Topological Classification of Complete Intersections *

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Abstract- We show that the homeomorphism type of the complete intersection $X_n(d_1, d_2, \cdots, d_r)$ is determined by its Pontryagin classes, Euler characteristic and the product $d_1 d_2 \cdots d_r$, if $d_1 d_2 \cdots d_r$ has no prime factors less than $\frac{n+3}{2}$. We also prove that, when n is even the homotopy type of $X_n(d_1, d_2, \cdots, d_r)$ is determined by its Euler charateristic, signature and the product $d_1 d_2 \cdots d_r$, in the case $d_1 d_2 \cdots d_r$ has no prime factors less than $\frac{n+3}{2}$. The latter confirms a conjecture of Libgober and Wood [8].

Sur les classifications topologiques des intersections complètes

Résumé- Nous montrons que le type d'homeomorphisme d'une intersection complète $X_n(d_1, d_2, \cdots, d_r)$ est déterminé par ses classes de Pontryagin, son nombre d'Euler, et le product $d_1 d_2 \cdots d_r$ dans le cas que $d_1 d_2 \cdots d_r$ n'a pas des facteurs primes moins de $\frac{n+3}{2}$. Nous montrons aussi que quand n est paire, le type d'homotopie de $X_n(d_1, d_2, \cdots, d_r)$ est déterminé par son nombre d'Euler, des mêmes signatures et le product $d_1 d_2 \cdots d_r$ dans le cas $d_1 d_2 \cdots d_r$ n'a pas des facteurs primes moins de $\frac{n+3}{2}$. Ce dernier résultat confirme une conjecture de Libgober et Wood [8].

Version française abrégée. Soit $X_n(\mathbf{d}) \subset CP^{n+r}$ une intersection complète, obtenne comme l'intersection transversale des r hypersurfaces dans CP^{n+r} définées par des polynomes de degré $\mathbf{d} = (d_1, d_2, \cdots, d_r)$. On appelle $d_1 \cdots d_r$ le degré total de $X_n(\mathbf{d})$.

Dans leur article [8], Libgober et Wood montrent que pour chaque mombre entier k, il y a k intersections complètes diffeomorphiques avec des multidegrés différents dans toutes dimensions.

Dans cette Note, nous annoncons quelques résultats sur le type d'homeomorphism et d'homotopie des intersections complètes.

*Supported in partial by Sonderforschungsbereich 343. The present version of this announcement was done during the author's visit in University of Bielefeld. The author would like to thank Prof. F.Waldhausen for his very warm hospitality.

Comme examples, on a

Théorèm 0.1 *Soient $X_n(\mathbf{d})$ et $X_n(\mathbf{d}')$ deux intersections complètes avec des mêmes degrés totals d. Supposons que d n'a pas des facteurs primes moins de $\frac{n+3}{2}$(i.e., Si $p|d$, alors, $2p \geq n+3$). Alors, $X_n(\mathbf{d})$ et $X_n(\mathbf{d}')$ sont homeomorphiques si et seulement si ils ont des mêmes classes de Pontryagin et des mêmes nombres d'Euler.*

Théorèm 0.2 *Soit n est un entier pair. Soient $X_n(\mathbf{d})$ et $X_n(\mathbf{d}')$ deux intersections complètes avec des mêmes degrés totals d. Supposons que d n'a pas des facteurs primes moins de $\frac{n+3}{2}$(i.e., Si $p|d$, alors, $2p \geq n+3$). Alors, $X_n(\mathbf{d})$ et $X_n(\mathbf{d}')$ sont equivalents homotopiques si et seulement si ils ont des mêmes nombres d'Euler et des mêmes signatures.*

On note que, théorème 0.2 confirme une conjecture de Libgober et Wood [8].

The purpose of this note is to announce several theorems about the homeomorphism and homotopy classification of complex complete intersections and to sketch out their proofs.

1. Nonsingular complete intersections- Let $X_n(\mathbf{d}) \subset CP^{n+r}$ denote a complete intersection, the transversal intersection of r hypersurfaces in CP^{n+r} defined by r homogeneous polynomials of multidegree $\mathbf{d} = (d_1, d_2, \cdots, d_r)$, we call the product $d_1 \cdots d_r$ the *total degree* and denote it by d. It is an easy fact that the total degree is a homotopy invariant of the complete intersection $X_n(\mathbf{d})$ if the dimension $n \geq 3$. Up to dimension $(n-1)$, the homolgy of a complete intersection can be read out from Lefschetz's hyperplane Theorem. It is a classical observation of R.Thom that the diffeomorphism type of a complete intersection is determined by the dimension n and its multidegree $\mathbf{d}$. In [8], Libgober-Wood proved that, for any integer k, there are k diffeomorphic complete intersections with different multidegrees in every dimension. It is a natural problem to ask when two complete intersections are diffeomorphic, homeomorphic and/or homotopy equivalent.

In the complex dimensions $n = 1$ and 3, the diffeomorphism classification of the complete intersections is known as a corollary of surface theory and the general classification of simply connected 6 manifolds [11]. For dimension $n = 2$, the topological and homotopy classification is well known by Freedman's work[3]. In dimension 4, the topological classification was completed in my joint paper with S.Klaus [2]. For higher dimensions, it seems very hard to obtain a complete answer about the classification problem of complete intersections up to homotopy or homeomorphism.

The first general results occured in the early 80's by a series works of Libgober and Wood. In [6], they proved that, when n is odd, the homotopy type of $X_n(\mathbf{d})$ depends only on its total degree and Euler charateristic provided d has no prime factors less than

$\frac{n+3}{2}$. They made a further conjecture in [8] , namely the same statement holds true in the case of n even. On the other hand, when the total degree has enough many small primes factors, 1985 Traving (c.f. [5]) proved that the diffeomorphism type of the a complete intersection depends only on its Pointryagin classes and Euler numbers by using M.Kreck's surgery program. In this note, we shall announce an affirmative answer to the conjecture mentioned above as well as several theorems about the homeomorphism classifiacation of complete intersections by appealing Sullivan's theory.

Let $n \geq 3$ and let x be a generator of $H^2(X_n(\mathbf{d}), \mathbf{Z}) \cong \mathbf{Z}$ the generator. Up to the middle dimension n, the cohomolgy ring of $X_n(\mathbf{d})$ is $\mathbf{Z}[x]$, and $x^n[X_n] = d$ with deg$x = 2$. Note that x gives a map $f_x : X_n(\mathbf{d}) \longrightarrow CP^\infty$, the stable tangent bundle of $X_n(\mathbf{d})$ is $f_x^*((n+r+1)H-(H^{d_1}+H^{d_2}+\cdots+H^{d_r}))$, where H denote the Hopf line bundle over CP^∞ and $H^i = H \otimes \cdots \otimes H(i\text{times})$. Thus the Pontryagin classes $P_i(X_n(\mathbf{d}))$ can be written as $p_i x^{2i}$, where p_i is an integer. We say that two complete intersections with the same total degree have the same Pontryagin classes iff these corresponding integers p_i agree. when n is even, it is equivalent to say that all of the corresponding Pontryagin numbers agree.

Theorem 0.1 *Let $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ be two complete intersections with the same total degree d. Suppose that d has no prime factors less than $\frac{n+3}{2}$ (i.e., if $p|d$, then $2p \geq n+3$). Then $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are homeomorphic to each other if and only if they have the same Pontryagin classes and Euler charateristic.*

By [8], when n is odd, $X_n(\mathbf{d})$ is always homeomorpic to the connected sum $K_n(\mathbf{d})\#N$ with $H_n(K_n(\mathbf{d})) = 0$, we say that $K_n(\mathbf{d})$ a core of $X_n(\mathbf{d})$. When n even, $X_n(\mathbf{d})$ can never be splitted as above so that the middle homology of the cores zero. However one has a "minimal" decomposition in the topological category such that rank$H_n(K_n(\mathbf{d})) \leq 5$ the minimum [8]. The minimal rank depends only on the intersection form and the total degree d. Also the homotopy type of cores is unique.

Theorem 0.2 *Let n be even, $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ be two complete intersections with the same total degree d. Suppose that d has no prime factors less than $\frac{n+3}{2}$. Then $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are homotopy equivalent if and only if they have the same Euler charateristic and signatures.*

In [8] it was proved that the cores of $X_n(p,p)$ and $X_n(p^2)$ are not homotopy equivalent if p is less than $\frac{n+3}{2}$. This implies that the condition that d has no prime factors less than $\frac{n+3}{2}$ is serious. By using K-theory one can get much stronger necessary condition for the homotopy equivalent of complete intersections with smaller prime factors.

Proposition 0.3 *Let $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ be two complete intersections with homotopy equivalent cores, l and l' denote the numbers of even entires in $\mathbf{d}$ and $\mathbf{d}'$ respectively.*

Then $2^{f(n)-1}$ *divides* $l-l'$, *where* $f(m)$ *is given by* $f(m+8)=f(m)+4$ *and* $f(1)=1, f(2)=f(3)=2, f(4)=f(5)=f(6)=f(7)=3, f(8)=4$.

We say that two manifolds M and N of dimension $2n$ are stable diffeomorphic (homeomorphic, homotopy equivalent) if there is integers m and m' such that $M\#m(S^n\times S^n)$ and $N\#m'(S^n\times S^n)$ are diffeomorphic(homeomorphic, homotopy equivalent).

Theorem 0.4 *Let* $X_n(\mathbf{d})$ *and* $X_n(\mathbf{d}')$ *be two stably diffeomorphic(homeomorphic, homotopy equivalent) complete intersections. Suppose that* $a_1, a_2, \cdots, a_k$ *are positive integers such that* $max\{a_1, a_2, \cdots, a_k\}\leq min\{\mathbf{d}, \mathbf{d}'\}$. *Then* $X_n(\mathbf{d}, a_1, a_2, \cdots, a_k)$ *and* $X_n(\mathbf{d}', a_1, a_2, \cdots, a_k)$ *are stably diffeomorphic(homeomorphic, homotopy equivalent).*

Remark-Let $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ are two diffeomorphic complete intersections with the same total Chern classes. The same conclusion holds true if one replaces "stable diffeomorphism" in the above theorem by diffeomorphism. Consequently, for any positive integer k, by adding k-copies of 2 to the multidegrees $\mathbf{d}$ and $\mathbf{d}'$ respectively we get the diffeomorphic complete intersections with same total Chern classes but different multidegrees.

2. Sketch of the proofs- Let M be a closed simply connected manifold of dimension $2n$, we say a codimension 2 submanifold N of M is <u>*taut*</u> if $(C, \partial C)$ is $(n-1)$ connected, where $C = M - \text{int } U$ and U is a regular neighborhood of N in M. By surgery technique it is known for a long time that every $(2n-2)$-dimensional homology class of M can be represented by a codimension 2 taut submanifold. M.Freedman [3] proved that these taut submanifolds are stable diffeomorphic to each other if they represent the same homology class.

The outline of the proof of Theorem 0.2. By [5], $X_{n+1}(\mathbf{d})$ and $X_{n+1}(\mathbf{d}')$ have homotopy equivalent cores since d has no prime factors less than $\frac{n+3}{2}$ and n even. Thus there exist integers r and r' such that $X_{n+1}(\mathbf{d})\#r(S^{n+1}\times S^{n+1})$ and $X_{n+1}(\mathbf{d}')\#r'(S^{n+1}\times S^{n+1})$ are homotopy equivalent. Let f be such a homotopy equivalence. By Quinn[9] prop1.3 we may assume that f is an almost canonical. Hence the maps $f|_{f^{-1}(X_n(\mathbf{d}'))}: f^{-1}(X_n(\mathbf{d}'))\to X_n(\mathbf{d}')$ and $f| : X_{n+1}(\mathbf{d})\#rS^{n+1}\times S^{n+1} - f^{-1}(X_n(\mathbf{d}'))\to X_{n+1}(\mathbf{d}')\#r'S^{n+1}\times S^{n+1} - X_n(\mathbf{d}')$ are n and $n+1$-connected respectively. Thus $f^{-1}(X_n(\mathbf{d}'))$ is a taut submanifold of $X_{n+1}(\mathbf{d})\#r(S^{n+1}\times S^{n+1})$ representing the dual Dx of the generator $x\in H^2(X_{n+1}(\mathbf{d})\#r(S^{n+1}\times S^{n+1}))$. Note that $X_n(\mathbf{d})\subset X_{n+1}(\mathbf{d})\#r(S^{n+1}\times S^{n+1})$ is also a taut submanifolds representing Dx. By Freedman[4] $X_n(\mathbf{d})$ and $f^{-1}(X_n(\mathbf{d}'))$ are stably diffeomorphic. Thereby there is an n-connected degree 1 map from $X_n(\mathbf{d})\#m(S^n\times S^n)$ to $X_n(\mathbf{d}')$ for some integer m. By the same method as in[7] prop3.3 it follows that $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ have homotopy equivalent cores. Therefore they are homotopy equivalent if the signatures and Euler charateristic agree.

Recall Sullivan's theory on surgery says that two homotopy equivalent $2n$-manifolds with n greater than 3 are homeomorphic if and only if they have the same codimension $2i$(here $n-i$ even) signatures defects and codimension $2j$(here $n-j$ odd) Arf defects which are determined by the Pontryagin classes and the total degree in our case. The proof of Theorem 0.1 can be done in the standard way.

The ouline of Proposition 0.3. Note that the stable normal bundle of the cores of $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$ have the same J-image in the J-group. By pulling back to the J-group of $CP^{\frac{n}{2}}$ we obtain the same element in $J(CP^{\frac{n}{2}})$, where $CP^{\frac{n}{2}}$ is regarded as the submanifold of the cores of $X_n(\mathbf{d})$ and $X_n(\mathbf{d}')$. The group $J(CP^{\frac{n}{2}})$ is well understood and our conclusion follows easily.

The outline of the proof of Theorem 0.4. By induction we need only to show the theorem where $k=1$. Suppose that there are integers r and r' such that $X_{n+1}(\mathbf{d})\#r(S^{n+1}\times S^{n+1})$ and $X_{n+1}(\mathbf{d}')\#r'(S^{n+1}\times S^{n+1})$ are homotopy equivalent. Let f be such a homtopy equivalence. Notice that $X_{n-1}(\mathbf{d},a)$ represents the dual of $ax\in H^2(X_n(\mathbf{d})\#r(S^{n+1}\times S^{n+1}))$ in $X_n(\mathbf{d})$. Similar statement holds true for $X_{n-1}(\mathbf{d}',a)$. By Quinn[9] prop1.3 we can deform f such that the restriction map $f: f^{-1}(X_{n-1}(\mathbf{d}',a))\to X_{n-1}(\mathbf{d}',a)$ is $(n-1)$-connected degree 1 map. By Freedman[4], $f^{-1}(X_{n-1}(\mathbf{d}',a))$ is stable diffeomorphic to $X_{n-1}(\mathbf{d},a)$. Recall [10] that $X_n(\mathbf{d},a)$ can be regarded as an a-fold branched cover over $X_n(\mathbf{d})$ with branch set $X_{n-1}(\mathbf{d},a)$. Now we may construct an a-fold branch cover W over $X_n(\mathbf{d})$ with branch set $f^{-1}(X_{n-1}(\mathbf{d}',a))$ by pulling back the canonical branch cover over $X_n(\mathbf{d}')$ with branch set $X_{n-1}(\mathbf{d}',a)$ so we obtain a map $F: W\to X_n(\mathbf{d}',a)$ which is n-connected and degree 1. Moreover, when n is even, W and $X_n(\mathbf{d}',a)$ have the same signature. Therefore W and $X_n(\mathbf{d}',a)$ are homotopy equivalent up to connected sum with some $S^n\times S^n$.

The next step is to construct an almost h-cobordism between W and $X_n(\mathbf{d},a)$, i.e, a cobordism V with boundary $W\sqcup X_n(\mathbf{d},a)$ so that the relative homology groups $H_i(V,X_n(\mathbf{d},a))$ and $H_i(V,W)$ vanishes except in the middle dimension. Moreover, every embedded spheres in V of dimension n has trivial normal bundle. Then it is easy to show that W and $X_n(\mathbf{d})$ are stable diffeomorphic. Combining the first step the proof follows.

The details will appear in [1].

Addendum: I was informed very recently by M.Kreck that Stephan Stolz has a unpublished result, namely, under the same condition in Theorem 0.1 the complete intersections are diffeomorphic which was done along different line.

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