ON THE DOMAIN OF ATTRACTION TO THE NORMAL LAW AND CONVERGENCE OF SELFNORMALIZED SUMS

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Abstract

It is proved that if the selfnormalized sums for i.i.d. random variables with the law of X are asymptotically standard normal, and if there is, additionally, convergence of some moments, then X is in the domain of attraction of the normal law. It is also proved that if X is in the domain of attraction of some α–stable law, $\mathbf{E}X=0$ for $1<\alpha\leq 2$, X symmetric for $\alpha=1$, then all the moments of the selfnormalized sums converge, in particular showing that the previous result has a converse. A characterization of the domain of attraction of the normal law in terms of the asymptotic neglibility of the maximum term of the sum with respect to the sum centered at its median is also presented.

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1. Introduction. Let $X, X_i,\ i \in \mathbf{N}$, be independent, identically distributed random variables and let

$$S_n = \sum_{i=1}^{n} X_i, \quad V_n^2 = \sum_{i=1}^{n} X_i^2, \quad n \in \mathbf{N}.$$

Logan et *al.* conjecture that 'S_n/V_n is asymptotically normal if [and perhaps only if] X is in the domain of attraction of the normal law' (and X is centered). The 'if' part of this conjecture follows rather easily from basic principles (Raikov's theorem) but the parenthetical 'only if' is unknown in the non–symmetric case. Griffin and Mason (1991) attribute to Roy Erickson a beautiful proof of the fact that, for X *symmetric*, convergence in distribution of the selfnormalized sums S_n/V_n to the standard normal law implies that X is in the domain of attraction of the normal law ($X \in DAN$). The proof relies heavily on symmetry, and the non–symmetric case seems to be, perhaps surprisingly, quite harder. The present paper is devoted to this problem.

Our main result states that, for X integrable and centered, if $\mathcal{L}(S_n/V_n) \to_w N(0,1)$ and $\sup_n \mathbf{E}(S_n/V_n)^6 < \infty$, then X is indeed in the domain of attraction of the normal law (uniform boundedness of the sixth moment can be replaced by some mild regularity on the law of X). We should remark that this constitutes a characterization of the domain of attraction of the normal law because we also prove (and this is our second main result) that, if X is in the domain of attration of any α–stable law, assuming $\mathbf{E}X = 0$ for $1 < \alpha \le 2$ and X symmetric for $\alpha = 1$, then *all* the moments of the selfnormalized sums converge. This is another aspect of the general fact that *selfnormalization improves convergence in the CLT*. Perhaps the most important instance of this fact occurs in the paper by Bentkus and Götze (1994) where they show that this is the case for the speed of convergence of the cdf of selfnormalized sums to that of the standard normal distribution when $X \in DAN$.

When we started this research we were unaware of O'Brien's result to the effect that $X \in DAN$ if and only if $\max_{i \le n} |X_i|/V_n \to 0$ in probability, a fact generally known in the symmetric case (e.g., Griffin and Mason, 1991), that we needed for the proof of the main result. We independently obtained a proof of this fact, as well as of the following related result, which seems of independent interest: $X \in DAN$ if and only if $\max_{i \le n} |X_i|/|S_n - m_n| \to 0$ in probability, where m_n denotes a median of S_n. That is, normal convergence is characterized by the maximum term being asymptotically negligible in probability with respect to the sum *centered at its median* (any of its medians). This paper also contains two other, perhaps less interesting, characterizations of the domain of attaction of the normal law.

In order to treat S_n/V_n we must deal with the quantities $\Lambda_m := (n!/(n-m)!)\mathbf{E}\big[X_1 \cdots X_m/V_n\big]$. To this effect we use the simple fact that $\int_0^\infty \lambda^{r-1} \exp(-\lambda a) d\lambda = (r-1)!/a^r$ ($r \in \mathbf{N}$ and $a > 0$) to express the quantities Λ_m as averages of products of expected values of the form $\mathbf{E}f(X)$. In a sense, this works as a de Finetti representation for the *finite* family of exchangeable variables $X_1/V_n, \ldots, X_n/V_n$. This allows, in addition, to obtain a Hölder type inequality, $\Lambda_4 \le \Lambda_2^{1/2} \Lambda_6^{1/2}$, which is crucial in the proof of the main result (thus the sixth moment!). These are the two main new ingredients in our proofs.

Obviously, the problem of characterizing the domain of attraction of the normal law by the CLT for selfnormalized sums is interesting in its own right. In fact, some of our original motivation for looking at it has to do with the observation that an affirmative answer would imply the remarkable fact that a sharp bound of Bentkus and Götze (1994, Theorem 1.4) for $\sup_{x\in\mathbf{R}}|\Pr\{S_n/V_n < x\} - \Phi(x)|$, that depends rather explicitly on the law of X, would tend to zero *whenever* $\mathcal{L}(S_n/V_n) \to_w N(0,1)$.

Contents of the paper: Section 2 contains the theorem on convergence of moments in the selfnormalized CLT, in Section 3 we obtain the chartacterization of DAN by the asymptotic neglibility of the maximum term and, in Section 4, the main result about the relationship between DAN and the CLT for selfnormalized sums.

Finally, here is some notation. As in this introduction, in this article, $X_i, i \in \mathbf{N}$, will mean i.i.d. random variables, with the law of X, $S_n = \sum_{i=1}^n X_i$, $V_n^2 = \sum_{i=1}^n X_i^2$, and the law of any random variable Y will often be denoted by $\mathcal{L}(Y)$. We will use, with some abuse, the notational convention $\frac{X}{Y} := \frac{X}{Y} I_{Y\neq 0}$, already used implicitly when writing S_n/V_n. We also write $\tilde{X}_{n,i} := \frac{X_i}{V_n} I_{X_i\neq 0}$, so that

$$\frac{S_n}{V_n} := \frac{S_n}{V_n} I_{V_n\neq 0} = \sum_{i=1}^n \tilde{X}_{n,i}.$$

$X \in DA(\alpha)$ will mean 'X is in the domain of attraction of an α–stable law'. But we may also write DAN for $DA(2)$, that is, for the domain of attraction of the normal law. $X \in DA(\alpha)$ automatically implies that X is nondegenerate.

2. Convergence of moments in the selfnormalized central limit theorem. Logan et *al.* (1973) show that selfnormalized sums of i.i.d. random variables in the domain of attraction of α–stable laws, $0 < \alpha \leq 2$, converge in distribution to limit laws which, among other important properties, have Gaussian–like tails. It is thus natural to ask whether these limit theorems for selfnormalized sums can be complemented with convergence of moments. In this section the question is answered in the affirmative.

2.1. THEOREM. *Let $X \in DA(\alpha)$ for some $0 < \alpha \leq 2$, and suppose further that X is symmetric if $\alpha = 1$ and that $\mathbf{E}X = 0$ if $1 < \alpha \leq 2$. For each α, let Z_α be a random variable whose law is the limit law of the normalized sums S_n/V_n. Then,*

$$\lim_{n\to\infty} \mathbf{E}\Big|\frac{S_n}{V_n}\Big|^k = \mathbf{E}|Z_\alpha|^k \tag{2.1}$$

for all $k > 0$.

PROOF. By uniform integrability it suffices to prove

$$\sup_n \mathbf{E}\Big(\sum_{i=1}^n \tilde{X}_{n,i}\Big)^m < \infty \tag{2.2}$$

for m even.

For X symmetric (2.2) follows directly from Khinchine's inequality (in this case, simply, direct computation of expectations of even moments of Rademacher sums): if X is symmetric then the variables $|X|$ and $X/|X|$ are independent and the latter is Rademacher, that is, $\Pr\{X/|X| = 1\} = \Pr\{X/|X| = -1\} = 1/2$; then, setting $\varepsilon_i = X_i/|X_i|$ and denoting by $\mathbf{E}_\varepsilon$ conditional expectation given the sequence $\{|X_i|\}$, we have

$$\begin{aligned}\sup_{n\in\mathbf{N}} \mathbf{E}\Big|\frac{S_n}{V_n}\Big|^m &= \sup_{n\in\mathbf{N}} \mathbf{E}\mathbf{E}_\varepsilon\Big|\sum_{i=1}^n \tilde{X}_{n,i}\varepsilon_i\Big|^m \\ &\le c_m \sup_{n\in\mathbf{N}} \mathbf{E}\Big[\sum_{i=1}^n \tilde{X}_{n,i}^2\Big]^{\frac{m}{2}} \\ &= c_m \sup_{n\in\mathbf{N}} \Pr\{V_n \neq 0\} \\ &= c_m,\end{aligned}$$

where c_m is the constant provided by Khinchin's inequality.

We now consider the remaining cases. For ease of notation we set $I_n^m = \{(i_1, \dots, i_m) : i_k \text{ integers}, 1 \le i_k \le n, i_k \neq i_\ell \text{ if } k \neq \ell\}$. By Newton's identities (e.g. Jacobson, 1951, page 110) $\big(\sum_{i=1}^n \tilde{X}_{n,i}\big)^m$ is a linear combination of $\sum_{I_n^m} \tilde{X}_{n,i_1}\cdots\tilde{X}_{n,i_m}$ and of products of the form $\prod_{r=1}^s(\sum_{i=1}^n \tilde{X}_{n,i}^{k_r})^{\ell_r}$ with $\sum_{r=1}^s k_r\ell_r = m$ and not all the k_r equal to one. Hence, since moreover $|\sum_{i=1}^n \tilde{X}_{n,i}^{k_r}| \le 1$ if $k_r \ge 2$, we can use induction to conclude that the proof of (2.2) reduces to showing

$$\sup_n \mathbf{E}\Big(\sum_{I_n^m} \tilde{X}_{n,i_1}\cdots\tilde{X}_{n,i_m}\Big) < \infty \tag{2.3}$$

for all m even. Using the elementary fact that $\int_0^\infty \lambda^{r-1}\exp(-\lambda a)d\lambda = (r-1)!/a^r$ for all $r \in \mathbf{N}$ and $a > 0$, we can write

$$\begin{aligned}\mathbf{E}\Big(\sum_{I_n^m} \tilde{X}_{n,i_1}\cdots\tilde{X}_{n,i_m}\Big) &= \frac{1}{(\frac{m}{2}-1)!}\mathbf{E}\Big[\int_0^\delta \sum_{I_m^n} X_{i_1}I_{X_{i_1}\neq 0}\cdots X_{i_m}I_{X_{i_m}\neq 0}\lambda^{\frac{m}{2}-1}e^{-\lambda V_n^2}d\lambda\Big] \\ &\quad + \frac{1}{(\frac{m}{2}-1)!}\mathbf{E}\Big[\int_\delta^\infty \sum_{I_m^n} X_{i_1}I_{X_{i_1}\neq 0}\cdots X_{i_m}I_{X_{i_m}\neq 0}\lambda^{\frac{m}{2}-1}e^{-\lambda V_n^2}d\lambda\Big] \\ &= (I) + (II), \end{aligned}\tag{2.4}$$

for $\delta > 0$ to be specified below. Now,

$$\begin{aligned}(II) &= \mathbf{E}\Big[\frac{\sum_{I_n^m} X_{i_1}I_{X_{i_1}\neq 0}\cdots X_{i_m}I_{X_{i_m}\neq 0}}{V_n^m}e^{-\delta V_n^2}\sum_{r=0}^{\frac{m}{2}-1}\frac{(\delta V_n^2)^r}{r!}\Big] \\ &\le 2^{\frac{m}{2}-1}n^{\frac{m}{2}}\mathbf{E}e^{-\frac{\delta}{2}V_n^2} \\ &= 2^{\frac{m}{2}-1}n^{\frac{m}{2}}\Big[\mathbf{E}e^{-\frac{\delta}{2}X^2}\Big]^n.\end{aligned}$$

By hypothesis there exists $\tau > 0$ such that $\Pr\{|X| \le \tau\} < 1$ and therefore,

$$\mathbf{E}e^{-\frac{\delta}{2}X^2} \le \Pr\{|X| \le \tau\} + e^{-\frac{\delta\tau^2}{2}}\Pr\{|X| > \tau\} := \sigma_\delta < 1.$$

Hence,

$$(II) \le 2^r n^{\frac{m}{2}} \sigma_\delta^n \to 0 \quad \text{as} \quad n \to \infty, \tag{2.5}$$

for all $\delta > 0$.

To estimate (I) in (2.4) we consider three cases: $\mathbf{E}X^2 < \infty$, $X \in DA(\alpha)$ for $1 < \alpha \le 2$ and $\mathbf{E}X^2 = \infty$, and $X \in DA(\alpha)$ for $0 < \alpha < 1$.

Case 1: $\mathbf{E}X^2 < \infty$ *and* $\mathbf{E}X = 0$. Letting θ denote a random variable uniform on $[0,1]$ independent of X, and denoting by c a finite constant independent of n that may be different at each appearance, we have the following two estimates:

$$\begin{aligned}
\left|\mathbf{E}Xe^{-\lambda X^2}\right| &= \left|\lambda \mathbf{E}\left[X^3 e^{-\lambda\theta X^2}\right]\right| \\
&\le \lambda\big(\mathbf{E}X^2\big)\mathbf{E}\big[\sup_x |x| e^{-\lambda\theta x^2}\big] \\
&\le \lambda^{\frac{1}{2}} \frac{e^{-\frac{1}{2}}}{\sqrt{2}} \big(\mathbf{E}X^2\big)\mathbf{E}\theta^{-\frac{1}{2}} \\
&= c\lambda^{\frac{1}{2}},
\end{aligned}$$

and, for $\lambda \le \delta$, there exists $c_\delta > 0$ such that

$$\begin{aligned}
\mathbf{E}e^{-\lambda X^2} &= \int_0^\infty \Pr\{X^2 < t\}\lambda e^{-\lambda t} dt \\
&= 1 - \lambda \int_0^\infty \Pr\{X^2 \ge t\} e^{-\lambda t} dt \\
&\le 1 - \lambda \int_0^\infty \Pr\{X^2 \ge t\} e^{-\delta t} dt \\
&= 1 - c_\delta \lambda.
\end{aligned}$$

Therefore, for $n > m$ and all $\delta > 0$,

$$\begin{aligned}
(I) &= \frac{1}{(\frac{m}{2}-1)!}\frac{n!}{(n-m)!}\int_0^\delta \lambda^{\frac{m}{2}-1}\big(\mathbf{E}Xe^{-\lambda X^2}\big)^m \big(\mathbf{E}e^{-\lambda X^2}\big)^{n-m} d\lambda \\
&\le cn^m \int_0^\delta \lambda^{m-1}(1-c'\lambda)^{n-m} d\lambda \\
&\le cn^m \int_0^\delta \lambda^{m-1} e^{-c'(n-m)\lambda} d\lambda \le c,
\end{aligned}$$

where c depends on m and δ, but not on n (in the first identity, the interchange of integrals is justified because $|\tilde{X}_{n,1}\cdots\tilde{X}_{n,m}| \le 1$). Now, estimating the expected value in (2.4) by means of this inequality and the limit (2.5), we obtain (2.3).

Case 2: $X \in DA(\alpha)$, $1 < \alpha \le 2$, $\mathbf{E}X^2 = \infty$ *and* $\mathbf{E}X = 0$. Since

$$\frac{t^2 \Pr\{|X| > t\}}{\mathbf{E}X^2 I_{|X| \le t}} \to 2 - \alpha \tag{2.6}$$

as $t \to \infty$ and the function $\mathbf{E}X^2 I_{|X|\le t}$ is regularly varying at infinity with exponent $2-\alpha$ (slowly varying for $\alpha = 2$), we have that the function

$$L(x) := \int_0^x t \Pr\{|X| > t\} dt$$

(note $2L(x) = \mathbf{E}X^2 I_{|X|\le x} + x^2 \Pr\{|X| > x\}$) is slowly varying with the same exponent (slowly varying for $\alpha = 2$) and increases to infinity with x. Moreover,

$$tL'(t) = t^2 \Pr\{|X| > t\} = \varepsilon(t)L(t), \tag{2.7}$$

where $\varepsilon(t) \to 2-\alpha$ as $t \to \infty$. We begin with the following estimate, valid for u positive and small (meaning that for every $\delta > 0$ the inequalities hold for $u \le \delta$ with constants c that depend on δ):

$$\begin{aligned}
\mathbf{E}|X|^3 e^{-uX^2} &= \int_0^\infty (3x^2 - 2ux^4)e^{-ux^2}\frac{L'(x)}{x}dx \\
&\le \int_0^{\sqrt{\frac{3}{2u}}} 3xe^{-ux^2}L'(x)dx \\
&= 3xe^{-ux^2}L(x)\Big|_0^{\sqrt{\frac{3}{2u}}} - \int_0^{\sqrt{\frac{3}{2u}}}(3-6ux^2)e^{-ux^2}L(x)dx \\
&\le c\frac{1}{\sqrt{u}}L\Big(\frac{1}{\sqrt{u}}\Big) + \int_0^{\sqrt{\frac{3}{2u}}} 6ux^2e^{-ux^2}L(x)dx \\
&= c\frac{1}{\sqrt{u}}L\Big(\frac{1}{\sqrt{u}}\Big) + \frac{1}{\sqrt{u}}\int_0^{\sqrt{\frac{3}{2}}} 6y^2e^{-y^2}L\Big(\frac{y}{\sqrt{u}}\Big)dy \\
&\le c\frac{1}{\sqrt{u}}L\Big(\frac{1}{\sqrt{u}}\Big),
\end{aligned}$$

where the fact that $L(3/2\sqrt{u}) \le cL(1/\sqrt{u})$ (for some finite constant c depending on δ) is used in the last step. Hence, for small λ,

$$\begin{aligned}
\Big|\mathbf{E}Xe^{-\lambda X^2}\Big| &= \Big|\lambda \mathbf{E}\big[X^3 e^{-\lambda\theta X^2}\big]\Big| \\
&\le c\lambda\int_0^1 \frac{1}{\sqrt{\lambda\theta}}L\Big(\frac{1}{\sqrt{\lambda\theta}}\Big)d\theta \\
&= 2c\int_{\frac{1}{\sqrt{\lambda}}}^\infty \frac{L(v)}{v^2}dv \\
&\le c\sqrt{\lambda}L\Big(\frac{1}{\sqrt{\lambda}}\Big),
\end{aligned} \tag{2.8}$$

where the last inequality follows from the properties of regularly varying functions (e.g., Feller, 1971, Theorem 1, VIII.9). Moreover, also for small λ,

$$\mathbf{E}e^{-\lambda X^2} = 1 - \lambda\int_0^\infty \frac{L'(\sqrt{t})}{\sqrt{t}}e^{-\lambda t}dt$$

$$
\begin{aligned}
&= 1 - 2\lambda^2 \int_0^\infty L(\sqrt{t}) e^{-\lambda t} dt \\
&\le 1 - 2\lambda^2 \int_{\frac{1}{2\lambda}}^{\frac{1}{\lambda}} L(\sqrt{t}) e^{-\lambda t} dt \\
&= 1 - 2\lambda \int_{\frac{1}{2}}^{1} L\Big(\sqrt{\frac{u}{\lambda}}\Big) e^{-u} du \\
&\le 1 - c\lambda L\Big(\frac{1}{\sqrt{\lambda}}\Big), \qquad (2.9)
\end{aligned}
$$

just as above. Then, by (2.8) and (2.9),

$$
(I) \le cn^m \int_0^\delta \lambda^{m-1} L\Big(\frac{1}{\sqrt{\lambda}}\Big)^m \Big(1 - c\lambda L\Big(\frac{1}{\sqrt{\lambda}}\Big)\Big)^{n-m} d\lambda.
$$

Setting

$$
t = \lambda L\Big(\frac{1}{\sqrt{\lambda}}\Big)
$$

we have, by (2.7),

$$
dt = \Big[L\Big(\frac{1}{\sqrt{\lambda}}\Big) - \frac{1}{2\sqrt{\lambda}} L'\Big(\frac{1}{\sqrt{\lambda}}\Big)\Big] d\lambda = \Big(1 - \frac{1}{2}\varepsilon\Big(\frac{1}{\sqrt{\lambda}}\Big)\Big) L\Big(\frac{1}{\sqrt{\lambda}}\Big) d\lambda, \qquad (2.10)
$$

and the coefficient of $d\lambda$ is positive for all λ small enough (as $\varepsilon(t) < 2$ for all t such that $\mathbf{E}X^2 I_{|X|\le t} > 0$). With this change of variables in the above integral we obtain that for δ small enough there is c finite, independent of n, such that, for $n \ge m$,

$$
(I) \le c' n^m \int_0^{\delta L(1/\sqrt{\delta})} t^{m-1} e^{-c''(n-m)t} dt \le c.
$$

(2.3) follows from this as in the previous case.

Case 3: $X \in DA(\alpha)$, $0 < \alpha \le 1$. The only difference with the previous case is that now the estimate (2.8) can be obtained in a more direct way because $L(x)$ varies regularly with exponent $2 - \alpha > 1$ (note also that $L(x)$ is of the order of x^2 at zero):

$$
\begin{aligned}
\mathbf{E}|X| e^{-\lambda X^2} &= \int_0^\infty (1 - 2\lambda x^2) e^{-\lambda x^2} \frac{L'(x)}{x} dx \\
&\le \int_0^{\frac{1}{\sqrt{2\lambda}}} \frac{e^{-\lambda x^2}}{x} L'(x) dx \\
&\le \frac{e^{-\lambda x^2} L(x)}{x}\Big|_0^{\frac{1}{\sqrt{2\lambda}}} + (2\lambda + 1) \int_0^{\frac{1}{\sqrt{2\lambda}}} \frac{L(x)}{x^2} dx \\
&\le c\sqrt{\lambda} L\Big(\frac{1}{\sqrt{\lambda}}\Big). \qquad (2.8')
\end{aligned}
$$

$\square$

2.2. REMARK. It is worthwhile to determine the minimal conditions on the law of X under which the previous proof still gives uniform boundedness of the k–th moments of the selfnormalized sums, for all k (i.e., (2.2)). We are particularly interested in this for the case $\mathbf{E}|X| < \infty$, $\mathbf{E}X = 0$. Simple examination of the previous proof, Case 2, shows that the only properties of the non–decreasing function L that are used are:

(i) For each $0 < c < 1$ there is $d < \infty$ such that $L(x) \le dL(cx)$, for large x (in (2.8) and in (2.9));

(ii) $\limsup_{n\to\infty} \dfrac{\int_a^\infty x^{-2}L(x)dx}{L(a)/a} < \infty$ in (2.8).

(i) is seen to hold as follows:

$$\begin{aligned} L(x) &= \mathbf{E}X^2 I_{|X|\le x} + x^2 \Pr\{|X| > x\} \\ &\le \mathbf{E}X^2 I_{|X|\le cx} + 2x^2 \Pr\{|X| > cx\} \\ &\le \frac{2}{c^2} L(cx). \end{aligned}$$

Regarding condition (ii) we note that the proof of Theorem 3.1 in Cuzick et *al.* (1995) applied with $k = 1$ and $b^*(t) = t$, implies that there is a constant $c < \infty$ such that, for all $d > 1$,

$$\int_{2^d}^\infty \mathbf{E}X^2 I_{|X|\le t} dt \le c \int_{2^{d-1}} t^2 \Pr\{|X| \ge t\} dt.$$

Together with (i), this reduces condition (ii) to the following condition:

$$\limsup_{a\to\infty} \frac{\int_a^\infty \Pr\{|X| > x\}dx}{a^{-1}\mathbf{E}X^2 I_{|X|\le a} + a \Pr\{|X| > a\}} < \infty, \tag{2.11}$$

a one sided regular or slow variation type condition on the law of $|X|$. Note that this condition is always satisfied when $X \in DA(\alpha)$, $1 < \alpha \le 2$. We have thus proved that: *If* $\mathbf{E}|X| < \infty$, $\mathbf{E}X = 0$ *and the law of* X *satisfies condition (2.11), then*

$$\sup_n \mathbf{E}\big|S_n/V_n\big|^k < \infty \tag{2.2}$$

for all $k > 0$. It would be interesting to determine if even condition (2.11) is superfluous.

3. Some characterizations of the domain of attraction of the normal law. The following proposition, whose idea is described in one form or another in Lévy's (1937) book, seems to be new in the non–symmetric case.

3.1. THEOREM. *X belongs to the domain of attraction of the normal law ($X \in DAN$) if and only if*

$$\frac{\max_{1\le i\le n} |X_i|}{|S_n - m_n|} \to 0 \;\text{ in pr.} \tag{3.1}$$

as $n \to \infty$, where: X_i are i.i.d. with the law of X, $S_n = \sum_{i=1}^n X_i$ and m_n is a median of S_n.

Proof. If $X \in DAN$ it is well known that there exists a sequence $b_n \nearrow \infty$ such that both:

$$\frac{S_n - \mathbf{E}S_n}{b_n} \to_w N(0,1) \tag{3.2}$$

and

$$\frac{\max_{1\le i\le n}|X_i|}{b_n} \to 0 \ \text{ in pr.} \tag{3.3}$$

By (3.2), for all $\tau > 0$ we have that eventually $\Pr\{S_n - \mathbf{E}S_n > -\tau b_n\} > 1/2$ and $\Pr\{S_n - \mathbf{E}S_n < \tau b_n\} > 1/2$. Therefore, for all $\tau > 0$, eventually,

$$\mathbf{E}S_n - \tau b_n \le m_n \le \mathbf{E}S_n + \tau b_n,$$

that is,

$$\frac{m_n - \mathbf{E}S_n}{b_n} \to 0. \tag{3.4}$$

In particular, if $X \in DAN$ then, by (3.2) and (3.4), $\mathcal{L}\big((S_n - m_n)/b_n\big) \to_w N(0,1)$ which, by (3.3), implies

$$\frac{\max_{1\le i\le n}|X_i|}{|S_n - m_n|} = \frac{\max_{1\le i\le n}|X_i|/b_n}{|S_n - m_n|/b_n} \to 0 \ \text{ in pr.}$$

giving the 'if' part of the proposition.

The 'only if' part is more interesting. It obviously reduces to showing that

$$\mathbf{E}X^2 = \infty \text{ and } X \notin DAN \ \Rightarrow \frac{\max_{1\le i\le n}|X_i|}{|S_n - m_n|} \not\to 0 \ \text{ in pr.} \tag{3.5}$$

By the well known DAN criterion, there exists $\delta > 0$ and $t_n \nearrow \infty$ such that

$$\frac{t_n^2 \Pr\{|X| > t_n\}}{\mathbf{E}X^2 I_{|X|\le t_n}} > \delta \tag{3.6}$$

for all $n \in \mathbf{N}$. For $\alpha > 0$ to be specified below, let $k_n \nearrow \infty$, $k_n \in \mathbf{N}$, satisfy

$$k_n \Pr\{|X| > t_n\} \approx \alpha \tag{3.7}$$

(e.g. we can take k_n to be the natural number that makes $k_n \Pr\{|X| > t_n\}$ closest to α). Then, a classical elementary inequality gives

$$\Pr\Big\{\max_{1\le i\le k_n}|X_i| > t_n\Big\} \underset{\sim}{\in} \Big[\frac{\alpha}{1+\alpha}, \alpha\Big], \tag{3.8}$$

with self–explanatory and somewhat abusive notation. (3.6) and (3.7), together with the fact that $\mathbf{E}X^2 = \infty$, imply that we also have

$$\frac{k_n}{t_n^2}\mathrm{Var} X I_{|X|\le t_n} \lesssim \frac{\alpha}{\delta}. \tag{3.9}$$

Let us set

$$S'_{k_n} = \sum_{i=1}^{k_n} X_i I_{|X_i| \le t_n}.$$

Then, for $c \in (0, \infty)$ to be specified below, we have

$$\begin{aligned}
&\Pr\Big\{\max_{1\le i\le k_n} |X_i| > \frac{1}{c+1}|S_{k_n} - \mathbf{E}S'_{k_n}|, \max_{1\le i\le k_n} |X_i| > t_n\Big\} \\
&\quad \ge \Pr\Big\{\max_{1\le i\le k_n} |X_i| > t_n, |S'_{k_n} - \mathbf{E}S'_{k_n}| \le ct_n, |S_{k_n} - S'_{k_n}| \le \max_{1\le i\le k_n} |X_i|\Big\} \\
&\quad \ge \Pr\big\{\max_{1\le i\le k_n} |X_i| > t_n\big\} + \Pr\big\{|S'_{k_n} - \mathbf{E}S'_{k_n}| \le ct_n\big\} \\
&\qquad\qquad + \Pr\big\{|S_{k_n} - S'_{k_n}| \le \max_{1\le i\le k_n} |X_i|\big\} - 2. \qquad (3.10)
\end{aligned}$$

The first summand of the last term in (3.10) can be estimated by (3.8) and the second, by (3.9) and Chebyshev. For the third we use the binomial estimate

$$Bin_{k,p}\{x \ge r\} \le p^r \binom{k}{r} \le \Big(\frac{ekp}{r}\Big)^r$$

and obtain

$$\Pr\big\{|S_{k_n} - S'_{k_n}| > \max_{1\le i\le k_n} |X_i|\big\} \le \Pr\big\{\#[|X_i| > t_n : i \le k_n] \ge 2\big\} \lesssim \Big(\frac{e\alpha}{2}\Big)^2.$$

Combining these estimates with (3.10) yields

$$\Pr\Big\{\max_{1\le i\le k_n} |X_i| > \frac{1}{c+1}|S_{k_n} - \mathbf{E}S'_{k_n}|, \max_{1\le i\le k_n} |X_i| > t_n\Big\} \gtrsim \frac{\alpha}{1+\alpha} - \frac{\alpha}{\delta c^2} - \frac{e^2\alpha^2}{4}. \qquad (3.11)$$

To be specific, taking $c = 10/\sqrt{\delta}$ and $\alpha = 0.1$ gives

$$\begin{aligned}
&\Pr\Big\{\max_{1\le i\le k_n} |X_i| > \frac{1}{c+1}|S_{k_n} - \mathbf{E}S'_{k_n}|, \max_{1\le i\le k_n} |X_i| > t_n\Big\} \\
&\qquad \gtrsim \frac{1}{1.1}\Big[0.1 - 0.0011 - \frac{0.099}{4}\Big] = \frac{0.0739}{1.1} > 0. \qquad (3.11')
\end{aligned}$$

Next, in order to replace $|S_{k_n} - \mathbf{E}S'_{k_n}|$ by $|S_{k_n} - m_{k_n}|$, we note that, for any K,

$$\begin{aligned}
\Pr\big\{|S_{k_n} - \mathbf{E}S'_{k_n}| > Kt_n\big\} &\lesssim \frac{4\alpha}{\delta K^2} + \Pr\Big\{|S_{k_n} - S'_{k_n}| > \frac{Kt_n}{2}\Big\} \\
&\le \frac{4\alpha}{\delta K^2} + \Pr\big\{\#[X_i > t_n : i \le k_n] \ge 1\big\} \\
&\approx \frac{4\alpha}{\delta K^2} + 1 - \Big(1 - \frac{\alpha}{k_n}\Big)^{k_n} \lesssim \frac{4\alpha}{\delta K^2} + 1 - e^{-\alpha} \\
&\le \frac{4\alpha}{\delta K^2} + \alpha.
\end{aligned}$$

The number α chosen in (3.11) being less than $1/2$, we can choose K so that the number $\frac{4\alpha}{\delta K^2} + \alpha$ is also smaller than $1/2$. The previous string of inequalities then shows that, for such K,

$$|m_{k_n} - \mathbf{E}S'_{k_n}| \leq Kt_n. \tag{3.12}$$

Now, if $\max_{1\leq i\leq k_n} |X_i| > t_n$ and $|S_{k_n} - \mathbf{E}S'_{k_n}|/\max_{1\leq i\leq k_n} |X_i| < c+1$ as in (3.10), then, by equation (3.12), $|S_{k_n} - m_{k_n}|/\max_{1\leq i\leq k_n} |X_i| < c+1+K$. This implies, by (3.10) and (3.11'),

$$\Pr\Big\{\max_{1\leq i\leq k_n} |X_i| > t_n, \frac{\max_{1\leq i\leq k_n} |X_i|}{|S_{k_n} - m_{k_n}|} > \frac{1}{c+1+K}\Big\} \gtrsim \frac{0.0739}{1.1} > 0,$$

for all $n \in \mathbf{N}$, showing that $\max_{1\leq i\leq n} |X_i|/|S_n - m_n| \not\to 0$ in pr. The implication (3.5) is thus proved.

□

This theorem adds precision, in the not necessarily symmetric case, to a classical statement, which should now read as follows: '$X \in DAN$ iff the largest summand, $\max_{1\leq i\leq n} |X_i|$, is asymptotically negligible, in probability, with respect to the sum S_n *centered at its median*'.

The next theorem shows that we can replace $S_n - m_n$ by *the square root of the sum of the squares of the variables* in (3.1). This is a known result (O'Brien, 1980) but we sketch here a proof along the lines of the previous one.

3.2. THEOREM. (O'Brien) *With the previous notation, $X \in DAN$ if and only if*

$$\lim_{n\to\infty} \frac{\max_{1\leq i\leq n} |X_i|}{V_n} = 0 \text{ in pr.} \tag{3.13}$$

PROOF. If $\mathbf{E}X^2 < \infty$ then (3.13) actually holds a.s. due to the law of large numbers (upon dividing numerator and denominator by $\sqrt{n}$). If $X \in DAN$ and $\mathbf{E}X^2 = \infty$ then there is a sequence $b_n \nearrow \infty$ such that

$$\lim_{n\to\infty} \mathcal{L}\Big(\frac{\sum_{i=1}^n (X_i - \mathbf{E}X)}{b_n}\Big) = N(0,1), \tag{3.14}$$

and it is classical that, moreover, $n\Pr\{|X| > b_n\} \to 0$ and $b_n^{-2} n\mathbf{E}X^2 I_{|X|\leq b_n} \to 1$ (see, for example, Araujo and Giné, 1980, Ch. 2, Cor. 4.8 and Lemma 6.3). Hence, for all $\varepsilon > 0$ and for all n sufficently large so that $|b_n^{-2} n\mathbf{E}X^2 I_{|X|\leq b_n} - 1| \leq \varepsilon/2$, we have

$$\begin{aligned}\Pr\Big\{\Big|\frac{\sum_{i=1}^n X_i^2}{b_n^2} - 1\Big| > \varepsilon\Big\} &\leq n\Pr\Big\{|X| > b_n\Big\} \\ &\quad + \Pr\Big\{\Big|\sum_{i=1}^n X_i^2 I_{|X_i|\leq b_n} - n\mathbf{E}X^2_{|X_i|\leq b_n}\Big| > b_n\varepsilon/2\Big\} \\ &\leq n\Pr\Big\{|X| > b_n\Big\} + \frac{4n}{\varepsilon^2 b_n^2}\mathbf{E}X^2 I_{|X_i|\leq b_n} \to 0.\end{aligned}$$

Therefore, by (3.14),

$$\frac{\sum_{i=1}^n X_i^2}{b_n^2} \to 1 \text{ in pr.} \tag{3.15}$$

(This simple fact, together with its converse, is known as Raikov's theorem.)

To prove the converse, we can assume, as in the previous theorem, that $\mathbf{E}X^2 = \infty$ and that $X \notin DAN$. Taking δ, α, t_n and k_n as in the proof of Theorem 3.1, (3.9) is equivalent to

$$\frac{k_n}{t_n^2}\mathbf{E}X^2 I_{|X|\le t_n} \lesssim \frac{\alpha}{\delta} \tag{3.9$'$}$$

since $\big(\mathbf{E}|X|I_{|X|\le t_n}\big)^2/\mathbf{E}X^2 I_{|X|\le t_n} \to 0$ if $\mathbf{E}X^2 = \infty$. Then, setting

$$V'_{k_n} = \Big(\sum_{i=1}^{k_n} X_i^2 I_{|X|\le t_n}\Big)^{\frac{1}{2}},$$

we obtain, as in (3.10),

$$\begin{aligned}\Pr\Big\{\max_{1\le i\le n} X_i^2 > \frac{1}{c+1}V_{k_n}^2, \max_{1\le i\le n}|X_i| > t_n\Big\} &\ge \Pr\{\max_{1\le i\le n}|X_i| > t_n\}\\ &+ \Pr\{V_{k_n}'^2 \le ct_n^2\} + \Pr\{V_{k_n}^2 - V_{k_n}'^2 \le \max_{1\le i\le n} X_i^2\} - 2. \quad (3.10')\end{aligned}$$

The fist summand is estimated by (3.8), the second, by Markov's inequality and (3.9') (as $\Pr\{V_{k_n}'^2 > ct_n^2\} \lesssim \alpha/(c\delta)$), and the third, by

$$\Pr\{V_{k_n}^2 - V_{k_n}'^2 > \max_{1\le i\le n} X_i^2\} \le \Pr\{\#[X_i > t_n : i \le k_n] \ge 2\} \lesssim \Big(\frac{e\alpha}{2}\Big)^2,$$

just as in the previous proof. So, proceeding as in (3.11) and (3.11') (with c instead of c^2), we obtain that $\max_{1\le i\le n}|X_i|/V_n \not\to 0$ in probability.

□

Let $\bar{X}_n = \sum_{i=1}^n X_i/n$. Since $V_n^2 \ge \sum_{i=1}^n (X_i - \bar{X}_n)^2$, it follows from the previous proposition that

$$\frac{\max_{1\le i\le n}|X_i|}{\big[\sum_{i=1}^n (X_i - \bar{X}_n)^2\big]^{\frac{1}{2}}} \to 0 \text{ in pr.} \tag{3.16}$$

implies $X \in DAN$. If $\mathbf{E}X^2 < \infty$, then (3.16) is obviously satisfied. Suppose now $X \in DAN$ and $\mathbf{E}X^2 = \infty$. Then, if $\{b_n\}$ is a monotone norming sequence for which (3.2) holds, we have $b_n^2/n \to \infty$ (this is obvious from the fact that $nb_n^{-2}\mathbf{E}X^2 I_{|X|\le b_n} \to 1$). Now,

$$\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{V_n^2} = 1 - \frac{\frac{1}{n}S_n^2}{V_n^2} \tag{3.17}$$

and $V_n^2 = O_P(b_n^2)$ by (3.15) whereas

$$\frac{1}{n}S_n^2 = \frac{1}{n}\big[(S_n - \mathbf{E}S_n)^2 + 2(\mathbf{E}S_n)(S_n - \mathbf{E}S_n) + (\mathbf{E}S_n)^2\big]$$

is $O_P(b_n^2/n)$ if X is centered, and $O_P(n)$ otherwise. Therefore, the quotient at the left side of equation (3.17) tends to 1. We have thus proved:

3.3. COROLLARY. *$X \in DAN$ if and only if*

$$\frac{\max_{1\le i\le n}|X_i|}{\left[\sum_{i=1}^n (X_i-\bar{X}_n)^2\right]^{\frac{1}{2}}} \to 0 \text{ in pr.} \tag{3.16}$$

¿From a theoretical point of view, Theorem 3.1 seems to be the most interesting of the last three statements above. But we will use Theorem 3.2 to prove the main result in this paper, in Section 4.

Finally, here is another characterization of DAN in terms of sums of squares.

3.4. THEOREM. *Let X, X_i, X_i', $i \in \mathbf{N}$, be i.i.d. non–degenerate random variables. Then, $X \in DAN$ if and only if*

$$\frac{\sum_{i=1}^n X_i^2}{\sum_{i=1}^n X_i'^2} \to 1 \text{ in pr.} \tag{3.18}$$

PROOF. Without loss of generality we can replace X_i by $X_i + I_{X_i=0}$ and likewise for X_i', so that neither the numerator nor the denominator in (3.18) ever vanish. If $\mathbf{E}X^2 < \infty$ we have convergence a.s. in (3.18) as a consequence of the LLN. If $X \in DAN$ and $\mathbf{E}X^2 = \infty$ then (3.18) is a consequence of Raikov's theorem, mentioned above.

Suppose (3.18) holds. We may as well assume $\mathbf{E}X^2 = \infty$. Then,

$$\log\big(\sum_{i=1}^n X_i^2\big) - \log\big(\sum_{i=1}^n X_i'^2\big) \to 0 \text{ in pr.}$$

Hence, the sequence of random variables $\left\{\log\big(\sum_{i=1}^n X_i^2\big)\right\}$ is shift tight, that is, there is a sequence of numbers $\{b_n\}$ such that the sequence

$$\left\{\log\big(\sum_{i=1}^n X_i^2\big) - \log b_n^2\right\} \tag{3.19}$$

is stochastically bounded (e.g. Araujo and Giné,1980, page 27). Note that $b_n \to \infty$ necessarily. The following general fact is obvious: If the sequence $\{\xi_n\}$ is independent from the sequence $\{\xi_n'\}$, both are tight and $\xi_n - \xi_n' \to 0$ in pr., then the sequence of vectors $\{(\xi_n, \xi_n')\}$ is tight and all its subsequential limits (Z, Z') satisfy $Z - Z' = 0$, Z and Z' independent, hence $Z = Z' = \textit{constant}$ a.s., and moreover, the set of these constants is bounded (as the subsequential limits of a tight sequence form a tight set). Hence, all the subsequential limits of (3.19) are constants, and these constants form a bounded set. Consider a subsequence for which there is convergence, say $\{n'\}$. Then

$$\frac{\sum_{i=1}^{n'} X_i^2}{b_{n'}^2} \to c \text{ in pr.}$$

for some $c \in (0,\infty)$. Since $b_n \to \infty$ we can apply the general central limit theorem, degenerate case (i.e., general LLN) to obtain that for all $\delta > 0$:

$$n' \Pr\{X^2 > \delta b_{n'}^2\} \to 0 \quad \text{and} \quad \frac{n'}{b_{n'}^2} \mathbf{E} X^2 I_{X^2 \le \delta b_{n'}^2} \to c.$$

But these conditions, in the case $\mathbf{E}X^2 = \infty$, that we are assuming, imply

$$\mathcal{L}\left(\frac{\sum_{i=1}^{n'} (X_i - \mathbf{E} X_i I_{X_i^2 \le \delta b_{n'}^2})}{b_{n'}}\right) \to_d N(0,c). \tag{3.20}$$

Therefore, if X satisfies condition (3.18), then X belongs to the domain of partial attraction of the normal law and, moreover, there exists a sequence $b_n \to \infty$ such that the limit (3.20) holds along a subsequence of every subsequence. We easily conclude from this, by the convergence of types theorem, that X belongs to the domain of partial attraction of only one type of laws, the normal laws. Then, a result of Jain and Orey (1980) implies that X actually belongs to the domain of attraction of the normal law.

□

Condition (3.18) can be restated in other ways, such as

$$\frac{\sum_{i=1}^{n/2} X_i^2}{\sum_{i=1}^{n} X_i^2} \to \frac{1}{2} \text{ in pr.} \tag{3.18$'$}$$

4. A characterization of the domain of attraction of the normal law by the CLT for selfnormalized sums. As mentioned in the Introduction, the implication

$$X \in DAN \Longrightarrow \mathcal{L}\Big(\frac{S_n - \mathbf{E}S_n}{V_n}\Big) \to N(0,1) \tag{4.1}$$

follows almost trivially from basic principles (Raikov's theorem): This is just the fact that (3.14) implies (3.15), which, as shown above, is an immediate consequence of the general normal convergence criterion. Thus, we should ask whether

$$\mathcal{L}\Big(\frac{S_n - m_n}{V_n}\Big) \to_w N(0,1) \quad \Longrightarrow \quad X \in DAN,$$

or, more modestly, whether

$$\mathbf{E}X = 0 \;\; \textit{and} \;\; \mathcal{L}\Big(\frac{S_n}{V_n}\Big) \to_w N(0,1) \quad \Longrightarrow \quad X \in DAN.$$

For X symmetric this conjecture was shown to be true by Roy Erickson (Griffin and Mason, 1991). Here we make some remarks for the case of centered not necessarily symmetric X as explained in the Introduction.

Since in one of the two theorems that follow we use Erickson's result, and in the other one we follow its proof quite closely, we sketch it here for the reader's convenience.

4.1. THEOREM. *(*Erickson*) If X is* symmetric *then*

$$X \in DAN \quad \textit{if and only if} \quad \mathcal{L}\Big(\frac{S_n}{V_n}\Big) \to_w N(0,1). \tag{4.2}$$

PROOF. The direct part has already been proved ((4.1)). Let Z be a $N(0,1)$ variable. Then, $S_n/V_n \to Z$ in law implies that

$$\mathbf{E}\Big(\frac{S_n^4}{V_n^4} \wedge M\Big) \to_{n\to\infty} \mathbf{E}(Z^4 \wedge M) \to_{M\to\infty} \mathbf{E}Z^4 = 3,$$

and therefore, that

$$\liminf_{n\to\infty} \mathbf{E}\Big[\Big(\frac{S_n}{V_n}\Big)^4 \wedge M\Big] \geq 3. \tag{4.3}$$

As mentioned at the beginning of the proof of Theorem 2.1, by symmetry, the variables $\varepsilon_i := X_i/|X_i|$ are independent Rademacher, and independent from the variables $|X_i|$. Hence,

$$\mathbf{E}\Big[\frac{X_1X_2^3}{V_n^4}\Big] = \Big[\mathbf{E}\frac{|X_1||X_2|^3}{V_n^4}\Big](\mathbf{E}\varepsilon_1)(\mathbf{E}\varepsilon_2^3) = 0.$$

Likewise, $\mathbf{E}\big[X_1X_2X_3^2/V_n^4\big] = 0$ and $\mathbf{E}\big[X_1X_2X_3X_4/V_n^4\big] = 0$. Therefore,

$$\begin{aligned}
\mathbf{E}Z_n^4 &= \mathbf{E}\Big[\frac{\sum_{i=1}^n X_i^4}{V_n^4} + 3\frac{\sum_{1\leq i\neq j\leq n} X_i^2X_j^2}{V_n^4}\Big] \\
&= \mathbf{E}\Big[\frac{3\big(\sum_{i=1}^n X_i^2\big)^2 - 2\sum_{i=1}^n X_i^4}{V_n^4}\Big] \\
&= 3I_{V_n\neq 0} - 2n\mathbf{E}\Big(\frac{X_1^4}{V_n^4}\Big).
\end{aligned} \tag{4.4}$$

Equations (4.3) and (4.4) imply

$$n\mathbf{E}\Big(\frac{X_1^4}{V_n^4}\Big) \to 0,$$

which in turn gives that, for all $\varepsilon > 0$,

$$\Pr\Big\{\frac{\max_{1\leq i\leq n}|X_i|}{V_n} > \varepsilon\Big\} \leq n\Pr\Big\{\frac{|X_1|}{V_n} > \varepsilon\Big\} \leq \frac{n}{\varepsilon^4}\mathbf{E}\Big(\frac{X_1^4}{V_n^4}\Big) \to 0.$$

The result now follows from Theorem 3.2.

$\square$

Our main result is as follows.

4.2. Theorem. *Let X, X_i, $1 \le i \le \infty$, be i.i.d. real random variables. If*

$$\mathcal{L}\Big(\frac{S_n}{V_n}\Big) \to_w N(0,1) \tag{4.5}$$

and

$$\sup_n \mathbf{E}\Big|\frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}}\Big|^6 < \infty, \tag{4.6}$$

then $X \in DAN$ and $\mathbf{E}X = 0$ (and conversely).

Proof. The converse is contained in Theorem 3.1. We now assume that X satisfies (4.5) and (4.6). We set

$$\Lambda_m := \Lambda_m(n) = \mathbf{E}\Big(\sum_{I_n^m} \tilde{X}_{n,i_1} \dots \tilde{X}_{n,i_m}\Big),\ m \le n.$$

Since, by (4.6),

$$1 = \lim_{n\to\infty} \mathbf{E}\big(\sum_{i=1}^n \tilde{X}_{n,i}\big)^2 = 1 + \lim_{n\to\infty} \mathbf{E}\Big(\sum_{I_n^2} \tilde{X}_{n,i}\tilde{X}_{n,j}\Big),$$

we have

$$\lim_{n\to\infty} \Lambda_2(n) = 0. \tag{4.7}$$

Next, we note the following simple, but useful, relations:

$$\sum_{I_n^2} \tilde{X}_{n,i}^2 \tilde{X}_{n,j}^2 = 1 - \sum_{i=1}^n \tilde{X}_{n,i}^4,$$

$$\Lambda_2 = \mathbf{E}\Big(\sum_{(i,j)\in I_n^2} \tilde{X}_{n,i}\tilde{X}_{n,j} \sum_{k=1}^n \tilde{X}_{n,k}^2\Big) = \mathbf{E}\Big(\sum_{I_n^3} \tilde{X}_{n,i}\tilde{X}_{n,j}\tilde{X}_{n,k}^2\Big) + 2\mathbf{E}\Big(\sum_{I_n^2} \tilde{X}_{n,i}\tilde{X}_{n,j}^3\Big),$$

and

$$\begin{aligned}\big(\sum_{i=1}^n \tilde{X}_{n,i}\big)^4 = \sum_{i=1}^n \tilde{X}_{n,i}^4 + 3\sum_{I_n^2} \tilde{X}_{n,i}^2\tilde{X}_{n,j}^2 + 4\sum_{I_n^2} \tilde{X}_{n,i}\tilde{X}_{n,j}^3 \\ + 6\sum_{I_n^3} \tilde{X}_{n,i}\tilde{X}_{n,j}\tilde{X}_{n,k}^2 + \sum_{I_n^4} \tilde{X}_{n,i}\tilde{X}_{n,j}\tilde{X}_{n,k}\tilde{X}_{n,\ell}.\end{aligned}$$

They imply that, *assuming the limits exist,*

$$\lim_{n\to\infty} \mathbf{E}\big(\sum_{i=1}^n \tilde{X}_{n,i}\big)^4 = 3 - 2\lim_{n\to\infty} \mathbf{E}\sum_{i=1}^n \tilde{X}_{n,i}^4 - 8\lim_{n\to\infty} \mathbf{E}\sum_{I_n^2} \tilde{X}_{n,i}\tilde{X}_{n,j}^3 + \lim_{n\to\infty} \Lambda_4(n). \tag{4.8}$$

By (4.5) and (4.6), the limit at the left of (4.8) is 3. Hence, if we show that the last two limits at the right of (4.8) are 0, we will have proved:

$$\lim_{n\to\infty} n\mathbf{E}\Big(\frac{X_1}{V_n}\Big)^4 = \lim_{n\to\infty} \mathbf{E}\sum_{i=1}^n \tilde{X}_{n,i}^4 = 0, \tag{4.9}$$

which, as in the proof of Theorem 4.1, implies that $X \in DAN$ by Theorem 3.2. It then follows from (4.5), (3.14) and (3.15) that, in addition, X is centered.

For $\lambda > 0$, let us set

$$\varphi(\lambda) := \mathbf{E}\exp(-\lambda X^2).$$

Let r be an even number and let $r = r_1 + \ldots + r_m$, r_i non–negative integers. Then, using the same elementary identity as in (2.4) and Fubini's theorem (note $|\tilde{X}_{n,i}| \le 1$), we have

$$\begin{aligned}
\mathbf{E}\Big(\sum_{I_m^n} \tilde{X}_{n,i_1}^{r_1}\cdots\tilde{X}_{n,i_m}^{r_m}\Big) &= \frac{n!}{(n-m)!}\mathbf{E}\big(\tilde{X}_{n,1}^{r_1}\cdots\tilde{X}_{n,m}^{r_m}\big)\\
&= \frac{n!}{(n-m)!}\mathbf{E}\Big[\frac{X_1^{r_1}I_{X_1\neq 0}\cdots X_m^{r_m}I_{X_m\neq 0}}{V_n^{\frac{r}{2}}}\Big]\\
&= \frac{n!}{(n-m)!}\frac{1}{(\frac{r}{2}-1)!}\mathbf{E}\Big[\int_0^\infty \lambda^{\frac{r}{2}-1}\exp(-\lambda V_n^2)X_1^{r_1}I_{X_1\neq 0}\cdots X_m^{r_m}I_{X_m\neq 0}\,d\lambda\Big]\\
&= \frac{n!}{(n-m)!}\frac{1}{(\frac{r}{2}-1)!}\int_0^\infty \prod_{i=1}^m\Big(\lambda^{r_i/2}\mathbf{E}\big(X^{r_i}\exp(-\lambda X^2)\big)\Big)\ \frac{\varphi^{n-m}(\lambda)}{\lambda}\ d\lambda.
\end{aligned} \tag{4.10}$$

In particular, for m even,

$$\Lambda_m = \frac{n!}{(n-m)!}\frac{1}{(\frac{m}{2}-1)!}\int_0^\infty \Big(\frac{\lambda^{\frac{1}{2}}\mathbf{E}\big(X\exp(-\lambda X^2)\big)}{\varphi(\lambda)}\Big)^m\ \frac{\varphi^n(\lambda)}{\lambda}\ d\lambda. \tag{4.10$'$}$$

Equation (4.10') allows us to apply Hölder's inequality to obtain that, for $n \ge 6$,

$$0 \le \Lambda_4 \le 4\Lambda_2^{\frac{1}{2}}\Lambda_6^{\frac{1}{2}}. \tag{4.11}$$

By Newton's identities, the variable $\sum_{I_n^6} \tilde{X}_{n,i_1}\ldots\tilde{X}_{n,i_6}$ is a fixed (i.e., independent of n) linear combination of terms of the form $(\sum_{i=1}^n \tilde{X}_{n,i}^{r_1})\cdots(\sum_{i=1}^n \tilde{X}_{n,i}^{r_k})$ where r_i are positive integers adding up to 6. Since $\sum_{i=1}^n |\tilde{X}_{n,i}|^r \le 1$ if $r > 1$, it follows that the absolute value of any such product is dominated by $|\sum_{i=1}^n \tilde{X}_{n,i}|^\ell$, where $\ell \le 6$ is the number of exponents r_i which equal 1. Therefore, condition (4.6) implies $\sup_n \Lambda_6(n) < \infty$. Combining this and the limit (4.7) with inequality (4.11) gives

$$\lim_{n\to\infty} \Lambda_4(n) = 0, \tag{4.12}$$

which is one of the two limits we needed to compute in order to complete the proof.

By (4.10) and Hölder,

$$
\begin{aligned}
&\big|\mathbf{E}\sum_{I_n^2}\tilde{X}_{n,i}\tilde{X}_{n,j}^3\big| \\
&\quad= \frac{n!}{(n-2)!}\left|\int_0^\infty \Big(\lambda^{\frac{1}{2}}\mathbf{E}\big(X\exp(-\lambda X^2)\big)\Big)\Big(\lambda^{\frac{3}{2}}\mathbf{E}\big(X^3\exp(-\lambda X^2)\big)\Big)\ \frac{\varphi^{n-2}(\lambda)}{\lambda}\ d\lambda\right| \\
&\quad\le \frac{n!}{(n-2)!}\left[\int_0^\infty \Big(\lambda^{\frac{1}{2}}\mathbf{E}\big(X\exp(-\lambda X^2)\big)\Big)^2\frac{\varphi^{n-2}(\lambda)}{\lambda}\ d\lambda\right]^{\frac{1}{2}} \\
&\qquad\qquad\qquad\times\left[\int_0^\infty \Big(\lambda^{\frac{3}{2}}\mathbf{E}\big(X^3\exp(-\lambda X^2)\big)\Big)^2\frac{\varphi^{n-2}(\lambda)}{\lambda}\ d\lambda\right]^{\frac{1}{2}} \\
&\quad= 2^{\frac{1}{2}}\Lambda_2^{\frac{1}{2}}(n)\big[\mathbf{E}\sum_{I_n^2}\tilde{X}_{n,i}^3\tilde{X}_{n,j}^3\big]^{\frac{1}{2}}.
\end{aligned}
$$

Since $\sum_{I_n^2}|\tilde{X}_{n,i}^3\tilde{X}_{n,j}^3| \le \big(\sum_{i=1}^n|\tilde{X}_{n,i}|^3\big)^2 \le 1$, it follows from the previous inequality and the limit (4.7) that

$$
\lim_{n\to\infty}\mathbf{E}\sum_{I_n^2}\tilde{X}_{n,i}\tilde{X}_{n,j}^3 = 0. \tag{4.13}
$$

As argued above, the limits (4.12) and (4.13) imply the limit (4.9) and, with it, the theroem.

□

Remark 2.2 shows that the sixth moment condition is implied by the regularity condition (2.11). We thus have:

4.3. Corollary. *Suppose $\mathbf{E}|X| < \infty$, $\mathbf{E}X = 0$ and that condition (2.11) holds for X. Then,*

$$
\mathcal{L}\Big(\frac{S_n - m_n}{V_n}\Big) \to_w N(0,1) \quad\Longrightarrow\quad X \in DAN.
$$

Finally, we observe that, if $S_{n/2}/V_n$ and $(S_n - S_{n/2})/V_n$ converge jointly to $1/\sqrt{2}$ times the standard normal distribution of $\mathbf{R}^2$, then we can symmetrize and therefore obtain that $X \in DAN$:

4.4. Theorem. *A necessary and sufficient condition for $X \in DAN$ and $\mathbf{E}X = 0$ is that*

$$
\mathcal{L}\Big(\frac{\sum_{i=1}^{[n/2]}X_i}{V_n}, \frac{\sum_{i=[n/2]+1}^{n}X_i}{V_n}\Big) \to_w N\Big(0, \begin{pmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\Big). \tag{4.14}
$$

Proof. If $X \in DAN$ and $\mathbf{E}X = 0$ then (4.14) is a simple consequence of (3.15), which allows replacement of V_n by b_n. To prove the converse, it suffices to show that if $\{\varepsilon, \varepsilon_i\}$

is a Rademacher sequence (that is, these variables are i.i.d. and $\Pr\{\varepsilon = 1\} = 1 - \Pr\{\varepsilon = -1\} = 1/2$) independent of the sequence $\{X_i\}$, then

$$\mathcal{L}\Big(\frac{\sum_{i=1}^{n} \varepsilon_i X_i}{V_n}\Big) \to_w N(0,1). \tag{4.15}$$

To see this, we note that if (4.15) holds then, εX being symmetric, we can apply Theorem 4.1 and conclude that $\varepsilon X \in DAN$. But then, obviously, also $X \in DAN$ (the DAN criterion for X only depends on the distribution of $|X|$). Moreover, as we have already observed above, (4.5) and $X \in DAN$ implies $\mathbf{E}X = 0$.

So, let us assume that (4.14) holds. To prove (4.15) we begin by showing that if $\tau_n = o(n)$, $\tau_n \in \mathbf{N}$, then

$$\frac{S_{\tau_n}}{V_n} \to 0 \ \text{ in pr.} \tag{4.16}$$

By writing

$$\frac{S_{\tau_n}}{V_n} = \frac{S_{\tau_n}}{V_{\tau_n}} \frac{V_{\tau_n}}{V_n},$$

and taking into account that the sequence $\big\{\frac{S_{\tau_n}}{V_{\tau_n}}\big\}$ is stochastically bounded by hypothesis, it suffices to see

$$\frac{V_{\tau_n}^2}{V_n^2} \to 0 \ \text{ in pr.} \tag{4.16$'$}$$

This sequence is tight as it is bounded by 1, and so is the sequence of vectors

$$\Big\{\Big(\frac{V_{\tau_n}^2}{V_n^2}, \frac{V_{2\tau_n}^2 - V_{\tau_n}^2}{V_n^2}, \ldots, \frac{V_{k\tau_n}^2 - V_{(k-1)\tau_n}^2}{V_n^2}\Big)\Big\}_{n=1}^{\infty}$$

for each k since eventually the sum of their coordinates is smaller than 1. The coordinates of these vectors are not independent, but they are equidistributed. We conclude that, if Y_1 has the law of a subsequential limit of the laws of $\frac{V_{\tau_n}^2}{V_n^2}$, then for each k there are random variables $Y_2, \ldots, Y_k$ (a different collection for each k) with the same distribution as Y_1 and such that $\sum_{i=1}^{k} Y_i \le 1$. Hence, $\mathbf{E}Y_1 \le k^{-1}$ for all k, that is, $Y_1 = 0$ a.s. (as $Y_1 \ge 0$), proving (4.16').

Let now $A_n = \{i \le n : \varepsilon_i = 1\}$. By the law of the iterated logarithm, $|\#A_n - n/2| = O\big((n \log\log n)^1/2\big)$ a.s. If we let $\tau_n = |\#A_n - [n/2]|$, which is $o(n)$ a.s., then we have that, for all n,

$$\Big(\frac{\sum_{i\in A_n} X_i}{V_n}, \frac{\sum_{i\in A_n^c} X_i}{V_n}\Big) \stackrel{\mathcal{L}}{=} \Big(\frac{\sum_{i=1}^{[\frac{n}{2}]} X_i}{V_n} \pm \frac{\sum_{i=[\frac{n}{2}]+1}^{[\frac{n}{2}]+\tau_n} X_i}{V_n}, \frac{\sum_{i=[\frac{n}{2}]+1}^{n} X_i}{V_n} \mp \frac{\sum_{i=[\frac{n}{2}]+1}^{[\frac{n}{2}]+\tau_n} X_i}{V_n}\Big).$$

By (4.16) this last sequence of random vectors is weak convergence equivalent to the sequence whose general term is

$$\Big(\frac{\sum_{i=1}^{\frac{n}{2}} X_i}{V_n}, \frac{\sum_{i=\frac{n}{2}+1}^{n} X_i}{V_n}\Big).$$

By hypothesis, this sequence has limit $N\big(0, \frac{1}{2} Id\big)$. We have thus proved that, if Z_1 and Z_2 are two independent, centered normal variables each with variance 1/2, then,

$$\mathcal{L}\left(\frac{\sum_{i=1}^n \varepsilon_i X_i}{V_n}\bigg|\{\varepsilon_i\}\right) \approx_{\text{a.s.}} \mathcal{L}\left(\frac{\sum_{i=1}^{\frac{n}{2}} X_i}{V_n} - \frac{\sum_{i=\frac{n}{2}+1}^{n} X_i}{V_n}\right) \to_w \mathcal{L}(Z_1 - Z_2).$$

Therefore, the limit (4.15) holds, and the theorem is proved.

□

(4.14) should be compared to (3.18'). It would be interesting if (3.18') could be deduced directly from (4.14), or even better, from weaker versions of (4.14) such as

$$\mathcal{L}\Big(\frac{\sum_{i=1}^{[n/2]} X_i}{V_n}\Big) \to N\big(0, \frac{1}{2}\big) \ \text{ and } \ \mathcal{L}\Big(\frac{\sum_{i=1}^{n} X_i}{V_n}\Big) \to N\big(0, 1\big).$$

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