# Extremal problems for affine cubes of integers

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**Abstract**

A collection $H$ of integers is called an *affine $d$-cube* if there exist $d+1$ positive integers $x_0, x_1, \ldots, x_d$ so that

$$H = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq \{1, 2, \ldots, d\} \right\}.$$

We address both density and Ramsey-type questions for affine $d$-cubes.

Regarding density results, upper bounds are found for the size of the largest subset of $\{1, 2, \ldots, n\}$ not containing an affine $d$-cube. In 1892 Hilbert published the first Ramsey-type result for affine $d$-cubes by showing that for any positive integers $r$ and $d$, there exists a least number $n = h(d, r)$ so that for any $r$-coloring of $\{1, 2, \ldots, n\}$, there is a monochromatic affine $d$-cube. Improvements for upper and lower bounds on $h(d, r)$ are given for $d > 2$.

## 1 Introduction

In this section, we give a very brief survey of results related to partition and density problems for affine cubes of integers. For a more detailed survey of these extremal problems, see [12], for example. In Section 2, we state two main results (Theorems 2.1 and 2.2) and contrast them to known bounds. The proofs of these follow in the remaining sections, including some tools using extremal graph theory and a partition theorem for sets of integers not containing any arithmetic progression of length three.

For a set $X$ we use the standard notations $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ and $[X]^s = \{S \subset X : |S| = s\}$. It will often be convenient to use $X = [n] =$

$[1,n] = \{1, 2, \ldots, n\}$. An arithmetic progression of length $k$ will be denoted by $AP_k$.

## 1.1 Arithmetic progressions

One of the most well-known (Ramsey-type) partition theorem for sets of integers was published In 1927 [28] van der Waerden published his celebrated partition theorem for integers (see also [29] for history of its proof).

**Theorem 1.1** (van der Waerden) *For positive integers $d$ and $r$, there is a least number $n = W(d,r)$ so that for any partition of $[n]$ into $r$ classes, one class contains an $AP_d$.*

For more proofs of van der Waerden's theorem we refer the reader to any of, for example, [3], [7], [11], [15], [16], or [24].

At present, there is not much more known about general upper bounds for the function $\mathrm{W}(d,r)$, except that it is primitive recursive (see [24]). Behrend [1] proved a (lower bound) density result for subsets of $[n]$ not containing any $AP_3$. (See [8] and [9] for related results.)

**Theorem 1.2** (Behrend) *There exists a constant $c$ so that for $m$ sufficiently large, there exists an $AP_3$-free set $B \subset [m]$ with*

$$|B| \geq me^{-c\sqrt{\ln m}} = m^{1-o(1)}.$$

## 1.2 Affine cubes

In 1892, Hilbert [18] proved the first non-trivial partition Ramsey type theorem. This theorem, which preceded the celebrated Schur and van der Waerden theorems, states that in any finite coloring of the set of positive integers, there exists a monochromatic "affine $d$-cube" (defined below).

**Definition 1.3** *A collection $H$ of integers is called a* $d$-dimensional affine cube*, or more simply, an* affine $d$-cube *if there exist $d+1$ positive integers $x_0, x_1, \ldots, x_d$ so that*

$$H = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq [d] \right\}.$$

*If all sums are distinct, that is, if $|H| = 2^d$, then we say that the affine $d$-cube is* replete.

If an affine $d$-cube $H$ is generated by $x_0, x_1, \ldots, x_d$, then we write $H = H(x_0, x_1, \ldots, x_d)$. For example, $H(1,1,1) = \{1,2,3\}$, while a replete affine 2-cube is $H(1,3,9) = \{1,4,10,13\}$. We note that $H(x_0, x_1, \ldots, x_d)$ may differ from, say, $H(x_1, x_0, \ldots, x_d)$.

The following finite version of Hilbert's theorem follows from Hilbert's original result by compactness.

**Theorem 1.4** (Hilbert) *For every $r$, $d$, there exists a least number $h(d,r)$ so that for every coloring*

$$\chi : [h(d,r)] \to [1,r],$$

*there exists an affine $d$-cube monochromatic under $\chi$.*

The original statement of the theorem asserted that if one colored the positive integers with finitely many colors, then one color class contained a (monochromatic) $d$-cube. Theorem 1.4 follows from van der Waerden's Theorem since an $AP_{d+1}$ given by $\{a, a+k, a+2k, \ldots, a+dk\}$ is also an affine $d$-cube $H(a,k,k,\ldots,k)$.

There have been many, in some sense, strengthened versions of Hilbert's result by eliminating the $x_0$ in Definition 1.3. For a set $\{x_1, \ldots, x_d\}$, the set of all finite non-empty sums $\{\sum_{i\in I} x_i : I \subset [1,d], I \neq \emptyset\}$ is sometimes called a *projective $d$-cube* (see, for example, [20]) or *finite sum* set. For an infinite set $\{x_1, x_2, \ldots\}$, the set of all finite non-empty sums is said to be an infinite-dimensional projective cube or a *Hindman set*.

Rado [21], and later independently, Sanders [22] and Folkman (see [11] or [13]), showed that for any $r$ and $d$, there exists a least number $n = \mathrm{FRS}(r,d)$ so that for any partition of $[1,n]$ into $r$ classes, one class contains a projective $d$-cube. The case $d = 2$ is the celebrated Schur's Theorem [23]. Hindman [19] settled a conjecture of Graham and Rothschild [14] by showing that upon any finite coloring of the positive integers, there exists a monochromatic infinite-dimensional projective cube. Any known upper bounds on $\mathrm{FRS}(r,d)$ are tower functions (see [27] or [12] for discussion) and for large $d$, there is a huge gap between upper and lower bounds; as we will see, one can do much better for $h(d,r)$.

## 2 Results

In 1969, Szemerédi [26] proved that if a set $A$ of positive integers has positive upper density, (i.e., $\overline{\lim_{n\to\infty}}|A \cap [1,n]| > 0$) then $A$ contains an $AP_4$. In [26] (Lemma $p(\delta, l)$, p. 93), Szemerédi gave a density version of Theorem 1.4; explicit bounds which follow directly from his argument are also mentioned in [11] (cf. also [16]).

**Theorem 2.1** (Szemerédi) *For each $d$ there exists a constant $c$ so that for sufficiently large $n$, if $A \subset [n]$ and*

$$|A| \geq cn^{1-\frac{1}{2^d}},$$

*then $A$ contains an affine $d$-cube.*

We note that the "2" in the exponent has been typographically misprinted in at least one article [2] as an "e".

In Section 3 we give a new proof of Szemerédi's theorem based on extremal hypergraphs.

In [2] it was shown that $h(2,r) = (1+o(1))r^2$; the lower bound uses Singer sets (cyclic difference sets arising from a finite projective plane, see also [25]) and the upper bound follows from well known bounds for $B_2$-sets, or Sidon sets (see also [6]). Also in [2], it was noted that there exist constants $c_1$ and $c_2$ so that

$$r^{c_1 d} \leq h(d,r) \leq r^{c_2^d}, \tag{1}$$

where $c_2 \sim 2.6$ follows from Hilbert's original proof (using Fibonacci numbers).

Here, for $d > 2$, we improve both bounds found in (1).

**Theorem 2.2** *There exists a constant $c_4$, so that for any integers $d \geq 3$ and $r \geq 2$,*

$$r^{(1-o(1))(2^d-1)/d} \leq h(d,r) \leq c_4 r^{2^d}$$

*where $o(1) \to 0$ as $r \to \infty$.*

In Section 4 the upper bound in Theorem 2.2 is proved by showing that if $[1, c_4 r^{2^d}]$ is partitioned into $r$ parts, then at least one part is so large (by the pigeon-hole principle) that the density version (Theorem 2.1) guarantees an

affine $d$-cube in that part. The remainder of Section 4 is devoted to the proof of the lower bound in Theorem 2.2 . By definition, to prove that $n \leq h(d,r)$, we need to show that there exists an $r$-coloring of the set $[1,n]$ which prevents monochromatic affine $d$-cubes. Indeed, our proof relies on finding, for a given $d$ and $n$, as small as possible value for such an $r$, a "lower coloring bound".

# 3 Hypergraphs and Theorem 2.1

A *$d$-uniform hypergraph* is a pair $G = (V, \mathcal{E}) = (V(G), \mathcal{E}(G))$, with vertex set $V$ and hyperedge set $\mathcal{E} \subset [V]^d$. Note that by this definition, each $d$-set from $V$ may occur only once as a hyperedge. For pairwise disjoint sets $X_1, X_2, \ldots, X_d$, let

$$G = (X_1, X_2, \ldots, X_d, \mathcal{E}(G))$$

denote a $d$-partite $d$-uniform hypergraph on vertex set $V(G) = \cup_{i=1}^{d} X_i$ and edge set $\mathcal{E}(G) \subseteq [V(G)]^d$, where for each $E \in \mathcal{E}(G)$ and each $i = 1, \ldots, d$, $|E \cap X_i| = 1$ holds; the sets $X_1, \ldots, X_d$ will be called *partite* sets.

Let $K^{(d)}(n_1, n_2, \ldots, n_d)$ denote the complete $d$-partite $d$-uniform hypergraph on $\sum_{i=1}^{d} n_i$ vertices, partitioned into sets of sizes $n_1, n_2 \ldots, n_d$, and having $\prod_{i=1}^{d} n_i$ edges, each edge containing exactly one vertex from each partite set. The complete $d$-partite $d$-uniform hypergraph with two vertices in each partite set will be denoted by $K^{(d)}(2, 2, \ldots, 2)$. For any $d$-uniform hypergraph $H$, the maximum number of $d$-hyperedges in any $H$-free hypergraph on $n$ vertices is denoted by $\mathrm{ex}(n, H)$.

In 1964 Erdős [5] (cf. equation (4.2) in [10]) showed that there exists a universal constant $c < 1$ so that for each $d$ and sufficiently large $m$,

$$\mathrm{ex}(m, K^{(d)}(2, 2, \ldots, 2)) \leq c m^{d - \frac{1}{2^{d-1}}}. \tag{2}$$

For $d > 2$, at present there is still a gap between the lower and upper bounds for $\mathrm{ex}(n, K^{(d)}(2, 2, \ldots, 2))$ (see [17] for discussion).

**Proof of Theorem 2.1:** Fix $d \geq 2$. For a fixed $n$ (large, but to be determined later), and $A \subset [1,n]$ with $|A| \geq c_1 n^{1 - \frac{1}{2^d}}$ define a $d$-partite $d$-uniform graph $G = (X_1, X_2, \ldots, X_d, \mathcal{E}(G))$ as follows. Each partite set $X_i$ will be a distinct copy of the integers $[-\sqrt{n}, \sqrt{n}]$; hence $|X_1| = |X_2| = \ldots = |X_d| = 2\sqrt{n} + 1$. For any number $q \in [0, n-1]$, let $(s(q), t(q))$ be the unique pair of integers satisfying $0 \leq s(q), t(q) \leq \sqrt{n} - 1$ so that

$q = s(q)\sqrt{n} + t(q)$. We can now define the edge set $\mathcal{E}(G)$: a $d$-tuple of integers $(z_1, z_2, \ldots, z_d) \in [-\sqrt{n}, \sqrt{n}]^d$ is a hyperedge if and only if $q \in A$ and $\sum_{i=1}^{d-1} z_i = s(q)$ and $z_d = t(q)$.

For any one $q \in A$, there are at least $c_2(\sqrt{n})^{d-2}$ hyperedges, and so in all, $G$ contains at least

$$c_1 n^{1-1/2^d} \cdot c_2(\sqrt{n})^{d-2} = c(\sqrt{n})^{d-\frac{1}{2^{d-1}}}$$

hyperedges in all. Now let $\sqrt{n} = m$ be so large that (2) holds, guaranteeing a copy of $K^{(d)}(2, 2, \ldots, 2)$ in $G$. Fix a copy of $K^{(d)}(2, 2, \ldots, 2)$ on vertices $y_1, z_1, y_2, z_2, \ldots, y_d, z_d$, where for each $i$, $\{y_i, z_i\} \subset X_i$ and $y_i < z_i$. It is not difficult to check that such a copy corresponds to the affine $d$-cube $H(x_0, x_1, \ldots, x_d)$, where $x_0 = y_d + \sqrt{n}\sum_{i=1}^{d-1} y_i$ and for each $i = 1, \ldots, d-1$, $x_i = (z_i - y_i)\sqrt{n}$ and $x_d = z_d - y_d$. $\Box$

# 4 Bounds on $h(d, r)$; proof of Theorem 2.2

We begin by observing that the upper bound for $h(d, r)$ in Theorem 2.2 is the trivial one obtained by the associated density result. With $n = cr^{2^d}$, coloring $[n]$ with $r$ colors, one color class contains at least $c'n^{1-1/2^d}$ elements, large enough so that Theorem 2.1 guarantees an affine $d$-cube in that color class. Hence $h(d, r) \leq cr^{2^d}$.

On the other hand, the lower bound for $h(d, r)$ requires more development. We devote the remainder of this section to it. One eventual aim is to partition $[n]$ into (as few as possible) $d$-cube-free sets in order to give a lower bound for $h(d, r)$. The following simple idea will be useful.

**Lemma 4.1** *If a finite collection $X$ of distinct positive integers does not contain any replete affine $d$-cubes and does not contain any arithmetic progressions of length three, then $X$ does not contain any affine $d$-cubes.*

**Proof:** Let $X$ satisfy the assumptions. Suppose $H = H(x_0, x_1, \ldots, x_d) \subset X$ is an affine $d$-cube, but is not replete. Then there exist $I \subset [d]$ and $J \subset [d]$, $I \neq J$, so that

$$x_0 + \sum_{i \in I} x_i = x_0 + \sum_{j \in J} x_j,$$

giving

$$\sum_{i \in I \setminus J} x_i = \sum_{j \in J \setminus I} x_j \neq 0.$$

In this case,

$$x_0 + \sum_{i \in I \cap J} x_i, \quad x_0 + \sum_{i \in I} x_i, \quad x_0 + \sum_{i \in I \cup J} x_j,$$

is an arithmetic progression, contrary to our assumption. $\Box$

We partition $[n]$ into cube-free sets in two stages. First, we will partition $[n]$ into sets none of which contain any three term arithmetic progressions. We then refine this partition into sets not containing any replete affine $d$-cubes. In this case, each resultant set will not contain any $AP_3$, nor any replete affine $d$-cubes, and then we apply Lemma 4.1.

## 4.1 $AP_3$-free partitioning

Modifying Behrend's [1] proof (cf. [12], Theorem 6.6, lower bound) gives a partition result.

**Theorem 4.2** *For sufficiently large $n$, there exists a partition $[n] = X_1 \cup \ldots \cup X_q$, $q < e^{3\sqrt{\ln n}}$ so that each $X_i$ is $AP_3$-free and $|X_i| \leq n/e^{\ln 2\sqrt{\ln n}}$.*

**Proof:** Set $d = \lfloor e^{\sqrt{\ln n}} \rfloor$ (not to be confused with the dimension $d$ of the affine cube as in the rest of this paper), $k = \lceil \frac{\ln(n+1)}{\ln(2d)} \rceil - 1$ and for each $x \in [n]$, write

$$x = \sum_{i=0}^{k} x_i (2d)^i,$$

where for each $i$, $0 \leq x_i \leq 2d-1$. Note that due to our choice of $k$, $(2d)^k < n+1 \leq (2d)^{k+1}$ holds. Put

$$X_{n,d} = \{x \in [n] : 0 \leq x_i \leq d-1 \text{ for all } i\}.$$

For $I \subset [0,k]$, set

$$y_I = \sum_{i \in I} d(2d)^i$$

and

$$Y_I = y_I + X_{n,d} = \{y_I + x : x \in X_{n,d}\}.$$

This gives $Y_\emptyset = X_{n,d}$ and $|Y_I| = |X_{n,d}|$ for every $I \subseteq [0,k]$. For $x = \sum_{i=0}^{k} x_i (2d)^i$, let

$$I = I(x) = \{i \in [0,k] : x_i \geq d\};$$

then

$$x - y_{I(x)} = \sum_{i \notin I(x)} x_i (2d)^i + \sum_{i \in I(x)} (x_i - d)(2d)^i$$

is an element of $X_{n,d}$, and so $[n] = \bigcup_{I \subseteq [0,k]} Y_I$ is a partition. For an integer $s$ set

$$X_{n,d,s} = \left\{ x \in X_{n,d} : \sum_{i=0}^{k} x_i^2 = s \right\}$$

and for each $I \subseteq [0,k]$, put

$$Y_{I,s} = y_I + X_{n,d,s} = \{ y_I + x : x \in X_{n,d,s} \}.$$

By definition, $X_{n,d} = \cup_s X_{n,d,s}$ and so also $Y_I = \cup_s Y_{I,s}$ are partitions, where there are $(d-1)^2(k+1)+1$ choices for $s$ (including zero) and $2^{k+1}$ choices for $I$. Thus,

$$[n] = \bigcup_{I \subseteq [0,k]} \bigcup_{s=0}^{(d-1)^2(k+1)} Y_{I,s}$$

is a partition of $[n]$ into at most $q = 2^{k+1}((d-1)^2(k+1)+1)$ parts (each corresponding to a sphere in $(k+1)$-dimensional Euclidean space).

For the case where $k = 1$ (two coordinates), the different $Y_I$'s can be thought of as translations of a square in a quadrant of the cartesian plane (see Figure 1); the similar notion holds in $k+1$ dimensional Euclidean space. We claim that each part $Y_{I,s}$ is a collection of integers which does not contain an arithmetic progression of length three. Since each $Y_{I,s} = y_I + X_{n,d,s}$ is a translate of $X_{n,d,s}$, it suffices to show that $X_{n,d,s}$ does not contain an arithmetic progression. This has been, however, shown by Behrend [1]. Here, we give the proof for completeness:

Suppose $x = \sum x_i (2d)^i$, $z = \frac{x+y}{2} = \sum z_i (2d)^i$, and $y = \sum y_i (2d)^i$ are distinct elements of $X_{n,d,s}$. For each $i$, since $x_i$, $y_i$, and $z_i$ are all less than $d$, $x_i + y_i < 2d$, so no carrying occurs, that is, $\frac{x_i + y_i}{2} = z_i$. However,

$$s = \sum x_i^2 = \sum y_i^2 = \sum \left( \frac{x_i + y_i}{2} \right)^2$$

implies $\sum x_i^2 + \sum y_i^2 = \sum (x_i + y_i)^2/2$, which implies $\sum (x_i - y_i)^2 = 0$ and hence $x = y$, a contradiction.

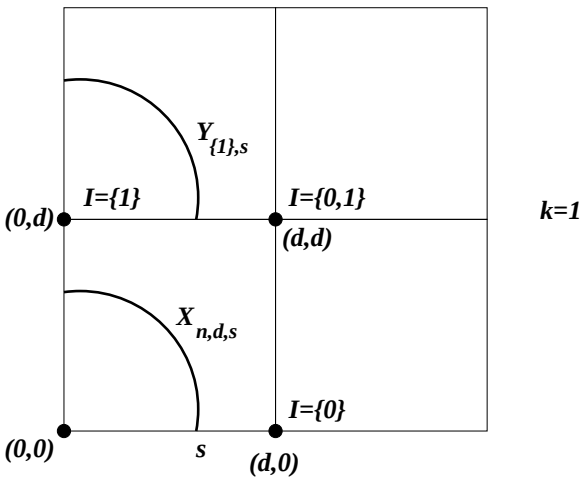


Figure 1: Partitioning integers written in base 2d

With the above choice of $d$ (and hence also $k$) we minimize the order of magnitude of $q = 2^{k+1}((d-1)^2(k+1)+1)$, the number of partition classes. Note that $k \sim \frac{\ln(n+1)}{\sqrt{\ln n}} \sim \sqrt{\ln n}$ and therefore

$$\begin{aligned} q &\sim 2^{k+1}d^2k \\ &= 2 \cdot 2^{\sqrt{\ln n}} e^{2\sqrt{\ln n}} \sqrt{\ln n} \\ &= 2 \cdot e^{(\ln 2+2)\sqrt{\ln n}} \sqrt{\ln n} \\ &< e^{3\sqrt{\ln n}} \end{aligned}$$

for sufficiently large $n$. The last inequality in the statement of the theorem follows from the crude upper bound $|Y_{I,s}| \leq |Y_I| \sim n/2^{k+1}$. $\square$

## 4.2 Replete-cube-free, density upper bound

The proof of the following result employs the standard deletion technique (cp. [4]).

**Lemma 4.3** *For each $d \geq 2$ and every set $X$ of positive integers there exists an $A \subset X$ with*

$$|A| \geq \frac{1}{8}|X|^{1-\frac{d}{2^d-1}}.$$

*which does not contain any replete affine $d$-cubes.*

**Proof of Lemma 4.3:** Fix $d \geq 2$, $X$ and let

$$p = |X|^{-\frac{d}{2^d-1}}.$$

Without loss of generality, we can assume that $X$ is large enough so that $\frac{1}{8}|X|^{1-\frac{d}{2^d-1}} > 2^d - 1$, because if not, then any set $A$ of at most $2^d - 1$ elements would satisfy the theorem. So, for example, we can assume that $p|X| > 8(2^d - 1)$.

Let $Y$ be a random subset of $X$ whose elements are chosen independently with probability $p$.

Since any affine $d$-cube is determined uniquely by $d+1$ distinct integers in $X$ (the smallest of which plays the role of $x_0$, the rest the roles of $x_0 + x_1, \ldots, x_0 + x_d$ in the definition of affine $d$-cube), the expected number of replete affine $d$-cube's in $Y$ is easily bound above by

$$\binom{|X|}{d+1} p^{2^d}.$$

Therefore (by Markov's inequality), with probability at least $1/2$, the number of replete affine $d$-cubes in $Y$ does not exceed

$$2\binom{|X|}{d+1} p^{2^d} < \frac{1}{3}|X|p. \tag{3}$$

On the other hand, the number of elements in any random subset $Y$ is a binomially distributed random variable with expectation $|X|p$ and easy reasoning gives that therefore

$$\text{Prob}(|Y| \geq \left\lfloor \frac{1}{2}|X|p \right\rfloor) > \frac{1}{2}. \tag{4}$$

Hence there exists an instance of $A^* \subset X$ satisfying both of the above events; fix such an $A^*$. Due to equations (3) and (4), deleting an element from each replete affine $d$-cube in $A^*$, we get a set $A$ with no replete affine $d$-cube such that

$$|A| \geq |A^*| - 2\binom{|X|}{d+1} > \left\lfloor \frac{1}{2}|X|p \right\rfloor - \frac{1}{3}|X|p > \frac{1}{8}|X|p,$$

where the last inequality follows from the restriction on $|X|p$. $\square$

## 4.3 Partitioning into replete-cube-free sets

To prove a lower bound for $h(d,r)$, we show a way to partition any set into subsets, each of which does not contain a replete affine $d$-cube; a greedy coloring algorithm is used.

**Lemma 4.4** *For each $d \geq 2$, there exists a constant $c$ so that for any sufficiently large set $X$ of positive integers, there exists a partition $X = A_0 \cup A_1 \cup \ldots \cup A_r$ into $r+1 \leq c|X|^{\frac{d}{2^d-1}}$ colors so that no color class contains a replete affine $d$-cube.*

**Proof of Lemma 4.4:** Set $c = \frac{12(2^d-1)}{d \ln 2}$; we will give an $r+1$-coloring $X = A_0 \cup A_1 \cup \ldots \cup A_r$ with no replete affine $d$-cube in any one color class $A_i$. By Lemma 4.3, there is a set $A_1 \subset X$ containing no replete affine $d$-cube and

$$\frac{1}{8}\left(\frac{|X|}{2}\right)^{1-\frac{d}{2^d-1}} \leq \frac{1}{8}|X|^{1-\frac{d}{2^d-1}} \leq |A_1| \leq |X|/2.$$

Since $|X \backslash A_1| \geq |X|/2$, by Lemma 4.3 (applied to sets of size $|X|/2$), we can choose $A_2 \subseteq X \backslash A_1$ containing no replete affine $d$-cube and

$$\frac{1}{8}\left(\frac{|X|}{2}\right)^{1-\frac{d}{2^d-1}} \leq |A_2| \leq \frac{|X|}{2}.$$

Continue this process, recursively choosing (pairwise disjoint) sets

$$A_1, A_2, \ldots, A_{r_1},$$

where for each $j = 1, \ldots, r_1$, $A_j \subset X$ is free of replete affine $d$-cubes,

$$A_j \subset X \backslash \bigcup_{i=1}^{j-1} A_i, \tag{5}$$

with $r_1$ the least so that

$$|X \backslash \bigcup_{i=1}^{r_1} A_i| < \frac{|X|}{2}.$$

Without loss of generality, we can also assume that

$$|X \backslash \bigcup_{i=1}^{r_1} A_i| \geq \frac{|X|}{4} \tag{6}$$

since we can limit our greed by possibly taking $A_{r_1}$ smaller than maximal size.

Since for each $j < r_1$,

$$\left| X \setminus \bigcup_{i=1}^{r_1} A_i \right| \geq \frac{|X|}{2},$$

we can ensure by Lemma 4.3 that

$$\frac{1}{8}\left(\frac{|X|}{2}\right)^{1-\frac{d}{2^d-1}} \leq |A_j|.$$

Thus due to (6) we can infer that

$$r_1 \leq \frac{\frac{3}{4}|X|}{\frac{1}{8}\left(\frac{|X|}{2}\right)^{1-\frac{d}{2^d-1}}} = 12\left(\frac{|X|}{2}\right)^{\frac{d}{2^d-1}}.$$

Again, by Lemma 4.3 (with $|X|/4$ as the set size), choose successively $A_{r_1+1}$, $A_{r_1+2}, \ldots, A_{r_1+r_2}$, where for each $j = r_1+1, \ldots, r_1+r_2$, the relation (5) holds and

$$\frac{1}{8}\left(\frac{|X|}{4}\right)^{1-\frac{d}{2^d-1}} \leq |A_j| \leq \frac{|X|}{4},$$

with $r_2$ the smallest so that

$$\left| X \setminus \bigcup_{i=1}^{r_1+r_2} A_i \right| < \frac{|X|}{4}$$

while

$$\frac{|X|}{8} < \left| X \setminus \bigcup_{i=1}^{r_1+r_2} A_i \right|.$$

In this case,

$$r_2 \leq \frac{\frac{3}{8}|X|}{\frac{1}{8}\left(\frac{|X|}{4}\right)^{1-\frac{d}{2^d-1}}} = 12\left(\frac{|X|}{4}\right)^{\frac{d}{2^d-1}}.$$

Continuing this greedy process using $1 + r_1 + r_2 + \cdots r_k$ colors, where $k = \lfloor \log_2 |X| \rfloor - d + 1$, we color all but at most $2^d - 1$ elements of $X$; these

remaining we call the color class $A_0$. Thus using $1 + r_1 + r_2 + \cdots r_k$ colors one can color $X$ with no color class containing a replete affine $d$-cube. Since

$$\begin{aligned}\sum_{i=1}^{k} r_i &\leq 12\sum_{i=1}^{\infty}\left(\frac{|X|}{2^i}\right)^{\frac{d}{2^d-1}} \\ &= \frac{12}{2^{\frac{d}{2^d-1}}}|X|^{\frac{d}{2^d-1}} \\ &< 12\frac{2^d-1}{d\ln 2}|X|^{\frac{d}{2^d-1}}\end{aligned}$$

we are done.□

## 4.4 Final step in proof of lower bound for $h(d,r)$

We now prove for $d \geq 2$, and $r \geq 2$ that $r^{\frac{2^d-1}{d}(1-o(1))} \leq h(d,r)$, where $o(1)$ tends to zero as $r$ increases.

**Proof of lower bound in Theorem 2.2:** Fix $n$ large. It suffices to provide a partition of $[n]$ into at most $n^{\frac{d+o(1)}{2^d-1}}$ parts, each part affine $d$-cube-free. By Theorem 4.2, let $[n] = X_1 \cup \ldots \cup X_q$ be a partition into at most $q = e^{3\sqrt{\ln n}}$ parts, each $X_i$ not containing any three term arithmetic progression, and each $|X_i| \leq n/2^{\sqrt{\ln n}}$. For each $i = 1, \ldots q$, by Lemma 4.4, partition $X_i = A_{i,1}, \ldots, A_{i,k(i)}$,

$$k(i) \leq \frac{12}{2^{\frac{d}{2^d-1}}}|X_i|^{\frac{d}{2^d-1}} < \frac{12}{2^{\frac{d}{2^d-1}}}\left(n/2^{\sqrt{\ln n}}\right)^{\frac{d}{2^d-1}}$$

into parts, none of which contains a replete affine $d$-cube. For each $i = 1, \ldots, q$ and for each $j = 1, \ldots, k(i)$, $A_{i,j}$ contains neither a three term arithmetic progression nor a replete affine $d$-cube, so by Lemma 4.1, each $A_{i,j}$ is affine $d$-cube-free. The total number of partitions is at most

$$\begin{aligned}q \cdot \max k(i) &\leq e^{3\sqrt{\ln n}} \cdot \frac{12}{2^{\frac{d}{2^d-1}}}(n/2^{\sqrt{\ln n}})^{\frac{d}{2^d-1}} \\ &= n^{\frac{d+o(1)}{2^d-1}}. \quad \square\end{aligned}$$

We conclude by noting that we have proved Theorem 2.2 for $d \geq 2$. This is, however, only an improvement over known results for $d \geq 3$. (cf [2])

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