SUBGROUP GROWTH OF FREE PRODUCTS

Thomas Müller

Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld

Introduction

For a group Γ denote by $s_\Gamma(n)$ the number of subgroups of index n in Γ. If Γ is finitely generated (as we will always suppose) or of finite (subgroup) rank then $s_\Gamma(n)$ is finite for all n. The concept of *subgroup growth*, which has evolved during the last decade, provides a framework for investigations concerning the arithmetic properties of the sequence $\{s_\Gamma(n)\}_1^\infty$ or related subgroup counting functions and their connection with the algebraic structure of the group Γ. Like Milnor's concept of word growth it can be thought of as a chapter in "asymptotic group theory", where one studies groups from the point of view of the asymptotic behaviour of some of their invariants. Some of the major developments concerning the function $s_\Gamma(n)$ are described in Lubotzky's Galway notes [L1]; cf. also [GSS], [L2], [LM], [LMS], [dS] and the references therein.

The purpose of the present paper is to investigate the growth behaviour and the asymptotics of $s_\Gamma(n)$ in the case when Γ is a free product of the form

$$\Gamma = \mathop{*}_{\sigma=1}^{s} G_\sigma \, * \, F_r, \quad 0 \leq r,\ s < \infty, \quad 1 < m_\sigma = |G_\sigma| < \infty. \tag{1}$$

Here, the G_σ are arbitrary finite groups and F_r denotes the free group of rank r. Our main result (Theorem 1) gives a complete asymptotic expansion for $s_\Gamma(n)$ in terms of the numbers r, s and the local data m_σ and $s_{G_\sigma}(d_\sigma)$ for $\sigma = 1, \ldots, s$ and $d_\sigma | m_\sigma$. We also consider the question as to what kind of information on the decomposition (1) of such a group Γ is contained in the function $s_\Gamma(n)$ or its asymptotics, and we show that this combinatorial information determines Γ up to finitely many isomorphism types; cf. Theorem 3. Our results depend on the recent solution in [M5] of another problem from "asymptotic group theory" concerning finite group actions:

(2) *Given a finite group G, how does the number of finite group actions ${}_GX$ behave as $|X| \to \infty$?*

This problem is of considerable interest in its own right; those results on (2) needed in our context are presented in §1 together with some historical comments. For proofs and more details on the history of (2) the reader is referred to [M5].

The study of the subgroup growth in "large" groups originated with the work of T. Radó and M. Hall in the late 1940's on Schreier systems and their associated subgroups in free groups; cf. [HR] and [H1-2]. In [H2], M. Hall using the method developed in [HR] computed $s_\Gamma(n)$ for free groups Γ of finite rank. Some of the latent implications of Hall's and Radó's work were explored by Dey [D], who in particular generalized Hall's formula to free products. Since then a number of results on $s_\Gamma(n)$ or related subgroup counting functions have been obtained for some specific groups, mostly in the case when Γ is free or the classical modular group; cf. for example [N], [St], [I], [GIR]; however, a general theory for the subgroup growth of virtually free groups has evolved only rather recently.

For a finitely generated virtually free group Γ denote by $b_\Gamma(\lambda)$ the number of free subgroups of index λm_Γ in Γ, where m_Γ is the least common multiple of the orders of the finite subgroups in Γ. In [M1-3] an explicit estimate on the growth of the difference function $b_\Gamma(\lambda+1) - b_\Gamma(\lambda)$ and a full asymptotic expansion for $b_\Gamma(\lambda)$ are obtained in terms of a presentation of Γ as fundamental group of a finite graph of finite groups. Furthermore, the question as to what kind of information on the structure of Γ is contained in the number of free subgroups of finite index (i.e. in the function b_Γ together with the constant m_Γ) or in the asymptotics of this number is treated with the help of a certain new invariant, the *type* $\tau(\Gamma)$ of Γ; moreover, it is shown that this combinatorial information on free subgroups determines the group Γ up to finitely many isomorphism classes.

On the other hand, not much appears to be known for Γ virtually free about the arithmetic properties of the function $s_\Gamma(n)$. In fact, aside from the case of a free group the only result of interest in the present context seems to be Newman's asymptotic formula [N, Theorem 4] for the modular group. He shows that

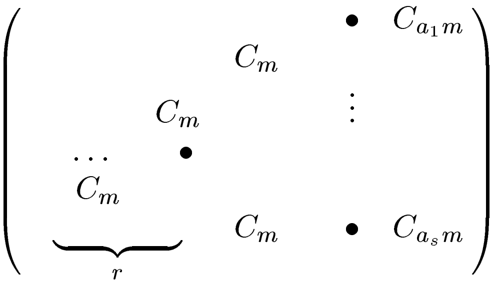
$$(3) \qquad s_{\mathrm{PSL}(2,\mathbb{Z})}(n) \sim (12\pi e^{1/2})^{-1/2}\, n^{n/6}\, \exp\Big(-\frac{n}{6} + n^{1/2} + n^{1/3} + \frac{1}{2}\,\log\, n\Big).$$

In this paper we will derive results on the function $s_\Gamma(n)$ similar in nature and degree of explicitness to those on $b_\Gamma(\lambda)$ in the case when Γ is of the form (1). It appears that this class of groups (call it $\mathcal{F}$) is precisely the largest class of virtually free groups admitting in a uniform way this rather explicit type of results. Consider for instance the sequence of groups $\Gamma_m = C_{2m} \underset{C_m}{*} C_{3m}$, i.e. $\Gamma_1 = \Gamma \cong \mathrm{PSL}(2,\mathbb{Z})$, $\Gamma_2 \cong \mathrm{SL}(2,\mathbb{Z})$, etc. Then one can show that the relation

$$\frac{s_{\Gamma_m}(n)}{s_\Gamma(n)} = 1 + \mathcal{O}(n^{-K}) \quad (n \to \infty) \tag{4}$$

holds for all integers $m,\ K \geq 1$. Indeed, there is nothing special about the modular group here. The same result (4) holds for example, if one replaces $\mathrm{PSL}(2,\mathbb{Z})$ by any cyclic cover $\Gamma = C_{a_1} * \cdots * C_{a_s} * F_r$ in $\mathcal{F}$ such that $\chi(\Gamma) < 0$, and then lifts to a sequence of groups of the form

(5) $\Gamma_m = \pi_1$



, $m = 1, 2, \ldots$.

This type of phenomenon will be pursued more systematically in a separate paper. As a consequence of (4) the asymptotic expansion provided by Theorem 1 for the function $s_\Gamma(n)$ of the base group $\Gamma = \Gamma_1$ carries over without change to the groups Γ_m in (5); in particular, the finiteness result Theorem 3(b) breaks down if we attach one such "hair" $\{\Gamma_m\}_1^\infty$ to the class $\mathcal{F}$. This seems to indicate that for virtually free groups Γ not in $\mathcal{F}$ the growth of $s_\Gamma(n)$ is in general not sufficiently controlled by global invariants of Γ, and it is not even clear whether a meaningful asymptotic result always exists for the function $s_\Gamma(n)$ of these groups.

I feel that this is a good opportunity to express my sincere gratitude to Alex Lubotzky. Large parts of the work reported in [M5] and the present paper were carried out while I held a Landau Fellowship at the Hebrew University of Jerusalem at his invitation. The enthusiasm he has shown for my work ever since we first met in St. Andrews in 1989 and his constant encouragement have helped me to get through some difficult times.

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1. Counting Finite Group Actions

Question (2) has, at least in the case when G is cyclic, received a fair amount of attention since the early 1950's; cf. in particular [CHM], [MW1], [MW2], [Wi] and the references therein. From an analytic point of view, (2) is a special case of the problem to obtain explicit asymptotic information on the coefficients of entire functions of the form $e^{P(z)}$ for real polynomials $P(z)$. In this form the problem was studied already by Pólya [Po] in connection with his investigation concerning the zeros of the derivatives of certain functions. His result is however not sufficiently explicit for the purposes of (2). In [MW1-2] Moser and Wyman developed a different method, which enabled them to obtain explicit asymptotic formulas in the case when G is of prime order. Their result is that

$$|\mathrm{Hom}\,(C_p, S_n)| \sim K_p\, n^{(1-1/p)n}\ \exp\left(-\ \frac{p-1}{p}\, n\ +\ n^{1/p}\right)\quad (n\to\infty), \tag{6}$$

where $K_p = p^{-1/2}$ for primes $p > 2$ and $K_2 = 2^{-1/2}e^{-1/4}$. More recently, Wilf [Wi] has found a (somewhat implicit) asymptotic formula for cyclic G. In [M5], building on the work of Harris and Schoenfeld [HS], a complete asymptotic expansion for the coefficients of entire functions of the form $e^{P(z)}$ is established, which is explicit in n and the polynomial $P(z)$. As a special case we obtain an asymptotic expansion for the number $|\mathrm{Hom}\ (G, S_n)|$ of actions of an arbitrary finite group G on an n-set in terms of the order $|G| = m$ and the subgroup numbers $s_G(d)$ for $d|m$. In order to state this result we first have to give some definitions. Let $P(z) = \sum_{\mu=1}^m c_\mu z^\mu \in \mathbb{R}[z]$ be a polynomial of degree m. Define a sequence $\gamma_1, \gamma_2, \ldots$ by

$$\gamma_\mu = \beta_1^{-(\mu+1)} \sum_{\substack{\alpha\geq 0,\ \nu\geq 1\\ \alpha m+\nu=\mu+1}} \binom{-\nu/m}{\alpha} \frac{\beta_\nu}{\nu}\,,$$

where $\beta_\nu = \langle z^{\nu-1},\ \left(\sum_{\mu=1}^m \mu c_\mu z^{m-\mu}\right)^{\nu/m}\rangle$. Next, introduce a sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots$ via the following hierarchy of definitions:

(i) $$\psi_{\alpha,\lambda} := (m^2c_m)^{-\alpha} \sum_{\substack{\mu_1+\cdots+\mu_\alpha=\alpha m-\lambda \\ 1\le\mu_1,\ldots,\mu_\alpha\le m/2}} \mu_1^2\ldots\mu_\alpha^2 c_{\mu_1}\ldots c_{\mu_\alpha},$$

(ii)
$$\chi_{\mu,k,\kappa} := \Big(\frac{m^2c_m}{2}\Big)^{-(k+\kappa)} \sum_{\substack{j_1+\cdots+j_\kappa=2k \\ j_1,\ldots,j_\kappa\ge1}} \sum_{\substack{\mu_1+\cdots+\mu_\kappa=\mu \\ 1\le\mu_1,\ldots,\mu_\kappa\le m}} \prod_{\ell=1}^{\kappa}\left[\frac{\mu_\ell c_{\mu_\ell}}{j_\ell+2}\left((-1)^{j_\ell}+\binom{\mu_\ell-1}{j_\ell+1}\right)\right],$$

(iii)
$$\widetilde{\chi}_{\nu,k,\kappa} := \chi_{m(k+\kappa)-\nu,k,\kappa} + \sum_{\substack{\lambda,\mu\ge1 \\ m(k+\kappa)+\lambda=\mu+\nu}} \sum_{\alpha\ge1}(-1)^\alpha\binom{k+\kappa+\alpha-1}{\alpha}\psi_{\alpha,\lambda}\chi_{\mu,k,\kappa},$$

(iv) $$\chi_\nu := \sum_{k\ge1}\sum_{\kappa=1}^{2k}(-1)^{k+\kappa}\,2^{1-2(k+\kappa)}\,\frac{(2k+2\kappa-1)!}{\kappa!(k+\kappa-1)!}\,\widetilde{\chi}_{\nu,k,\kappa},$$

(v) $$\widetilde{\chi}_\nu := \chi_\nu + \sum_{\beta\ge1}\sum_{\lambda\ge1}\binom{\beta}{\lambda}\chi_\beta \sum_{\substack{\nu_1+\cdots+\nu_\lambda=\nu-\beta \\ \nu_1,\ldots,\nu_\lambda\ge m/2}} \gamma_{\nu_1}\ldots\gamma_{\nu_\lambda},$$

(vi) $$\mathcal{A}_\nu := \sum_{\lambda\ge1}(-1)^{\lambda-1}\,\frac{1}{\lambda} \sum_{\substack{\nu_1+\cdots+\nu_\lambda=\nu \\ \nu_1,\ldots,\nu_\lambda\ge1}} \widetilde{\chi}_{\nu_1}\ldots\widetilde{\chi}_{\nu_\lambda}.$$

Finally, we define a sequence $\mathcal{B}_1,\mathcal{B}_2,\ldots$ by

(7)

$$\mathcal{B}_\nu = \mathcal{B}_\nu(P) :=$$

$$\mathcal{A}_\nu + mc_m\sum_{\lambda\ge1}(-1)^\lambda\,\frac{1}{\lambda}\,\Bigg(\frac{\psi_{\lambda,\nu}}{2mc_m} - \sum_{\substack{\nu_1+\cdots+\nu_\lambda=m+\nu \\ \nu_1,\ldots,\nu_\lambda\ge m/2}}\gamma_{\nu_1}\ldots\gamma_{\nu_\lambda}\Bigg)$$

$$+\sum_{1\le\mu\le m}\sum_{\lambda\ge1}(-1)^\lambda\binom{\mu+\lambda-1}{\lambda}\,c_\mu \sum_{\substack{\nu_1+\cdots+\nu_\lambda=\mu+\nu \\ \nu_1,\ldots,\nu_\lambda\ge m/2}}\gamma_{\nu_1}\ldots\gamma_{\nu_\lambda}$$

$$+\sum_{\mu\ge0}\sum_{\alpha\ge1}\sum_{\lambda\ge1}(-1)^{\alpha+\lambda}\,\frac{1}{2\lambda}(m^2c_m)^{-\lambda} \sum_{\substack{\nu_1+\cdots+\nu_\alpha=\nu-\mu \\ \nu_1,\ldots,\nu_\alpha\ge m/2}}\gamma_{\nu_1}\ldots\gamma_{\nu_\alpha}\sum\prod_{\ell=1}^{\lambda}\left[\mu_\ell^2c_{\mu_\ell}\binom{\mu_\ell+\alpha_\ell-1}{\alpha_\ell}\right],$$

the last sum being extended over those (2λ)-tuples $(\mu_1,\ldots,\mu_\lambda,\alpha_1,\ldots,\alpha_\lambda)$ for which $\mu_1+\cdots+\mu_\lambda=\lambda m-\mu$, $\alpha_1+\cdots+\alpha_\lambda=\alpha$, $1\le\mu_1,\ldots,\mu_\lambda\le m$, $\alpha_1,\ldots,\alpha_\lambda\ge 0$ and $(\mu_1,\alpha_1),\ldots,(\mu_\lambda,\alpha_\lambda)\ne(m,0)$.

Associating with each finite group G its polynomial

$$P_G(z) := \sum_{d|m} \frac{s_G(d)}{d}\, z^d, \quad |G| = m$$

we can now state our solution of problem (2).

Let G be a finite group of order $m\ge 2$. Then there exist constants $\widetilde{\mathcal{C}}_\nu=\widetilde{\mathcal{C}}_\nu(G)$ such that the number $|\mathrm{Hom}\,(G,S_n)|$ of G-actions on an n-set has the asymptotic expansion

$$\tag{8} \begin{aligned} |\mathrm{Hom}\,(G,S_n)| \approx K_G\, n^{(1-1/m)n} \exp\Big(-\frac{m-1}{m}\,n + \sum_{\substack{d|m\\ d<m}} \frac{s_G(d)}{d}\, n^{d/m}\Big) \\ \times \Big\{1+\sum_{\nu=1}^{\infty}\widetilde{\mathcal{C}}_\nu(G)\, n^{-\nu/m}\Big\} \qquad (n\to\infty), \end{aligned}$$

where

$$K_G := \begin{cases} m^{-1/2}; & m \text{ odd} \\ m^{-1/2}\exp\Big(-\frac{(s_G(m/2))^2}{2m}\Big); & m \text{ even.} \end{cases}$$

Moreover, the $\widetilde{\mathcal{C}}_\nu$'s satisfy the (formal) identity

$$\tag{9} 1+\sum_{\nu=1}^{\infty}\widetilde{\mathcal{C}}_\nu(G)\, z^{-\nu} = \exp\Big(\sum_{\nu=1}^{\infty}\widetilde{\mathcal{B}}_\nu(G)\, z^{-\nu}\Big),$$

where

$$\widetilde{\mathcal{B}}_\nu(G) := \mathcal{B}_\nu(P_G) + \begin{cases} \frac{m^2 B_{\nu/m+1}}{\nu(\nu+m)}; & \nu\equiv m\ (2m) \\ 0; & \text{otherwise} \end{cases}$$

with $B_{\nu/m+1}$ a Bernoullian number and $\mathcal{B}_\nu(P_G)$ as given by (7).

The main term in (8) yields the asymptotic formula

$$(10)\quad |\mathrm{Hom}\,(G,S_n)| \sim K_G\, n^{(1-1/m)n} \exp\Big(-\ \frac{m-1}{m}\, n\ +\ \sum_{\substack{d|m\\ d<m}} \frac{s_G(d)}{d}\, n^{d/m}\Big) \quad (n\to\infty),$$

which is a direct generalization of Moser and Wyman's formula (6). The identity (9) is used in [M5] to calculate the coefficients $\widetilde{\mathcal{C}}_\nu(G)$ occurring in (8) for $\nu \leq 3$; to obtain more precision is, of course, only a matter of calculation.

Denote by T_n the number of C_2-actions on a set of cardinality n. In [CHM] Chowla et al. showed among other things that the quotient T_n/T_{n-1} is asymptotically equal to $n^{1/2}$, [CHM, Theorem 3], and conjectured that, more precisely, an asymptotic expansion of the form

$$(11)\qquad T_n/T_{n-1} \approx n^{1/2}\Big\{1+\sum_{\nu=1}^{\infty} \widetilde{Q}_\nu\, n^{-\nu/2}\Big\} \quad (n\to\infty)$$

holds with appropriate constants $\widetilde{Q}_\nu$. That this is indeed the case was first shown by Moser and Wyman; cf. [MW1, 2.12]. They observed that

$$T_n = \frac{H_n(i/\sqrt{2})}{i^n\ 2^{n/2}}\ ,$$

where H_n is the n^{th} Hermite polynomial, and then used a classical result of generalized Hilb's type for Hermite polynomials [Sz, Theorem 8.22.7] to prove (11), obtaining the first two coefficients explicitly as $\widetilde{Q}_1 = \frac{1}{2}$ and $\widetilde{Q}_2 = -\frac{1}{8}$. Later, the conjecture was extended to cyclic groups to the effect that the quotient $|\mathrm{Hom}\,(C_m,S_n)|/|\mathrm{Hom}\,(C_m,S_{n-1})|$ should have an asymptotic expansion for fixed m as $n\to\infty$ of the form $n^{1-1/m}$ times a Poincaré series in $n^{-1/m}$ whose coefficients depend only on m; but as far as we are aware no further progress was made. However, such a statement is indeed true, even for arbitrary finite groups, and this fact will become important for us later on. In §3, when investigating the asymptotic behaviour of the number $s_\Gamma(n)$ of subgroups of index n in a free product Γ of the form (1), we will, as a first step, expand the function

$(n-1)!\ \frac{s_\Gamma(n)}{|\mathrm{Hom}\ (\Gamma, S_n)|}$ in terms of the functions

$$\frac{n!}{(n-k)!}\ \frac{|\mathrm{Hom}(\Gamma, S_{n-k})|}{|\mathrm{Hom}(\Gamma, S_n)|} = \Big(\frac{n!}{(n-k)!}\Big)^{1-r} \prod_{\sigma=1}^{s} \prod_{\nu=1}^{k} \frac{|\mathrm{Hom}(G_\sigma, S_{n-k+\nu-1})|}{|\mathrm{Hom}(G_\sigma, S_{n-k+\nu})|} \qquad (k = 1, 2, \dots),$$

which form an asymptotic sequence, provided that $\chi(\Gamma) < 0$; cf. Proposition 2. In order to shift from these gauge functions to the asymptotic sequence $\{n^{-\nu/m_\Gamma}\}_{\nu=0}^{\infty}$, we will then need good asymptotic knowledge not only of the function $|\mathrm{Hom}\ (\Gamma, S_n)|$, but also of the quotient $|\mathrm{Hom}\ (G, S_{n-1})|/|\mathrm{Hom}\ (G, S_n)|$ for an arbitrary finite group G. As a consequence of (8) and (9) we have the following.

Let G *be a finite group of order* $m \geq 2$. *Then there exist constants* $\widetilde{Q}_\nu = \widetilde{Q}_\nu(G)$ *such that as* $n \to \infty$

$$|\mathrm{Hom}\ (G, S_n)|\ /\ |\mathrm{Hom}\ (G, S_{n-1})| \approx n^{1-1/m}\Big\{1 + \sum_{\nu=1}^{\infty} \widetilde{Q}_\nu(G)\ n^{-\nu/m}\Big\}. \tag{12}$$

Moreover, the $\widetilde{Q}_\nu$*'s satisfy the identity*

$$1\ +\ \sum_{\nu=1}^{\infty} \widetilde{Q}_\nu(G)\ z^{-\nu} = \exp\ \Big(\sum_{\nu=1}^{\infty} Q_\nu(G)\ z^{-\nu}\Big), \tag{13}$$

where

$$Q_\nu = Q_\nu(G) := -\sum_{1 \leq \lambda < \nu/m} \binom{\nu/m - 1}{\lambda} \mathcal{B}_{\nu - m\lambda}(P_G)$$

$$+ \begin{cases} -\ \frac{m(\nu+m-2)}{2\nu(\nu+m)}\ ; & \nu \equiv 0\ (m) \\ (-1)^{(\nu-m+d)/m} \binom{d/m}{(\nu+d)/m}\ \frac{s_G(d)}{d}\ ; & \nu \equiv m - d\ (m),\ d|m,\ d < m \\ 0; & \text{otherwise.} \end{cases}$$

Using (13), the coefficients $\widetilde{Q}_\nu(G)$ are calculated explicitly in [M5] for $\nu \leq m+3$. æ

2. The Triangle Δ_Γ

As before, denote by $\mathcal{F}$ the class of free products of the form (1), i.e. the smallest class of groups containing all finite groups and the infinite-cyclic group, which is closed under the operation of forming the free product of two (and hence finitely many) of its members. With each group $\Gamma \in \mathcal{F}$ we associate an infinite triangular array $\Delta_\Gamma = \big(H_k^n(\Gamma)\big)_{0 \leq k \leq n}$ with entries defined by

$$H_k^n(\Gamma) = \binom{n}{k}^{-1} \frac{|\mathrm{Hom}\,(\Gamma, S_n)|}{|\mathrm{Hom}\,(\Gamma, S_k)|\;|\mathrm{Hom}\,(\Gamma, S_{n-k})|}\,, \quad 0 \leq k \leq n.$$

For the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z}) \cong C_2 * C_3$ for example the triangle Δ_Γ looks as follows:

$$\begin{array}{c}
1 \\
1 \qquad 1 \\
1 \qquad 1 \qquad 1 \\
1 \qquad 2 \qquad 2 \qquad 1 \\
1 \qquad \tfrac{15}{8} \qquad \tfrac{15}{4} \qquad \tfrac{15}{8} \qquad 1 \\
1 \qquad \tfrac{91}{75} \qquad \tfrac{91}{40} \qquad \tfrac{91}{40} \qquad \tfrac{91}{75} \qquad 1 \\
1 \qquad \tfrac{171}{91} \qquad \tfrac{57}{25} \qquad \tfrac{171}{80} \qquad \tfrac{57}{25} \qquad \tfrac{171}{91} \qquad 1 \\
1 \qquad \tfrac{754}{399} \qquad \tfrac{174}{49} \qquad \tfrac{377}{175} \qquad \tfrac{377}{175} \qquad \tfrac{174}{49} \qquad \tfrac{754}{399} \qquad 1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{array}$$

By definition, Δ_Γ is symmetric, $H_k^n(\Gamma) = H_{n-k}^n(\Gamma)$, and satisfies the boundary condition $H_0^n(\Gamma) = 1,\ n \geq 0$. In fact, the similarity of Δ_Γ with Pascal's triangle goes beyond this observation, and, for reasons which will become apparent in the next paragraph, the investigation of the growth behaviour and the asymptotics of $s_\Gamma(n)$ requires the discussion of a certain combinatorial property of the triangle Δ_Γ.

For Pascal's triangle the sum of the reciprocal values of the entries in the interior of a row converges to zero as the row number goes to infinity. More precisely, we have for each fixed integer $K \geq 1$ that

$$\sum_{k=K}^{n-K} \binom{n}{k}^{-1} = \mathcal{O}(n^{-K}) \quad (n \to \infty).$$

For $K = 1, 2, \dots$ put

$$A_\Gamma^{(K)}(n) := \sum_{k=K}^{n-K} \big(H_k^n(\Gamma)\big)^{-1}.$$

Then the sum $A_\Gamma^{(K)}(n)$ also tends to zero for fixed K as $n \to \infty$ in an appropriately rapid fashion depending on K, provided that $\chi(\Gamma) < 0$, i.e. if Γ contains a non-abelian free subgroup. More precisely, we have the following.

Proposition 1. *Suppose that* $\chi(\Gamma) < 0$ *and let* $K \geq 1$ *be any fixed natural number. Then*

$$A_\Gamma^{(K)}(n) = \mathcal{O}\big(n^{K\chi(\Gamma)}\big) \quad (n \to \infty). \tag{14}$$

Proof. Let Γ be as in (1). From (10) and Stirling's formula we find that

$$h_\Gamma(n) := |\operatorname{Hom}(\Gamma, S_n)| \,/\, (n!) \sim L_\Gamma\, n^{-1}\, \Phi_\Gamma(n) \quad (n \to \infty), \tag{15}$$

where

$$L_\Gamma := (2\pi)^{\frac{r-1}{2}} (m_1 \dots m_s)^{-1/2} \ \exp\Big(-\sum_{\substack{\sigma \\ 2|m_\sigma}} \frac{\big(s_{G_\sigma}(m_\sigma/2)\big)^2}{2m_\sigma}\Big) \tag{16}$$

and

$$\Phi_\Gamma(n) := n^{-\chi(\Gamma)n} \exp\Big(\chi(\Gamma)n \ + \sum_{\sigma=1}^{s} \sum_{\substack{d_\sigma | m_\sigma \\ d_\sigma < m_\sigma}} \frac{s_{G_\sigma}(d_\sigma)}{d_\sigma}\, n^{d_\sigma/m_\sigma} \ + \ \frac{r+1}{2} \ \log\ n\Big). \tag{17}$$

(15) together with the symmetry of the triangle Δ_Γ implies in particular that for fixed $K \geq 1$

$$A_\Gamma^{(K)}(n) = \mathcal{O}\Big(\sum_{K \leq k \leq n/2} \exp\big(\Psi_\Gamma(k) + \Psi_\Gamma(n-k) - \Psi_\Gamma(n)\big)\Big) \tag{18}$$

with an implied constant depending only on Γ, where

$$\Psi_\Gamma(x) := -\chi(\Gamma) x \log x \; + \sum_{\sigma=1}^{s} \sum_{\substack{d_\sigma | m_\sigma \\ d_\sigma < m_\sigma}} \frac{s_{G_\sigma}(d_\sigma)}{d_\sigma} \, x^{d_\sigma / m_\sigma} \; + \; \frac{r-1}{2} \log x, \quad x > 0.$$

Now we claim the following.

(19) *Given* K, *there exists a constant* $\mathcal{C} = \mathcal{C}(\Gamma, K) \geq K - \chi(\Gamma)^{-1}$ *such that the function* $\Psi_\Gamma(x)$ *satisfies*

$$\Psi_\Gamma(k) + \Psi_\Gamma(n-k) \leq \Psi_\Gamma(\mathcal{C}) + \Psi_\Gamma(n - \mathcal{C})$$

for $\mathcal{C} \leq k \leq n/2$ *and* $n \geq 2\,\mathcal{C}$.

To see this, observe first that under the assumption $\chi(\Gamma) < 0$ the second derivative of $\Psi_\Gamma(x)$ satisfies $\Psi''_\Gamma(x) \geq 0$ for large x, hence $\Psi'_\Gamma(x)$ is an increasing function for $x \geq x_0$ with some constant $x_0 = x_0(\Gamma)$. Choose the constant $\mathcal{C}$ in such a way that (i) $\Psi'_\Gamma(x)$ is increasing for $x \geq \mathcal{C}$, (ii) $\mathcal{C} \geq K - \chi(\Gamma)^{-1}$ and (iii) $\mathcal{C} \in \mathbb{Z}$. Assume that $\mathcal{C} \leq k \leq n/2 - 1$. By the mean value theorem we can find real numbers ε_1 and ε_2 (depending on k and n) such that $0 \leq \varepsilon_i \leq 1$ and

$$\begin{aligned} \Psi_\Gamma(k+1) - \Psi_\Gamma(k) &= \Psi'_\Gamma(k + \varepsilon_1) \\ \Psi_\Gamma(n-k) - \Psi_\Gamma(n-k-1) &= \Psi'_\Gamma(n-k-1+\varepsilon_2). \end{aligned}$$

Using our assumption that $\mathcal{C} \leq k \leq n/2 - 1$ and the fact that $\Psi'_\Gamma(x)$ is increasing for $x \geq \mathcal{C}$ we find that

$$\begin{aligned} \Psi_\Gamma(k+1) - \Psi_\Gamma(k) &= \Psi'_\Gamma(k + \varepsilon_1) \\ &\leq \Psi'_\Gamma(n-k-1+\varepsilon_2) \\ &= \Psi_\Gamma(n-k) - \Psi_\Gamma(n-k-1), \end{aligned}$$

i.e. $\Psi_\Gamma(k+1) + \Psi_\Gamma(n-k-1) \leq \Psi_\Gamma(k) + \Psi_\Gamma(n-k)$. Our claim (19) follows now by iteration.

Splitting the right-hand side of (18) and applying (19) we obtain for $n \geq 2\,\mathcal{C}$

$$A_\Gamma^{(K)}(n) = \mathcal{O}\Big(\sum_{K \leq k < \mathcal{C}} \exp\ \big(\Psi_\Gamma(k) + \Psi_\Gamma(n-k) - \Psi_\Gamma(n)\big)\Big)$$

$$+\, \mathcal{O}\Big(\sum_{\mathcal{C} \leq k \leq n/2} \exp\big(\Psi_\Gamma(\mathcal{C}) + \Psi_\Gamma(n-\mathcal{C}) - \Psi_\Gamma(n)\big)\Big).$$

Moreover, if k remains bounded, we have that

$$\Psi_\Gamma(k) + \Psi_\Gamma(n-k) - \Psi_\Gamma(n) = \chi(\Gamma)k\log n + \mathcal{O}(1) \quad (n \to \infty) \tag{20}$$

with implied constants depending on Γ and K. This follows readily from the definition of $\Psi_\Gamma(x)$. We conclude from (20) that as $n \to \infty$

$$\sum_{K \leq k < C} \exp\big(\Psi_\Gamma(k) + \Psi_\Gamma(n-k) - \Psi_\Gamma(n)\big) = \sum_{K \leq k < \mathcal{C}} \exp\big(\chi(\Gamma)k\log n + \mathcal{O}(1)\big)$$

$$= \mathcal{O}\big(n^{K\chi(\Gamma)}\big),$$

since $\chi(\Gamma) < 0$. Similarly, we find that

$$\sum_{\mathcal{C} \leq k \leq n/2} \exp\big(\Psi_\Gamma(\mathcal{C}) + \Psi_\Gamma(n-\mathcal{C}) - \Psi_\Gamma(n)\big) = \sum_{\mathcal{C} \leq k \leq n/2} \exp\big(\chi(\Gamma)\mathcal{C}\log n + \mathcal{O}(1)\big)$$

$$= \mathcal{O}\big(n^{1+\mathcal{C}\chi(\Gamma)}\big) = \mathcal{O}\big(n^{K\chi(\Gamma)}\big),$$

since $\chi(\Gamma) < 0$ and $\mathcal{C}$ was such that $\mathcal{C} \geq K - \chi(\Gamma)^{-1}$. Hence, the estimate (14) holds with implied constants depending on Γ and K.

$\square$

Remark. Proposition 1 tells us in particular that $A_\Gamma(n) := A_\Gamma^{(1)}(n) \to 0$ as $n \to \infty$, provided that $\chi(\Gamma) < 0$. On the other hand, $A_{C_\infty}(n) = n - 1$, and one infers from (15) that $A_\Gamma(n)$ is at least of order $\mathcal{O}(\sqrt{n}\,)$ if $\Gamma = C_2 * C_2$ and grows exponentially if $|\Gamma| < \infty$. Hence, the following statements on a group $\Gamma \in \mathcal{F}$ are equivalent:

(i) $\chi(\Gamma) < 0$.

(ii) $\lim_{n\to\infty} A_\Gamma(n) = 0.$

(iii) $A_\Gamma(n)$ remains bounded.

This is the analogue of [M4, Theorem 3] for the triangle Δ_Γ. Proposition 1 itself is analogous to [M2, Theorem 2].

In other respects the triangle Δ_Γ tends to behave less satisfactory, from a combinatorial point of view, than its counterpart $\Delta(\underline{X}) = \big(V_\mu^\lambda(\underline{X})\big)_{0\leq\mu\leq\lambda}$, the triangle associated with a finite order graph $\underline{X}$, which was introduced in [M1] in connection with the problem of counting free subgroups of finite index. For example, while "most" of the triangles $\Delta(\underline{X})$ have the unimodal property

$$V_{\mu-1}^\lambda(\underline{X}) < V_\mu^\lambda(\underline{X}) \quad \text{for} \quad 0 < \mu < \frac{\lambda+1}{2} \quad \text{and} \quad \lambda \geq 2,$$

this is not true for the triangle Δ_Γ.[1] Here the situation is more complicated. It follows in particular from (12) that there exists a constant $A_\Gamma > 0$ such that

$$\frac{H_k^n(\Gamma)}{H_{k-1}^n(\Gamma)} \geq A_\Gamma \left(\frac{n-k+1}{k}\right)^{-\chi(\Gamma)}, \quad 0 < k \leq n.$$

In fact one can take $A_\Gamma := \prod_{\sigma=1}^s \big(A_1(G_\sigma)/A_2(G_\sigma)\big)$, where $A_1(G_\sigma)$ and $A_2(G_\sigma)$ are chosen according to (12) in such a way that the inequalities

$$A_1(G_\sigma)\, n^{1-1/m_\sigma} \leq |\operatorname{Hom}(G_\sigma, S_n)| \,/\, |\operatorname{Hom}(G_\sigma, S_{n-1})| \leq A_2(G_\sigma)\, n^{1-1/m_\sigma}$$

hold for all $n \geq 1$. In the case $\chi(\Gamma) < 0$ one concludes from this for example that the quotients

$$\frac{H_k^n(\Gamma)}{H_{k-1}^n(\Gamma)} \quad \text{with} \quad 0 < k < \frac{n+1}{2} \quad \text{and} \quad n \geq 2$$

are at least bounded away from zero and that Δ_Γ contains "many" unimodal sequences. To be more precise, given any natural number k_1, we will have $H_{k-1}^n(\Gamma) < H_k^n(\Gamma)$ for

[1] For the definition of the triangle $\Delta(\underline{X})$ and an account of its connection with the growth behaviour and the asymptotics of the function $b_\Gamma(\lambda)$ for virtually free groups Γ we refer to [M1]; cf. also [M2, Sect. 4]. A more systematic investigation of finite order graphs and their associated triangles is contained in [M4]; compare in particular [M4, Theorem 1] for a precise statement concerning the unimodal property of the triangle $\Delta(\underline{X})$.

$0 < k \leq k_1$ and all $n \geq n_1 := \lfloor k_1(1 + A_\Gamma^{1/\chi(\Gamma)})\rfloor$. Hence, for each $n \geq n_1$,

$$1 = H_0^n(\Gamma) < H_1^n(\Gamma) < \cdots < H_{k_1}^n(\Gamma) > H_{n-k_1+1}^n(\Gamma) > \cdots > H_{n-1}^n(\Gamma) > H_n^n(\Gamma) = 1$$

is a unimodal sequence of length $2k_1 + 1$.

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3. Growth Behaviour and Asymptotics of the Function $s_\Gamma(n)$

Let Γ be an $\mathcal{F}$-group. By Dey's formula [D, Theorem (6.10)] $s_\Gamma(n)$ is related to the function $h_\Gamma(n) = |\operatorname{Hom}(\Gamma, S_n)| / (n!)$ via the transformation formula

$$\sum_{k=1}^{n} h_\Gamma(n-k)\, s_\Gamma(k) = n\, h_\Gamma(n), \quad n \geq 1. \tag{21}$$

In order to discuss the asymptotic behaviour of the function $s_\Gamma(n)$ adopt, for a moment, the following more formal point of view.

Consider two sequences $1 = h_0,\ h_1,\ h_2, \ldots$ and $s_1,\ s_2,\ s_3, \ldots$ of real numbers satisfying a relation

$$\sum_{k=1}^{n} h_{n-k}\, s_k = c\, n\, h_n, \quad n \geq 1 \tag{22}$$

with some constant $c > 0$. We require $s_n \geq 0$ and $h_n > 0$ for $n \geq 1$. Define the *triangle* $\Delta = (H_k^n)_{0 \leq k \leq n}$ *associated with the transformation* (22) by

$$H_k^n := \frac{h_n}{h_k\, h_{n-k}}, \quad 0 \leq k \leq n,$$

and for each fixed integer $K \geq 1$ put

$$A_n^{(K)} := \sum_{k=K}^{n-K} (H_k^n)^{-1}.$$

If the sequence $\{h_n\}_0^\infty$ is growing in an appropriately rapid fashion then the following criterion can be used to transfer information concerning the asymptotic behaviour of the h_n's to the sequence s_n.

Lemma. *Suppose that for some integer* $K \geq 1$ *as* $n \to \infty$

(i) $h_{n-1} = o(h_n)$

(ii)$_K$ $A_n^{(K)} = \mathcal{O}(h_{n-K}/h_n)$.

Then the sequence s_n *satisfies the relation*

$$\frac{s_n}{c\,n} = \sum_{k=0}^{K-1} d_k\, h_{n-k} \;+\; \mathcal{O}(h_{n-K}) \quad (n \to \infty),$$

where d_k *is the coefficient of* z^k *in the series* $\left(\sum_{n=0}^{\infty} h_n z^n\right)^{-1}$.

This is a consequence of the main results in [Be]; cf. also [M2, Sect. 2].

Now assume that $\chi(\Gamma) < 0$, i.e. Γ contains a non-abelian free subgroup. The sequences $\{h_\Gamma(n)\}_0^\infty$ and $\{s_\Gamma(n)\}_1^\infty$ are related via the transformation formula (22) with $c = 1$, and (12) implies in particular that

$$h_\Gamma(n-1)/h_\Gamma(n) \sim n^{\chi(\Gamma)} \quad (n \to \infty).$$

Hence, under the assumption $\chi(\Gamma) < 0$, condition (i) of the lemma is satisfied. Also, for fixed $K \geq 1$

$$h_\Gamma(n-K)/h_\Gamma(n) = \prod_{k=1}^{K} h_\Gamma(n-K+k-1)/h_\Gamma(n-K+k) \sim n^{K\chi(\Gamma)},$$

and condition (ii)$_K$ takes the (equivalent) form

$$A_\Gamma^{(K)}(n) = \mathcal{O}\big(n^{K\chi(\Gamma)}\big) \quad (n \to \infty).$$

But, in virtue of Proposition 1, this estimate holds for every fixed positive integer K, provided that $\chi(\Gamma) < 0$. Consequently, the lemma applies for all $K \geq 1$, and we obtain the following.

Proposition 2. *Suppose that* $\chi(\Gamma) < 0$. *Then for each fixed integer* $K \geq 1$ *we have a* $\mathcal{O}$*-formula*

$$(23) \qquad \frac{s_\Gamma(n)}{n\, h_\Gamma(n)} = 1 \; + \; \sum_{k=1}^{K-1} d_k(\Gamma)\, \frac{h_\Gamma(n-k)}{h_\Gamma(n)} \; + \; \mathcal{O}\big(n^{K\chi(\Gamma)}\big) \quad (n \to \infty),$$

where the $d_k(\Gamma)$*'s are the coefficients of the series* $\left(\sum_{n=0}^\infty h_\Gamma(n) z^n\right)^{-1}$.

Combining (8), (9), (12) and (13) with Proposition 2 now yields an asymptotic expansion for the function $s_\Gamma(n)$ in terms of the numbers s, r of finite respectively infinite-cyclic free factors in the decomposition (1) of Γ and the local data $m_\sigma = |G_\sigma|$ and $s_{G_\sigma}(d_\sigma)$ for $\sigma = 1, \ldots, s$ and $d_\sigma | m_\sigma$. By iteration we obtain from (12) and (13) that for fixed $k \geq 1$ as $n \to \infty$

$$(24) \qquad |\mathrm{Hom}(G, S_n)| / |\mathrm{Hom}(G, S_{n-k})| \approx n^{k(1-1/m)} \exp\Big(\sum_{\nu=1}^\infty Q_\nu^{(k)} n^{-\nu/m}\Big),$$

where

$$Q_\nu^{(k)} = Q_\nu^{(k)}(G) := k\, Q_\nu(G) + \sum_{1 \leq \lambda < \nu/m} \binom{\nu/m - 1}{\lambda} Q_{\nu - \lambda m}(G)\, \mathfrak{s}_\lambda(k-1)$$

$$- \begin{cases} \frac{(m-1)\mathfrak{s}_{\nu/m}(k-1)}{\nu}\,; & \nu \equiv 0\ (m) \\ 0; & \text{otherwise} \end{cases}$$

and $\mathfrak{s}_\alpha(\beta) := \sum_{\kappa=1}^\beta \kappa^\alpha$ with $\alpha, \beta \in \mathbb{Z}_+$. From (24) we find that for fixed $k \geq 1$

$$(25) \qquad h_\Gamma(n-k)/h_\Gamma(n) \approx n^{k\chi(\Gamma)} \Theta_\Gamma^{(k)}(n^{1/m_\Gamma}) \quad (n \to \infty),$$

where $\Theta_\Gamma^{(k)}(z)$ is the Poincaré series in z^{-1} given by

$$\Theta_\Gamma^{(k)}(z) := \exp\Big(\sum_{\lambda=1}^\infty \frac{(r-1)\, \mathfrak{s}_\lambda(k-1)}{\lambda}\, z^{-\lambda m_\Gamma} \; - \; \sum_{\sigma=1}^s \sum_{\lambda=1}^\infty Q_\lambda^{(k)}(G_\sigma)\, z^{-\lambda m_\Gamma / m_\sigma}\Big)$$

and m_Γ denotes the least common multiple of the m_σ. Consider the expansion

$$n! \approx (2\pi)^{1/2}\, n^{n+1/2}\, e^{-n} \Big\{1 + \sum_{\nu=1}^{\infty} \mathfrak{c}_\nu\, n^{-\nu}\Big\} \quad (n \to \infty) \tag{26}$$

of factorials derived from Stirling's asymptotic expansion of $\log \Gamma(z)$. As is well-known the coefficients $\mathfrak{c}_\nu$ in (26) can be expressed in terms of Bernoullian numbers via the identity

$$\exp\Big(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)}\, z^{-(2k-1)}\Big) = 1\ +\ \sum_{\nu=1}^{\infty} \mathfrak{c}_\nu\, z^{-\nu}. \tag{27}$$

From (8), (9), (26) and (27) we obtain the expansion

$$h_\Gamma(n) \approx L_\Gamma\, n^{-1}\, \Phi_\Gamma(n)\, \Theta_\Gamma(n^{1/m_\Gamma}) \quad (n \to \infty), \tag{28}$$

where L_Γ and $\Phi_\Gamma(n)$ are as in (16) and (17), and $\Theta_\Gamma(z)$ is the Poincaré series in z^{-1} given by

$$\Theta_\Gamma(z) := \exp\Big(\sum_{k=1}^{\infty} \frac{(r-1)B_{2k}}{2k(2k-1)}\, z^{-(2k-1)m_\Gamma}\ +\ \sum_{\sigma=1}^{s} \sum_{\nu=1}^{\infty} \widetilde{\mathcal{B}}_\nu(G_\sigma)\, z^{-\nu m_\Gamma/m_\sigma}\Big).$$

Finally, observe that $m_\Gamma \chi(\Gamma) \in \mathbb{Z}$. In fact, this is a general phenomenon valid for all groups Γ of finite homological type; cf. [Br1, Theorem 5] or [Br2, Chapter IX]. In the case of a finitely generated virtually free group Γ however this can be seen as follows. Define the *(free) rank* $\mu(\Gamma)$ of Γ to be the rank of a free subgroup of index m_Γ in Γ. The existence of such a subgroup follows for example from [Se, II.2.6, Lemma 10] or [M1, formulae (9) and (3)]. Then observe that $\mu(\Gamma)$ is connected with the Euler characteristic of Γ via

$$\mu(\Gamma) + m_\Gamma \chi(\Gamma) = 1, \tag{29}$$

which shows in particular that $\mu(\Gamma)$ is well-defined. Taking into account (25), (28), (29) and Proposition 2 we now obtain the main result of this paper.

Theorem 1. *Let* Γ *be as in* (1) *and assume that* $\chi(\Gamma) < 0$. *Then there exist constants* $\widetilde{\mathcal{D}}_\nu = \widetilde{\mathcal{D}}_\nu(\Gamma)$ *such that the function* $s_\Gamma(n)$ *has the asymptotic expansion*

$$s_\Gamma(n) \approx L_\Gamma\ \Phi_\Gamma(n) \left\{1 + \sum_{\nu=1}^{\infty} \widetilde{\mathcal{D}}_\nu(\Gamma)\ n^{-\nu/m_\Gamma}\right\} \quad (n \to \infty). \tag{30}$$

Moreover, the $\widetilde{\mathcal{D}}_\nu$*'s satisfy the identity*

$$1\ +\ \sum_{\nu=1}^{\infty} \widetilde{\mathcal{D}}_\nu(\Gamma)\ z^{-\nu} = \Theta_\Gamma(z) \left\{1\ +\ \sum_{k=1}^{\infty} d_k(\Gamma)\ z^{-k(\mu(\Gamma)-1)}\ \Theta_\Gamma^{(k)}(z)\right\} \tag{31}$$

with $d_k(\Gamma) = \langle z^k,\ \big(\sum_{n=0}^{\infty} h_\Gamma(n)\ z^n\big)^{-1}\rangle$ *and* $\Theta_\Gamma(z)$, $\Theta_\Gamma^{(k)}(z)$ *as defined above.*

Corollary. *Let* Γ *be as in* (1) *and suppose that* $\chi(\Gamma) < 0$. *Then*

$$s_\Gamma(n) \sim L_\Gamma\ \Phi_\Gamma(n) \quad (n \to \infty), \tag{32}$$

where L_Γ *and* $\Phi_\Gamma(n)$ *are given by* (16) *and* (17).

Given a group $\Gamma \in \mathcal{F}$ of rank $\mu(\Gamma) \geq 2$, the coefficients $\widetilde{\mathcal{D}}_\nu(\Gamma)$ occurring in the expansion (30) are readily calculated from the identity (31) to whatever precision is needed. Note in particular that

$$d_k(\Gamma) = \Big\langle z^k,\ \Big(\sum_{n=0}^{\infty} (n!)^{r+s-1} \prod_{\sigma=1}^{s} \langle z^n,\ \exp(P_{G_\sigma}(z))\rangle\ z^n\Big)^{-1}\Big\rangle \tag{33}$$

can be calculated once r, s and the local data m_σ and $s_{G_\sigma}(d_\sigma)$ are known. As an example let us refine Newman's asymptotic formula (3) for the modular group $\Gamma =$ PSL $(2, \mathbb{Z})$, which involves a relative error of order $\mathcal{O}(n^{-1/6})$. We have $m_\Gamma = 6$, $\chi(\Gamma) = -1/6$ and

$$L_\Gamma\ \Phi_\Gamma(n) = (12\pi e^{1/2})^{-1/2}\ n^{n/6}\ \exp\Big(-\ \frac{n}{6}\ + n^{1/2} + n^{1/3} + \frac{1}{2}\ \log\ n\Big).$$

From the fact that

$$\widetilde{\mathcal{B}}_1(C_2) = \frac{7}{24}\,, \qquad \widetilde{\mathcal{B}}_2(C_2) = -\,\frac{7}{48}$$
$$\widetilde{\mathcal{B}}_1(C_3) = -\,\frac{1}{6}\,, \quad \widetilde{\mathcal{B}}_2(C_3) = \frac{1}{3}\,, \quad \widetilde{\mathcal{B}}_3(C_3) = -\,\frac{1}{6}$$

we find that

$$\begin{aligned}\Theta_\Gamma(z) = &\ \exp\big(-\ \frac{1}{6}z^{-2} + \frac{7}{24}z^{-3} + \frac{1}{3}z^{-4} - \frac{19}{48}z^{-6} + 0\cdot z^{-7} + \dots\big)\\ = &\ 1 - \frac{1}{6}z^{-2} + \frac{7}{24}z^{-3} + \frac{25}{72}z^{-4} - \frac{7}{144}z^{-5} - \frac{4247}{10368}z^{-6} + \frac{175}{1728}z^{-7} + \dots\,.\end{aligned}$$

For $k \le 7$ the $d_k(\Gamma)$'s are given by (33) as $d_1(\Gamma) = d_3(\Gamma) = -1$, $d_2(\Gamma) = 0,\ d_4(\Gamma) = -\ \frac{3}{4}$, $d_5(\Gamma) = \frac{19}{20}$, $d_6(\Gamma) = -\ \frac{11}{5}$ and $d_7(\Gamma) = -\ \frac{31}{28}$. Moreover, from the fact that

$$Q_1(C_2) = \frac{1}{2}\,, \quad Q_2(C_2) = -\,\frac{1}{4}$$
$$Q_1(C_3) = 0, \qquad Q_2(C_3) = \frac{1}{3}\,, \quad Q_3(C_3) = -\,\frac{1}{3}$$

we find that

$$\begin{aligned}\Theta_\Gamma^{(1)}(z) &= 1 - \frac{1}{2}\,z^{-3} - \frac{1}{3}\,z^{-4} + \frac{17}{24}\,z^{-6} + \dots\\ \Theta_\Gamma^{(3)}(z) &= 1 - \frac{3}{2}\,z^{-3} - z^{-4} + \dots\\ \Theta_\Gamma^{(4)}(z) &= 1 - 2z^{-3} + \dots\\ \Theta_\Gamma^{(5)}(z) &= 1 + 0\cdot z^{-1} + 0\cdot z^{-2} + \dots\\ \Theta_\Gamma^{(6)}(z) &= 1 + 0\cdot z^{-1} + \dots\\ \Theta_\Gamma^{(7)}(z) &= 1 + \dots\end{aligned}$$

and Theorem 1 gives the result

$$\begin{aligned} s_{\mathrm{PSL}(2,\mathbb{Z})}(n) = (12\pi e^{1/2})^{-1/2}\, n^{n/6}\, \exp\Big(-\frac{n}{6} + n^{1/2} + n^{1/3} + \frac{1}{2}\log n\Big) \\ \times \Big\{1 - n^{-1/6} - \frac{1}{6}\, n^{-1/3} - \frac{13}{24}\, n^{-1/2} - \frac{7}{36}\, n^{-2/3} + \frac{253}{240}\, n^{-5/6} \\ - \frac{67963}{51840}\, n^{-1} + \frac{203719}{362880}\, n^{-7/6} + \mathcal{O}(n^{-4/3})\Big\}. \end{aligned}$$

From (32) and Stirling's formula we also obtain an estimate on the growth behaviour of the difference function $s_\Gamma(n+1) - s_\Gamma(n)$.

Theorem 2. *Let* Γ *be as in Theorem* 1 *and choose a constant* ρ *with* $0 < \rho < 1$. *Then there exists a positive integer* $n_1 = n_1(\Gamma, \rho)$ *such that for* $n \geq n_1$

$$\begin{aligned} s_\Gamma(n+1) - s_\Gamma(n) > \rho\, (2\pi)^{\chi(\Gamma)/2}\, L_\Gamma\, (n!)^{-\chi(\Gamma)} \\ \times \exp\Big(\sum_{\sigma=1}^{s} \sum_{\substack{d_\sigma \mid m_\sigma \\ d_\sigma < m_\sigma}} \frac{s_{G_\sigma}(d_\sigma)}{d_\sigma}\, n^{d_\sigma/m_\sigma} + \frac{1-\chi(\Gamma)+r}{2}\log n\Big); \end{aligned} \tag{34}$$

in particular, the function $s_\Gamma(n)$ *is strictly increasing for sufficiently large* n, *provided that* $\chi(\Gamma) < 0$.

This is an analogue of [M1, Theorem 4] for the function $s_\Gamma(n)$. Note that for small values of n the function $s_\Gamma(n)$ need indeed not be increasing; compare for example Newman's table of the number of subgroups of index n in $\mathrm{PSL}\,(2,\mathbb{Z})$ for $1 \leq n \leq 100$ in [N].

Remark. Let Γ be as in (1) and suppose that $\chi(\Gamma) < 0$. By [M1, Theorem 5] the number $b_\Gamma(\lambda)$ of free subgroups of index $m_\Gamma\lambda$ in Γ is asymptotically equal to

$$m_\Gamma\big((2\pi m_\Gamma)^{r-1} m_1 \ldots m_s\big)^{1/2}\, (m_\Gamma\lambda)^{-m_\Gamma\chi(\Gamma)\lambda}\, \exp\Big(m_\Gamma\chi(\Gamma)\lambda + \frac{r+1}{2}\log\lambda\Big).$$

Comparing this with (32) we find for the Laplacian probability of picking a free group out of the set of all subgroups of index $m_\Gamma \lambda$ in Γ that as $\lambda \to \infty$

$$\frac{b_\Gamma(\lambda)}{s_\Gamma(m_\Gamma \lambda)} \sim m_1 \dots m_s \, \exp\Big(-\sum_{\sigma=1}^{s} \sum_{\substack{d_\sigma | m_\sigma \\ d_\sigma < m_\sigma}} \frac{s_{G_\sigma}(d_\sigma)}{d_\sigma} (m_\Gamma \lambda)^{d_\sigma / m_\sigma} + \sum_{\substack{\sigma \\ 2 | m_\sigma}} \frac{\big(s_{G_\sigma}(m_\sigma/2)\big)^2}{2 m_\sigma}\Big);$$

in particular, this probability goes to zero as the index increases, but it does so rather slowly compared to the growth of the sample space.

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4. A Combinatorial Classification of $\mathcal{F}$-Groups

Let Γ be an $\mathcal{F}$-group. From Theorem 2 we infer in particular that $s_\Gamma(n) \neq 0$ for all sufficiently large n, provided that $\chi(\Gamma) < 0$. Since this is also true if $\chi(\Gamma) = 0$ we can define a relation $\sim_a$ between two groups $\Gamma_1, \Gamma_2 \in \mathcal{F}$ by

$$\Gamma_1 \sim_a \Gamma_2 :\Longleftrightarrow s_{\Gamma_1}(n) \sim s_{\Gamma_2}(n) \quad (n \to \infty),$$

provided that both Γ_1 and Γ_2 are infinite. We extend this definition to the whole class $\mathcal{F}$ by setting

$$\Gamma_1 \sim_a \Gamma_2 :\Longleftrightarrow |\Gamma_1| = |\Gamma_2| (:= m) \quad \text{and} \quad s_{\Gamma_1}(d) = s_{\Gamma_2}(d) \quad \text{for} \quad d | m$$

if one of the groups involved is finite. Clearly, this defines an equivalence relation on $\mathcal{F}$ with the property that

$$\Gamma_1 \sim \Gamma_2 \Longrightarrow \Gamma_1 \sim_a \Gamma_2,$$

where $\sim$ is the equivalence relation on $\mathcal{F}$ given by

$$\Gamma_1 \sim \Gamma_2 :\Longleftrightarrow s_{\Gamma_1}(n) = s_{\Gamma_2}(n) \quad \text{for all} \quad n \geq 1.$$

Let G_1 and G_2 be two non-isomorphic finite groups such that $|G_1| = |G_2| = m$ and $s_{G_1}(d) = s_{G_2}(d)$ for all $d | m$, and for $i = 1, 2$ let $\Gamma_i = G_i * C_2$. Then Γ_1 and Γ_2 are not isomorphic; on the other hand, for a finite group G

$$\sum_{n=0}^{\infty} \frac{|\,\mathrm{Hom}\,(G, S_n)|}{n!}\, z^n \;=\; \exp\,\Big(\sum_{d|m} \frac{s_G(d)}{d}\, z^d\Big), \quad m = |G|, \tag{35}$$

which together with Dey's formula implies that $s_{\Gamma_1}(n) = s_{\Gamma_2}(n)$ for $n \geq 1$. This shows that the equivalence relation $\sim$ is strictly coarser than the one given by isomorphism, even when restricted to large $\mathcal{F}$-groups. Such pairs $\{G_1, G_2\}$ are found for example among the groups of order pq^2, where p and q are primes such that $3 < p < q$ and $q \equiv 1 \bmod p$; cf. [R]. Non-isomorphic p-groups having isomorphic subgroup lattices provide another source of such pairs of groups. We remark that both formula (35) and Dey's formula as well as Wohlfahrt's result [Wo] are contained as special cases in [M1, Prop. 1], which in turn is a special case of the exponential principle for combinatorial functors as developed in [DM]. No characterization in terms of structural invariants is known for the relation $\sim$; however (32) implies such a characterization for the coarser equivalence relation $\sim_a$. From this characterization we also obtain a finiteness result on $\sim_a$-equivalence classes and hence a fortiori for $\sim$-classes.

Theorem 3. (a) *Let Γ_1 and Γ_2 be two $\mathcal{F}$-groups with decompositions $\Gamma_i = \mathop{*}\limits_{\sigma_i=1}^{s(\Gamma_i)} G_{\sigma_i}^{(i)} * F_{r(\Gamma_i)}$ $(i = 1, 2)$, and let $m_{\sigma_i}^{(i)}$ denote the order of $G_{\sigma_i}^{(i)}$. Then we have $\Gamma_1 \sim_a \Gamma_2$ if and only if Γ_1 and Γ_2 satisfy the following conditions:*

(i) $\chi(\Gamma_1) = \chi(\Gamma_2)$.

(ii) $m_{\Gamma_1} = m_{\Gamma_2} (:= m)$.

(iii) $\prod\limits_{\sigma_1=1}^{s(\Gamma_1)} m_{\sigma_1}^{(1)} = \prod\limits_{\sigma_2=1}^{s(\Gamma_2)} m_{\sigma_2}^{(2)}$.

(iv) $r(\Gamma_1) = r(\Gamma_2)$.

(v) $\sum\limits_{\substack{\sigma_1 \\ d|m_{\sigma_1}^{(1)}}} \dfrac{s_{G_{\sigma_1}^{(1)}}(m_{\sigma_1}^{(1)}/d)}{m_{\sigma_1}^{(1)}} = \sum\limits_{\substack{\sigma_2 \\ d|m_{\sigma_2}^{(2)}}} \dfrac{s_{G_{\sigma_2}^{(2)}}(m_{\sigma_2}^{(2)}/d)}{m_{\sigma_2}^{(2)}}$

for all integers d with $d \geq 2$ and $d|m$.

(vi) $\displaystyle\sum_{\substack{\sigma_1 \\ 2|m_{\sigma_1}^{(1)}}} \frac{\left(s_{G_{\sigma_1}^{(1)}}(m_{\sigma_1}^{(1)}/2)\right)^2}{m_{\sigma_1}^{(1)}} = \sum_{\substack{\sigma_2 \\ 2|m_{\sigma_2}^{(2)}}} \frac{\left(s_{G_{\sigma_2}^{(2)}}(m_{\sigma_2}^{(2)}/2)\right)^2}{m_{\sigma_2}^{(2)}}$.

(b) *A* $\sim_a$ *-equivalence class of* $\mathcal{F}$*-groups contains only a finite number of non-isomorphic groups.*

Proof. (a) Suppose that $\chi(\Gamma_1),\ \chi(\Gamma_2) < 0$. By (32) $\Gamma_1 \sim_a \Gamma_2$ is then equivalent to the assertion that

$$q_n = 1 + o(1) \quad (n \to \infty), \tag{36}$$

where

$$\begin{aligned} q_n := \exp\Big(&- \big(\chi(\Gamma_1) - \chi(\Gamma_2)\big) n \log n + \big(\chi(\Gamma_1) - \chi(\Gamma_2)\big) n \\ &+ \sum_{d \geq 2} d \Big(\sum_{\substack{\sigma_1 \\ d|m_{\sigma_1}^{(1)}}} \frac{s_{G_{\sigma_1}^{(1)}}(m_{\sigma_1}^{(1)}/d)}{m_{\sigma_1}^{(1)}} - \sum_{\substack{\sigma_2 \\ d|m_{\sigma_2}^{(2)}}} \frac{s_{G_{\sigma_2}^{(2)}}(m_{\sigma_2}^{(2)}/d)}{m_{\sigma_2}^{(2)}} \Big) n^{1/d} \\ &+ \frac{r(\Gamma_1) - r(\Gamma_2)}{2} \log n + \log\left(L_{\Gamma_1}/L_{\Gamma_2}\right)\Big). \end{aligned} \tag{37}$$

Since the functions of n involved in the exponent on the right-hand side of (37) represent a set of linearly independent vectors in the space $\mathbb{R}^{\mathbb{Z}_+}/\{\text{null sequences}\}$, (36) implies (i) and (iv) as well as

$$\sum_{\substack{\sigma_1 \\ d|m_{\sigma_1}^{(1)}}} \frac{s_{G_{\sigma_1}^{(1)}}(m_{\sigma_1}^{(1)}/d)}{m_{\sigma_1}^{(1)}} = \sum_{\substack{\sigma_2 \\ d|m_{\sigma_2}^{(2)}}} \frac{s_{G_{\sigma_2}^{(2)}}(m_{\sigma_2}^{(2)}/d)}{m_{\sigma_2}^{(2)}}, \quad d \geq 2, \tag{38}$$

and

$$L_{\Gamma_1} = L_{\Gamma_2}. \tag{39}$$

Observe that, assuming (38), a given prime power divides m_{Γ_1} if and only if it divides m_{Γ_2}, i.e. $m_{\Gamma_1} = m_{\Gamma_2}$. Therefore, (38) is equivalent to the conjunction of (ii) and (v).

Also, by Schneider's transcendence theorem [Sc] the conjunction of (iv) and (39) is equivalent to (iii), (iv) and (vi).

It remains to deal with the case when one of the groups (say Γ_1) is finite or virtually infinite-cyclic. If Γ_1 is finite the equivalence $\Gamma_1 \sim_a \Gamma_1 \Longleftrightarrow$ (i) - (vi) follows immediately from the definition of the relation $\sim_a$. Thus we are left with the case when Γ_1 is virtually infinite-cyclic. But then Γ_1 is isomorphic to C_∞ or the infinite dihedral group $C_2 * C_2$, and the assumption $\Gamma_1 \sim_a \Gamma_2$ implies that Γ_2 is also infinite and that $s_{\Gamma_2}(n) = \mathcal{O}(n)$, since $s_{C_2 * C_2}(n) = 2\lfloor n/2 \rfloor + 1$; cf. for example [GIR, formula (12)]. In view of Theorem 2 this estimate is inconsistent with the assumption that $\chi(\Gamma_2) < 0$, hence Γ_2 must also be virtually infinite-cyclic and $\Gamma_1 \sim_a \Gamma_2$ implies in fact that $\Gamma_1 \cong \Gamma_2$, in particular conditions (i) - (vi) hold for Γ_1 and Γ_2. Conversely, suppose that Γ_1 and Γ_2 satisfy conditions (i) -(vi) and that Γ_1 is virtually infinite-cyclic. Then, by condition (i), the group Γ_2 is also virtually infinite-cyclic, and by (iv) we have $\Gamma_1 \cong \Gamma_2$, hence a fortiori $\Gamma_1 \sim_a \Gamma_2$.

(b) Consider the $\sim_a$ -class $\mathcal{E}$ generated by a group $\Gamma_0 \in \mathcal{F}$. By part (a) m_{Γ_0}, $r(\Gamma_0)$ and the Euler characteristic $\chi(\Gamma_0)$ are invariants of $\mathcal{E}$. Decompose each $\Gamma \in \mathcal{E}$ in the form (1), say $\Gamma = G_1 * \cdots * G_s * F_{r(\Gamma_0)}$ with non-trivial finite groups G_σ and some $0 \leq s < \infty$, and write the Euler characteristic of Γ as

$$\chi(\Gamma_0) = \chi(\Gamma) = \sum_{\sigma=1}^{s} \Big(\frac{1}{|G_\sigma|} - 1\Big) - r(\Gamma_0) + 1.$$

Since $G_\sigma \neq 1$ we have $|G_\sigma|^{-1} - 1 \leq -1/2$, and inserting this we find that $s \leq 2\big(1 - r(\Gamma_0) - \chi(\Gamma_0)\big)$; and since each $|G_\sigma|$ is bounded by m_{Γ_0} it follows indeed that $\mathcal{E}$ decomposes into finitely many isomorphism classes.

$\square$

Remarks. 1. By conditions (ii) and (v) the number $M_\Gamma := \max_\sigma |G_\sigma|$ and its multiplicity in the decomposition (1) of a (non-free) group $\Gamma \in \mathcal{F}$ are invariant under $\sim_a$ -equivalence. Also, if Γ and Γ' are cyclic covers such that $\Gamma \sim_a \Gamma'$, then the number ν_d of σ such that $|G_\sigma| = d$ is the same in both groups for all $2 \leq d \mid m_\Gamma$.

2. Theorem 3(b) can be refined to yield an estimate on the number $\varepsilon(\Gamma)$ of isomorphism classes into which the $\sim_a$ -class generated by a group $\Gamma \in \mathcal{F}$ decomposes. Denoting by $\iota(n)$ the number of (isomorphism classes of) groups of order n, a slight refinement of the above argument leads to the inequality

$$\varepsilon(\Gamma) \leq \delta_{m_\Gamma,\pi_\Gamma}\delta_{m_\Gamma,1/(r(\Gamma)+\chi(\Gamma))}\iota(m_\Gamma) + \sum_{s=2}^{\lfloor 2(1-r(\Gamma)-\chi(\Gamma))\rfloor} \sum_{\substack{2\leq\nu_1\leq\cdots\leq\nu_s\\ \nu_1\ldots\nu_2=\pi_\Gamma\\ [\nu_1,\ldots,\nu_s]=m_\Gamma\\ \Sigma 1/\nu_\sigma=\chi(\Gamma)+r(\Gamma)+s-1}} \iota(\nu_1)\ldots\iota(\nu_s)$$

in terms of the class invariants m_Γ, $r(\Gamma)$, $\chi(\Gamma)$ and $\pi_\Gamma := \Pi_\sigma m_\sigma$, which can be made more explicit by using known estimates for the function $\iota(n)$; cf. for example [Py].

To conclude, we give examples of free products $\Gamma_1, \Gamma_2 \in \mathcal{F}$ such that $s_{\Gamma_1} \neq s_{\Gamma_2}$ but $s_{\Gamma_1}(n) \sim s_{\Gamma_2}(n)$ as $n \to \infty$. Let $\mathfrak{Q}$ be the quaternion group and denote by G_0 the group of order 12 generated by two elements X, Y subject to the defining relations $X^3 = Y^4 = 1$ and $XY = YX^2$. G_0 contains exactly one element of order 2, has one subgroup of order 3, three subgroups of order 4 and one subgroup of order 6. Fix a positive integer η and put

$$\Gamma_1(\eta) := \underbrace{C_8 * \cdots * C_8}_{2\eta} * \underbrace{G_0 * \cdots * G_0}_{3\eta} \quad \text{and} \quad \Gamma_2(\eta) := \underbrace{\mathfrak{Q} * \cdots * \mathfrak{Q}}_{2\eta} * \underbrace{C_{12} * \cdots * C_{12}}_{3\eta}.$$

Conditions (i) - (vi) are readily checked and we conclude from Theorem 3 that for each $\eta \in \mathbb{N}$ indeed

$$s_{\Gamma_1(\eta)}(n) \sim s_{\Gamma_2(\eta)}(n) \quad (n \to \infty).$$

Note also that the Euler characteristic $\chi\big(\Gamma_i(\eta)\big)$ strictly decreases as η increases, hence (again by Theorem 3) pairs of groups $\{\Gamma_1(\eta), \Gamma_2(\eta)\}$ constructed to different values of η belong to different classes with respect to the equivalence relation $\sim_a$. On the other hand we find from Dey's formula (21) that $\Gamma_1(\eta)$ contains exactly $2^{5\eta} - 1$ subgroups of index 2 while $s_{\Gamma_2(\eta)}(2) = 2^{7\eta} - 1$. It follows that the three equivalence relations on the class $\mathcal{F}$ given by (i) isomorphism, (ii) the property that $s_{\Gamma_1} = s_{\Gamma_2}$ and (iii) the relation $\sim_a$ are all distinct.

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