ON MACLANE COHOMOLOGY FOR THE RING OF INTEGERS

VINCENT FRANJOU AND TEIMURAZ PIRASHVILI

1. Introduction

For a ring R, let $\mathcal{F}(R)$ be the category whose objects are the functors from the category of finitely generated free left R-modules to the category of left R-modules, and whose maps are the natural transformations; one such functor is the inclusion functor I. The MacLane cohomology of the ring R with coefficient in a functor T of $\mathcal{F}(R)$ is defined by

$$\mathrm{HML}^*(R,T) = \mathrm{Ext}^*_{\mathcal{F}(R)}(I,T)\ .$$

When T is just tensoring with a R-R-bimodule M, we write $\mathrm{HML}^*(R,M)$ for $\mathrm{HML}^*(R,-\otimes_R M)$. By [8], this is compatible with MacLane's original definition [10].

There is a dual notion, MacLane homology, known [6, 12] to be isomorphic to the topological Hochschild homology in the sense of Bökstedt [2] and stable K-theory in the sense of Waldhausen [9]. Using sophisticated topological methods Bökstedt has calculated $\mathrm{THH}(\mathbb{Z})$ and $\mathrm{THH}(\mathbb{Z}/p\mathbb{Z})$, for all primes p [3]. Bökstedt's result for $\mathbb{Z}/p\mathbb{Z}$ is closely related to L. Breen's calculations [4]. Since, V. Franjou, J. Lannes and L. Schwartz [7] have given algebraic computations of $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ and $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, Sym^k)$, where Sym^k denotes the k-th symmetric power; their results include the module structure over $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ defined by the Yoneda product.

The purpose of this paper is to give a short and elementary algebraic calculation of $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$ which includes the Yoneda ring structure. We prove:

Theorem. *Let $\Gamma(x)$ be the free divided-powers algebra on a generator x of degree 2. The algebra $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$ is isomorphic to the quotient of $\Gamma(x)$ by its ideal generated by the class x.*

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This theorem is proved in section 4, using a spectral sequence whose first line is the desired $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$, and which converges to zero. This spectral sequence is constructed in section 3. Its E_2-term (but the first line) is computed by the corollary 2.2.

2. MacLane cohomology with coefficients in symmetric or exterior powers

For an abelian group A, we denote by Sym^*A the symmetric algebra generated by A. Our first task is the computation of $\mathrm{HML}^*(\mathbb{Z}, Sym^n)$ as a right module over $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ for each n but for $n=1$.

For a given prime p, the rings maps:

$$\mathrm{HML}^*(\mathbb{Z},\mathbb{Z}) \to \mathrm{HML}^*(\mathbb{Z};\mathbb{Z}/p\mathbb{Z}) \leftarrow \mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$$

will be used to relate the module structures over these different rings. Let us recall from [7, par. 9] the computation of the right-hand side map.

We let Λ denote the following graded algebra:

$$\Lambda := \mathbb{Z}/p\mathbb{Z}[e_0, ..., e_h, ...]/({e_h}^p; h \geq 0)\ ,$$

where e_h is a class of degree $2p^h$. Let Λ_k be the factoralgebra of Λ by the ideal generated by $e_0, ..., e_{k-1}$. There exist isomorphisms of graded algebras:

$$\begin{aligned} \mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) &\cong \Lambda && [7, \text{Théorème } 7.3],\\ \mathrm{HML}^*(\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) &\cong \Lambda_1 \otimes \Lambda(\xi_1) && [7, \text{par. } 9.2], \end{aligned}$$

where ξ_1 is a class of degree $2p-1$ and $\Lambda(\xi)$ denotes the exterior algebra on a generator ξ. Under these isomorphisms, the map

$$\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z},\mathbb{Z}/p\mathbb{Z}) \to \mathrm{HML}^*(\mathbb{Z};\mathbb{Z}/p\mathbb{Z})$$

is the composite: $\Lambda \twoheadrightarrow \Lambda_1 \hookrightarrow \Lambda_1 \otimes \Lambda(\xi_1)$.

Proposition 2.1. *Let n be an integer, $n \geq 2$.*

If n is not a prime power, then: $\mathrm{HML}^*(\mathbb{Z}, Sym^n) = 0$.

If $n = p^h$ is a power of a prime p, the right module structure over $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$ *factors through* $\mathrm{HML}^*(\mathbb{Z};\mathbb{Z}/p\mathbb{Z})$, *making* $\mathrm{HML}^*(\mathbb{Z}, Sym^n)$ *a free Λ_h-module on a class of degree 1. In particular:*

$$\mathrm{HML}^i(\mathbb{Z}, Sym^n) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } i \equiv 1 \pmod{2n} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By [11] (or by [7, lemme 0.4], see also the appendix of [1]) we know that $\mathrm{HML}^*(\mathbb{Z}, Sym^i \otimes Sym^j)$ is 0, for $i,\ j \geq 1$. Thus by [5, 10.9], $\mathrm{HML}^*(\mathbb{Z}, Sym^n)$ is annihilited by the binomial coefficient $\binom{n}{i}$, for $1 \leq i \leq n-1$.

By [5, 10.10] we first get: $\mathrm{HML}^*(\mathbb{Z}, Sym^n) = 0$ if $n \geq 2$ is not a prime power; now fixing a prime p and setting $n = p^k \geq 2$ we get as well that $\mathrm{HML}^*(\mathbb{Z}, Sym^n)$ is annihilited by p.

On the other hand, for any $T \in \mathcal{F}(\mathbb{Z}/p\mathbb{Z})$, one has an exact couple of right $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$-modules [7, par. 9]:

$$\begin{array}{ccc} \mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, T) & \xrightarrow{i} & \mathrm{HML}^*(\mathbb{Z}, \widehat{T}) \\ \pi \nwarrow & & \swarrow \delta \\ & \mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, T) & \end{array}$$

where $\widehat{T} \in \mathcal{F}(\mathbb{Z})$ is defined by $\widehat{T}(X) := T(X/pX)$; the map δ is of degree -1, and $\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ acts on $\mathrm{HML}^*(\mathbb{Z}, \widehat{T})$ via the ring-map:

$$\mathrm{HML}^*(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{HML}^*(\mathbb{Z}; \mathbb{Z}/p\mathbb{Z}) \ .$$

The map π, of degree 2, is the product by the class e_0. When T is the symmetric power Sym^n, $n = p^h$, we know [7, corollaire 7.4] that $\mathrm{HML}^i(\mathbb{Z}/p\mathbb{Z}, Sym^n)$ is isomorphic to Λ_h, so that $\pi = 0$. The exact couple reduces to a short exact sequence:

$$0 \to \Lambda_h \xrightarrow{i} \mathrm{HML}^i(\mathbb{Z}, Sym^n \otimes \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\delta} \Lambda_h \to 0$$

which splits for degree reasons.

Finally, since the multiplication by p yields the zero map on $\mathrm{HML}^*(\mathbb{Z}, Sym^n)$, the statement follows from the cohomology long exact sequence associated to the following short exact sequence of coefficients:

$$0 \to Sym^n \xrightarrow{p} Sym^n \to Sym^n \otimes \mathbb{Z}/p\mathbb{Z} \to 0 \ .$$

□

Corollary 2.2. *Let n be an integer, $n \geq 2$.*

If n is not a prime power, then: $\mathrm{HML}^*(\mathbb{Z}, \Lambda^n) = 0$.

If $n = p^h$ is a power of a prime p, the right module structure over $\mathrm{HML}^*(\mathbb{Z})$ *factors through* $\mathrm{HML}^*(\mathbb{Z}; \mathbb{Z}/p\mathbb{Z})$, *making* $\mathrm{HML}^*(\mathbb{Z}, \Lambda^n)$ *a free Λ_h-module on a class of degree n. In particular:*

$$\mathrm{HML}^i(\mathbb{Z}, \Lambda^n) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{if } \ i = tn \ , \quad t \ \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Apply the cancellation result [11] for tensor products to the Koszul complex

$$0 \to \Lambda^n \to \cdots \Lambda^{n-i} \otimes Sym^i \to \cdots \to Sym^n \to 0 \ .$$

□

3. A spectral sequence

In this paragraph, we set up a spectral sequence that will enable us to compute $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$ as an algebra.

Proposition 3.1. *There is a first quadrant spectral sequence of right* $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$*-modules with*

$$II_2^{s,t} = \mathrm{HML}^s(\mathbb{Z},\Lambda^{t+1})$$

converging to $\mathbb{Z}$ *concentrated in degree* 0.

Proof. For a free abelian group A, let $\mathcal{I}(A)$ be the augmentation ideal of the group ring of A and let $B^*(A)$ be the bar-complex:

$$\cdots \to \mathcal{I}(A)^{\otimes i} \to \cdots \mathcal{I}(A)^{\otimes 2} \to \mathcal{I}(A)$$

For any integer k, there is a natural isomorphism from its homology $Tor_k^{\mathbb{Z}[A]}(\mathbb{Z},\mathbb{Z})$ to $\Lambda^k(A)$.

Our spectral sequence arises from the "dual" of the bar-complex. For an abelian group A, define $DA := \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Z})$. By applying the functor $\mathrm{Hom}_{\mathcal{F}(\mathbb{Z})}(I,-)$ on the complex DB^*D, one gets two hypercohomology spectral sequences with

$$I_1 = \mathrm{HML}^*(\mathbb{Z}, DB^*D)$$
$$II_2^{s,t} = \mathrm{HML}^s(\mathbb{Z},\Lambda^{t+1}) \ .$$

The first one collapses at I_1 (though $\mathcal{I}^{\otimes i}$ is a projective in $\mathcal{F}(\mathbb{Z})$, this is not the case that $D\mathcal{I}^{\otimes i}D$ is an injective in $\mathcal{F}(\mathbb{Z})$; however, $D\mathcal{I}^{\otimes i}D$ is a direct factor of $\mathbb{Z}^{\mathrm{Hom}(-,\mathbb{Z}^i)}$, and $\mathrm{Ext}^k_{\mathcal{F}(\mathbb{Z})}(I,\mathbb{Z}^{\mathrm{Hom}(-,\mathbb{Z}^i)}) = \mathrm{Ext}^k_{\mathbb{Z}}(I(\mathbb{Z}^i),\mathbb{Z}) = 0$ for $k>0$). The result follows. $\square$

We deduce the first consequences below.

Corollary 3.2. *The group* $\mathrm{HML}^n(\mathbb{Z},\mathbb{Z})$ *is finite for every positive integer* n, *and zero for odd* n. *Furthermore, for any prime* p, *if* $n = 2p^k t$, $(p,t)=1$, *the order of the* p*-component of the group* $\mathrm{HML}^n(\mathbb{Z},\mathbb{Z})$ *is* p^k.

Proof. Looking at the spectral sequence and the corollary 2.2, the last statement only needs an explanation. By dimension reason all nontrivial differentials lands to $t=0$. Hence all of them are monomorphisms. In total degree $2p^kt-1$, only the k terms $\mathrm{HML}^{2p^kt-p^l}(\mathbb{Z},\Lambda^{p^l})$, $1\le l\le k$, have a non-trivial p-component, and they are just $\mathbb{Z}/p\mathbb{Z}$. $\square$

4. The Yoneda product

Let $\Gamma(x)$ be the free divided-powers algebra on a generator x of degree 2. Recall that $\Gamma(x)$ is the subring of $\mathbb{Q}[x]$, generated by $x^{[n]} = \dfrac{x^n}{n!}$, $n \geq 1$.

Theorem 4.1. *The algebra* $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$ *is isomorphic to the quotient of the divided-powers algebra* $\Gamma(x)$ *by its ideal generated by the degree 2 class* x.

Proof. Both algebras are connected and are finite groups in every positive dimension, hence they are isomorphic to the tensor product of their p-components over all primes p. This makes it enough to prove the isomorphism of each p-component. By the Bockstein exact sequence, the kernel of

$$p : \mathrm{HML}^i(\mathbb{Z},\mathbb{Z}) \to \mathrm{HML}^i(\mathbb{Z},\mathbb{Z})$$

is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or is 0, hence the p-component of $\mathrm{HML}^i(\mathbb{Z},\mathbb{Z})$ is a cyclic group or is 0. Comparing this fact with Corollary 3.2, we conclude that this group is simply $\mathbb{Z}/p^k\mathbb{Z}$, where $i = 2p^k t$ and $(t,p) = 1$. From this follows that both algebras in question have the same underlying abelian groups. In order to construct an isomorphism of rings, we need the following

Lemma 4.2. *Let p be a prime. For each positive integer k, a generator a_k can be chosen in* $\mathrm{HML}^{2p^k}(\mathbb{Z},\mathbb{Z})$ *to satisfy:*

$$a_k^p = pa_{k+1}\ , \quad k \geq 1\ .$$

To prove the lemma, we consider the spectral sequence of the proposition 3.1. We now are able to determine it entirely. In particular, for every odd t, the differential

$$d_p : II_2^{tp,p-1} \cong II_p^{tp,p-1} \to II_p^{(t+1)p,0} \cong II_2^{(t+1)p,0}$$

is an injection

$$\mathbb{Z}/p\mathbb{Z} \simeq \mathrm{HML}^{tp}(\mathbb{Z},\Lambda^{tp}) \to \mathrm{HML}^{(t+1)p}(\mathbb{Z},\mathbb{Z})\ .$$

Starting with a generator x in $\mathrm{HML}^p(\mathbb{Z},\Lambda^p)$, we chose $a_1 := d_p(x)$. Working by induction, suppose now that $a_1, \dots\ a_k$ are chosen, $k \geq 1$. Then, in the group $\mathrm{HML}^{2p^{k+1}}(\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z}/p^{k+1}\mathbb{Z}$, one has:

$$p^{k-1}a_k^p = p^{k-2}a_{k-1}^p a_k^{p-1} = \cdots = a_1^p a_2^{p-1} \dots a_k^{p-1} = d_p(x)a_1^{p-1} \dots a_k^{p-1}\ .$$

Note that the image of a generator a_i, $i \geq 1$, under the map $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z}) \to \mathrm{HML}^*(\mathbb{Z},\mathbb{Z}/p\mathbb{Z})$ is a non-zero multiple of e_i. As d_p is a right $\mathrm{HML}^*(\mathbb{Z},\mathbb{Z})$-module map, we find that $p^{k-1}a_k^p$ is, up to a unit, equal to $d_p(x\ e_1^{p-1} \dots e_k^{p-1})$, which is non zero. However, $p^k a_k^p$ is zero, and there exist a generator a_{k+1} to complete the induction step.

One then can uniquely extend the correspondance $x^{[p^k]} \mapsto a_k$, $k \geq 1$, to a ring homomorphism between the p-components, which is an isomorphism. $\square$

References

1. S. Betley and T. Pirashvili: *Stable K-theory as a derived functor.* J. Pure and Appl. Algebra, 96 (1994), 245-258
2. M. Bökstedt: *Topological Hochschild homology.* Preprint (1985)
3. M. Bökstedt: *The topological Hochschild homology of* $\mathbb{Z}$ *and* $\mathbb{Z}/p$. Preprint (1985)
4. L. Breen: *Extensions du groupe additif.* Publ. IHES 48, (1978), 39-125
5. A. Dold, D. Puppe: *Homologie nicht-additiver Functoren. Anwendungen.* Ann. Ins. Fourier, 11 (1961), 201-312
6. B. I. Dundas, R. McCarthy: *Stable K-theory and topological Hochschild homology.* Ann. of Math. 140 (1994), 685-701; *Erratum*, 142 (1995), 425-426
7. V. Franjou, J. Lannes, L. Schwartz: *Autour de la cohomologie de MacLane des corps finis.* Invent. Math. 115 (1994), 513-538
8. M. Jibladze, T. Pirashvili: *Cohomology of algebraic theories.* J. Algebra 137 (1991), 253-296
9. C. Kassel: *La K-théorie stable.* Bull. Soc. Math. France 110 (1982), 381-416
10. S. MacLane: *Homologie des anneaux et des modules.* In: CBRM, Colloque de topologie algebrique, Louvain (1956), 55-80
11. T. Pirashvili: *Higher additivizations.* Proc. Math. Inst. Tbilisi, 91 (1988), 44-54
12. T. Pirashvili, F. Waldhausen: *MacLane homology and topological Hochschild homology.* J. Pure and Appl. Algebra 82 (1992), 81-98

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Université Paris-Sud, Mathématiques, bât. 425, F-91405 Orsay cedex, France

Math. Inst., Aleksidze str. 1, Tbilisi 380093, Republic of Georgia