_{l_q}, V_{r_q}\rangle$ does not exceed $2\,s \le 2m$. Thus, this product is independent of at least one of variables $V_1, \ldots, V_{2m+1}$, say V_b. Therefore, by conditioning,

$$J \le \left(\mathbf{E}\prod_{q=1}^{s}|V_{l_q}|\ |V_{m_q}|\right)\Big|\ \mathbf{E}\,\exp\big\{\, i\,\langle V_b, s\rangle\,\big\}\Big|,$$

since

$$\Big|\mathbf{E}_{V_b}\ \exp\big\{\, i\,\langle V, s\rangle\,\big\}\Big| = \Big|\mathbf{E}_{V_b}\ \exp\big\{\, i\,\langle V_b, s\rangle\,\big\}\Big| = \Big|\ \mathbf{E}\,\exp\big\{\, i\,\langle V_b, s\rangle\,\big\}\Big|.$$

Applying the geometric-arithmetic mean inequality we get

$$\mathbf{E}\prod_{q=1}^{s}|V_{l_q}|\ |V_{r_q}| \le c\sum_{q=1}^{s}\Big(\ \mathbf{E}\,|V_{l_q}|^{2\,s} + \mathbf{E}\,|V_{r_q}|^{2\,s}\Big) \le c\ \mathbf{E}\,|V|^{2\,s}$$

since by Jensen's inequality

$$\mathbf{E}\,|X|^{2\,s} = \mathbf{E}\,|X + \mathbf{E}\,Y|^{2\,s} = \mathbf{E}\,\big|\mathbf{E}_Y\,(X+Y)\big|^{2\,s} \le \mathbf{E}\,|X+Y|^{2\,s}$$

provided that X and Y are independent and $\mathbf{E}\,Y = 0$. Collecting these estimates, we obtain (3.1).

The proof of (3.2) is a little bit more involved. We have

$$\Big|(1 - T^{-2}\Delta_\tau)^m\Big(u(\tau)g(\tau/R)\Big)\Big| \le c\sum_{s=0}^{m} T^{-2\,s}\,I_s,$$

where

$$I_s = \Big| \Delta_\tau^s \big(u(\tau)\, g(\tau/R) \big) \Big|.$$

Differentiating the product we get

$$I_s \le c \sum_{|\alpha|+|\beta|=2\,s} \Big| \partial_1^{\alpha_1} \dots \partial_k^{\alpha_k} u(\tau) \Big|\, R^{-|\beta|} \, \Big| \big(\partial_1^{\beta_1} \dots \partial_k^{\beta_k} g \big)(\tau/R) \Big|.$$

Here $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\beta = (\beta_1, \dots, \beta_k)$ denote non-negative integer multi-indices with $|\alpha| = \alpha_1 + \dots + \alpha_k$. Recall that $g(s) = \big(1 + |s|^2\big)^{-m}$. Therefore

$$\Big| \big(\partial_1^{\beta_1} \dots \partial_k^{\beta_k} g \big)(s) \Big| \le c(k,m)\, g(s).$$

Furthermore, writing the random variable $U = (U_{(1)}, \dots, U_{(k)})$ in coordinates of $\mathbf{R}^k$, we have

$$\Big| \, \partial_1^{\alpha_1} \dots \partial_k^{\alpha_k} u(\tau) \, \Big| = \Big| \, \mathbf{E}\, U_{(1)}^{\alpha_1} \dots U_{(k)}^{\alpha_k} \exp\big\{ \, i \, \big\langle U, \tau \big\rangle \, \big\} \Big|.$$

The product $U_{(1)}^{\alpha_1} \dots U_{(k)}^{\alpha_k}$ is a polynomial of order $|\alpha| \le 2\,s \le 2m$ in U. Thus arguing similarly as in the proof of (3.1) we obtain

$$\Big| \, \mathbf{E}\, U_{(1)}^{\alpha_1} \dots U_{(k)}^{\alpha_k} \exp\big\{ \, i \, \big\langle U, \tau \big\rangle \, \big\} \Big| \le c \, \mathbf{E}\, |U|^{|\alpha|} \sum_{1 \le a \le 2m+1} \Big| \, \mathbf{E}\, \exp\big\{ \, i \, \big\langle U_a, s \big\rangle \, \big\} \, \Big|,$$

and (3.2) follows since

$$\sum_{s=0}^{m} \sum_{|\alpha|+|\beta|=2\,s} T^{-2\,s}\, R^{-|\beta|}\, \mathbf{E}\, |U|^{|\alpha|} \le c\, A\, C.$$

Proof of Lemma 3.2. We shall derive the result from Lemma 3.6. Let $X \in \mathbf{R}^k$ denote a random variable with non-negative characteristic function. It is sufficient to show that

$$J \stackrel{\text{def}}{=} \int_{\mathbf{R}^k} g(s/R)\, \mathbf{E}\, \exp\big\{ \, i \, \big\langle X, s \big\rangle \, \big\}\, ds \le c(k,m)\, R^k \; \mathbf{P}\, \big\{ \, |R\, X| \le 1 \, \big\}. \tag{3.3}$$

Introduce the concentration function

$$Q(X, 1) = \sup_{s \in \mathbf{R}^k} \; \mathbf{P}\, \big\{ \, |X - s|_\infty \le 1 \, \big\}.$$

Assume that a function $p(s) = p_1(s_1) \dots p_k(s_k)$ can be represented as a product such that $p_j : \mathbf{R} \to \mathbf{R}$ are even, non-negative, non-increasing on $[0, \infty)$ functions such

that $p_j(t) = p_j(0)$, for $|t| \le 1$, for all $1 \le j \le k$. It is known that (see Zaitsev (1988a), Lemma 5.3, or (1988b), Lemma 2.6)

$$\int_{\mathbf{R}^k} p(s)\,\mathbf{E}\exp\bigl\{\,i\,\bigl\langle Z,s\bigr\rangle\,\bigr\}\,ds \le c(k)\,Q(Z,1)\int_{\mathbf{R}^k} p(s)\,ds, \tag{3.4}$$

for any random variable Z with non-negative characteristic function.

Let us apply the bound (3.4) to the integral in (3.3). Define $p(s) = \theta(s_1)\ldots\theta(s_k)$, where the function $\theta : \mathbf{R} \to \mathbf{R}$ satisfies

$$\theta(t) = 1, \text{ for } |t| \le 1, \qquad \text{and} \qquad \theta(t) = 2/(1+t^2), \text{ for } |t| \ge 1.$$

Then $g(s) \le c(k,m)\,p(s)$ and a change of the variable of integration together with (3.4) implies $J \le c\,R^k\,Q(RX,1)$. To finish the proof of (3.3) we should replace the norm $|\cdot|_\infty$ by $|\cdot|$. To this end notice that $Q(X,1) \le c(k)\,\delta^{-k}\,Q(X,\delta)$, for any $0 < \delta \le 1$. Furthermore, the inequality

$$Q(X,1) \le c(k)\,\mathbf{P}\,\bigl\{\,|X|_\infty \le 1\,\bigr\}$$

(see Zaitsev (1988a)) holds for any random variable X with non-negative characteristic function.

Proof of Corollary 3.3. Obviously

$$\mathbf{P}\,\bigl\{\,|RU_a| \le 1\,\bigr\} \le\ \mathbf{P}\,\bigl\{\,\overline{R}^2\,|U_a/R| \le 1\,\bigr\}$$

and the result follows from Lemma 3.2 by an application of

$$\mathbf{P}\,\bigl\{\,|X| \le 1\,\bigr\} \le c(k)\int_{|s|\le 1}\Bigl|\ \mathbf{E}\ \exp\bigl\{\,i\,\bigl\langle X,s\bigr\rangle\,\bigr\}\Bigr|\,ds,$$

see Esseen (1968), Lemma 6.1.

4. Technical Lemmas

The following symmetrization inequality (see Bentkus and Götze (1994a), Lemma 3.1) improves a well-known inequality due to Götze (1979).

Lemma 4.1. *Let $L, C \in \mathbf{R}^k$. Let Z, U, V and W denote independent random variables taking values in $\mathbf{R}^k$. Denote by*

$$P(x) = \bigl\langle Qx, x\bigr\rangle + \bigl\langle L, x\bigr\rangle + C, \qquad \text{for } x \in \mathbf{R}^k,$$

a real-valued polynomial of second order. Then

$$2\Bigl|\ \mathbf{E}\ \exp\bigl\{\,it\,P(Z+U+V+W)\,\bigr\}\Bigr|^2 \le\ \mathbf{E}\ \exp\bigl\{\,2it\,\bigl\langle Q\widetilde{Z},\widetilde{U}\bigr\rangle\,\bigr\} + \mathbf{E}\ \exp\bigl\{\,2it\,\bigl\langle Q\widetilde{Z},\widetilde{V}\bigr\rangle\,\bigr\}.$$

We shall need the following auxiliary bound for the convergence rate in the CLT in finite dimensional and Hilbert spaces for expectations of smooth functions.

Lemma 4.2. *Assume that X is symmetric. Let a function $\varphi : \mathbf{R}^k \to \mathbf{R}$ be four times Frechet differentiable. Then there exists an absolute constant c such that*

$$\big|\, \mathbf{E}\, \varphi(S_N) - \mathbf{E}\, \varphi(G) \,\big| \le c\, \beta_4\, N^{-1} \sup_{x \in \mathbf{R}^k} \big\|\, \varphi^{(4)}(x) \,\big\|.$$

Proof. The result can be proved like Theorem 1.2 in Bentkus et al (1991). For the sake of completeness we include a sketch of the proof. Denote

$$W_j = N^{-1/2}\, (X_1 + \cdots + X_{j-1} + G_{j+1} + \cdots + G_N).$$

Then

$$\big|\, \mathbf{E}\, \varphi(S_N) - \mathbf{E}\, \varphi(G) \,\big| \le \delta_1 + \cdots + \delta_N,$$

where, by the Taylor expansions in powers of $X_j/\sqrt{N}$ and $G_j/\sqrt{N}$,

$$\delta_j = \big|\, \mathbf{E}\, \varphi(W_j + X_j/\sqrt{N}) - \mathbf{E}\, \varphi(W_j + G_j/\sqrt{N}) \,\big| \le c\, N^{-2}\, \mathbf{E}\, |X|^4 \sup_{x \in \mathbf{R}^k} \big\|\, \varphi^{(4)}(x) \,\big\|.$$

Lemma 4.3. *Assume that X is symmetric. Let $\delta > 0$ and $e \in \mathbf{R}^k$. Then there exists an absolute constant c such that*

$$2\, \mathbf{P}\, \big\{\, |S_N - e| \le 2\delta \,\big\} \ge \mathbf{P}\, \big\{\, |G - e| \le \delta \,\big\}, \quad \textit{for } N \ge c\, \delta^{-4} \beta_4 \big/ \mathbf{P}\, \big\{\, |G - e| \le \delta \,\big\}.$$

Proof. We shall apply Lemma 4.2. Consider a function $\varphi : \mathbf{R}^k \to \mathbf{R}$ with infinitely many bounded derivatives such that

$$0 \le \varphi \le 1, \qquad \varphi(x) = 1, \text{ for } |x| \le 1, \qquad \varphi(x) = 0, \text{ for } |x| \ge 2.$$

Applying Lemma 4.2 we have

$$2\, \mathbf{P}\, \big\{\, |S_N - e| \le 2\delta \,\big\} \ge 2\, \mathbf{E}\, \varphi\Big(\, \frac{S_N - e}{\delta} \,\Big) \ge 2\, \mathbf{E}\, \varphi\Big(\, \frac{G - e}{\delta} \,\Big) - c\, N^{-1} \delta^{-4} \beta_4$$

$$\ge 2\, \mathbf{P}\, \big\{\, |G - e| \le \delta \,\big\} - \mathbf{P}\, \big\{\, |G - e| \le \delta \,\big\} = \mathbf{P}\, \big\{\, |G - e| \le \delta \,\big\}$$

provided N satisfies the condition of the Lemma.

5. On discretization of expectations

Assume that the symmetrization $\widetilde{U}$ of a random variable $U \in \mathbf{R}^k$ satisfies

$$\mathbf{P}\left\{ |\widetilde{U} - e| \le \delta \right\} \ge p, \tag{5.1}$$

for some $\delta \ge 0$, $0 < p \le 1$ and $e \in \mathbf{R}^k$. Let $z_1, z_2, \ldots \in \mathbf{R}^k$ denote a (non-random) sequence such that

$$|z_j - e| \le \delta, \qquad \text{for all } j = 1, 2, \ldots \tag{5.2}$$

Recall that $\varepsilon_1, \varepsilon_2, \ldots$ denote i.i.d. symmetric Rademacher random variables.

For a bounded and measurable function $H : \mathbf{R}^k \to \mathbf{B}$ with values in a Banach space $\big(\mathbf{B}, |\cdot|_{\mathbf{B}}\big)$, define the norm $|H|_\infty = \sup_x \big|H(x)\big|_{\mathbf{B}}$.

Lemma 5.1. *Let $V_n = U_1 + \cdots + U_n$ be the sum of n independent copies of U such that (5.1) is fulfilled, and let $W \in \mathbf{R}^k$ denote a random variable independent of V_n. Write $r = \lceil np/2 \rceil$ and $T = \varepsilon_1 z_1 + \cdots + \varepsilon_r z_r$. Then, for any $d \ge 0$, we have*

$$\big|\,\mathbf{E}\,H(2V_n + W)\,\big| \le c_d\,(pn)^{-d}\,|H|_\infty + \sup_Z \sup_b \big|\,\mathbf{E}\,H(T + W + b)\,\big|$$

where $\sup_Z$ *(respectively,* $\sup_b$*) denotes the supremum over all sequences $z_1, z_2, \ldots$ of type (5.2) (respectively, over all $b \in \mathbf{R}^k$).*

Proof. Replacing H by the function $x \mapsto \mathbf{E}_W\,H(x+W)$, we can assume that $W = 0$.

Let $\alpha_1, \alpha_2, \ldots$ be a sequence of i.i.d. random variables such that

$$\mathbf{P}\left\{\alpha_1 = 0\right\} = \mathbf{P}\left\{\alpha_1 = 1\right\} = 1/2.$$

Then the sum

$$V^\alpha = \sum_{j=1}^n \big(\alpha_j\,U_j + (1-\alpha_j)\overline{U}_j\big)$$

has the same distribution as V_n. Notice that the random variables $\varepsilon_1, \varepsilon_2, \ldots$ defined by $\varepsilon_j = 2\alpha_j - 1$ are i.i.d. symmetric Rademacher random variables. Introduce the sum $V^\varepsilon = \varepsilon_1\widetilde{U}_1 + \cdots + \varepsilon_n\widetilde{U}_n$. Using the relation $\alpha_j = \frac{\varepsilon_j}{2} + \frac{1}{2}$ we may write $V^\alpha = \frac{1}{2}\,V^\varepsilon + b$, for some $b = b_n(U_1, \ldots, U_n)$, which is independent of $\varepsilon_1, \ldots, \varepsilon_n$. Conditioning we have

$$\big|\,\mathbf{E}\,H(2V_n)\,\big| \le \mathbf{E}\,|\psi|, \qquad \text{where} \quad \psi = \mathbf{E}_\varepsilon\,H(V^\varepsilon + b),$$

and where $\mathbf{E}_\varepsilon$ denotes the partial integration with respect to the distribution of $\varepsilon_1, \varepsilon_2, \ldots$

Introduce the independent events

$$A(l) = \bigl\{\, |\widetilde{U}_l - e| \le \delta \,\bigr\}, \qquad\qquad l = 1, 2, \ldots .$$

Notice that $\mathbf{P}\bigl\{ A(l) \bigr\} \ge p$ by the assumption (see (5.1)). Consider the event

$$B_n = \Bigl\{ \text{at least } \ \frac{np}{2} \ \text{ of events } A(1), \ldots, A(n) \text{ occur} \Bigr\},$$

and introduce Bernoulli random variables $\xi_l = \mathbf{I}\bigl\{ A(l) \bigr\}$. For the complement B_n^c of the event B_n, we have

$$\mathbf{P}\bigl\{ B_n^c \bigr\} = 1 - \mathbf{P}\bigl\{ B_n \bigr\} \le c_d\, (pn)^{-d}, \qquad\qquad \text{for any } d \ge 0. \tag{5.3}$$

Indeed, if $pn \le 1$, then (5.3) follows from the trivial bound $\mathbf{P}\bigl\{ B_n^c \bigr\} \le 1$. If $pn \ge 1$, then the Chebyshev inequality implies (denote $\eta_j = \xi_j - \mathbf{E}\,\xi_j$)

$$\mathbf{P}\bigl\{ B_n^c \bigr\} = \mathbf{P}\Bigl\{ \xi_1 + \cdots + \xi_n < \frac{np}{2} \Bigr\} \le c_d\, (pn)^{-2d}\, \mathbf{E}\,(\eta_1 + \cdots + \eta_n)^{2d},$$

and (5.3) follows from the inequality (see, e. g., Petrov (1987), Ch. 3, Th. 19)

$$\mathbf{E}\,(\eta_1 + \cdots + \eta_n)^{2d} \le c_d \Bigl(n\, \mathbf{E}\,|\eta_1|^{2d} + \bigl(n\, \mathbf{E}\,\eta_1^2 \bigr)^d \Bigr),$$

which is valid for any i.i.d. mean zero random variables $\eta_1, \ldots, \eta_n$. Notice that in our case $\mathbf{E}\,|\eta_1|^{2d} \le c_d\, p$.

Consequently, (5.3) yields

$$\mathbf{E}\,|\psi| \le c_d\, (pn)^{-d}\, |H|_\infty + \mathbf{E}\,\mathbf{I}\bigl\{ B_n \bigr\}\, \bigl|\, \psi \,\bigr|,$$

and it remains to show that

$$\mathbf{I}\bigl\{ B_n \bigr\}\, \bigl|\, \psi \,\bigr| \le \sup_Z \sup_b \bigl|\, \mathbf{E}\, H(T + b) \,\bigr|.$$

If the event B_n does not occur, then $\mathbf{I}\bigl\{ B_n \bigr\}\, |\psi| = 0$. If B_n occurs then at least $np/2$ of events $A(l)$ occur, say $A(l_1), \ldots, A(l_a)$ with some $a \ge np/2 \ge r$. Reorder random variables $\varepsilon_l\, \widetilde{U}_l$, $1 \le l \le n$, so that $A(1), \ldots, A(a)$ occur. Then we may write

$$\mathbf{I}\bigl\{ B_n \bigr\}\, |\psi| = \mathbf{I}\bigl\{ B_n \bigr\}\, \bigl|\, \mathbf{E}_\varepsilon\, H(\varepsilon_1 \widetilde{U}_1 + \cdots + \varepsilon_a \widetilde{U}_a + b + b_1) \,\bigr|$$

with some b_1 independent of $\varepsilon_1, \dots, \varepsilon_a$. Conditioning on ε_l with $l > r$, and replacing

$$\varepsilon_{r+1}\widetilde{U}_{r+1} + \dots + \varepsilon_a \widetilde{U}_a + b_1 + b$$

by the variable b of the Lemma, we conclude the proof.

Assume that the symmetrization of a random variable $Y \in \mathbf{R}^k$ satisfies

$$\mathbf{P}\left\{ |\widetilde{Y} - e_j| \le \delta \right\} \ge p, \qquad \mathbf{P}\left\{ |\widetilde{Y} - Qe_j| \le \delta \right\} \ge p, \tag{5.4}$$

for all $1 \le j \le s$. Let

$$\Upsilon_K = Y_1 + \dots + Y_K,$$

where $Y_1, Y_2, \dots$ denote independent copies of Y.

The next Lemma 5.2 allows in $\mathbf{E}\, H(\Upsilon_K)$ to replace the sum Υ_K by a sum of discrete bounded random variables of a very special type. Introduce the sums

$$\Lambda_r = Z_1 + \dots + Z_r, \qquad \Lambda'_r = Z'_1 + \dots + Z'_r, \tag{5.5}$$

where $Z_1, \dots, Z_r,\ Z'_1, \dots, Z'_r \in \mathbf{R}^k$ denote independent (non-identically distributed!) random variables of class $\Gamma(\delta;\, e_1, \dots, e_s)$.

Lemma 5.2. *Let $r = \lceil pK/(4s) \rceil$. Then, for any $d \ge 0$, we have*

$$\left| \mathbf{E}\, H(2\Upsilon_K + b) \right| \le c_d(s)\,(pK)^{-d}\,|H|_\infty + \sup_{\Lambda} \sup_{b} \left| \mathbf{E}\, H(\Lambda_r + Q\Lambda'_r + b) \right|,$$

where the supremum $\sup_{\Lambda}$ *(respectively* $\sup_{b}$*) is taken over all random variables Λ_r and Λ'_r of type* (5.5) *(respectively, over all $b \in \mathbf{R}^k$).*

Proof. Let $n = \lceil K/(2s) \rceil$. Introducing i.i.d. random variables $V_j, \overline{V}_j$ such that their common distribution is equal to $\mathcal{L}(Y_1 + \dots + Y_n)$, we may write

$$Y_1 + \dots + Y_K = V_1 + \dots + V_s + \overline{V}_1 + \dots + \overline{V}_s + b_1,$$

where b_1 is independent of $V_j, \overline{V}_j$, $1 \le j \le s$. Therefore

$$\left| \mathbf{E}\, H(2\Upsilon_K + b) \right| \le \sup_{b} \left| \mathbf{E}\, H(2V_1 + \dots + 2V_s + 2\overline{V}_1 + \dots + 2\overline{V}_s + b) \right|.$$

To prove the Lemma, it suffices to apply Lemma 5.1 (in two steps) $2s$ times to the last expectation.

Step 1—replacement of $2V_1 + \dots + 2V_s$ by a sum Λ_r of type (5.5). Choosing $e = e_1$ in Lemma 5.1, using assumption (5.4) we can estimate the expectation over $2V_1$ by a

supremum of expectations over $T_{e_1} = \varepsilon_1 z_1 + \cdots + \varepsilon_r z_r$, plus an error which is bounded from above by $c_d\,(pn)^{-d}\,|H|_\infty$. The vectors z_j satisfy $|e_1 - z_j| \le \delta$. Repeating these estimates with V_j and $e = e_j$, for $2 \le j \le s$, we can replace the expectation over the sum $2\,V_1 + \cdots + 2\,V_s$ by supremum of expectations over $T_{e_1} + \cdots + T_{e_s}$. The total additional error accumulates to

$$c_d(s)\,(pn)^{-d}\,|H|_\infty \le c_d(s)\,(pK)^{-d}\,|H|_\infty.$$

But the supremum of expectations over $T_{e_1} + \cdots + T_{e_s}$ equals to the supremum of expectations over Λ_r.

Step 2—replacement of $2\,\overline{V}_1 + \cdots + 2\,\overline{V}_s$ by sums Λ_r' of type (5.5). We shall repeat Step 1 with $\overline{V}_j$ replaced by V_j, and with Qe_j replaced by e_j, $1 \le j \le s$. This leads to an estimate of expectations over $2\,\overline{V}_1 + \cdots + 2\,\overline{V}_s$ by a supremum of expectations over $T_{Qe_1} + \cdots + T_{Qe_s}$, plus an error, which is bounded from above as in Step 1. To conclude the proof it remains to notice that the supremum of expectations over $T_{Qe_1} + \cdots + T_{Qe_s}$ equals to the supremum of expectations over $Q\Lambda_r'$. Indeed, for any $1 \le l \le s$, the sum T_{Qe_l} can be written as $T_{Qe_l} = \varepsilon_1 z_1 + \cdots + \varepsilon_r z_r$ with some z_j such that $|Qe_l - z_j| \le \delta$. Since $Q^2 = I$, the inequality $|Qe_1 - z_j| \le \delta$ is equivalent to $|e_1 - Qz_j| \le \delta$.

Let $H_t : \mathbf{R}^k \to \mathbf{B}$ denote a family of bounded functions indexed by $t \in \mathbf{R}$ such that the function $(t, x) \mapsto H_t(x)$ is measurable. In Lemmas 5.3 and 5.4 we shall apply the discretization procedure to the integrals $\int |\,\mathbf{E}\,H_t(S_N)|\,dt$.

Lemma 5.3. *Assume that the conditions of Lemma* 5.1 *hold. Then, for any* $d \ge 0$,

$$\int\limits_{A \le |t| \le B} \big|\,\mathbf{E}\,H_t(2V_n + W)\,\big|\,dt \le I + c_d\,(pn)^{-d} \int\limits_{A \le |t| \le B} |H_t|_\infty\,dt,$$

where

$$I = \sup_Z \sup_b \int\limits_{A \le |t| \le B} \big|\,\mathbf{E}\,H_t(T + W + b)\,\big|\,dt.$$

Proof. We shall proceed similarly as in the proof of Lemma 5.1. We can assume that $W = 0$. Then

$$\int_{A \le |t| \le B} \big|\,\mathbf{E}\,H_t(2V_n)\,\big|\,dt \le I_1, \qquad I_1 \stackrel{\text{def}}{=} \mathbf{E} \int_{A \le |t| \le B} |\psi|\,dt,$$

where

$$\psi = \mathbf{E}_\varepsilon\,H_t\Big(\frac{1}{2}\,V^\varepsilon + b\Big).$$

Define the event B_n similarly as in the proof of Lemma 5.1. Then

$$I_1 \le \mathbf{E}\, I_2 + c_d\,(p\,n)^{-d} \int_{A\le|t|\le B} |H_t|_\infty\, dt,$$

where

$$I_2 = \mathbf{I}\big\{\, B_n \,\big\} \int_{A\le|t|\le B} |\psi|\, dt.$$

Thus it remains to show that $I_2 \le I$. If the event B_n does not occur, there is nothing to prove. If it occurs then we may argue as in the proof of Lemma 5.1.

Lemma 5.4. *Under the condition and in notation of Lemma* 5.2*,*

$$\int_{A\le|t|\le B} \big|\, \mathbf{E}\, H_t(2\,\Upsilon_K + b)\,\big|\, dt \le I + c_d(s)\,(pK)^{-d} \int_{A\le|t|\le B} |H_t|_\infty\, dt,$$

where

$$I = \sup_{\Lambda} \sup_{b} \int_{A\le|t|\le B} \big|\, \mathbf{E}\, H_t(\Lambda_r + Q\Lambda_r' + b)\,\big|\, dt.$$

Proof. The proof of the Lemma repeats the proof of Lemma 5.2. The only difference is that instead of Lemma 5.1 it is necessary to use Lemma 5.3.

For integrals of the characteristic functions, Lemma 5.4 implies

Corollary 5.5. *Let* $0 < A \le B$*. Then, for* $r = \lceil pK/(4\,s)\,\rceil$ *and any* $d \ge 0$*,*

$$\int\limits_{A\le|t|\le B} \Big|\, \mathbf{E}\, \exp\big\{\, 2it\,Q[\Upsilon_K + b]\,\big\}\,\Big|\, \frac{dt}{|t|} \le I + c_d(s)\,(pK)^{-d}\, \log\frac{B}{A}\,,$$

where

$$I = \sup_{\Lambda} \sup_{b} \int\limits_{A\le|t|\le B} \sqrt{\varphi(4\,t\,r)}\; \frac{dt}{|t|}\,,$$

and $\varphi(t) = \Big|\, \mathbf{E}\, \exp\big\{\, itr^{-1}\,Q[\Lambda_r + QvL_r' + b]\,\big\}\,\Big|^2$.

Proof. It is sufficient to choose $H_t(x) = \frac{1}{t}\, \mathbf{E}\, \exp\big\{\, 4it\,Q[(x+b)/2]\,\big\}$ in Lemma 5.4.

6. Bounds for characteristic functions

The main result of the Section is the following

Lemma 6.1. *There exists a positive constant c_s such that*

$$\Big|\, \mathbf{E}\, \exp\big\{\, it\, Q[S_N - a]\,\big\}\,\Big| \le c_s\, M_s\big(pt; p^2\, N/\beta_4\big),$$

where p is defined by (2.1) *and the function M_s by* (2.2).

Lemma 6.1 is a corollary of the following Theorem 6.3, which is valid without any moment assumptions.

Let $e_1, \dots, e_s \in \mathbf{R}^k$ denote an orthonormal system. Assume that the symmetrization of a random variable $Y \in \mathbf{R}^k$ satisfies

$$\mathbf{P}\,\big\{\, |\widetilde{Y} - e_j| \le \delta\,\big\} \ge p, \qquad \mathbf{P}\,\big\{\, |\widetilde{Y} - Qe_j| \le \delta\,\big\} \ge p, \qquad\qquad \delta = \frac{1}{50\, s}\,, \tag{6.1}$$

for all $1 \le j \le s$. Let

$$\Upsilon_K = Y_1 + \dots + Y_K,$$

where $Y_1, Y_2, \dots$ denote independent copies of Y.

Remark 6.2. The Gaussian random variable G satisfies (2.1). Therefore, for any sufficiently large natural number m, the sum S_N can be represented as

$$S_N = (2\, m/N)^{1/2}\, \big(Y_1 + \dots + Y_K\big) + b = (2\, m/N)^{1/2}\, \Upsilon_K + b,$$

where the i.i.d. random variables $Y_1, \dots, Y_K$ satisfy (6.1), b is independent of Υ_K, and $K = \big\lceil N/(2\, m)) \big\rceil$. More precisely, it suffices that

$$m \ge c_s\, \beta_4/p, \tag{6.2}$$

with the sufficiently large constant c_s. Indeed, we can write

$$X_1 + \dots + X_N = Y_1 + \dots + Y_K + b = \Upsilon_K + b,$$

where $Y_1, \dots, Y_K$ are independent copies of Y such that

$$\mathcal{L}(Y) = \mathcal{L}\Big((2m)^{-1/2}\, (X_1 + \dots + X_m)\Big).$$

Condition (2.1) for G together with the CLT implies that $\widetilde{Y}$ satisfies (6.1) (with $p/2$ instead of p), for m from (6.2) (see Lemma 4.3).

Theorem 6.3. *Assume that* $Y \in \mathbf{R}^k$ *satisfies* (6.1). *Then*

$$\Big| \mathbf{E} \exp\big\{ it K^{-1} Q[\Upsilon_K - a] \big\} \Big| \le c_s \, M_s\big(pt; pK\big).$$

Proof of Lemma 6.1. By Remark 6.2, conditioning on b and Theorem 6.3, we obtain

$$\begin{aligned} \sup_a \Big| \mathbf{E} \exp\big\{ it\, Q[S_N - a] \big\} \Big| &\le \sup_b \Big| \mathbf{E} \exp\big\{ 2itm\, N^{-1} Q[\Upsilon_K + b] \big\} \Big| \\ &\le c_s \, M_s\big(2\,t\,m p N^{-1} K; pK/2\big) \le c_s \, M_s\big(pt; p^2 N/\beta_4\big), \end{aligned}$$

provided we choose the minimal m such that (6.2) holds and $K = \lceil N/m \rceil$. By an application of (2.3), we get rid of constants by arguments of the function M_s.

To prove Theorem 6.3 we need auxiliary Lemmas 6.4 and 6.5. Lemma 6.4 is a preparatory step for an application of the double large sieve in Lemma 6.5. In Section 7 we shall extend the methods of this Section for the proof of the multiplicative inequality.

In the proof of the next Lemma we shall combine discretization (see Lemma 5.2) with the symmetrization (see Lemma 4.1). An application of the geometric-arithmetic mean inequality will reduce the problem to the i.i.d. case.

Assume that W and W' are independent random variables of class $\Gamma\,(\delta;\, e_1, \dots, e_s)$. Let $W_1, \dots, W_r,\ W_1', \dots, W_r'$ denote independent copies of W and W' respectively. Introduce the sums

$$U = \widetilde{W}_1 + \dots + \widetilde{W}_q, \qquad\qquad V = \widetilde{W}_1' + \dots + \widetilde{W}_q'. \tag{6.3}$$

Lemma 6.4. *Denote* $q = \lceil pK/(8\,s) \rceil$ *and* $\beta = 8\,|t|\,K^{-1}$. *Then*

$$\Big| \mathbf{E} \exp\big\{ it\, K^{-1} Q[\Upsilon_K - a] \big\} \Big| \le c_d(s)\,(pK)^{-d} + \sup_{U,V} \sqrt{\mathbf{E} \exp\big\{ i\beta \,\big\langle U, V \big\rangle \big\}},$$

where $\sup\limits_{U,V}$ *denotes the supremum over all distributions of* U *and* V *given by* (6.3).

Proof. Choose $H(x) = \exp\big\{ 4it\, K^{-1} Q[(x-a)/2] \big\}$. Lemma 5.2 implies

$$\Big| \mathbf{E} \exp\big\{ it\, K^{-1} Q[\Upsilon_K - a] \big\} \Big| \le c_d(s)\,(pK)^{-d} + \sup_Z \sup_b |\varphi|,$$

where $\varphi = \mathbf{E} \exp\big\{ 4it\, K^{-1} Q[\Lambda_r + Q\Lambda_r' + b] \big\}$. Here $\sup\limits_Z$ (respectively $\sup\limits_b$) is taken over all random variables Λ_r and Λ_r' defined by (5.5) (respectively $b \in \mathbf{R}^k$).

To bound $|\varphi|^2$ we shall apply the symmetrization Lemma 4.1. Split $\Lambda_r = T+T'+b$ and $\Lambda'_r = R + R' + b'$ so that each of the sums T, T', R, R' contains q independent summands. Notice that in expectations below we may replace t by $|t|$, due to the symmetry of random variables. Then

$$2\,|\varphi|^2 \le \mathbf{E}\,\exp\big\{\, i\beta\,\big\langle \widetilde{T}, Q^2\widetilde{R}\big\rangle\,\big\} + \mathbf{E}\,\exp\big\{\, i\beta\,\big\langle \widetilde{T}, Q^2\widetilde{R}'\big\rangle\,\big\}.$$

Notice that $Q^2 = I$. Furthermore,

$$\sup_Z\ \mathbf{E}\,\exp\big\{\, i\beta\,\big\langle \widetilde{T}, \widetilde{R}\big\rangle\,\big\} = \sup_Z\ \mathbf{E}\,\exp\big\{\, i\beta\,\big\langle \widetilde{T}, \widetilde{R}'\big\rangle\,\big\}.$$

Thus,

$$|\varphi|^2 \le \sup_Z\ \mathbf{E}\,\exp\big\{\, i\beta\,\big\langle \widetilde{T}, \widetilde{R}\big\rangle\,\big\}.$$

Applying the geometric-arithmetic mean inequality we have

$$\begin{aligned}
\mathbf{E}\,\exp\big\{\, i\beta\,\big\langle \widetilde{T}, \widetilde{R}\big\rangle\,\big\} &= \mathbf{E}\,\prod_{j=1}^{q} \mathbf{E}_{\widetilde{W}_j}\,\exp\big\{\, i\beta\,\big\langle \widetilde{W}_j, \widetilde{R}\big\rangle\,\big\} \\
&\le \frac{1}{q}\sum_{j=1}^{q} \mathbf{E}\,\Big(\mathbf{E}_{\widetilde{W}_j}\,\exp\big\{\, i\beta\,\big\langle \widetilde{W}_j, \widetilde{R}\big\rangle\,\big\}\Big)^q \\
&\le \sup_Y\ \mathbf{E}\,\Big(\mathbf{E}_{\widetilde{W}}\,\exp\big\{\, i\beta\,\big\langle \widetilde{W}, \widetilde{R}\big\rangle\,\big\}\Big)^q \\
&= \sup_U\ \mathbf{E}\,\exp\big\{\, i\beta\,\big\langle U, \widetilde{R}\big\rangle\,\big\}. \qquad (6.4)
\end{aligned}$$

Arguing as in (6.4), we replace $\widetilde{R}$ by V in the last expectation, that concludes the proof of the Lemma.

Lemma 6.5. *Let U and V denote random variables defined by* (6.3). *Then*

$$\mathbf{E}\,\exp\big\{\, i\beta\,\big\langle U, V\big\rangle\,\big\} \le c_s\, M_s^2(\beta q; q), \qquad \textit{for } \beta \in \mathbf{R}. \qquad (6.5)$$

Proof. To prove the Lemma we shall apply the double large sieve.

There exists an s-dimensional subspace $\mathbf{L} \subset \mathbf{R}^k$ such that U (respectively V) can be written as sums of q independent copies of a random variable $\widetilde{W} \in \mathbf{L}$ (respectively, of $\widetilde{W}' \in \mathbf{L}$) such that

$$\widetilde{W} = \widetilde{\varepsilon}_1 z_1 + \cdots + \widetilde{\varepsilon}_s z_s \qquad \text{(respectively } \widetilde{W}' = \widetilde{\varepsilon}_1 z'_1 + \cdots + \widetilde{\varepsilon}_s z'_s),$$

where $z_j, z_j' \in \mathbf{L}$ satisfy

$$|z_j - e_j| \le \delta, \qquad |z_j' - e_j| \le 3\delta, \qquad 1 \le j \le s. \tag{6.6}$$

Recall that $U = \widetilde{W}_1 + \cdots + \widetilde{W}_q$ is the sum of q independent copies of the random variable $\widetilde{W}$,

$$\widetilde{W} = \widetilde{\varepsilon}_1 z_1 + \cdots + \widetilde{\varepsilon}_s z_s,$$

where the (non-random) $z_j \in \mathbf{R}^k$ satisfy (6.6). Thus we can choose $\mathbf{L}$ to be the linear span of $z_1, \ldots, z_s$. The condition (6.6) together with $\delta = \frac{1}{50\,s}$ and the orthonormality of the system $e_1, \ldots, e_s$ ensures that the linear subspace $\mathbf{L} \subset \mathbf{R}^k$ has dimension s. The sum V is the sum of q independent copies of a random variable, say $\widetilde{Y}$, such that $\widetilde{Y} = \widetilde{\varepsilon}_1 y_1 + \cdots + \widetilde{\varepsilon}_s y_s$, where $y_1, \ldots, y_s \in \mathbf{R}^k$ satisfy $|y_j - e_j| \le \delta$. Introduce the orthogonal projector $P : \mathbf{R}^k \to \mathbf{L}$. Then $x = Px = P^2 x$, for all $x \in \mathbf{L}$, and $\langle U, V\rangle = \langle U, PV\rangle$. Thus we may replace V by $PV \in \mathbf{L}$ such that $W' = PY$ and $z_j' = Py_j$. Hence, the inequality (6.6) for z_j' follows since P is an orthogonal projector and both z_j and y_j are near to e_j.

The inequalities (6.6) imply that, for all $z \in \mathbf{L}$,

$$\sum_{j=1}^{s} \langle z_j, z\rangle^2 \ge \tfrac{|z|^2}{4} \qquad \text{and} \qquad \sum_{j=1}^{s} \langle z_j', z\rangle^2 \ge \tfrac{|z|^2}{4}. \tag{6.7}$$

Indeed, any $z \in \mathbf{L}$ may be represented as $z = \sum_{j=1}^{s} a_j z_j$, with some real a_j since $\mathbf{L}$ is the linear span of $z_1, \ldots, z_s$. Therefore (6.7) can be proved by straightforward calculations comparing $|z|^2$ and sums in (6.7) with the norm $|z|_0^2 = \sum_{j=1}^{s} a_j^2$.

We return now to the proof of (6.5). Without loss of generality we can assume that $\beta \ge 0$. Write $\tau^2 = \beta q$, $\tau \ge 0$. Thus we have to prove that

$$\mathbf{E}\, \exp\bigl\{ i\tau^2 q^{-1} \langle U, V\rangle \bigr\} \le c_s\, M_s^2(\tau^2; q). \tag{6.8}$$

We can also suppose that $\tau \ge 1$. Indeed, if $0 \le \tau \le 1$ then (6.8) follows from the trivial bound $\mathbf{E}\exp\bigl\{ i\tau^2 q^{-1} \langle U, V\rangle \bigr\} \le 1$. For $\tau \ge 1$, let us apply the double large sieve bound of Corollary 3.3 for the s-dimensional space $\mathbf{L}$. Replace U and V in this Corollary by $\tau q^{-1/2} U$ and $\tau q^{-1/2} V$. Choose

$$m = s, \qquad R = T = \tau, \qquad R_0^2 = T_0^2 = \sqrt{q}/4.$$

Let $r = \bigl\lceil q/(2\,s+1) \bigr\rceil$. Split

$$U = U_1 + \cdots + U_{2\,s+2}, \qquad V = V_1 + \cdots + V_{2\,s+2}$$

so that U_j (respectively V_j) are i.i.d., for $1 \le j \le 2\,s+1$, with distributions equal to

$$\mathcal{L}(\widetilde{Y}_1 + \cdots + \widetilde{Y}_r) \qquad \Big(\text{respectively } \mathcal{L}(\widetilde{Y}_1 + \cdots + \widetilde{Y}_r)\Big).$$

The constants A, B, C and D of Corollary 3.3 satisfy

$$A \le c_s, \qquad B \le c_s, \qquad C \le c_s, \qquad D = \tau^{2\,s}.$$

For example, by the triangle inequality and the well-known moment inequalities for sums of real valued random variables, we have

$$A = 1 + \mathbf{E}\,\big|\,q^{-1/2}\,U\,\big|^{2\,s} \le 1 + c_s \sum_{j=1}^{s} |z_j|^{2\,s} q^{-s}\,\mathbf{E}\,\big|\,\varepsilon_1 + \cdots + \varepsilon_q\,\big|^{2\,s} \le c_s$$

since $|z_j| \le 2$. Therefore Corollary 3.3 implies

$$\mathbf{E}\,\exp\big\{\,i\tau\,q^{-1/2}\,\big\langle U, V\big\rangle\,\big\} \le c_s\,\tau^s\,I\,\bar{I} \tag{6.9}$$

with

$$I = \int\limits_{\mathbf{L}\cap\{|z|\le 1\}} \mathbf{E}\,\exp\Big\{\,\frac{i\overline{R}^2}{\sqrt{q}}\,\big\langle Z, z\big\rangle\Big\}\,dz, \qquad \bar{I} = \int\limits_{\mathbf{L}\cap\{|z|\le 1\}} \mathbf{E}\,\exp\Big\{\,\frac{i\overline{R}^2}{\sqrt{q}}\,\big\langle Z', z\big\rangle\Big\}\,dz,$$

where $\overline{R}^{\,2} = \min\big\{\,\tau^2;\,\sqrt{q}/4\,\big\}$, and Z and Z' are defined by

$$Z = \widetilde{W}_1 + \cdots + \widetilde{W}_r, \qquad Z' = \widetilde{W}'_1 + \cdots + \widetilde{W}'_r.$$

The bound (6.9) implies (6.5) provided that we show that

$$I \le c_s\,\max\Big\{\,\frac{1}{\tau^{2\,s}}\,;\;\frac{1}{q^{s/2}}\,\Big\}, \qquad \bar{I} \le c_s\,\max\Big\{\,\frac{1}{\tau^{2\,s}}\,;\;\frac{1}{q^{s/2}}\,\Big\}. \tag{6.10}$$

Thus it remains to prove (6.10). We shall estimate I. The estimate of $\bar{I}$ is similar. We have

$$\mathbf{E}\,\exp\big\{\,i\overline{R}^{\,2}q^{-1/2}\,\big\langle Z, z\big\rangle\,\big\} = \Big(\,\mathbf{E}\,\exp\big\{\,i\overline{R}^{\,2}q^{-1/2}\,\big\langle \widetilde{W}, z\big\rangle\,\big\}\Big)^q.$$

Recall that $\widetilde{W}$ denotes the random variable $\widetilde{\varepsilon}_1 z_1 + \cdots + \widetilde{\varepsilon}_s z_s$. Therefore

$$\mathbf{E}\,\exp\big\{\,i\overline{R}^{\,2}q^{-1/2}\,\big\langle \widetilde{W}, z\big\rangle\,\big\} = \prod_{j=1}^{s}\,\mathbf{E}\,\exp\big\{\,i\,\xi_j\,\big\}, \tag{6.11}$$

where ξ_j has the distribution of $\widetilde{\varepsilon}\,\overline{R}^{\,2}\,q^{-1/2}\,\langle z_j, z\rangle$. The symmetric random variable ξ_j is bounded,

$$|\xi_j| \le 4\,\overline{R}^{\,2}\,q^{-1/2} \le 1, \qquad \text{since } |z| \le 1 \text{ and } |z_j| \le 2.$$

Hence the inequality $\cos x \le 1 - x^2/4$, for $|x| \le 1$, implies

$$\mathbf{E}\,\exp\{\,i\,\xi_j\,\} = \mathbf{E}\,\cos\xi_j \le 1 - \frac{1}{4}\,\mathbf{E}\,\xi_j^2 \le \exp\Big\{-\frac{1}{4}\,\mathbf{E}\,\xi_j^2\Big\}. \tag{6.12}$$

Notice that $\mathbf{E}\,\xi_j^2 = 2\overline{R}^{\,4}\,\langle z_j, z\rangle^2/q$. Therefore (6.12) together with (6.7) implies

$$\prod_{j=1}^{s}\,\mathbf{E}\,\exp\{\,i\xi_j\,\} \le \exp\Big\{-\frac{1}{4}\sum_{j=1}^{s}\,\mathbf{E}\,\xi_j^2\Big\} \le \exp\Big\{-\frac{\overline{R}^{\,4}\,|z|^2}{8q}\,\Big\}.$$

This bound combined with (6.11) yields (recall that $r = \lceil\, q/(2\,s+1)\,\rceil$)

$$I \le \int_{\mathbf{L}} \exp\Big\{-\frac{r}{8q}\,\overline{R}^{\,4}\,|z|^2\Big\}\,dz = \tfrac{c_s}{\overline{R}^{\,2\,s}} \le c_s\,\max\Big\{\tfrac{1}{\tau^{2\,s}};\ \tfrac{1}{q^{s/2}}\Big\},$$

that concludes the proof of (6.10) and of the Lemma.

Proof of Theorem 6.3. Let us apply Lemma 6.4 and Lemma 6.5 with $q = \lceil\, pK/(8\,s)\,\rceil$. We have

$$\Big|\,\mathbf{E}\,\exp\{\,it\,K^{-1}\,Q[\Upsilon_K - a]\,\}\,\Big| \le c_d(s)\,(p\,K)^{-d} + c_s M_s(\beta q;\,q).$$

By (2.3), $M_s(\beta q;\,q) \le c_s M_s(pt;\,pK)$. Notice that $\inf_t\,M_s(pt;\,pK) = (p\,K)^{-s/4}$. Choosing $d = s/4$ completes the proof of the Theorem.

7. The multiplicative inequality

The main result of the section is the multiplicative inequality of Lemma 7.1 for characteristic functions of discrete random variables. Combined with discretization described in Section 5, the multiplicative inequality can be applied to bound integrals over general characteristic functions.

Let independent (non-identically distributed!) r.v. $Z_1, \dots, Z_r,\ Z_1', \dots, Z_r'$ be of class $\Gamma\,(\delta;\,e_1, \dots, e_s)$. Introduce the sums

$$\Lambda_r = Z_1 + \cdots + Z_r, \qquad \Lambda_r' = Z_1' + \cdots + Z_r'. \tag{7.1}$$

Denote

$$\varphi(t) = \Big|\,\mathbf{E}\,\exp\{\,itr^{-1}\,Q[\Lambda_r + Q\Lambda_r' + b]\,\}\,\Big|^2.$$

Lemma 7.1. *Assume that $\delta = 1/(50\,s)$ and that $e_1, \dots, e_s$ is an orthonormal system in $\mathbf{R}^k$. Then, for $\tau \in \mathbf{R}$, we have*

$$\varphi(\tau+\gamma)\,\varphi(\tau-\gamma) \le c_s\, M_s^2(\gamma;\, r), \qquad \text{for any} \quad \gamma > 0.$$

Introduce the function

$$\psi(\tau) = \Big|\, \mathbf{E}\, \exp\big\{\, i\tau\, Q[\Lambda_{4\,r} + Q\Lambda'_{4\,r} + b]\,\big\}\,\Big|.$$

For the proof of Lemma 7.1 we need the following

Lemma 7.2. *Let $\tau \in \mathbf{R}$. Then, for any $d \ge 0$,*

$$\psi(\tau+\gamma)\,\psi(\tau-\gamma) \le I + c_d(s)\, r^{-d/2}, \qquad \text{for all} \quad \gamma \ge 0,$$

with

$$I = \sup_{\Lambda} \sup_{b,\, b_0} \Big|\, \mathbf{E}\, \exp\big\{\, i\tau\, \big\langle Q\Lambda_r + \Lambda'_r, b_0\big\rangle + 2i\gamma\, Q[\Lambda_r + Q\Lambda'_r + b]\,\big\}\,\Big|,$$

where $\sup_\Lambda$ denotes the supremum over the class of distributions of Λ_r and Λ'_r defined in (7.1).

Proof. Notice that

$$Q[x] - Q[y] = \big\langle Q(x+y), x-y\big\rangle, \qquad 2\,Q[x] + 2\,Q[y] = Q[x+y] + Q[x-y]. \quad (7.2)$$

For an arbitrary random variable ξ, let $\bar\xi$ denote an independent copy of ξ. Assume that that the random variable $\xi - \bar\xi$ is measurable with respect to a σ-algebra $\mathcal{A}$. Writing

$$\theta(\tau) = \Big|\, \mathbf{E}\, \exp\big\{\, i\tau\, Q[\xi + b]\,\big\}\,\Big|,$$

we have

$$\begin{aligned}
\theta(\tau+\gamma)\,\theta(\tau-\gamma) &= \Big|\, \mathbf{E}\, \exp\big\{\, i\,(\tau+\gamma)\, Q[\xi+b] - i\,(\tau-\gamma)\, Q[\bar\xi + b]\,\big\}\,\Big| \\
&= \Big|\, \mathbf{E}\, \exp\big\{\, i\tau\, Q[\xi+b] - i\tau\, Q[\bar\xi+b] + i\gamma\, Q[\xi+b] + i\gamma\, Q[\bar\xi+b]\,\big\}\,\Big| \\
&= \Big|\, \mathbf{E}\, \exp\Big\{\, i\tau\, \big\langle Q\xi + Q\bar\xi + 2Qb,\, \xi - \bar\xi\big\rangle + \frac{i\gamma}{2}\, Q[\xi + \bar\xi + 2b] + \frac{i\delta}{2}\, Q[\xi - \bar\xi]\,\Big\}\,\Big| \\
&\le \mathbf{E}\, \Big|\, \mathbf{E}\, \Big(\exp\Big\{\, i\tau\, \big\langle Q\xi + Q\bar\xi + 2Qb,\, \xi - \bar\xi\big\rangle + \frac{i\gamma}{2}\, Q[\xi + \bar\xi + 2b]\,\Big\}\,\Big|\, \mathcal{A}\Big)\,\Big|, \quad (7.3)
\end{aligned}$$

by (7.2) and taking of the conditional expectation with respect to the σ-algebra $\mathcal{A}$.

The random variables Z_j and Z'_j in (7.1) may be written as

$$Z_j = \varepsilon_{j1} z_{j1} + \cdots + \varepsilon_{js} z_{js}, \qquad Z'_j = \varepsilon'_{j1} z'_{j1} + \cdots + \varepsilon'_{js} z'_{js},$$

where ε_{jl}, ε'_{jl} denote i.i.d. Rademacher random variables, which are independent in aggregate, and z_{jl}, z'_{jl} denote non-random vectors in $\mathbf{R}^k$ such that

$$|z_{jl} - e_l| \le \delta, \qquad |z'_{jl} - e_l| \le \delta,$$

for all possible values of j and l. Introduce the random variables $\eta_{jl} = \varepsilon_{jl} - \bar\varepsilon_{jl}$ and $\eta'_{jl} = \varepsilon'_{jl} - \bar\varepsilon'_{jl}$ and consider the random matrices η and η' with elements η_{jl} and η'_{jl} respectively, $1 \le j \le 4r$, $1 \le l \le s$. We choose $\mathcal{A} = \sigma(\eta, \eta')$—the σ-algebra generated by η, η', take

$$\xi = \Lambda_{4r} + Q\Lambda'_{4r},$$

and apply (7.3). This yields

$$\psi(\tau+\gamma)\,\psi(\tau-\gamma) \le J, \tag{7.4}$$

where

$$J = \mathbf{E}\,\Big|\,\mathbf{E}\,\Big(\exp\Big\{i\tau\,\big\langle Q\xi + Q\bar\xi + 2\,Qb, \xi - \bar\xi\big\rangle + \frac{i\gamma}{2}\,Q[\xi + \bar\xi + 2b]\Big\}\,\Big|\,\eta, \eta'\Big)\Big|.$$

Given η, η' (that is, given two matrices with elements 0, 2, -2) the random variables $\varepsilon_{jl} + \bar\varepsilon_{jl}$ and $\varepsilon'_{jl} + \bar\varepsilon'_{jl}$ are independent in aggregate. Moreover, for given η_{jl}, we may write $\varepsilon_{jl} + \bar\varepsilon_{jl} = 2\alpha_{jl}\,\varepsilon_{jl}$, where $\alpha_{jl} = 1$, if $\eta_{jl} = 0$, and $\alpha_{jl} = 0$, if $|\eta_{jl}| = 2$. Similarly $\varepsilon'_{jl} + \bar\varepsilon'_{jl} = 2\alpha'_{jl}\,\varepsilon'_{jl}$. Using the notation introduced above, we have

$$\xi + \bar\xi = \Lambda_{4r} + \overline{\Lambda}_{4r} + Q\Lambda'_{4r} + Q\overline{\Lambda}'_{4r}$$

with

$$\Lambda_{4r} + \overline{\Lambda}_{4r} \stackrel{\text{def}}{=} \sum\nolimits_{l=1}^{s} 2\,T_l, \qquad T_l \stackrel{\text{def}}{=} \sum\nolimits_{j=1}^{4r} \alpha_{jl}\,\varepsilon_{jl}\,z_{jl},$$

and a similar formula holds for $\Lambda'_{4r} + \overline{\Lambda}'_{4r}$

Notice that the random variables α_{jl}, α'_{jl} are i.i.d. in aggregate and

$$\mathbf{P}\,\big\{\,\alpha_{jl} = 0\,\big\} = \mathbf{P}\,\big\{\,\alpha_{jl} = 1\,\big\} = 1/2.$$

For $1 \le l \le s$, introduce the indicator functions

$$\mathbf{I}_l = \mathbf{I}\Big\{\,\sum\nolimits_{j=1}^{4r} \alpha_{jl} \ge r\,\Big\}, \qquad \mathbf{I}'_l = \mathbf{I}\Big\{\,\sum\nolimits_{j=1}^{4r} \alpha'_{jl} \ge r\,\Big\}.$$

The random variables $\mathbf{I}_l$, $\mathbf{I}'_l$, for $1 \le l \le s$, are i.i.d. and

$$\mathbf{E}\,(1 - \mathbf{I}_l) = \mathbf{E}\,(1 - \mathbf{I}'_l) = \mathbf{P}\,\Big\{\,\sum\nolimits_{j=1}^{4r} \alpha_{j1} < r\,\Big\} \le c_d\,r^{-d},$$

for any $d \geq 0$.

Thus, the integral J in (7.4) can be bounded by

$$J \leq c_d(s)\, r^{-d} + \mathbf{E}\, J_1,$$

where

$$J_1 = \mathbf{I}_1 \mathbf{I}'_1 \ldots \mathbf{I}_s \mathbf{I}'_s\, \mathbf{E}\, \exp\Big\{ i\tau \,\big\langle Q\xi + Q\bar{\xi} + 2\,Qb, \xi - \bar{\xi}\big\rangle + \frac{i\gamma}{2}\, Q[\xi + \bar{\xi} + 2b]\Big\}.$$

To conclude the proof of the Lemma it remains to show that, for given (non-random) η and η',

$$J_1 \leq \sup_Z \sup_{b_0} \sup_b \Big|\, \mathbf{E}\, \exp\big\{\, i\tau \,\big\langle Q\Lambda_r + \Lambda'_r, b_0\big\rangle + 2i\delta\, Q[\Lambda_r + Q\Lambda'_r + b]\,\big\}\,\Big|. \tag{7.5}$$

If $\mathbf{I}_1 \mathbf{I}'_1 \ldots \mathbf{I}_s \mathbf{I}'_s = 0$ then $J_1 = 0$ and (7.5) is obvious. If $\mathbf{I}_1 \mathbf{I}'_1 \ldots \mathbf{I}_s \mathbf{I}'_s \neq 0$ then we have $\sum_{j=1}^{4r} \alpha_{jl} \geq r$. This means that at least r of α_{jl}, $1 \leq j \leq 4r$, are equal to 1, that is, each of the sums T_l and T'_l contains at least r summands $\varepsilon_{jl} z_{jl}$. Taking precisely r of these summands and conditioning on the remaining ones, we derive (7.5).

Proof of Lemma 7.1. By an application of Lemma 7.2, we have

$$\varphi(\tau + \gamma)\, \varphi(\tau - \gamma) \leq 2\, I + c_d(s)\, r^{-d/2}$$

where $\tau = t\, r^{-1}$, and

$$I = \sup_{\Lambda} \sup_{b,\, b_0} \Big|\, \mathbf{E}\, \exp\big\{\, i\tau \,\big\langle Q\Lambda_{r/4} + \Lambda'_{r/4}, b_0\big\rangle + 2i\gamma\, r^{-1}\, Q[\Lambda_{r/4} + Q\Lambda'_{r/4} + b]\,\big\}\,\Big|^2.$$

By an application of the symmetrization inequality of Lemma 4.1 we get rid of the linear term $\big\langle Q\Lambda_{r/4} + \Lambda'_{r/4}, b_0\big\rangle$ in the exponent, and similarly to the proof of Lemma 6.4 we obtain

$$I \leq \sup_{\Lambda}\ \mathbf{E}\, \exp\big\{\, 2i\gamma \,\big\langle \widetilde{\Lambda}_{r/8}, \Lambda'_{r/8}\big\rangle\,\big\}.$$

The arithmetic-geometric mean inequality yields similar as in the proof of Lemma 6.4

$$I \leq \sup_{U,\, V}\ \mathbf{E}\, \exp\big\{\, 2i\gamma\, r^{-1} \,\big\langle U, V\big\rangle\,\big\},$$

where U (respectively V) is the sum of $r/8$ independent copies of a random variable, say Z, of class $\Gamma\,(\delta;\, e_1, \ldots, e_s)$. Applying Lemma 6.5, we get $I \leq c_s\, M_s^2(\gamma; r)$, and

$$\varphi(\tau + \gamma)\, \varphi(\tau - \gamma) \leq c_s\, r^{-d} + c_s\, M_s^2(\gamma; r).$$

To conclude the proof of the Lemma it is sufficient to choose $d = s/2$ and to notice that $r^{-s/2} = \inf_\gamma\ M_s^2(\gamma;\, r)$.

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RATES FOR QUADRATIC FORMS

Vidmantas Bentkus
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld 1
Germany
E-mail address: bentkus@mathematik.uni-bielefeld.de

Friedrich Götze
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld 1
Germany
E-mail address: goetze@mathematik.uni-bielefeld.de