Regular two-graphs from the even unimodular lattice $E_8 \oplus E_8$

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Abstract

This is a revised version of the first part of [8]. Each two-graph is in one-to-one correspondence with a set of equiangular lines. This implies that a two-graph is represented by a system of vectors of equal odd norm with mutual inner products ± 1. This is a special odd system, i.e. a set of vectors with odd inner products. The construction of obtaining odd systems from doubly even lattices introduced in [5], [6] is applied to the even unimodular lattice $E_8 \oplus E_8$ multiplied by $\sqrt{2}$. For the odd system of 456 vectors obtained by this construction, we study odd subsystems of 36 vectors spanning equiangular lines, i.e. subsystems representing regular two-graphs on 36 points. A subsystem of vectors representing a two-graph generates a sublattice of the lattice $E_8 \oplus E_8$. These sublattices are distinguished by sets of lattice vectors of norm 2. These sets are root systems. Hence the set of all two-graphs from $E_8 \oplus E_8$ is partitioned into families of two-graphs with the same root system. It is proved here that there are 7 families with the following root systems: $\emptyset$, A_1^7, $A_2A_3^3$, A_6A_7, $A_1^9D_4$, $A_1A_3D_4D_6$, D_7E_7. The two-graphs with the 4 first root systems relate to the 23 Steiner triple systems with a head. We prove that the 100 first two-graphs of the list of two-graphs obtained by E.Spence by use a computer [10] are exactly all nonisomorphic two-graphs from $E_8 \oplus E_8$.

1 Introduction

A two-graph is a set of *coherent* triples such that every 4-set contains an even number of coherent triples. But it is more important for us that each two-graph is represented by a set of equiangular lines. If the acute angle between lines is equal to $arccos\frac{1}{m}$, then we can choose a vector of norm (squared length) m along each line such that inner products of chosen vectors is equal to ± 1. If v_1, v_2, v_3 are vectors along 3 lines, then these 3 lines compose a coherent triple if $(v_1v_2)(v_2v_3)(v_3v_1) = -1$, where v_iv_j is the inner product of vectors v_i and v_j.

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Sets of equiangular lines of a sufficient large size in a space of fixed dimension exist only if m is an odd number. There is a special bound on the maximal number $n(m,d)$ of equiangular lines at the angle $arccos\frac{1}{m}$ in a space of dimension d:

$$n(m,d) \leq \frac{d(m^2-1)}{m^2-d}.$$

This bound is achieved if and only if the corresponding two-graph is *regular*. A two-graph is regular if every pair of points belongs to the same number of coherent triples.

This is a great problem to classify all regular two-graphs $\mathcal{T}(m,d)$ with parameters m and d, in particular, to find a number $N_m(d)$ of all nonisomorphic two-graphs with the same parameters.

For $m=3$, a regular two-graph $\mathcal{T}(3,d)$ exists only for $d=5,6$ and 7, and $\mathcal{T}(3,d)$ is unique in each dimension, i.e. $N_3(d)=1$ for $d=5,6,7$.

We are interested here in regular two-graphs $\mathcal{T}(5,15)$ with $m=5$ and $d=15$, when $n(5,15)=36$. A set of equiangular lines at angle $arccos\frac{1}{5}$ representing a regular two-graph $\mathcal{T}(5,d)$ may exist only in dimensions $d=$ 5, 10, 13, 15, 19, 20, 21, 22 and 23. Regular two-graphs $\mathcal{T}(5,d)$ are known for all these $d \neq 19,20$. It is known also the number $N_5(d)$ of all nonisomorphic two-graphs $\mathcal{T}(5,d)$ for $d=5,10,13,23$, namely, $N_5(5)=N_5(10)=N_5(23)=1$, $N_5(13)=4$.

Seidel mentions in [9] at first time results of E.Spence on computing nonisomorphic two-graphs on 36 points. Spence find that $N_5(15) \geq 227$. The number 227 is composed of 11 two-graphs from Latin squares of order 6, of 80 two-graphs from Steiner triple systems of order 15, and 136 new ones discovered by use of a computer [10]. In [9] Seidel describes a large subclass of the Spence' family. Every two-graph of this subclass is related to a 2-(10,4,2) design, and it is shown in [8] that it is a special gluing of the unique two-graphs $\mathcal{T}(5,10)$ and $\mathcal{T}(3,5)$.

We show in this paper that every two-graph $\mathcal{T}(5,15)$ from $E_8 \oplus E_8$ is a similar gluing of the unique two-graphs $\mathcal{T}(3,7)$ and $\mathcal{T}(7,7)$. More precisely, a projection of the set of equiangular lines representing a two-graph from $E_8 \oplus E_8$ onto a 7-dimensional space X is a set of equiangular lines representing $\mathcal{T}(3,7)$. A restriction of the set of equiangular lines onto another 7-dimensional space orthogonal to X is a set of equiangular lines representing $\mathcal{T}(7,7)$.

An inspection of the list of two-graphs computed by Spence [10] shows that the new two-graphs discovered by Spence which are not related to 2-(10,4,2) designs are two-graphs from $E_8 \oplus E_8$.

Denote by a class D and a class E the classes of two-graphs which are gluing $\mathcal{T}(5,10)$ with $\mathcal{T}(3,5)$, and $\mathcal{T}(3,7)$ with $\mathcal{T}(7,7)$, and can be obtained from the lattices D_{16}^+ and $E_8 \oplus E_8$, respectively.

All in all we have the following 4 main classes of two-graphs on 36 points:

1) 80 two-graphs from Steiner triple systems on 15 points, the class STS,
2) 11 two-graphs from Latin squares of order 6, the class LSQ,
3) 100 two-graphs of the class D from D_{16}^+,
4) 100 two-graphs of the class E from $E_8 \oplus E_8$.

The numbers of two-graphs in each class are obtained from the list of 227 two-graphs in [10]. The Latin square class is disjoint with all other classes. The 3 classes D, E, and from Steiner triple systems are not disjoint. The classes D and E have 41 two-graphs in common. The class E contains 23 two-graphs from STS with a head. The class D contains 7 two-graphs from STS with a head.

2 Odd systems and lattices

A set of vectors of an odd norm m with ± 1-inner products spanning equiangular lines is a special case of an odd system. An *odd system* $\mathcal{V}$ is a set of vectors v such that the inner product vv' of any (may be equal) two vectors of $\mathcal{V}$ is an odd integer. (We denote the inner product of two vectors v and v' by its juxtaposition vv'.) The inner product $v^2 = vv$ of a vector v with itself is called *norm* of v. Hence norms of all vectors of an odd system are odd. An odd system is called *uniform* (of norm m) if norms of all its vectors are equal (to m). As we use here only uniform odd systems, in what follows, we omit sometimes the word uniform.

We call an odd system *regular* if it represents a regular two-graph. We consider also *reduced* odd systems such that from two opposite vectors only one belongs to the odd system. Call an interchanging of a subset of vectors of a reduced odd system by its opposite *switching* of the odd system. Similarly, we call the operation of changing the sign of a vector v by *switching* v.

We call odd systems $\mathcal{V}$ and $\mathcal{V}'$ isomorphic if there is a bijection $\phi : \mathcal{V} \to \mathcal{V}'$ such that $\phi(v_1 v_2) = \phi(v_1)\phi(v_2)$. Obviously, isomorphic odd systems are equicardinal.

Let $\mathcal{U}$ be an odd system related to a two-graph (i.e. spanning equiangular lines). Since $vv' = \pm 1$ for $v, v' \in \mathcal{U}$, $v \neq \pm v'$, we can introduce a graph $G(\mathcal{U})$ with $\mathcal{U}$ as the set of its vertices. Two vertices v, v' of $G(\mathcal{U})$ are adjacent if and only if $vv' = -1$. If $\mathcal{U}$ relates to a regular two-graph, then $G(\mathcal{U})$ is a Taylor distance-regular graph of diameter 3 (see [1]).

Let $\mathcal{U}$ be a reduced odd system. A switching of $\mathcal{U}$ corresponds to a switching of $G(\mathcal{U})$. Fix $v_0 \in \mathcal{U}$. By a switching, we can isolate v_0, i.e. $G(\mathcal{U}^{sw}) = \{v_0\} \cup H_0$, where $H_0 = G(\mathcal{U}_0)$ and $\mathcal{U}_0 = \mathcal{U}^{sw} - \{v_0\}$. If $\mathcal{U}$ relates to a regular two-graph, then H_0 is a strongly regular graph. The (± 1)-adjacency matrix A of H_0 has minimal eigenvalue $-m$. Hence the matrix $mI + A$ is positive semidefinite, and it is the Gram matrix of the set of vectors of $\mathcal{U}_0$. For example, H_0 has parameters (35,16,6,8) for a regular two-graph $\mathcal{T}(5, 15)$ on 36 points.

A lattice L of dimension n is a free Abelian group of rank n of vectors. A lattice is called *integral* if inner products of all its vectors are integral. An integral lattice is called *even* if norms of all its vectors are even. An even lattice L is called *doubly even* if $\frac{1}{\sqrt{2}}L$ is even. Norms of all vectors of a doubly even lattice are multiples of 4, and all inner products are even. Hence minimal norm of a nonzero vector of a doubly even lattice L is not less than 4. The set L_4 of all vectors of norm 4 of L is, up to the multiple $\sqrt{2}$, a *root system*. Hence below we call a vector of norm 4 a *root*.

Each root system is a direct sum of irreducible root systems, called its *components*. A root system is called *irreducible* if it cannot be partitioned into two subsystems such that roots of one of these systems are orthogonal to all roots of other. All irreducible root systems are known. These are A_n, D_n and E_m, where n and m are dimensions of the corresponding root systems, and $m = 6, 7, 8$. Following to [4], we denote a root system consisting of components R_1, R_2,..., R_k by the product $R_1R_2...R_k$. In particular, a sum of k equal components R is denoted by R^k. A lattice generated by a root system is called *root lattice*, but it is denoted by the direct sum of corresponding components. For example, the root lattice $E_8 \oplus E_8$ is generated by the root system E_8^2.

In [5] and [6], a construction of uniform odd systems from a doubly even lattice is introduced. Here we describe this construction for uniform odd systems of norm 5.

Let L be a doubly even lattice, and let L_8 be the set of all $a \in L$ of norm 8. Let $c \in L$ have norm 12. We set

$$\mathcal{A}(c) = \{a \in L_8 : ac = 6\}.$$

It is easy to see that $a \in \mathcal{A}(c)$ implies $a^* = c - a \in \mathcal{A}(c)$, and $aa^* = -2$. Conversely, any two vectors $a, a' \in L_8$ with $aa' = -2$ provide a vector $c = a + a'$ of norm 12.

For $a \in \mathcal{A}(c)$, define

$$v(a) = a - \frac{1}{2}c.$$

Then we have $v(a)v(a') = aa' - 3$. Since inner products of all $a \in L$ are even, the inner product $v(a)v(a')$ is odd. In particular, $v^2(a) = 5$. In other words, the set

$$\mathcal{V}(c) = \{v(a) : a \in \mathcal{A}(c)\}$$

is a uniform odd system of norm 5.

The construction can be reversed. Let c be a vector of norm 12, which is orthogonal to the space spanned by an odd system $\mathcal{V}$ of norm 5. Then the vector $a(v) = v + \frac{1}{2}c$ have norm 8, and $a(v) + a(-v) = c$. Let L be the lattice linearly generated by $a(v)$ for all $v \in \mathcal{V}$. Then $a(v) \in L_8$. Hence the odd system $\mathcal{V}(c)$ from this lattice contains the original odd system $\mathcal{V}$ as a subsystem. It is proved in [6] that $\mathcal{V} = \mathcal{V}(c)$ if and only if $\mathcal{V}$ is *closed*.

Now we define the closure of an odd system. This notion is very useful for to distinguish nonisomorphic odd systems (and two-graphs). Consider the following lattices generated by an odd system $\mathcal{V}$:

$$L^q(\mathcal{V}) = \{u : u = \sum_{v \in \mathcal{V}} z_v v,\ \sum_{v \in \mathcal{V}} z_v \equiv q \pmod 2,\ z_v \in \mathbf{Z}\},\ q = 0, 1.$$

Let $\mathcal{V}$ be uniform and of norm 5. It is proved in [5] that $L^0(\mathcal{V})$ is a doubly even lattice, and the affine lattice $L^1(\mathcal{V}) = v + L^0(\mathcal{V})$ is a translation of $L^0(\mathcal{V})$. $L^1(\mathcal{V})$ is an odd system and $u^2 \equiv 1 \pmod 4$ for all $u \in L^1(\mathcal{V})$. Let $L^1_k(\mathcal{V})$ be the set of all vectors of $L^1(\mathcal{V})$ of norm k. Obviously, $\mathcal{V} \subseteq L^1_5(\mathcal{V})$.

The convex hull $\mathrm{conv}L^1_5(\mathcal{V})$ of endpoints of all vectors from $L^1_5(\mathcal{V})$ is very often a Delaunay polytope of the lattice $L^0(\mathcal{V})$. In particular, it is so if $\mathcal{V}$ is not pillar, i.e. there is no vector e of the norm 1 such that $ev = \pm 1$ for all $v \in \mathcal{V}$. The conditions when $\mathrm{conv}L^1_5(\mathcal{V})$ is a Delaunay polytope is given in [6].

Definition. The uniform odd system $L_5^1(\mathcal{V})$ is called the *closure* of the odd system $\mathcal{V}$. The odd system $\mathcal{V}$ is called *closed* if $\mathcal{V} = L_5^1(\mathcal{V})$. Sometimes we denote the closure of $\mathcal{V}$ by cl$\mathcal{V}$. $\square$

Recall that if $\mathcal{V}$ has the form $\mathcal{V}(c)$ for some c, then $\mathcal{V}$ is closed.

Suppose that $\mathcal{U}$ is a maximal uniform odd system of norm 5 spanning equiangular lines, i.e. $uu' = \pm 1$ for distinct $u, u' \in \mathcal{U}$. If $\mathcal{U}$ is not closed, then, for each $v \in$cl$\mathcal{U} - \mathcal{U}$, there is $u \in \mathcal{U}$ such that $vu = 3$. Then the vector $v - u$ has norm 4, i.e. it is a root. Let $R(\mathcal{U})$ be the set of all roots obtained by such a way.

Lemma 1 *The set $R(\mathcal{U})$ is the root system of all roots of the lattice $L^0(\mathcal{U})$, i.e. $R(\mathcal{U}) = L_4^0(\mathcal{U})$.*

Proof. Obviously, $R(\mathcal{U})$ is contained in the root system of $L^0(\mathcal{U})$. Let r be a root of $L^0(\mathcal{U})$. Then $ru = 0, \pm 2$ for all $u \in \mathcal{U}$, and if $ru = 2$ then $v = u - r \in$cl$\mathcal{U}$, i.e. $r = v - u \in R(\mathcal{U})$. Since all roots of $L^0(\mathcal{U})$ lie in the space spanned by $\mathcal{U}$, for every root r, there is $u \in \mathcal{U}$ such that $ru = 2$. $\square$

If $\mathcal{U}$ and $\mathcal{U}'$ represent isomorphic two-graphs and are not reduced (reduced), then they are isomorphic (switching equivalent to isomorphic odd systems, respectively).

The following obvious proposition helps to distinguish nonisomorphic odd systems spanning equiangular lines, and therefore non isomorphic two-graphs.

Proposition 1 *Let $\mathcal{U}$ and $\mathcal{U}'$ be d-dimensional odd systems representing two-graphs $\mathcal{T}$ and $\mathcal{T}'$ with the same parameters $(5, d)$. Then $\mathcal{T}$ and $\mathcal{T}'$ are not isomorphic if $R(\mathcal{U}) \neq R(\mathcal{U}')$.* $\square$

3 Two-graphs from the lattice $E_8 \oplus E_8$

Recall that there are exactly two nonisomorphic even unimodular lattices in dimension 16, namely D_{16}^+ and $E_8 \oplus E_8$, where E_8 is a 8-dimensional root lattice. The root lattice E_8 is generated by its minimal vectors of norm 2 forming the root system E_8. We use the description of the root system E_8 given in [3]. In fact, the description is given in terms of vectors of norm 4, i.e. it gives $\sqrt{2}E_8$. We continue call the minimal vectors of norm 4 of the doubly even lattice $\sqrt{2}E_8$ by roots.

Let $V_8 = \{0\} \cup V_7$, and $V_7 = \{1, ..., 7\}$. Let h_i, $i \in V_8$, be 8 mutually orthogonal vectors of norm 1. Then roots of $\sqrt{2}E_8$ are

1) $\pm 2h_i$, $i \in V_8$,

2) $\sum_{i \in Q} \varepsilon_i h_i$, $\varepsilon_i \in \{\pm 1\}$, $|Q| = 4$, $Q \in S(3, 4, 8)$.

Here $S(3, 4, 8)$ is the Steiner system that has the following form. Let F_7 be 7 triples of the unique STS $S(2, 3, 7)$ on 7 points. Its triples are lines of the projective Fano plane $PG(2, 2)$. Each quadruple $Q \in S(3, 4, 8)$ has the form $Q = t \cup \{0\}$ or $Q' = V_7 - t = \overline{Q} := V_8 - Q$, where $t \in F_7$. If $Q \neq \overline{Q'}$, then $|Q \cap Q'| = 2$. In this case, $Q \Delta Q' \in S(3, 4, 8)$.

Let f_i, $i \in V_8$, be other 8 mutually orthogonal vectors of norm 1. All f_i are orthogonal to all h_j. Then the roots of the second copy of $\sqrt{2}E_8$ are given by the above expressions 1) and 2) with h_i changed by f_i.

The vectors of norms 8 and 12 in the lattice $\sqrt{2}(E_8 \oplus E_8)$ are sums of two and three, respectively, orthogonal roots of the lattice. Since the automorphism group of the root system E_8 is transitive on pairs of orthogonal roots, there are, up to symmetry, two types of vectors of norm 12: a sum of 3 orthogonal roots of the same copy of $\sqrt{2}E_8$, and a sum of two roots of one copy and of one root of the other copy of $\sqrt{2}E_8$.

A vector c of the first type gives a pillar odd system $\mathcal{V}(c)$. This means that vectors of $\mathcal{V}(c)$ have the form $\pm(e+r)$, where e is a vector of norm 1, and r is a root (of norm 4) which belongs to $E_8 \oplus E_8$ and is orthogonal to e and c (details see in [5]). A maximal reduced pillar odd system $\mathcal{U} \subseteq \mathcal{V}(c)$ spanning equiangular lines (i.e. representing a two-graph) contains less than 36 vectors, the number of points of a regular two-graph $\mathcal{T}(5, 15)$.

Hence we consider only the vectors c of the second type. Recall that all vectors c of the same type belong to the same orbit of the automorphism group of the lattice $\sqrt{2}(E_8 \oplus E_8)$.

We take c equal to

$$c_0 = h(Q) + h(\overline{Q}) + 2f_0 = h(V_8) + 2f_0.$$

Here and below we use the following denotation: for any set V, any $X \subseteq V$, any g_k, $k \in V$, we set

$$g(X) := \sum_{i \in X} g_i, \tag{1}$$

The set $\mathcal{A}(c_0)$ contains the following vectors:

1) 784 vectors $h(Q) + \sum_{i \in P} \varepsilon_i f_i$, $Q, P \in S(3, 4, 8)$, $0 \in P$, $\varepsilon_0 = 1$.

2) 56 vectors $a = h(Q) - 2h_i + 2f_0$, $i \in Q$, and 56 vectors $c_0 - a = h(Q) + 2h_i$, $i \notin Q$, $Q \in S(3, 4, 8)$.

3) 8 vectors $2h_i + 2f_0$, and 8 vectors $h(V_8) - 2h_i$, $i \in V_8$.

Recall that $v(a) = a - \frac{1}{2}c_0$. Then

$$\mathcal{V}(c_0) = \mathcal{V}_1 \cup \mathcal{V}_2,$$

where

$$\mathcal{V}_1 = \{h(Q) - \frac{1}{2}h(V_8) + \sum_{i \in P - \{0\}} \varepsilon_i f_i,\ Q, P \in S(3, 4, 8),\ 0 \in P\},$$

$$\mathcal{V}_2 = \pm\{h(Q) - \frac{1}{2}h(V_8) - 2h_i + f_0, i \in Q,\ \text{and } 2h_i - \frac{1}{2}h(V_8) + f_0, i \in V_8\}.$$

Recall that if $Q \ni 0$, then $Q = \{0\} \cup s$ with $s \in F_7$. For $s \in F_7$, $s \subset Q$, we define 7 vectors of norm 2 as follows:

$$w_s = h(Q) - \frac{1}{2}h(V_8) = h_0 + h(s) - \frac{1}{2}h(V_8).$$

Note that if Q does not contain 0, then $0 \in \overline{Q} = V_8 - Q$. Hence

$$h(Q) - \frac{1}{2}h(V_8) = -(h(\overline{Q}) - \frac{1}{2}h(V_8)) = -w_s \text{ for } s = \overline{Q} - \{0\}.$$

Similarly, for $P \ni 0$,we have $P = \{0\} \cup t$, $t \in F_7$. We set

$$v_s(t, \varepsilon) = w_s + \sum_{i \in t} \varepsilon_i f_i, \;\; s, t \in F_7.$$

In this notation, the odd system $\mathcal{V}_1$ takes the form

$$\mathcal{V}_1 = \pm\{v_s(t, \varepsilon) : \varepsilon \in \{\pm 1\}^t, s, t \in F_7\}.$$

Using this explicit expression, it is not difficult to find that

$$R(\mathcal{V}_1) = D_7 E_7.$$

The roots of D_7 are $w_s \pm w_{s'}$, $s, s' \in F_7$. The roots of E_7 are $\pm 2f_i$, $\sum_{i \in Q} \varepsilon_i f_i$, $Q \in S(3, 4, 8)$, $0 \notin Q$. Note that the roots of E_7 are orthogonal to f_0.

Let $W = \frac{1}{2} \sum_{s \in F_7} w_s$. Then $W^2 = \frac{7}{2}$, and $W w_s = 1$ for all $s \in F_7$. It is easy to verify that $w_s w_t = 0$ for $s \neq t$, since $|s \cap t| = 1$ for distinct $s, t \in F_7$. Besides, $w_s h(V_8) = 0$. Hence the 8 vectors $h(V_8)$ and w_s, $s \in F_7$, form an orthogonal basis of the space spanned by h_i, $i \in V_8$. The vectors h_i can be expressed through $h(V_8)$ and w_s:

$$2h_0 = W + \frac{1}{4} h(V_8), \;\; 2h_i = -W + \sum_{s \ni i} w_s + \frac{1}{4} h(V_8).$$

Then the 128 vectors of $\mathcal{V}_2$ take the form $\pm(g + \frac{1}{2} \sum_{s \in F_7} \varepsilon_s w_s)$, where $\varepsilon_s \in \{\pm 1\}$, and there is an even number of minus signs. Hence we can redenote these vectors as $\pm u(S)$, where

$$u(S) = g + w(S),$$

$g = f_0 - \frac{1}{4} h(V_8)$ is the vector of norm $\frac{3}{2}$ orthogonal to all w_s, and $w(S) = \sum_{s \in S} w_s - W$.

Call a subset $S \subseteq F_7$ *odd* (*even*) if it has an odd (even) cardinality, respectively. Now the odd system $\mathcal{V}_2$ takes the form

$$\mathcal{V}_2 = \pm\{u(S) : S \subseteq F_7, \; S \text{ is odd}\}.$$

The root system of $\mathcal{V}_2$ is $R(\mathcal{V}_2) = D_7$. Since $R(\mathcal{V}_2) \subseteq R(\mathcal{V}_1)$, we have

$$R(\mathcal{V}(c_0)) = R(\mathcal{V}_1) = D_7 E_7.$$

These are roots of $E_8 \oplus E_8$ that are orthogonal to c_0.

We have $\frac{1}{2}|\mathcal{V}_1| = 392$, $\frac{1}{2}|\mathcal{V}_2| = 64$. Hence $\frac{1}{2}|\mathcal{V}(c_0)| = 456$.

It is easy to verify that $v_s(t, \varepsilon) u(S) = \pm 1$, i.e. $vv' = \pm 1$ for $v \in \mathcal{V}_1$ and $v' \in \mathcal{V}_2$. We seek a maximal odd subsystem $\mathcal{U} \subseteq \mathcal{V}(c_0)$ of vectors with all mutual inner products equal to ± 1. Of course, we have to find separately maximal subsets $\mathcal{U}_1 \subseteq \mathcal{V}_1$ and $\mathcal{U}_2 \subseteq \mathcal{V}_2$ such that $\mathcal{U}_1 \cup \mathcal{U}_2 = \mathcal{U}$. Recall that, for $m = 5$, $d = 15$, the special bound gives $\frac{1}{2}|\mathcal{U}| = 36$.

In what follows in this section, we consider only reduced odd systems $\mathcal{V}_1$ and $\mathcal{V}_2$ in the *canonical* form, when the vectors w_s and g in the vectors $v_s(t, \varepsilon)$ and $u(S)$, respectively, have positive signs. We preserve for canonical systems the same notations $\mathcal{V}_1$ and $\mathcal{V}_2$. Obviously, every reduced subsystem of $\mathcal{V}(c_0)$ can be made canonical by a switching.

3.1 Odd systems $\mathcal{U}_1$

Consider at first $\mathcal{V}_1$. Recall that $w_s^2 = 2$ and $w_s w_{s'} = 0$ for $s \neq s'$. We set $\delta(s,s') = 1$ if $s = s'$, and $\delta(s,s') = 0$ if $s \neq s'$. We have

$$v_s(t,\varepsilon)v_{s'}(t',\varepsilon') = 2\delta(s,s') + \sum_{i\in t\cap t'} \varepsilon_i \varepsilon'_i.$$

So, $v_s(t,\varepsilon)v_{s'}(t,\varepsilon) = 2\delta(s,s') + 3$. Since $vv' = \pm 1$ for distinct $v, v' \in \mathcal{U}_1$, this implies that, for each pair (t,ε), there is at most one s such that $v_s(t,\varepsilon) \in \mathcal{U}_1$. We denote this s by $s(t,\varepsilon)$.

We obtain that a map $s : (t,\varepsilon) \to s(t,\varepsilon) \in F_7$ corresponds to a set $\mathcal{U}_1 \subseteq \mathcal{V}_1$ spanning equiangular lines. Let

$$T_s(\mathcal{U}_1) = \{(t,\varepsilon) : s(t,\varepsilon) = s\}.$$

According to what was said above, the sets T_s are disjoint for distinct s.

Lemma 2 *For any $\mathcal{U}_1 \subseteq \mathcal{V}_1$ spanning equiangular lines, $|T_s(\mathcal{U}_1)| \leq 4$ for all $s \in F_7$.*

Proof. Let $T_s = T_s(\mathcal{U}_1)$. For $(t,\varepsilon), (t',\varepsilon') \in T_s$, we have $v_s(t,\varepsilon)v_s(t',\varepsilon') = 2 + \Sigma$, where

$$\Sigma = \sum_{i\in t\cap t'} \varepsilon_i \varepsilon'_i.$$

Note that Σ takes odd values. This implies that Σ should be equal either to –1 or –3. The case $\Sigma = -3$ is possible only if $t = t'$ and $\varepsilon = -\varepsilon'$. Then $T_s = \{(t,\varepsilon), (t,-\varepsilon)\}$. In fact, if there is another $(t'',\varepsilon'') \in T_s$, then $v_s(t,\pm\varepsilon)v_s(t'',\varepsilon'') = 2 \pm \sum_{i\in t\cap t''} \varepsilon_i \varepsilon''_i$ is equal to 3 for one of the signs $\pm$. So, $|T_s| = 2$ in this case .

Now, let $\Sigma = -1$. Then projections of vectors $v_s(t,\varepsilon)$ for $(t,\varepsilon) \in T_s$ on the space spanned by f_i, $1 \leq i \leq 7$, form an odd system of vectors of norm 3 with mutual inner products –1. But such a system contains at most 4 vectors. In fact, let $v_i^2 = 3$, $v_i v_j = -1$, $1 \leq i < j \leq k$. Then $0 \leq (\sum_i^k v_i)^2 = 3k - k(k-1)$, i.e. $k \leq 4$. Hence $|T_s| \leq 4$ in this case. $\square$

Since $|F_7| = 7$ and the sets T_s are disjoint for distinct s, Lemma 2 implies that $\mathcal{U}_1$ contains at most $4 \times 7 = 28$ pairs of opposite vectors.

By Lemma 2, if T_s contains more than two pairs (t,ε), then all vectors $\sum_{i\in t} \varepsilon_i f_i$, corresponding to pairs $(t,\varepsilon) \in T_s$, have mutual inner products –1.

Let $\mathcal{U}_1$ contain the maximal number 28 of vectors. Then the projection of $\mathcal{U}_1$ on the 7-dimensional space spanned by f_i, $i \in V_7$, is an odd system consisting 28 vectors

$$u(t,\varepsilon) = \sum_{i\in t} \varepsilon_i f_i$$

of norm 3 with mutual inner products ± 1. The vectors $u(t,\varepsilon)$ represent a two-graph $\mathcal{T}(3,7)$. The special bound gives $n(3,7) = 28$, i.e. the two-graph is the unique regular two-graph with parameters $(m,d) = (3,7)$.

The 28 vectors $u(t,\varepsilon)$ form a reduced odd system representing the two-graph $\mathcal{T}(3,7)$. Since for a fixed $t \in F_7$, there are 8 vectors $u(t,\varepsilon)$, the set of all $7 \times 8 = 56$ vectors $u(t,\varepsilon)$, $t \in F_7$, $\varepsilon \in \{\pm 1\}^t$ forms an odd system $\mathcal{W}(3,7)$ representing the two-graph $\mathcal{T}(3,7)$.

So we obtain that the odd system $\mathcal{U}_1$ uniquely determined by the following operations:

(i) choose a reduced odd system $\mathcal{U} \subseteq \mathcal{W}(3,7)$,

(ii) partition the 28 vectors of $\mathcal{U}$ into 7 groups each containing 4 vectors with mutual inner products -1,

(iii) relate each group to a vectors w_s, $s \in F_7$.

We call the result of these 3 operations by the map ϕ from $\mathcal{W}(3,7)$ to $\mathcal{U}_1$, and denote the obtained odd system $\mathcal{U}_1$ by $\mathcal{U}_1(\phi)$.

We can apply the map ϕ to an arbitrary odd system $\mathcal{W}$ representing $\mathcal{T}(3,7)$. Of course, $\mathcal{W}$ is isomorphic to $\mathcal{W}(3,7)$.

Let A be the adjacency matrix of the graph $G(\mathcal{U}_1(\phi))$. According to what was said above, the matrix A has order 28 and can be partitioned into 7×7 submatrices of order 4. The 7 diagonal submatrices are all 0 matrices. Any other submatrix has exactly two once in each row and each column.

3.2 Odd systems $\mathcal{U}_2$

Let $\mathcal{S}$ be a family of odd subsets of F_7. We set

$$\mathcal{U}_2(\mathcal{S}) = \{u(S) : S \in \mathcal{S}\}.$$

We want to find a family $\mathcal{S}$ such that $\mathcal{U}_2(\mathcal{S})$ is a maximal odd subsystem of $\mathcal{V}_2$ spanning equiangular lines.

Let S_0 have an even cardinality. Then the symmetric difference $S\Delta S_0$ is odd for any odd S. Hence the vector $u'(S) := u(S\Delta S_0)$ belongs to $\mathcal{V}_2$. Besides, $u'(S_1)u'(S_2) = u(S_1)u(S_2)$. Hence the odd systems $\mathcal{U}(\mathcal{S})$ and $\mathcal{U}(\mathcal{S}\Delta S_0) := \{u' : u \in \mathcal{U}(\mathcal{S})\}$ are isomorphic.

According to this, we can consider at first the case when $\mathcal{S}$ contains the odd set F_7.

Consider inner products of vectors $u(S)$:

$$u(S)u(T) = 5 - |S| - |T| + 2|S \cap T|. \tag{2}$$

For $S, T \in \mathcal{S}$, we have to have $u(S)u(T) = \pm 1$. For $S = F_7$, this condition implies $|T| = 1$ or 3.

Recall that maximal reduced odd systems $\mathcal{U}_1$ and $\mathcal{U}$ contain 28 and 36 vectors, respectively. Hence a maximal odd system $\mathcal{U}_2(\mathcal{S})$ contains 8 vectors. In other words, a maximal family $\mathcal{S}$ contains 8 odd sets.

Lemma 3 *A maximal family $\mathcal{S}$ with $F_7 \in \mathcal{S}$ does not contain sets of cardinality 1.*

Proof. Let S, T be distinct subsets of F_7 of cardinality 1. Then $|S \cap T| = 0$. Hence, for $|S| = |T| = 1$, (2) takes the form $u(S)u(T) = 3$. This implies that $\mathcal{S}$ contains at most one 1-set.

Suppose $\mathcal{S}$ contains a 1-set $S_0 = \{s\}$. For $|S| = 1$ and $|T| = 3$, (2) gives $u(S)u(T) = 2|S \cap T| + 1$. Hence $|S \cap T| = 0$ and $s \notin T$. So $\mathcal{S}$ consists of F_7, S_0 and some 3-sets T such that $s \notin T$. For 3-subsets T and T', (2) implies $|T \cap T'| \le 1$. If $|T \cap T'| = 0$, then $\mathcal{S}$ contains only four sets: F_7, S_0, T and T', because any other 3-set T'' has an intersection of cardinality 2 with T or T'. Hence $|T \cap T'| = 1$. But a 6-set contains at most four 3-subsets with mutual intersections of cardinality 1. Hence if $\mathcal{S}$ contains a 1-set, then it contains at most 6 sets. This implies that a maximal family $\mathcal{S}$ does not contain a set of cardinality 1. □

So, a maximal family $\mathcal{S}$ contains, besides F_7, only 3-sets. For 3-sets S, T, the equality (2) takes the form

$$u(S)u(T) = 2|S \cap T| - 1. \tag{3}$$

A maximal set of triples of a 7-set satisfying (3) contains 7 triples and forms a Steiner triple system on 7 points. All Steiner triple systems on 7 points are isomorphic. Their triples are lines of the Fano plane F_7. So, any two families $\mathcal{S}$ containing F_7 can be transformed each to other by a permutations of elements of F_7. As a basic family we take the family $\mathcal{S}_0$ containing F_7 and 3-sets

$$S_i := \{s \in F_7 : s \ni i\}, \ i \in V_7. \tag{4}$$

So

$$\mathcal{S}_0 = \{F_7\} \cup \mathcal{F}_7, \text{ with } \mathcal{F}_7 = \{S_i : i \in V_7\}.$$

Let $\beta : V_7 \to F_7$ be the bijection $i \to t_i$ given in (11) below. Then the bijection β transforms $t_k \subseteq V_7$ into $\beta(t_k) = S_k \subseteq F_7$, i.e. F_7 into $\mathcal{F}_7$. In other words, $\mathcal{F}_7 = \{\beta(t_i) : i \in V_7\}$.

We denote $\mathcal{U}_2(\mathcal{S}_0)$ by $\mathcal{U}_2^0$.

Lemma 4 *The odd system $\mathcal{U}_1(\phi) \cup \mathcal{U}_2(\mathcal{S})$ is isomorphic to a switching of an odd system $\mathcal{U}_1(\phi') \cup \mathcal{U}_2^0$ for some ϕ'.*

Proof. If the family $\mathcal{S}$ does not contain the set F_7, then we take an even set S_0 such that the family $\mathcal{S}\Delta S_0$ does contain F_7.

Note that if we change w_s by $-w_s$ for $s \in S_0$ in all vectors of an odd system $\mathcal{U} \subseteq \mathcal{V}(c_0)$, we obtain an odd system isomorphic to $\mathcal{U}$. We show that this change of signs of w_s, $s \in S_0$, transforms $\mathcal{U}_1(\phi) \cup \mathcal{U}_2(\mathcal{S})$ into a switching of $\mathcal{U}_1(\phi') \cup \mathcal{U}_2(\mathcal{S}\Delta S_0)$. The transformation $w_s \to -w_s$, $s \in S_0$, generates the following transformation of vectors $u(S)$: $u(S) \to u(S\Delta S_0)$. Hence the odd system $\mathcal{U}_2(\mathcal{S}\Delta S_0)$ can be obtained from $\mathcal{U}_2(\mathcal{S})$ by this map. Obviously, $\mathcal{U}_2(\mathcal{S})$ and $\mathcal{U}_2(\mathcal{S}\Delta S_0)$ are isomorphic.

Now recall the definition of vectors $v_s(t, \varepsilon) = w_s + \sum_{i \in t} \varepsilon_i f_i \in \mathcal{U}_1(\phi)$. The change of the signs of w_s transforms $v_s(t, \varepsilon)$ into $-v_s(t, -\varepsilon)$. Now we switch $-v_s(t, -\varepsilon)$ and transform $T_s = \{(t, \varepsilon)\}$ into $T'_s = \{(t, -\varepsilon)\}$ for $s \in S_0$. Obviously, after this transformation, we obtain, up to a switching, a canonical odd system $\mathcal{U}_1(\phi')$ with another map ϕ'.

Now we take a family $\mathcal{S}$ containing F_7 and make a permutation π of F_7 which transform $\mathcal{S}$ into $\mathcal{S}_0$, and simultaneously change correspondence of groups to $w_{\pi s}$ in (iii) of definition of ϕ. The assertion of the lemma follows. □

3.3 There are precisely 100 nonisomorphic two-graphs from $E_8 \oplus E_8$

According to Lemma 4, for to find all nonisomorphic two-graphs given by $E_8 \oplus E_8$, it is sufficient to consider the odd systems $\mathcal{U} = \mathcal{U}_1(\phi) \cup \mathcal{U}_2^0$. We denote the 8 vectors of $\mathcal{U}_2^0$ as follows:

$$u_0 = u(F_7) = g + W, \;\; u_i = u(S_i) = g - W + \sum_{s \ni i} w_s, \;\; i \in V_7.$$

The vector u_0 has the inner product $u_0 v = 1$ with all other $v \in \mathcal{U}$. In this case, the vertex u_0 is isolated in the graph $G(\mathcal{U})$, and the graph $G_0(\mathcal{U}) \equiv G(\mathcal{U} - \{u_0\})$ is a strongly regular graph with parameters (35,16,6,8).

The adjacency matrix of $G_0(\mathcal{U})$ has the following form

$$A = \begin{pmatrix} 0 & N \\ N^T & C \end{pmatrix}, \tag{5}$$

where C is the adjacency matrix of $\mathcal{U}_1(\phi)$ of order 28, and N is a matrix of order 7×28 which is obtained from the incidence 7×7 matrix of the Fano plane by complementing and by changing each 0 and 1 by a row of four 0 and four 1, respectively.

We use the list of two-graphs computed by Spence [10] for to prove that there are exactly 100 nonisomorphic two-graphs from the lattice $E_8 \oplus E_8$, i.e. two-graphs of the class E.

The Spence's list contains standard forms of the adjacency matrices A of graphs complementary to $G_0(\mathcal{U})$ in the decreasing lexicographical order. For brevity sake, we shall say that the Spence's list enumerate two-graphs in the decreasing lexicographical order, and call the adjacency matrix A by the adjacency matrix of the corresponding two-graph.

The Spence's list has the following properties:

(i) the adjacencies matrices of the first 100 two-graphs have equal the first 11 rows, and their complements are of the form (5),

(ii) the adjacency matrix of the two-graph number 101 has the 7th row distinct from the 7th rows of the adjacencies matrices of the first 100 two-graphs,

(iii) if there is a nonenumerated two-graph, then it is less than the two-graph number 225.

Note that (using the matrix A) it is not difficult to construct vectors of the odd system $\mathcal{U}$ representing any two-graph from the first 100 ones.

Proposition 2 *There are exactly 100 nonisomorphic regular two-graphs on 36 points from the lattice $E_8 \oplus E_8$, i.e. the class E contains exactly 100 two-graphs.*

Proof. Recall that each two-graph of the class E has the canonical representation by the odd system $\mathcal{U} = \mathcal{U}_1(\phi) \cup \mathcal{U}_2^0$. This representation provides the strongly regular graph $G_0(\mathcal{U})$ with the adjacency matrix of the form (5). The first 7 rows of the matrix correspond to the vectors of the odd system $\mathcal{U}_2^0 - \{u_0\}$.

We use the Spence's list [10]. It is not difficult to construct an odd system of the type $\mathcal{U}_1(\phi)\cup\mathcal{U}_2^0$ using the adjacency matrix of any of the first 100 two-graphs of the Spence's list. Hence, the first 100 two-graphs of the Spence's list belong to the class E.

Now we show that there is no other two-graph in the class E. Suppose to the contrary that there is a two-graph of the class E which is not in the Spence's list. The corresponding strongly regular graph has the adjacency matrix A_0 of the form (5). By the property (i) of the Spence's list, the first 7 rows of the matrices of the first 100 two-graphs of the Spence's list coincide with the first 7 rows of the complement of A_0. The property (ii) of the Spence's list implies that the adjacency matrix of the 101st is lexicographically less than the complement of the matrix A_0. But this contradicts to the property (iii) of the Spence's list. $\square$

3.4 Odd systems representing the two-graph $\mathcal{T}(3,7)$

The projection of the vector $u(S)$ on the space spanned by w_s, $s\in F_7$, is the vector $w(S):=\sum_{s\in S}w_s-W=\frac{1}{2}\sum_{s\in S}w_s-\frac{1}{2}\sum_{s\notin S}w_s$. The norm of the vector $\sqrt{2}w(S)$ equals 7. Moreover, for $u(S)\in\mathcal{U}_2$, all the corresponding 8 vectors $\sqrt{2}w(S)$ have mutual inner products –1 and span the 7-dimensional space with the basis $(w_s, s\in F_7)$. Hence these vectors represent a two-graph $\mathcal{T}(7,7)$. Since, for $(m,d)=(7,7)$ the special bound gives $n(7,7)=8$, the two-graph $\mathcal{T}(7,7)$ is the unique regular two-graph with these parameters.

Recall that the projection of $\mathcal{U}_1(\phi)$ on the 7-dimensional space with the basis $(f_i, i\in V_7)$ represent the unique two-graph $\mathcal{T}(3,7)$. Hence one can say that every two-graph $\mathcal{T}(5,15)$ obtained from the lattice $E_8\oplus E_8$ is a special gluing by the map ϕ of the unique two-graphs $\mathcal{T}(3,7)$ and $\mathcal{T}(7,7)$.

Let ε, ε', ε'' take values ±1. We call the triple $(\varepsilon,\varepsilon',\varepsilon'')$ *even* if the product $\varepsilon\varepsilon'\varepsilon''=1$. Otherwise the triple is called *odd*. There are 4 even triples and 4 odd triples. We set $\varepsilon^0=(1,1,1)$, $\varepsilon^1=(1,-1,-1)$, $\varepsilon^2=(-1,1,-1)$, $\varepsilon^3=(-1,-1,1)$. Let $0\le k\le 3$. Then ε^k is an even ε-triple, and $-\varepsilon^k$ is an odd ε-triple. Besides if we change the sign of one of the units in ε^k, we obtain an odd ε-triple $-\varepsilon^l$ for some l. For two ε-triples ε and ε' with the same support t, let $\varepsilon\varepsilon'=\sum_{i\in t}\varepsilon_i\varepsilon'_i$. Hence ε and ε' are of the same parity if and only if $\varepsilon\varepsilon'=-1$.

We use below sums of the type $u(t,\varepsilon^k)=\sum_{i\in t}\varepsilon^k_i f_i$. In such a sum, we consider $t\in F_7$ as an ordered triple ijl such that $1\le i<j<l\le 7$, and the orders of ε^k and t agree. For example, $\varepsilon^2=(-1,1,-1)$ in $\sum_{i\in 237}\varepsilon^2_i f_i$ means that $\varepsilon^2_2=-1$, $\varepsilon^2_3=1$, and $\varepsilon^2_7=-1$.

Consider the odd subsystem $\mathcal{W}^+\subseteq\mathcal{W}(3,7)$ consisting of 28 vectors $u(t,\varepsilon)$, $t\in F_7$, with all even ε. Recall that there are 4 vectors $u(t,\varepsilon^k)$, $0\le k\le 3$, with even ε^k for each $t\in F_7$.

Consider the graph $G(\mathcal{W}^+)$ with 28 vertices $u(t,\varepsilon)$. Recall that vertices $u,u'\in\mathcal{W}^+$ are adjacent in $G(\mathcal{W}^+)$ if and only if $uu'=-1$. The 4 vertices $u(t,\varepsilon^k)$, $0\le k\le 3$, for a fixed t form a clique of the graph $G(\mathcal{W}^+)$. Recall that the triangular graph $T(8)$ is a graph with P_8 as the set of vertices. Here and below P_8 is the set of all pairs of points of a 8-set. Two vertices of $T(8)$ are adjacent if and only if the corresponding pairs intersect.

Proposition 3 *The graph $G(\mathcal{W}^+)$ is the complement of the triangular graph $T(8)$.*

Proof. It is easy to verify that $G(\mathcal{W}^+)$ is a strongly regular graph with the parameters (28,15,6,10). These parameters have only 4 strongly regular graphs, namely, the complements of the triangular graph $T(8)$ and the 3 Chang graphs.

We show that $\overline{G(\mathcal{W}^+)} = T(8)$. Note that if we change the signs of some f_i in all vectors $u(t,\varepsilon^k)$, the adjacencies of $G(\mathcal{W}^+)$ do not change. We can transform any vertex $u(t,\varepsilon^k)$ into $u(t,\varepsilon^0)$ by changing signs of some f_i's. Hence w.l.o.g. we can consider the vertex $u(t,\varepsilon^0) = \sum_{i\in t} f_i$ as a general point. This vertex is adjacent in $\overline{G(\mathcal{W}^+)}$ to the 6-clique composed by the vertices $u(s,\varepsilon^0)$, $s \in F_7 - \{t\}$. But every vertex is adjacent to a 6-clique only in $T(8)$, but not in Chang graphs (see [1]). □

Recall that vertices of $T(8)$ are naturally labeled by pairs $ij \in P_8$. Using Proposition 3 we can label vectors of the odd system $\mathcal{W}^+$ by pairs $ij \in P_8$. Hence we denote these vectors of norm 3 by u_{ij}. Since $G(\mathcal{W}^+) = \overline{T(8)}$, we have

$$u_{ij}u_{kl} = \begin{cases} -1 & \text{if } \{ij\} \cap \{kl\} = \emptyset, \\ 1 & \text{if } |\{ij\} \cap \{kl\}| = 1. \end{cases}$$

3.5 Root systems of two-graphs from $E_8 \oplus E_8$

Recall that we call the class of all two-graphs from $E_8 \oplus E_8$ by the class E. Among the 100 two-graphs of the class E there is a unique two-graph $\mathcal{T}_0(5,15)$ with the empty root system. This is the two-graph nr.1 in the Spence's list.

Denote by $\mathcal{W}_0$ the odd system representing $\mathcal{T}_0(5,15)$. Each odd system representing a two-graph of the class E is a perturbation of the odd system $\mathcal{W}_0$. Moreover, an inspection of the Spence's list shows that the closure cl$\mathcal{U}$ of every odd system $\mathcal{U}$ representing a two-graph of the class E contains $\mathcal{W}_0$ as a subsystem. This fact allows characterize the root systems which are occur as root systems $R(\mathcal{U})$ of odd systems $\mathcal{U}$ representing two-graphs of the class E.

It is shown in [7] that $\mathcal{W}_0$ is closed, since it can be obtained by our construction from the Barnes-Wall lattice Λ_{16}. Namely, $\mathcal{W}_0 = \mathcal{V}(c)$ for every vector $c \in \sqrt{2}\Lambda_{16}$ of norm 12. Since the odd system $\mathcal{W}_0$ has no pair of vectors with the inner product ± 3, the root system of $\mathcal{W}_0$ is empty, i.e. $R(\mathcal{W}_0) = \emptyset$.

The odd system $\mathcal{W}_0 = \mathcal{U}_1(\phi_0) \cup \mathcal{U}_2^0$, where $\mathcal{U}_1(\phi_0)$ consists of vectors $v_t(t,\varepsilon^k)$, $t \in F_7$, $0 \le k \le 3$. For these vectors, we introduce the special denotation $u_t(\varepsilon^k)$. So

$$u_t(\varepsilon^k) = w_t + \sum_{i\in t} \varepsilon_i^k f_i,\ t \in F_7,\ 0 \le k \le 3. \tag{6}$$

Call minimal by inclusion dependencies of $\mathcal{W}_0$ by *circuits.* A circuit of $\mathcal{W}_0$ consists of 6 vectors such that a sum of these vectors or its opposites is equal to 0. In the graph $G(\mathcal{W}_0)$, a circuit generates a switching of a maximal clique (of size 6). In [8], the graph $G(\mathcal{W}_0)$ is considered in details.

Any 5 vectors of a circuit compose a *broken circuit.* Obviously, the sum of 5 vectors of a broken circuit is the 6th vector with an opposite sign. In other words, a broken circuit generates a vector of the lattice $L_1(\mathcal{W}_0)$.

An odd system $\mathcal{U}$ representing $\mathcal{T}(5,15)$ is obtained from $\mathcal{W}_0$ by a substitution of some vectors. In this case, if a vector v of a circuit is substituted by v', then this circuit ceases to be a dependency, it becomes a broken circuit. But this broken circuit generates v, i.e. v belongs to cl$\mathcal{U}$. Besides $vv' = \pm 3$, i.e. if $vv' = 3$, the vector $v - v'$ is a root of $R(\mathcal{U})$.

We use circuits of $\mathcal{W}_0$ of the following form

$$\{u_a, u_b, u_s(\varepsilon(s)) : s \in C_i\},$$

where $C_i = \overline{S_i}$ is the 4-set of triples $s \in F_7$ not containing $i \in V_7$, and the pair $a, b \in V_8$ is such that either $a = 0$, $b = i$ or the triple abi belongs to F_7, i.e. it is one of the triples of F_7 containing i. Each point $j \in V_7 - \{i\}$ belongs to exactly two $s, s' \in C_i$. Hence the ε-triples $\varepsilon(s)$ and $\varepsilon(s')$ are such that $\varepsilon_j(s) = \pm\varepsilon_j(s')$, where the sign – corresponds to all j in the case $(a,b) = (0,i)$, and to $j = a, b$ in the cases $(a,b) \neq (0,i)$, and the sign + corresponds to the other cases. The corresponding dependencies are

$$u_i - u_0 + \sum_{s \in C_i} u_s(\varepsilon(s)) = 0 \text{ and } u_a + \sum_{s \ni a, s \not\ni b} u_s(\varepsilon(s)) = u_b + \sum_{s \ni b, s \not\ni a} u_s(\varepsilon(s)). \qquad (7)$$

Broken circuits generate roots of $R(\mathcal{U})$ of the following types

$$w_s \pm w_{s'},\ s, s' \in F_7,\ 2f_i,\ i \in V_7,\ \sum_{i \in \overline{s}} \sigma_i f_i,\ \sigma \in \{\pm\}^{\overline{s}},\ s \in F_7,$$

where $\overline{s} = V_7 - s$. These roots are just the roots of the odd system $\mathcal{V}(c_0)$.

For $t \in F_7$ we consider the 4-set $V_4 = V_7 - t$ in details. Since F_7 is a Steiner triple system, each pair i, k of points of V_7 belongs to exactly one triple of F_7. In other words, the unordered pair ik determines uniquely a triple of F_7. Since there are 6 distinct pairs of points in V_4, we obtain all other 6 triples of F_7 distinct from t. These 6 triples are partitioned into 3 pairs of triples having the same intersection point with t. For $m \in t$, let $p(m)$ and $q(m)$ be the pair of triples with $p(m) \cap t = q(m) \cap t = \{m\}$. Since $p(m) \cap q(m) = \{m\}$, the triples $p(m)$ and $q(m)$ are determined by complementary pairs of points of the 4-set V_4. In other words, each point $m \in t$ determines uniquely both a partition of V_4 into complementary pairs ij, kl and the corresponding triples $p(m)$ and $q(m)$.

Call a quadruple $\sigma \in \{\pm 1\}^{V_4}$ *even* if $\prod_{j \in V_4} \sigma_j = 1$. There are 4 pair of opposite even quadruples. We set $\overline{t} = V_4 = V_7 - t$, and define the following root systems.

$$R(t) = \{\sum_{i \in \overline{t}} \sigma_i f_i : \sigma \in \{\pm 1\}^{\overline{t}} \text{ is even}\} \cup \{\pm(w_{p(m)} - w_{q(m)}) : m \in t\}, \qquad (8)$$

$$R^D(t) = \{\sum_{i \in \overline{t}} \sigma_i f_i : \sigma \in \{\pm 1\}^{\overline{t}}\} \cup \{\pm(w_{p(m)} \pm w_{q(m)}) : m \in t\}, \qquad (9)$$

$$R_0 = \{\pm 2f_i : i \in V_7\}. \qquad (10)$$

Recall that $S_a = \{s \in F_7 : s \ni a\}$, $C_a = F_7 - S_a$.

Lemma 5 *Let $\mathcal{U}$ be an odd system representing a two-graph of the class E. Let $cl\mathcal{U} \supseteq \mathcal{W}_0$. Then*

(i) if one of the roots of $R(t)$ belongs to $R(\mathcal{U})$, then $R(t) \subseteq R(\mathcal{U})$,

(ii) if at least one of the roots from each $R(t)$ and $R(t')$ belong to $R(\mathcal{U})$, then $\cup_{s\in S_a} R(s) \subseteq R(\mathcal{U})$, where $\{a\} = t \cap t'$,

(iii) if at least one of the roots from each $R(t)$, $R(t')$ and $R(t'')$ with $t \cap t' \cap t'' = \emptyset$ belong to $R(\mathcal{U})$, then $\cup_{s\in F_7} R(s) \subseteq R(\mathcal{U})$.

Proof. (i) For $m \in t$, let ε^k and ε^l with the supports $p(m)$ and $q(m)$, respectively, be such that $\varepsilon^k_m = \varepsilon^l_m$. Then we have the following identity

$$u_{p(m)}(\varepsilon^k) - u_{q(m)}(\varepsilon^l) = w_{p(m)} - w_{q(m)} + \sum_{i\in\overline{t}} \sigma_i f_i,$$

where $\sigma_i = \varepsilon^k_i$ for $i \in p(m) - \{m\}$, $\sigma_i = -\varepsilon^l_i$ for $i \in q(m) - \{m\}$. Since ε^k and ε^l are even, we obtain that σ is even, too.

By the definition of the lattice $L^0(\mathcal{U})$, the sum of any two vectors of $\mathcal{U}$ belongs to $L^0(\mathcal{U})$. Since any root of $R(\mathcal{U})$ is a vector of $L^0(\mathcal{U})$ of norm 4, the assertion of (i) is implied by the above identity.

(ii) By (i), $R(t) \cup R(t') \subseteq R(\mathcal{U})$. Let t'' be the third triple of S_a. Note that $\overline{t}\Delta\overline{t'} = \overline{t''}$, where Δ is the symmetric difference. For any σ'' with the support $\overline{t''}$, there are even σ and σ' with the supports $\overline{t}$ and $\overline{t'}$, respectively, such that

$$\sum_{i\in\overline{t''}} \sigma''_i f_i = \sum_{i\in\overline{t}} \sigma_i f_i + \sum_{i\in\overline{t'}} \sigma'_i f_i.$$

Hence the root $\sum_{i\in\overline{t''}} \sigma''_i f_i$ of $R(t'')$ belongs to $R(\mathcal{U})$. Now (i) implies (ii).

Similarly, (iii) is implied by (i) and (ii). $\Box$

Note that, for $k \neq 0$, $u_s(\varepsilon^0) - u_s(\varepsilon^k) = 2f_i + 2f_j \in L^0(\mathcal{U})$ for some $i, j \in s$. Hence one can prove the following lemma similarly as Lemma 5 was proved.

Lemma 6 *Let $\mathcal{U}$ be an odd system representing a two-graph of the class E. Let $cl\mathcal{U} \supseteq \mathcal{W}_0$. Then*

(i) if one of the roots of R_0 belongs to $R(\mathcal{U})$, then $R_0 \subseteq R(\mathcal{U})$,

(ii) if at least one of the roots from each R_0 and $R(t)$ belongs to $R(\mathcal{U})$, then $R_0 \cup R^D(t) \subseteq R(\mathcal{U})$,

(iii) if at least one of the roots from each R_0, $R(t)$ and $R(t')$ with $t \cap t' = \{a\}$ belong to $R(\mathcal{U})$, then $R_0 \cup_{s\in S_a} R^D(s) \subseteq R(\mathcal{U})$,

(iv) if at least one of the roots from each R_0, $R(t)$, $R(t')$ and $R(t'')$ with $t \cap t' \cap t'' = \emptyset$ belong to $R(\mathcal{U})$, then $R_0 \cup_{s\in F_7} R^D(s) \subseteq R(\mathcal{U})$. $\Box$

We have the following congruences:

$$R(t) \cong A_1^7, \ \cup_{t\in S_a} R(t) \cong A_2A_3^3, \ \cup_{t\in F_7} R(t) \cong A_6A_7,$$

$$R_0 \cong A_1^7, \ R_0 \cup R^D(t) \cong A_1^9 D_4, \ R_0 \cup_{t\in S_a} R^D(t) \cong A_1A_3D_4D_6, \ R_0 \cup_{t\in F_7} R^D(t) \cong D_7E_7.$$

Lemmas 5 and 6 imply the following assertion (cf. with Proposition 10 of [8]).

Proposition 4 *Any two-graph of the class E has one of the following root systems:*
(1) $\emptyset$,
(2) A_1^7,
(3) $A_2A_3^3$,
(4) A_6A_7,
(5) $A_1^9D_4$,
(6) $A_1A_3D_4D_6$,
(7) D_7E_7. $\square$

Note that cl$\mathcal{U}$ is uniquely determined by $\mathcal{W}_0$ and $R(\mathcal{U})$. Hence if $R(\mathcal{U}) = R(\mathcal{U}')$, then cl$\mathcal{U}$ =cl$\mathcal{U}'$, and if $R(\mathcal{U}) \cong R(\mathcal{U}')$, then cl$\mathcal{U}$ and cl$\mathcal{U}'$ are isomorphic.

4 Two-graphs from $E_8 \oplus E_8$ related to Steiner triple systems

4.1 Steiner triple systems with a head

Recall that a Steiner triple system (STS for brevity) on a set V is a family T of triples of points of V such that any pair of points of V is contained exactly in one triple of T. This implies that any two triples of T have at most one common point.

There is a unique STS F_7 on 7 points. There are 23 nonisomorphic STS on 15 points that have F_7 as a subsystem. The subsystem F_7 is called a *head* of an STS containing it.

Let V_7 be the ground set of the STS F_7, and let V_8 be the set of other points of an STS on 15 points with the head F_7. Each point s of V_7 occurs with a pair of points of V_8 in some $t \in T$. Recall that P_8 is the set of all $\binom{8}{2} = 28$ unordered pairs of points of V_8, i.e. the set of all edges of the complete graph K_8. Since the triples of T containing s intersect in the point s, the pairs of P_8 containing in triples with s are disjoint, i.e. the corresponding edges of K_8 form a factor. Hence the 7 points of V_7 determine a partition B of P_8 into 7 blocks $b \in B$, each containing 4 disjoint pairs of P_8. In other words, the 7 points determine a factorization of K_8. Two partitions (factorization) B and B' of P_8 (K_8, respectively) are isomorphic if there is a permutation of the set V_8 transforming the blocks (factors) of B into the blocks (factors) of B'. There are 6 nonisomorphic partitions of P_8 (i.e. factorizations of K_8) (see [12]).

So, each STS on 15 points with a head is determined by a partition B of P_8 into blocks, and by a bijection $\pi : V_7 \to B$ between the 7-set V_7 and the set of 7 blocks $b \in B$. The triples of such an STS are as follows:

there are 7 triples of F_7, the triples of the head,

there corresponds 4 triples sij, $ij \in \pi(s)$, to each point $s \in V_7$.

We have $7 + 4 \times 7 = 35$ triples in total.

Note that subsystems F_7 of all STS with a head in the list of [2] are not only isomorphic but are the same. We set $V_7 = \{1, 2, ..., 7\}$, $V_8 = \{8, 9, ..., 15\}$, and use the list of STS's with a head given in the extended version of [2]. The head F_7 of STS's of the list consists of the triples t_i, $i \in V_7$, where

$$t_1 = 123,\ t_2 = 145,\ t_3 = 167,\ t_4 = 246,\ t_5 = 257,\ t_6 = 347,\ t_7 = 356. \tag{11}$$

An inspection of the list of STS's with a head shows that the STS's nr.1, 2, 3, and 16 are determined by the same partition B_0 of P_8. The blocks of the partition B_0 are given in Table 1. The first number in each block is the point $s \in V_7$ that corresponds to this block in STS nr.1.

Table 1.

1	8	9	2	8	10	3	8	11	4	8	12	5	8	13	6	8	14	7	8	15
	10	11		9	11		9	10		9	13		9	12		9	15		9	14
	12	13		12	14		12	15		10	14		10	15		10	12		10	13
	14	15		13	15		13	14		11	15		11	14		11	13		11	12

If we identify the set of blocks with V_7 by the bijection of STS nr.1, then the bijection π is a permutation of blocks $b \in B_0$. The permutations are given in the table 2.

Table 2.

nr. of STS	π	root system
1	identity	$\emptyset$
2	(67)	A_1^7
3	(567)	$A_2A_3^3$
16	(3475)	A_6A_7

Every STS is represented by a reduced odd system $\{v_0\} \cup \mathcal{U}(T)$ such that if $v(t) \in \mathcal{U}(T)$ represents $t \in T$, then

$$v(t)v(t') = 2|t \cap t'| - 1. \tag{12}$$

Besides $v_0 v(t) = 1$ for all $t \in T$. (See, for example, [7], [8]). Note that $R(\{v_0\} \cup \mathcal{U}(T)) = R(\mathcal{U}(T))$. We denote $R(\mathcal{U}(T))$ by $R(T)$.

It is shown in [7] that 23 STS's with a head are partitioned into 4 families having the root systems shown in the last column of Table 2.

The family with $R(T) = \emptyset$ contains only one STS having nr.1 in the extended version of [2]. Taylor proves [11] that there is a unique two-graph $\mathcal{T}_0(5,15)$ with a doubly transitive automorphism group. The full automorphism group of $\mathcal{T}_0(5,15)$ is $Sp(6,2)$. It is shown in [2] that the two-graph $\mathcal{T}_0(5,15)$ relates to the unique Steiner triple system nr.1, triples of which are lines of a 3-dimensional projective space $PG(3,2)$ over the field GF_2.

The family with $R(T) = A_1^7$ contains also one STS, the STS nr.2. The corresponding two-graph is denoted by $\mathcal{T}_1(5,15)$.

The family with $R(T) = A_2A_3^3$ contains 5 STS's with numbers nr.3–7.

The family with $R(T) = A_6A_7$ contains 16 STS's with nr.8–22 and 61.

I take the opportunity to correct a misprint in Table 1 of [7], where, in row 4, nr. 61 should be instead of nr. 67. The row 5 of Table 1 in [7] should be corrected accordingly.

Using Table 1 we can give an explicit correspondence between the vectors u_{ij} and $u(t, \varepsilon^k)$ of the odd system $\mathcal{W}^+$. Let the 4 rows of Table 1 be numbered by k, $0 \le k \le 3$. Let ij lies in the row k of Table 1 and belongs to the block $s \in B_0$. (Recall that we identify B_0 and V_7). Then the vector u_{ij} corresponds to the vector $u(t_s, \varepsilon^k)$ where t_s, $s \in V_7$, is given in (11). We can say that the block $s \in B_0$ contains the 4 vectors $u(t_s, \varepsilon^k)$ for $0 \leq k \leq 3$.

4.2 Positive odd systems representing STS's with a head

In this section we consider odd systems $\mathcal{U}_1(\phi)$ of a special form when all ε in $v_s(t, \varepsilon)$ are even. In other words, we choose the reduced odd system $\mathcal{W}^+$ from $\mathcal{W}(3, 7)$ on the step (i) of the map ϕ. We call the obtained odd system $\mathcal{U}_1(\phi)$ *positive*.

Recall that by Proposition 3 there is a bijection between vectors $u(t, \varepsilon) \in \mathcal{W}^+$ and pairs $ij \in P_8$. The vector $u(t, \varepsilon)$ related to the pair ij is denoted by u_{ij}. The explicit correspondence between vectors $u(t, \varepsilon)$ and u_{ij} is given at the end of the previous section.

So we obtain that a positive odd system $\mathcal{U}_1(\phi)$ contains the vectors of the form $v_s(ij) = w_s + u_{ij}$, $s \in F_7$, $ij \in P_8$. Let $b_s = \{ij : v_s(ij) \in \mathcal{U}_1(\phi)\}$. Since $v_s(ij)v_s(kl) = w_s^2 + u_{ij}u_{kl} = 2 + u_{ij}u_{kl}$ should be equal to 1 or –1, we have to have $\{ij\} \cap \{kl\} = \emptyset$. Hence every set b_s contains 4 disjoint pairs. So we obtain again that every positive odd system $\mathcal{U}_1(\phi)$ is determined by a partition B of P_8 into 7 blocks of disjoint pairs and by a bijection π between the 7-set F_7 and blocks of the partition B. We denote it by $\mathcal{U}_1^+(\pi, B)$.

Call the odd system $\mathcal{U}^+(\pi, B) = \mathcal{U}_1^+(\pi, B) \cup \mathcal{U}_2^0$ *positive*, too. An odd system having $v_s(t, \varepsilon)$ with odd ε is called *nonpositive*. We show that a positive odd system represents a two-graph related to an STS with a head.

We relate to each vector of a positive odd system $\mathcal{U}^+(\pi, B)$ a triple t of an STS, and denote this vector by $v(t)$. We show that the obtained vectors satisfy (12).

We set $v_0 = u_0 = g + W$. As a ground set of this STS we take the union $F_7 \cup V_8$. Here V_8 is a 8-set whose pairs compose P_8. We set

$$v(S) = u(S) \text{ for a triple } S \subseteq F_7, S \in \mathcal{F}_7,$$

$$v(sij) = v_s(ij) = w_s + u_{ij}, \;\; s \in F_7, ij \in P_8.$$

By the construction, the set of 35 triples $S \in \mathcal{F}_7$, and sij, $s \in F_7$, $ij \in P_8$, forms an STS with the head $\mathcal{F}_7$. It is easy to verify, in particular using (3), that $v(S)$ and $v(sij)$ satisfy (12). Conversely, using the list of STS's of [2], we see that above formulas for $v(S)$ and $v(sij)$, where S and sij are triples of an STS with a head, provide an odd system $\mathcal{U}^+(\pi, B)$. So we obtain

Proposition 5 *Every positive odd system $\mathcal{U}^+(\pi, B)$ from $E_8 \oplus E_8$ represents a two-graph related to an STS on 15 points with a head, and every STS with a head is represented by a positive odd system $\mathcal{U}^+(\pi, B)$.*

Now we consider the odd system $\mathcal{U}(\pi) := \mathcal{U}^+(\pi, B_0) = \mathcal{U}_1^+(\pi, B_0) \cup \mathcal{U}_2^0$. Here π is a permutation of elements of $V_7 \cong B_0$. Recall that each block $s \in B_0$ contains the 4 vectors $u(t_s, \varepsilon^k)$, $0 \leq k \leq 3$, where t_s is given in (11). Hence the odd system $\mathcal{U}_1^+(\pi, B_0)$ consists of vectors $w_s + u(t_{\pi(s)}, \varepsilon^k)$.

According to Table 2, the odd system with π equal to identity represents the two-graph $\mathcal{T}_0(5,15)$. Recall that we denoted this odd system by $\mathcal{W}_0$. If π is the transposition (67), then $\mathcal{U}^+((67),B_0)$ represents the STS nr.2 and the two-graph $\mathcal{T}_1(5,15)$.

Using broken circuits of the odd system $\mathcal{U}^+(\pi,B_0)$, and Lemma 5, one can prove the following proposition. We set $R(\mathcal{U}^+(\pi,B_0))=R(\pi)$, and recall that the root system $R(t)$ is defined in (8).

Proposition 6 *Let, for $i\in V_7$, t_i is given by (11). Let $\pi_2=(ij)$, and let t be the third triple of S_{m_0}, where $\{m_0\}=t_i\cap t_j$. Let $\pi_3=(ijk)$ be such that $V_7-(t_i\cup t_j\cup t_k)=\{a\}$. Let $\pi_4=(ijkl)$ be such that $t_i\cup t_j\cup t_k\cup t_l=V_7$. Then*
(i) $R(\pi_2)=R(t)\cong A_1^7$,
(ii) $R(\pi_3)=\cup_{s\in S_a}R(s)\cong A_2A_3^3$,
(iii) $R(\pi_4)=\cup_{s\in F_7}R(s)\cong A_6A_7$. $\square$

Besides, we obtain that $\mathrm{cl}\mathcal{U}(\pi_2)$ contains, excepting the vectors of $\mathcal{U}(\pi_2)$, the vectors $u(D_i)$ for $D_i=F_7-(S_i\cup\{t\})$ and $i\in V_4$, $u_s(\varepsilon)$, $u_{s'}(\varepsilon)$, $v_{p(m)}(q(m),\varepsilon)$ and $v_{q(m)}(p(m),\varepsilon)$ for even ε, $m\in t$, $m\neq m_0$. So, we have $4+6\times 4=28$ additional vectors in $\mathrm{cl}\mathcal{U}(\pi_2)$.

It is proved in [8] that $\mathrm{cl}\mathcal{U}(\pi_2)$ is isomorphic to the odd system $\mathcal{W}_1=\mathrm{cl}\mathcal{U}^0(\{t\})$ consisting of 8 vectors u_i, $0\le i\le 7$, and 56 vectors $u_s(\varepsilon)$, $\varepsilon\in\{\pm 1\}^s$, $s\in F_7$. $\mathcal{U}^0(\{t\})$ represents the two-graph $\mathcal{T}(\{t\})$ (see below).

5 Two-graphs represented by nonpositive odd systems

5.1 Two-graphs $\mathcal{T}(S)$

Consider a reduced nonpositive odd system $\mathcal{U}_1(\phi)$ of the following type. Let $S\subseteq F_7$. We choose, on the step (i) of the map ϕ vectors $u(t,\varepsilon)\in\mathcal{W}(3,7)$ with odd (even) ε if $t\in S$ ($t\notin S$, respectively).

We denote the odd system $\mathcal{U}_1(\phi)$ for this map ϕ by $\mathcal{U}_1(S)$. The odd system $\mathcal{U}_1(S)$ consists of vectors $v_s(t,\varepsilon)=w_s+\sum_{i\in t}\varepsilon_i f_i$, where ε is either odd or even according to t belongs or do not belong to S. We denote by $\mathcal{U}(S)$ the union of $\mathcal{U}_1(S)$ with $\mathcal{U}_2^0$.

We define a transformation of S. Recall that there are 3 triples $s\in F_7$ containing a given point i. For each i, $1\le i\le 7$, consider triples $s\in S$ containing i. If there are 3 such triples, then delete them from S. If there are two such triples, then change them by the third triple containing i. If there is one or no triple containing i, then S is not transformed. Obviously, after such transformations for all i, we obtain S with either one or none of the triples. We call S *positive* if it is transformed into empty set, and *negative*, otherwise.

Proposition 7 *Let $S\subseteq F_7$ be a set of triples $s\in F_7$. Then $\mathcal{U}(S)$ is isomorphic to $\mathcal{U}(\emptyset)$ or $\mathcal{U}(\{s\})$ according to S is positive or negative, respectively.*

Proof. Note that if we change in $\mathcal{U}_1(S)$ vectors f_i for some i by $-f_i$, we obtain an isomorphic odd system. This change of the sign of f_i is equivalent to the change of the sign before ε_i^k for all k, $0 \le k \le 3$. In other words, the change f_i by $-f_i$ is equivalent to the change of the even ε-triple in $u_s(\varepsilon^k)$ with $s \ni i$ by an odd ε-triple (and conversely).

So, if the set $S \subseteq F_7$ contains 3 triples containing the point i, we can eliminate these triples from S simultaneously transforming f_i into $-f_i$. If the set S contains two triples with i, we can change the two triples by the third triple containing i and transforming the f_i by $-f_i$. Since we do not change vectors w_s, the assertion of the proposition follows. □

Now we consider odd systems of the following form. On the step (ii), we set 4 vectors $u(t, \varepsilon^k)$, $0 \le k \le 3$, with the same t in the same group. Hence each group is naturally labeled by an element $t \in F_7$. On the step (iii), we relate the group with the label t to the vector w_t. We denote the obtained odd system by $\mathcal{U}_1^0(S)$. The odd system $\mathcal{U}_1^0(S)$ consists of vectors $u_s(\varepsilon)$ of (6). We denote by $\mathcal{U}^0(S)$ the union of $\mathcal{U}_1^0(S)$ with $\mathcal{U}_2^0$. We have

$$\mathcal{U}^0(S) = \{u_i : 0 \le i \le 7;\ u_t(-\varepsilon^k), 0 \le k \le 3, t \in S;\ u_t(\varepsilon^k), 0 \le k \le 3, t \notin S;\ \}.$$

$\mathcal{U}^0(S)$ represents a regular two-graph $\mathcal{T}(5,15)$. Denote this two-graph by $\mathcal{T}(S)$.

According to Proposition 7, two-graphs $\mathcal{T}(S)$ are only of the following two types: $\mathcal{T}(\emptyset)$ and $\mathcal{T}(\{s\})$, $s \in F_7$.

Clearly, $\mathcal{U}^0(\emptyset) = \mathcal{W}_0$ is positive, and $\mathcal{T}(\emptyset) = \mathcal{T}_0(5,15)$ relates to the STS nr.1.

Obviously, there is a permutation of the set V_7 that transforms any triple $s \in F_7$ in any other $s' \in F_7$. This permutation generates an isomorphism of odd systems $\mathcal{U}^0(\{s\})$ and $\mathcal{U}^0(\{s'\})$ and two-graphs $\mathcal{T}(\{s\})$ and $\mathcal{T}(\{s'\})$. Denote by $\mathcal{T}_1$ the two-graph represented by any of these isomorphic odd systems. We show in [7] that $\mathcal{T}_1$ relates to the Steiner triple system having nr.2 in the extended version of [2], i.e. $\mathcal{T}_1 = \mathcal{T}_1(5,15)$.

Consider the odd system $\mathcal{U}^0(\{t\}) = \mathcal{U}_1^0(\{t\}) \cup \mathcal{U}_2^0$. The vectors of $\mathcal{U}(\{t\})$ distinct from the vectors of $\mathcal{W}_0$ are $u_t(-\varepsilon^k)$, $0 \le k \le 3$. Any circuit of (7) with $C_i \ni t$ generates $u_t(\varepsilon^k)$ for some k. Since there are 4 choices of $\varepsilon(s)$, we obtain $u_t(\varepsilon^k) \in$cl$\mathcal{U}(\{t\})$ for all k, $0 \le k \le 3$.

For $k \ne 0$, we have that the vector

$$u_t(\varepsilon^0) - u_t(-\varepsilon^k) = \sum_{j \in t} f_j + \sum_{j \in t} \varepsilon_j^k f_j = 2f_l$$

is a root. Here l is such that $\varepsilon_l^k = 1$ and $\varepsilon_j^k = -1$ for $j \ne l$. So, we obtain 3 roots $2f_j$ for $j \in t$.

Now, consider the vectors $u_s(\varepsilon^k) \in \mathcal{U}(\{t\})$ for $s \ne t$. Let $\{j\} = s \cap t$. Then the vector $u_s(\varepsilon^k) - 2\varepsilon_j^k f_j$ of norm 5 belongs to cl$\mathcal{U}(\{t\})$ and has the form $u_s(-\varepsilon^l)$ for some l. Then, as above, we can obtain the roots $2f_j$ for $j \in s$. Continuing in this way, we obtain the 7 mutually orthogonal roots $2f_j$, $j \in V_7$. In this case, cl$\mathcal{U}(\{t\})$ contains vectors $u_s(\varepsilon)$ for all $8 = 2^3$ ε-triples $\varepsilon \in \{\pm 1\}^s$.

It is shown in [8] that there is no other vector in cl$\mathcal{U}(\{t\})$, i.e. cl$\mathcal{U}(\{t\})$ contains the following 64 vectors: u_i, $i \in V_8$, and $u_s(\varepsilon)$, $\varepsilon \in \{\pm 1\}^s$, $s \in F_7$. We denote this odd system consisting of these 64 vectors by $\mathcal{W}_1$.

5.2 Two-graphs distinct from two-graphs related to STS's

It is proved in [8] (Proposition 10) that any two-graph related to 2-(10,4,2) designs has one of the following 6 root systems

$$\emptyset,\ A_1^7,\ A_2A_3^3,\ A_1^9D_4,\ A_1A_3D_4D_6,\ A_5D_{10}.$$

An inspection of the list of two-graphs computed by Spence [10] shows that all these root systems are in fact root systems of two-graphs related to 2-(10,4,2) designs, i.e. two-graphs from D_{16}^+.

Two-graphs with the first 5 root systems can be obtained as from D_{16}^+ so from $E_8 \oplus E_8$. This fact is seen very clearly in the Spence's list. Moreover, one can give explicit representations of these two-graphs by odd subsystems as of $\mathcal{V}(c_0)$ so of $\mathcal{V}(c_{II})$. Such two representations are given in [8] for the unique two-graphs with the root systems $\emptyset$ and A_1^7.

Recall that two-graphs with the root systems $\emptyset$, A_1^7, and $A_2A_3^3$ are related to Steiner triple systems with a head.

In this section we give examples of two-graphs from $E_8 \oplus E_8$ distinct from two-graphs related to Steiner triple systems.

For to represent a two-graph distinct from STS-two-graph, the odd system $\mathcal{U}(\phi)$ should be nonpositive. It means that, on the step (i) of the definition of the map ϕ, we choose a reduced odd subsystem $\mathcal{U} \subseteq \mathcal{W}(3,7)$ containing vectors $u(t,\varepsilon)$ with odd ε. We consider the following case. If $\mathcal{U}$ contains a vector $u(s,\varepsilon)$ with odd ε, then $\mathcal{U}$ contains all the 4 vectors $u(s,\varepsilon)$ with all 4 odd $\varepsilon = -\varepsilon^k$, $0 \le k \le 3$. According to Proposition 7 it is sufficient to consider the case when ε in $u(s,\varepsilon)$ is odd only for one $s \in F_7$

Let s is t_k from (11), and let π be such that $k = \pi(i)$ for $i \neq k$. Let $\mathcal{U}_s(\pi, B_0)$ be the odd system $\mathcal{U}^+(\pi, B_0)$, where the vectors $v_{t_i}(t_{\pi(i)}, \varepsilon)$ have odd ε if $t_{\pi(i)} = t_k = s$. We set $R_s(\pi) = R(\mathcal{U}_s(\pi, B_0))$. Using broken circuits of the odd system $\mathcal{U}_s(\pi, B_0)$ and Lemma 6, one can prove the following proposition. Recall that $R^D(t)$ and R_0 are defined in (9) and (10).

Proposition 8 *Let π_i, $i = 2,3,4$ be such as in Proposition 6. Then*
(i) $R_s(\pi_2) = R^D(t) \cup R_0 \cong A_1^9 D_4$,
(ii) $R_s(\pi_3) = \cup_{s \in S_a} R^D(s) \cup R_0 \cong A_1A_3D_4D_6$,
(iii) $R_s(\pi_4) = \cup_{s \in F_7} R^D(s) \cup R_0 \cong D_7E_7$. $\square$

6 Root systems of two-graphs on 36 points

Recall that the second even unimodular lattice D_{16}^+ provides two families of two-graphs, namely, the class STS consisting of two-graphs related to Steiner triple systems, and the class D of 100 two-graphs related to 2-(10,4,2) designs (see [8]). The two-graphs of the class STS have the following root systems: $\emptyset$, A_1^7, $A_2A_3^3$, A_6A_7, and A_{14}. The two-graphs related to 2-(10,4,2) designs have the following root systems: $\emptyset$, A_1^7, $A_2A_3^3$, $A_1^9D_4$, $A_1A_3D_4D_6$, A_5D_{10}.

An inspection of the Spence's list of 227 two-graphs shows that two-graphs from $E_8 \oplus E_8$, i.e. the two-graphs of the class E, having root systems distinct from A_6A_7 and D_7E_7 coincide with two-graphs of the class D.

Recall that we denote the class of two-graphs from Latin squares by LSQ, and that the two-graphs of the class LSQ have root systems A_1^9 and A_5^3 (see [8]).

We show the partition of all known regular two-graphs on 36 points into classes with the same root system in the following Table 3. The column "cardinality" of Table 3 shows the number of two-graphs having the corresponding root system.

The 100 two-graphs of the class D are marked in the Spence's list by letters **a**, **b**, and **c**.

The 80 two-graphs of the class STS are marked in the Spence's list by numbers in parentheses.

The 100 two-graphs of the class E have numbers from 1 to 100 in the Spence's list.

The two-graphs with the root systems $\emptyset$ and A_1^7 have the numbers 1 and 2 in the Spence's list. The next 4 two-graphs with the numbers 3–6 in the Spence's list have the root system $A_1^9D_4$. The unique two-graph with the root system A_1^9 (of the class LSQ) has the number 184 (89). (The number 89 in parentheses is the number of the two-graph in the list of two-graphs from Steiner triple systems and Latin Squares of the extended version of [2]).

Table 3.

Case	root system	cardinality	belong(s) to the class
1	$\emptyset$	1	STS, E, D
2	A_1^7	1	STS, E, D
3	$A_2A_3^3$	5	STS, E, D
4	A_6A_7	16	STS, E
5	A_{14}	57	STS
6	$A_1^9D_4$	4	E, D
7	$A_1A_3D_4D_6$	30	E, D
8	D_7E_7	43	E
9	A_5D_{10}	59	D
10	A_1^9	1	LSQ
11	A_5^3	10	LSQ

It seems to me that all regular two-graphs on 36 points are known. The lexicographical order of two-graphs in the Spence's list shows that all two-graphs of the classes E and D are known. The list contains all two-graphs related to Steiner triple systems and Latin squares. It is very unlikely that there is a new class of regular two-graphs on 36 points distinct from the classes STS, LSQ, E and D.

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