Splitting property in infinite posets

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Abstract

It is known that in every finite poset $\mathcal{P}$ any maximal antichain S with some *denseness* property may be partitioned into disjoint subsets S_1 and S_2, such that the union of the downset of S_1 with the upset of S_2 yields the entire poset: $\mathcal{D}(S_1) \cup \mathcal{U}(S_2) = \mathcal{P}$. Hereby we give analogues results for infinite posets.

1 Introduction

Let $\mathcal{P} = (P, <_P)$ be an arbitrary (finite or infinite) partially ordered set (poset) and let H be a subset of P. The *downset* $\mathcal{D}(H)$ of the subset H is:

$$\mathcal{D}(H) = \{x \in P : \exists s \in H (x \leq s)\}.$$

The *upset* of H is:

$$\mathcal{U}(H) = \{x \in P : \exists s \in H (s \leq x)\}.$$

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We introduce also the sets

$$\mathcal{D}^*(H) = \{x \in P : \exists s \in H(x < s)\}$$

and

$$\mathcal{U}^*(H) = \{x \in P : \exists s \in H(s < x)\}.$$

A subset $S \subset P$ is called *antichain* or *Sperner system* if no two elements of S are comparable. An antichain S is *maximal* if for every antichain $S' \subset P$, $S \subset S'$ implies $S = S'$. It is easy to see that S is a maximal antichain iff

$$\mathcal{D}(S) \cup \mathcal{U}(S) = P. \tag{1}$$

Definition A subset $H \subset P$ is called a *generator* if it satisfies condition (1) (S replaced by H.). We say that a generator H satisfies the *splitting property* if there exists a partition (H_1, H_2) of H into disjoint subsets such that

$$\mathcal{U}(H_1) \cup \mathcal{D}(H_2) = P \tag{2}$$

holds. Finaly a subset $H \subset P$ is called *dense* in the poset $\mathcal{P}$ if there is no *open interval* $\langle x, y\rangle = \{z \in P : x < z < y\}$) which intersects H in exactly one element.

In [1] R. Ahlswede, P.L. Erdős and N. Graham have shown that

Theorem 1 *Let S be a maximal, dense antichain in the finite poset $\mathcal{P}$. Then S satisfies the splitting property.*

In fact, there are a lot of finite posets where every maximal antichain satisfies the splitting property. One example for this is the Boolean algebra of n elements. On the other hand there was already in paper [1] posed the question: what can someone say about the infinite case. The first results in that direction were proved by R. Ahlswede and L. Khachatrian in [2]. In this paper we prove several results on splitting properties of infinite posets.

2 Results

In this paper a partially ordered set (poset) means always an arbitrary, finite or infinite one, except it is stated otherwise.

Definition Let S be a maximal antichain in the poset P. The well-ordering μ on S is called *closed* if for every element $z \in \mathcal{U}^*(S)$ there exists another (not necessaryly different) element $y \in \mathcal{U}^*(S)$ such that $y \leq z$ and there is a maximal element of S (concerning μ) which is comparable to y :

$$\exists \max_{\mu} \quad [\mathcal{D}(y) \cap S]\,. \tag{3}$$

Definition For the maximal antichain $S \subset P$ the splitting (S_1, S_2) is *(upper) minimal*, if there is no proper subset S_3 of S_2 such that $\mathcal{U}^*(S) = \mathcal{U}^*(S_3)$.

Theorem 2 *Let S be a maximal, dense antichain in P. Let the well ordering μ of S be closed. Then S has a minimal splitting.*

Proof At first we are going to determine a splitting then we will choose a minimal subset S_2 which still ensures the splitting.

Let an element $y \in \mathcal{U}^*(S)$ be *closed* if the maximum described in (3) exists for y. Then define the subposet $\mathcal{P}^*$ as follows: the elements are $P^* = \mathcal{D}^*(S) \cup S \cup \{y \in \mathcal{U}^*(S) : y \text{ is closed}\}$, and take the restricted partial order from $\mathcal{P}$. (We use the same notation for it.) It is quite clear that a minimal splitting of S concerning the poset $\mathcal{P}^*$ is also minimal splitting for $\mathcal{P}$.

Let

$$S^+ = \left\{ \max_{\mu} \quad [\mathcal{D}(y) \cap S] \; : \; y \in \mathcal{U}_{\mathcal{P}^*}(S) \right\} \tag{4}$$

Furthermore let $S^- = S \setminus S^+$. We claim that

$$\mathcal{D}^*(S^-) = \mathcal{D}^*(S). \tag{5}$$

To prove this assume the contrary. Then there is an $x \in \mathcal{D}^*(S)$ such that $x \notin \mathcal{D}^*(S^-)$. Let $s = \min_\mu [\mathcal{U}(x) \cap S]$. Since $s \notin S^-$ therefore there exists an $y \in \mathcal{U}_{\mathcal{P}^*}(S)$ such that the maximum in condition (3) is equal to s. But then

$$|\mathcal{D}(y) \cap S \cap \mathcal{U}(x)| = 1$$

which is a contradiction since S is dense. This proves (5). Therefore (S^-, S^+) is a splitting for $\mathcal{P}$.

Now we are going to choose a minimal subset S_2 of S^+ which still ensures a good splitting. Define the well ordering μ^* as the restriction of μ for S^+. It is easy to see that μ^* is closed on S^+. Indeed, μ is closed on S and each maximum taken from S is included to S^+. Let $S^+ = \{s_\xi : \xi < \alpha\}$ (that is $s_\xi <_{\mu^*} s_\eta$ iff $\xi < \eta < \alpha$). We define S_2 by transfinite induction. Suppose that $\beta < \alpha$ and for all $\gamma < \beta$ we already decided, whether s_γ belongs to S_2, or not. Let $s_\beta \in S_2$ if and only if

$$\exists y \in \mathcal{U}_{\mathcal{P}^*}(S) \qquad : \qquad s_\beta = \max_\mu \quad [\mathcal{D}(y) \cap S] .$$

and

$$\mathcal{D}(y) \cap \{s_\gamma \in S_2 : \gamma < \beta\} = \emptyset. \tag{6}$$

Furthermore let $S_1 = S \setminus S_2$. We claim that (S_1, S_2) is a minimal splitting in $\mathcal{P}^*$ (and therefore in $\mathcal{P}$ as well). The relation $\mathcal{U}^*(S_2) = \mathcal{U}^*(S^+)$ is clear by definition, so we have to prove the minimality of S_2. Assume the contrary. Then there exists $s_\beta \in S_2$ such that $\mathcal{U}^*(S_2) = \mathcal{U}^*(S_2 \setminus \{s_\beta\})$. By definition there is an $y \in \mathcal{U}^*(S^+)$ for which the desired maximum is s_β and for which, by condition (6), $y \notin \mathcal{U}^*(s_\xi)$ for all $\xi < \beta$. On the other hand for all $\xi < \alpha$ where $\beta < \xi$ we know that $y \notin \mathcal{U}^*(s_\xi)$ by the maximality of s_β. Therefore $y \notin \mathcal{U}^*(S_2 \setminus \{s_\beta\})$, a contradition. This proves the minimality of S_2 and finishes the proof of Theorem 2. $\square$

Theorem 1 is a consequence of this result, since in a finite poset every "well-ordering" on S is closed (because S is finite). However if we apply this proof to the finite case we get back the original proof of Theorem 1.

We give two applications of Theorem 2. The first one is a generalization of a theorem of Lovász on property B. The finite hypergraph $\mathcal{H}$ satisfies the so called *property B* iff there is a partition (X_1, X_2) of the underlying set X such that any edge of $\mathcal{H}$ intersects both classes. Lovász proved ([4], Problem 13.33) the following result:

Theorem 3 (Lovász, [4]) *Suppose that for the finite hypergraph $\mathcal{H}$ there are no two edges with exactly one element in common. Then $\mathcal{H}$ satisfies property B.*

Let S be an arbitrary (finite or infinite) set, and $\mathcal{A}$ and $\mathcal{B}$ be two set systems on it. $\mathcal{B}$ is called *closed* if there exists a well ordering μ on S such that every $B \in \mathcal{B}$ has a maximal element in μ. Furthermore the pair of these two set systems is called *dense* if there are no $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with precisely one element in common. Now we have

Corollary 4 *Let $\mathcal{A}$ and $\mathcal{B}$ be a dense set system pair on the set S where $\mathcal{B}$ is closed on S. Then*

(i) there is a partition (X_1, X_2) of S such that for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $X_1 \cap A \neq \emptyset$ and $X_2 \cap B \neq \emptyset$.

(ii) Furthermore the partiton can be chosen such a way, that X_2 is minimal. (No proper subset of it intersects every and each set in $\mathcal{B}$.)

Proof Let the poset $\mathcal{P}$ be defined as follows: the elements are $P = \mathcal{A} \cup S \cup \mathcal{B}$, where $\mathcal{U}^*(S) = \mathcal{B}$ and the relations are derived from the incidencies: $s < B$ iff $s \in B$ and $A < s$ iff $s \in A$. It is easy to see that this poset satisfies the conditions of Theorem 2, and the minimum splitting supplies the desired minimal partition (X_1, X_2). $\Box$

Since in the case of finite S every (well) ordering is closed, making $\mathcal{A}$ and $\mathcal{B}$ equal to $\mathcal{H}$ gives Lovász's result. Turning to the second application, J. Klimó proved (in [3]) the following result:

Corollary 5 *Let the set system $\mathcal{H}$ be a covering of the underlying set X. Suppose that $\mathcal{H}$ is closed, that is there exists a well ordering μ on $\mathcal{H}$ such that for every $x \in X$*

$$\exists \max_{\mu} \{H \in \mathcal{H} : x \in H\} \tag{7}$$

holds. Then $\mathcal{H}$ contains a minimal covering of X.

Proof In Corollary 4 let the underlying set S be equal to $\mathcal{H}$, let $\mathcal{A} = \emptyset$ and finally let $\mathcal{B}$ be equal to X. For every $x \in X$ let the subset B be the set in condition (7). Then the consequence of Corollary 4 supplies the required minimum cover. $\Box$

To be honest we must remark that applying the proof of Theorem 2 for this special case we get back the original proof of Klimó. We also remark, that in [2] Ahlswede and Khachatrian proved Corollary 4 (i) using Klimó's result.

In his paper Klimó found a necessary condition to be the covering $\mathcal{H}$ closed.

Theorem 6 (Klimó, [3], Theorem 9) *Let $\mathcal{H}$ be a covering of X such that there exists a finite constant k that for every $x \in X$*

$$|\{H \in \mathcal{H} : x \in H\}| \leq k. \tag{8}$$

Then $\mathcal{H}$ is closed.

Applying this result we can prove:

Lemma 7 *Let S be a maximal antichain in the poset $\mathcal{P}$. Assume that*

$$\text{in } \mathcal{U}^*(S) \text{ there exists an antichain } \bar{S} \text{ with } \mathcal{U}^*(\bar{S}) = \mathcal{U}^*(S) \tag{9}$$

and

$$\forall s \in S \quad : \quad \left|\mathcal{U}(s) \cap \bar{S}\right| \leq k. \tag{10}$$

Then there is a closed well ordering μ on S.

Proof For every $s \in S$ let $H(s) = \mathcal{U}(s) \cap \bar{S}$. Then $\mathcal{H} = \{H(s) : s \in S\}$ is a covering on $\bar{S}$. Due to condition (10), condition (8) holds for this covering therefore there is a well ordering μ on S which makes $\mathcal{H}$ closed. Consequently the well ordering μ is closed on S in the poset $\mathcal{P}$. □

Corollary 8 *Let S be a maximal dense antichain in $\mathcal{P}$ satisfying condition (9) and (10). Then S has a minimum splitting.*

Proof Application of Lemma 7 gives that there exists a closed well ordering μ. The application of Theorem 2 finishes the proof. □

Furthermore we give here another easy consequence of Theorem 2. Let $\mathbb{Z}$ denote the poset of all subsets of the natural numbers ordered by inclusions.

Corollary 9 *Let S be a maximal dense antichain in $\mathbb{Z}$ such that any element of $\mathcal{U}^*(S)$ is comparable some finite (ranked) element of S. Then S has a (minimal) splitting.*

To prove this statement it is enough to notify that taking any well ordering μ on S this well ordering is closed. Indeed, if $z \in \mathcal{U}^*(S)$ then take a finite s out of S with $s < z$ and take a finite subset y of z which contains s. Then there are just finitely many elements of S which is comparable to y, therefore the maximum in condition (3) does exist. Theorem 2 finishes the proof. □

Finally we give another application of Theorem 2.

Definition The subset H of $\mathcal{P}$ is d_1-*dense* iff for any $x, y \in P$ where $\langle x, y\rangle \cap H \neq \emptyset$ we have $|\langle x, y\rangle| \neq 1$. Ahlswede and Khachatrian proved in [2] that:

Theorem 10 (Ahlswede and Khachatrian) *Let S be a maximal d_1-dense antichain in the poset $\mathcal{P}$ satisfying condition (9). Let the well ordering μ be closed on S. Finally assume that*

$$\textit{in } \mathcal{D}^*(S) \textit{ there exists an antichain } \underline{S} \textit{ with } \mathcal{D}^*(\underline{S}) = \mathcal{D}^*(S). \tag{11}$$

Then S has a splitting.

We can improve this result simply to notify that applying Theorem 2 one can conclude that in Theorem 10 S has a minimal splitting as well.

It is interesting to remark that Theorem 2 and Theorem 10 have different consequences. For example let $\mathcal{P}$ be the chain of all rational numbers taking two copies of 0. If S consists of these two copies then Theorem 2 gives immediately that S has a splitting. Theorem 10 does not prove it. On the other hand Theorem 10 proves immediately that each maximal antichain in any finite Boolean algebra has a splitting since the existence of $\bar{S}$, $\underline{S}$ and the closed well ordering μ is obvious in every finite poset.

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