Topology of Dupin hypersurfaces with six distinct principal curvatures *

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Abstract

Let M be a Dupin hypersurface in the unit sphere S^{n+1} with six distinct principal curvatures. The main purpose of the present paper is to show M is either diffeomorphic to $SU(2) \times SU(2)/Q_8$ or homeomorphic to the tube around an embedded 5-dimensional complex Fermat hypersurface $X_5(2)$ in S^{13}, where $Q_8 \subset SU(2) = Sp(1)$ denotes the subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ and $X_5(2) = \{[z_0, z_1, \cdots z_6] \in CP^6 | z_0^2 + z_1^2 + \cdots + z_6^2 = 0\}$. Moreover, in the former case, all of the focal manifolds are diffeomorphic to $S^3 \times RP^2$; In the latter case, one of the focal manifolds is homeomorphic to $X_5(2)$.

1 Introduction

A closed n-dimensional submanifold in the unit sphere S^{n+1} is called a Dupin hypersurface if the multiplicities of the principal curvatures are constant and if each principal curvature function λ restricted to the leaves of the distribution $T_\lambda = \{x | Ax = \lambda x\}$ is constant [16]. The most important special case is isoparametric hypersurface, i.e., hypersurface with constant principal curvatures. Münzner [11] showed that, for an isoparametric hypersurface in the sphere, the number g of distinct principal curvatures must be 1, 2, 3, 4 or 6. This result was generalized by Thorbergsson [16] to the case of embedded closed Dupin hypersurface.

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In the situation of $g = 1$, 2 or 3, Dupin hypersurfaces are Lie equivalent to isoparametric hypersurfaces. The same conclusion does not hold for $g = 4$ or 6. It is a classical result of E.Cartan [2] that any isoparametric hypersurface with 1, 2 or 3 distinct principal curvatures is a homogeneous space. More precisely, it is either a tube around the veronese surface in S^4 or the tube around the standard embedded projective plane CP^2, HP^2 and QP^2 in S^7, S^{13} and S^{25} respectively. Consequently Dupin hypersurface with 1, 2 or 3 distinct principal curvatures is always diffeomorphic to a homogeneous space.

Using Clifford algebra, a family of infinite nonhomogeneous isoparametric hypersurfaces with four distinct principal curvatures was constructed by Ferus, Karcher and Münzner [6]. They are even not homeomorphic or homotopy equivalent to homogeneous spaces [18]. The topology and various restriction about the multiplicities of the principal curvatures of isoparametric hypersurfaces were explored in [1][4].

For the last case, i.e., $g = 6$, the multiplicities of the six distinct principal curvature are the same(we denote it by m) and equal 1 or 2 [7][1]. Dorfmeister and Naher [3] showed that isoparametric hypersurface with $g = 6$ and $m = 1$ is the homogeneous space $SU(3)/T^2$ by solving the Cartan-Münzner equations. Their methods can not work in the case of $m = 2$ or for Dupin hypersurface. Thus it is very natural to ask whether there exists nonhomogeneous Dupin hypersurface with $g = 6$ in geometry or topology.

The present paper is going to handle this question in topology. We will show that each Dupin hypersurface with $g = 6$ is either diffeomorphic to the homogeneous space $SU(2) \times SU(2)/Q_8$ or homeomorphic to the normal sphere bundle of an embedded $X_5(2)$ in S^{13} corresponding to $m = 1$ or 2. Moreover, we prove in the appendix that there are at most 3 different Dupin hypersurfaces with $g = 6, m = 2$ up to homeomorphism. We remind the, where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the subgroup of the quaternion $SU(2) = Sp(1)$. Except the homogeneous example G_2/T^2, we have no other example of Dupin hypersurface with $g = 6, m = 2$ and with different homeomorphism type. To be honest, I don't know how to prove that each Dupin hypersurface with $g = 6, m = 2$ is homeomorphic to G_2/T^2.

For reader's convenience, we recall the definition and the main properties of complex projective hypersurfaces as the follows.

Let $X_n(2) \subset \mathrm{CP}^{n+1}$ be the Fermat hypersurface of degree 2, i.e, $X_n(2) = \{(z_0, z_1, \cdots z_{n+1}) \in \mathrm{CP}^{n+1} | z_0^2 + z_1^2 + \cdots z_{n+1}^2 = 0\}$. The Lefschtz hyperplane section theorem and Poincaré duality implies that

$$H^i(X_n(2)) \cong Z \text{ if } i \text{ even} \ ; \ H^i(X_n(2)) \cong Z \text{ if } i \text{ odd} \ ;$$

Moreover, if n is odd, one obtain the cohomology ring $H^*(X_n(2)) \cong Z[x, z]/\{x^j = 2z\}$ here $n = 2j - 1$ and deg$x = 2$, deg$z = 2j$. The first Pontryagin class of $X_n(2)$ equal $(n-2)x^2$.

Theorem 1.1 *Let M be a Dupin hypersurface in the unit sphere S^{n+1} with six distinct principal curvatures. Then M is either diffeomorphic to $SU(3)/T^2$ or homeomorphic to the normal sphere bundle of an embedded $X_5(2)$ in S^{13}, where T^2 is the maximal torus subgroup of $SU(3)$. Moreover, in the former case each focal manifold is diffeomorphic to $S^3 \times RP^2$ and in the latter case, one of the focal manifold is homeomorphic to $X_5(2)$.*

Theorem 1.2 *There are at most 3 different compact Dupin hypersurfaces with $g = 6, m = 2$ up to homeomorphism.*

The organization of the paper is as follows. In section 2 we compute the cohomology rings and the Pontryagin classes of the focal manifolds when the multiplicity $m = 2$. In section 3 we study the $\mathrm{MO}\langle 8\rangle$-bordism theory for certain Thom spectrum, and combining this with Kreck's surgery program we prove that one of the focal manifolds of the Dupin hypersurface with $g = 6, m = 2$ is almost diffeomorphic to $X_5(2)$. In section 4 we complete the proof of Theorem 1.1. In the appendix, we build the Moore-Postnikov factorization for the fibration $BSO(3) \to BSO$ up to dimension 11 and give a classification theory of differential S^2-bundles over 10-complexes. As a corollary, we prove Theorem 1.2.

2 Cohomology rings and the Pontryagin classes of the focal manifolds when $m = 2$

Let M be a embedded compact Dupin hypersurface in the unit sphere S^{n+1}. Let $\lambda_1 < \lambda_2 < \cdots < \lambda_g$ be the principal curvature functions. Denote by $(m_1, m_2, \cdots, m_g)$ the corresponding set of multiplicities.

Let F_+(resp. F_-) be the set of focal points corresponding to the smallest(resp. largest) λ_i. Then [16] F_+ (resp. F_-) is a submanifold of S^{n+1} of codimension m_1+1(resp. m_g+1). Moreover, the focal maps $p_\pm : M \to F_\pm$ can be identify with the normal sphere bundles of $F_\pm$ in S^{n+1} and thus we have a decompositon of $S^{n+1} = D_+ \cup D_-$, where $D_\pm$ denote the disc bundles of $p_\pm$. By use this topological decomposition, some important properties of the Dupin hypersurfaces were explored. For instance, the number g must belongs to the set $\{1, 2, 3, 4, 6\}$ [16] and in case of $g = 6$, $m_1 = m_2 = \cdots = m_g$ is equal to 1 or 2, and hence the dimension $n = 6$ or 12 (cf. Grove and Halperin [7] or Abresch [1]).

The rational cohomology rings of Dupin hypersurfaces were determined by Grove and Halperin [7]. In this section, we will compute the integral cohomology rings and the Pontryagin classes of the focal manifolds for Dupin hypersurface with $g = 6$ and $m = 2$, which is essentially important for our purpose.

Theorem 2.1 *Let M be a Dupin hypersurface in the unit sphere S^{13} with $g = 6$ and $m = 2$. Let F_+ (resp. F_-) be the focal manifold corresponding to smallest (resp. largest) principal curvature. Then the cohomology rings of F_+ (F_- is the remainder) is either isomorphic to*

$$Z[x, z]/\{x^3 = 2z, z^2 = 0\} \text{ or}$$
$$Z[y, z', z'']/\{y^2 = 3z'; y^3 = 6z''; z''^2 = 0\},$$

where $degx = 2 = degy$ and $degz'' = 6 = degz$ and $degz' = 4$. The first Pontryagin class $p_1(F_+)$ is either $3x^2$ or z' correspondingly.

Proof: Let $p_\pm : M \to F_\pm$ be the normal sphere bundles of the focal manifolds $F_\pm$ in S^{13}. Note that $F_\pm$ are simply connected and $H^i(F_\pm) \cong Z$ if i is even and 0 otherwise [7]. Thus the Gysin exact sequences

$$H^{2i}(F_\pm) \xrightarrow{p_\pm^*} H^{2i}(M) \xrightarrow{\psi_\pm} H^{2i-2}(F_\pm)$$

is splitting exact. Let $e \in p_+^*(H^4(F_+)) \cong Z$ (resp. $f \in p_-^*(H^4(F_-)) \cong Z$) is a generator.

Denote by $x = \psi_+(f) \in H^2(F_+)$ and $y = \psi_-(e) \in H^2(F_-)$. Let $a = p_+^*(x)$ and $b = p_-^*(y)$. Then $x \in H^2(F_+)$ (resp. $y \in H^2(F_-)$) is a generator. A Mayer-Vietoris sequence argument for the splitting $S^{13} = D_+ \cup D_-$ shows that $H^2(M) = span_Z\{a, b\}$ and $H^4(M) = span_Z\{e, f\}$.

Let $p_+(x^2) = a^2 = ke$ and $p_-(y^2) = b^2 = lf$ for integers k and l. By [1], $a^2 = 1(mod 2)$ and $b^2 = 1(mod 2)$, thereby k and l are both odd. From the definition we have $\psi_+(a) = \psi_-(b) = 0$, $\psi_+(e) = \psi_-(f) = 0$. By the Gysin exact sequence we obtain

$$a^2 = p_+^*(x^2) = ke = p_-^*(\alpha) + a \cup p_-^*(\beta) \cdots \text{ (I)}$$

for some $\alpha \in H^4(F_-)$ and $\beta \in H^2(F_-)$. Moreover, $\psi_+(b) \in H^0(F_+)$ (resp. $\psi_-(a) \in H^0(F_-)$) is a generator.

Applying ψ_- and ψ_+ to the equation (I) we get $\beta = ky$ and $p_-^*(\alpha) = -kf$ respectively. Thus (I) becomes $ab = e + f$ by noting $k \neq 0$.

By the decomposition $S^{13} = D_+ \cup D_-$ again we have $H^6(M) = p_+^*(H^6(F_+)) \oplus p_-^*(H^6(F_-))$. Thus by Gysin exact sequence it folows that $\psi_- : p_+^*(H^6(F_+)) \to H^4(F_-)$ is a monomorphism. Therefore $a^3 = 0 \Longleftrightarrow \psi_-(a^3) = 0 \Longleftrightarrow k(1 - kl) = 0 \Longleftrightarrow kl = 1 \Longleftrightarrow b^3 = 0$. Furthermore, it implies that $z^5 = 0$ for each $z \in H^2(M)$. But this is impossible as the rational cohomology ring of M is $A_6(2) = \{u, v | u^6 = u^2 + 3v^2 = 0\}$ [7].

Now let's assume $kl \neq 1$, then $x^3 \neq 0$ and $y^3 \neq 0$. By duality we have $x^5 \neq 0$ and $y^5 \neq 0$. Note that $a^6 = b^6 = 0$ Thus

$$0 = \psi_+(b^6) = l^3\psi_+((-e + ab)^3) = l^3(l^2 + \frac{3}{k^2} - \frac{4l}{k})x^5,$$

and it follows that $kl = 1$ or $kl = 3$. The former case can not occur by our assumption.

Without loss of generality we assume that $k = 1$ and $l = 3$. Then $x^2 \in H^4(F_+)$ is a generator and $y^2 = 3u \in H^4(F_-)$, here u is a generator. By $\psi_-(a^3) = 2u$ it follows that $x^3 = 2z \in H^6(F_+)$, here z is a generator of degree 6. Thus the cohomology ring of F_+ is $Z[x, z]/\{x^3 = 2z, z^2 = 0\}$.

Similarly we conclude that $H^*(F_-) \cong Z[y, z']/\{y^2 = 3z'; y^3 = 6z'', z''^2 = 0\}$, here $\deg z' = 4$ and $\deg z'' = 6$.

Let α and β be as above, by Massey [10] the Pontryagin class of the bundle p_- is $p_1(p_-) = 4\alpha + \beta^2 = 4\alpha + y^2$. As $p_-^*(4\alpha + y^2) = -f$, thus $4\alpha + y^2 = -z'$ is a generator of $H^4(F_-)$.

Similarly $P_1(p_+) = -3x^2$.

Note that $p_1(F_\pm) = -p_1(p_\pm)$. This completes the proof. ♠

3 Homeomorphism classification of manifolds and bordism theory

The classification theory of CAT-manifolds(CAT=DIFF, PL or TOP) of dimension larger than 5 or equal to 5 was accomplished by surgery program in the sixties (cf. Wall [17]). Roughly saying, the theory claims that topology of a CAT manifold of dimension not less than 5 is determined by its homotopy type, stable normal bundle and normal invariant set. However, in practice, it is often impossible to discribe the homotopy type of a manifold so as to apply this theory. A useful extension of this theory was done by M.Kreck [9](cf. also Freedman[6] for even dimensional simply connected case), which is our tool to prove Theorem 1.1. For reader's convinience, we review this theory as follows.

Definition 3.1 *Let $\xi : B \to BCAT$ be a k-coconnected fibration(i.e. $\pi_i(B) \to \pi_i(BCAT)$ is an isomorphism for $i > k$ and an injection for $i = k$). Let $n = 2k$ or $2k - 1$. We say that a CAT n-manifold M is <u>of normal type ξ</u> if there is lifting $\tilde{\nu_M} : M \to B$ of the normal Gauss map $\nu_M : M \to BCAT$, which is a k-equivalence. $(M, \tilde{\nu_M})$ is called a <u>normal $(k-1)$-smoothing in ξ</u>.*

For $\xi : B \to B\mathrm{CAT}$, let $\Omega_*(\xi)$ denote the bordism classes of CAT manifolds equipped with a lift of the stable normal bundle ν to B. Each element in $\Omega_*(\xi)$ is represented by a pair $(M, \tilde{\nu} : M \to B)$, where $\xi \circ \tilde{\nu} = \nu : M \to B\mathrm{CAT}$.

Two normal $(k-1)$-smoothings $(M, \tilde{\nu})$ and $(M', \tilde{\nu}')$ in ξ are CAT equivalent if there is a CAT equivalence $f : M \to M'$ so that $\tilde{\nu}$ and $\tilde{\nu}' \circ f$ are fibre homotopic. Let $MSt_n(\xi)$ (where $n = 2k$ or $2k-1$) denote the s-cobordism classes of CAT closed n-dimensional manifolds with $(k-1)$-type ξ, and let $NSt_n(\xi)$ be the s-cobordism classes of closed CAT n-dimensional normal $(k-1)$-smoothings in ξ. There is a canonical projection $NSt_n(\xi) \to MSt_n(\xi)$.

Obstruction theory and the Moore-Postnikov factorization show that the normal $(k-1)$-type of M exists and is unique up to fibre homotopy equivalence. Making use of this one can prove that

Lemma 3.2 (Kreck) $MSt_n(\xi) \cong NSt_n(\xi)/Aut(\xi)$.

Let $Aut(\xi)$ denote the group of fibre homotopy classes of fibre homotopy equivalence of ξ. $Aut(\xi)$ acts on $NSt_n(\xi)$ and $\Omega_n(\xi)$ by $f(M, \tilde{\nu'}) = (M, f \circ \tilde{\nu'})$. There is a natural map $NSt_n(\xi) \to \Omega_n(\xi)$ which is equivariant with respect to the action of $Aut(\xi)$. This leads to a map

$$MSt_n(\xi) \to \Omega_n(\xi)/Aut(\xi).$$

Two $2k$-dimensional manifolds M and N are stably CAT-equivalent if there exist some nonnegative integers r and s such that the connected sum $M\#rS^k \times S^k$ is CAT equivalent to $N\#sS^k \times S^k$.

Theorem 3.3 (Kreck) *Let $\xi : B \to BCAT$ be a fibration with B connected and where the k-skeleton of B has the homotopy type of a finite complex. Let $n = 2k$ or $2k-1$ be not less than 4.*
(i) *The map*

$$MSt_n(\xi) \to \Omega_n(\xi)/Aut(\xi)$$

is surjective and elements in the same fibre are distinguished by an invariant in a group $L^{s,\tau}_{n+1}(\pi_1(B), w_1)$ modulo some indeterminacy.
(ii) *If n is even, then any two elements in the same fibre are stably CAT equivalent.*

For any integer m, the group $L^{s,\tau}_{2m}(\pi_1(B), w_1)$ fits into an exact sequence

$$0 \to L^s_{2m}(\pi_1(B), w_1) \to L^{s,\tau}_{2m}(\pi_1(B), w_1) \to Wh(\pi_1(B));$$

where $L^s_{2m}(\pi_1(B), w_1)$ denotes the Wall group and $Wh(\pi_1(B))$ the Whitehead group. Consequently, $L^{s,\tau}_{2m}(\{1\}), 0)$ is Z_2 for m odd and Z for m even.

Now let's look how this strategy works in our situation.

Let $H \searrow \mathrm{CP}^\infty$ be the complex Hopf line bundle, and let $p : \mathrm{BO}\langle 8\rangle \to BO$ be the 7-connected cover. Write $\mathrm{MO}\langle 8\rangle$ for the Thom spectrum of the universal bundle over $\mathrm{BO}\langle 8\rangle$. Consider the map

$$\xi = (-3H) \times p : \mathrm{CP}^\infty \oplus \mathrm{BO}\langle 8\rangle \to BO$$

Note that ξ is 5-connected. We shall see latter that a focal manifold with cohomology ring $Z[x,z]/\{x^3 = 2z, z^2 = 0\}$, say F_+, and $X_5(2)$ are both of normal type ξ. For applying Theorem 3.3 to show F_+ and $X_5(2)$ are homeomorphic, it is a key to compute the bordism group $\Omega_{10}(\xi)$. Note that the bordism group $\Omega_{10}(\xi)$ does not change if CP^∞ is replaced by CP^m for m large enough. So we will sometimes think of CP^m instead of CP^∞.

By Thom's transversality theory there is a natural transformation of functors

$$\Omega_{10}(\xi) \cong \pi^s_{10}(\mathrm{M}(-3\mathrm{H}) \wedge \mathrm{MO}\langle 8\rangle),$$

the latter denotes the stable homotopy group of the spectrum $\mathrm{M}(-3\mathrm{H}) \wedge \mathrm{MO}\langle 8\rangle$ (cf. Switzer [13] 12.30).

Let $\zeta \searrow \mathrm{CP}^m$ be a vector bundle of rank q. $M(\zeta)$ denotes its Thom spectrum. Consider the natural inclusion $CP^{m-1} \subset \mathrm{CP}^m$, which has the Hopf complex line bundle H as normal bundle. It is not difficult to show that the fibre inclusion $\mathbf{R}^q \to \zeta$ induces a cofibration of the Thom spectrums

$$S^0 \subset M(\zeta) \to \sum\nolimits^2 \wedge M(\zeta|_{CP^{m-1}} \oplus H).$$

This leads to a puppe type exact sequence of the $\mathrm{MO}\langle 8\rangle$-bordism theory.

Lemma 3.4 *The following is an exact sequence*

$$\pi^s_{10}(MO\langle 8\rangle) \to \pi^s_{10}(MO\langle 8\rangle \wedge M(-3H)) \xrightarrow{\tau} \pi^s_8(MO\langle 8\rangle \wedge M(-2H)) \xrightarrow{\delta} \pi^s_9(MO\langle 8\rangle).$$

Remark: Each homomorphism in the exact sequence is functorial.

Now we present a topological explanation of the homomorphism

$$\tau : \pi^s_{10}(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-3\mathrm{H})) \to \pi^s_8(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-2\mathrm{H}))$$

in the exact sequence above. Let $[M, \tilde{\nu}] \in \Omega_{10}(\xi)$. Consider the transversal preimage of $\tilde{\nu}^{-1}(CP^{m-1} \times \mathrm{BO}\langle 8\rangle) \subset M$, it is a codimension 2 submanifold with stable normal bundle $\gamma_M \oplus \tilde{\nu}^*(H)$. This construction leads to a well defined homomorphism

$$\Omega_{10}(\xi) \to \Omega_8(\xi').$$

Identifying $\pi^s_{10}(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-3\mathrm{H}))$ with $\Omega_{10}(\xi)$, $\pi^s_8(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-2\mathrm{H}))$ with $\Omega_8(\xi')$ where $\xi' = (-2H) \oplus p : \mathrm{CP}^\infty \times \mathrm{BO}\langle 8\rangle \to BO$, this gives the homomorphism τ. The same construction also works in the topological category.

Proposition 3.5

$$\pi^s_{10}(MO\langle 8\rangle \wedge M(-3H))/\pi^s_{10}(MO\langle 8\rangle) \cong Z \oplus Z \ \ and$$

$[M, \tilde{\nu}^*] \in \Omega_{10}(\xi)$ is ξ-bordant to a topological sphere iff
(1).$\tilde{\nu}^*(x^5) = 0$
(2). $\tilde{\nu}^*(x) \cup p_2(M) = 0$
where $x \in H^2(CP^\infty)$ is a generator and $p_2(M)$ the second Pontryagin class of M.

Proof: By lemma 3.4, $\pi^s_{10}(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-3\mathrm{H}))/\pi^s_{10}(\mathrm{MO}\langle 8\rangle) \cong \mathrm{Im}\tau$. As the Stiefel-Whitney classes $w_2(-2H) = 0$ and $w_4(-2H) \neq 0$, by using Adams spectral sequence it was showed in [5] that

$$\pi^s_8(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-2\mathrm{H}))/\pi^s_8(\mathrm{MO}\langle 8\rangle) \cong Z.$$

and the integer part is detected by the cohomology product. To show $\mathrm{Im}\tau \cong Z \oplus Z$, it is sufficient to prove the restriction of δ on the possible torsion subgroup of $\pi^s_8(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-2\mathrm{H}))$ (which is produced by the image of the homomorphism $\pi^s_8(\mathrm{MO}\langle 8\rangle) \to \pi^s_8(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-2\mathrm{H}))$) is injective.

Consider the inclusion $i : S^2 \to \mathrm{CP}^\infty$, it induces an commutative diagram of the Thom spectrum

$$\begin{array}{ccc} \Sigma^{-2}\mathrm{CP}^2 = M(i^*(H)) & \to & \mathrm{M}(-3\mathrm{H}) \\ \uparrow & & \uparrow \\ S^0 & = & S^0 \end{array}$$

which gives the following commutative diagram of the puppe exact sequences

$$\begin{array}{ccccccc} \cdots \to & \pi^s_{10}(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-3\mathrm{H})) & \xrightarrow{\tau} & \pi^s_8(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-2\mathrm{H})) & \xrightarrow{\delta} & \pi^s_9(\mathrm{MO}\langle 8\rangle) \\ \uparrow\cong & \uparrow & & \uparrow & & \cong\uparrow \\ \cdots \to & \pi^s_{10}(\mathrm{MO}\langle 8\rangle \wedge \Sigma^{-2}CP^2) & \xrightarrow{\tau} & \pi^s_8(\mathrm{MO}\langle 8\rangle) & \xrightarrow{\delta} & \pi^s_9(\mathrm{MO}\langle 8\rangle) \end{array}$$

The bottom is the long generalized homology exact sequence of the cofibration $S^0 \to S^0 \cup_\eta e^2 \to S^2$ where $\eta \in \pi_1(S^0)$ is a generator. Hence $\delta : \pi^s_8(\mathrm{MO}\langle 8\rangle) \to \pi^s_9(\mathrm{MO}\langle 8\rangle)$ is just the composition homomorphism with η whose restriction on torsion subgroup is injective by referring to Toda's table for the stable homotopy groups of sphere. Recall $\Omega^{MO\langle 8\rangle}_8 \cong \pi^s_8(\mathrm{MO}\langle 8\rangle) \cong Z \oplus Z_2$ and the integer part is detected by the signature . By the commutative diagram above it follows that $\pi^s_8(\mathrm{MO}\langle 8\rangle) \to \pi^s_8(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-2\mathrm{H}))$ is injective and the restriction of $\delta : \pi^s_8(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-3\mathrm{H})) \to \pi^s_9(\mathrm{MO}\langle 8\rangle)$ on torsion subgroup is injective. Thus $\mathrm{Im}\tau \cong Z \oplus Z$ and the first assertion follows.

Let $[M, \tilde{\nu}] \in \Omega_{10}(\xi) \cong \pi^s_{10}(\mathrm{MO}\langle 8\rangle \wedge \mathrm{M}(-3\mathrm{H}))$ and $\tilde{\nu}^*(x^5) = 0 = \tilde{\nu}^*(x) \cup p_2(M)$. Consider the transversal preimage $N = \tilde{\nu}^{-1}(\mathrm{CP}^{\infty-1} \times \mathrm{BO}\ \langle 8\rangle) \subset M$. It is easy to prove that sig$N = 0$ and $\tilde{\nu}^*(x^4)[N] = 0$. By [5] again $[N, \tilde{\nu}] = \tau[M, \tilde{\nu}]$ is ξ' bordant to a homotopy sphere, a torsion element. On the other hand, Imτ is torsion free. Thus $\tau[M,\tilde{\nu}]=0$ and hence $[M,\tilde{\nu}]$ is ξ bordant to element of $\Omega_{10}^{MO\langle 8\rangle}\cong\pi_{10}^{s}(\mathrm{MO}\langle 8\rangle$ which is represented by ten dimensional homotopy sphere. This completes the proof. ♠

Let $X_5(2)=\{[z_0,\cdots z_6]\in \mathrm{CP}^6|z_0^2+\cdots+z_6^2=0\}$ be the degree 2 Fermat hypersurface, and let $y\in H^2(X_5(2))$ be the hyperplane class. Note that $y^5[X_5(2)]=2$, the first Pontryagin class $p_1(X_5(2))=3y^2$ and $p_2(X_5(2))=9y^4$. Combining Theorem 2.1 it follows that one of the focal manifolds, say F_+, has the same cohomology rings and the first Pontryagin classes. Since F_+ is a codimension 3 submanifold of the sphere, the second Pontryagin class of the normal bundle vanishes. Thereby $p_2(F_+)=9x^4$. The following theorem concludes that F_+ and $X_5(2)$ are in fact homeomorphic to each other.

Theorem 3.6 *Let F_+ be the focal manifold of a Dupin hypersurface in S^{13} with $g=6$. Suppose that the cohomology ring $Z[x,z]/\{x^3=2z,z^2=0\}$. Then F_+ is homeomorphic to the 5 dimensional degree 2 Fermat hypersurface $X_5(2)$ in CP^6(In fact they are almost diffeomorphic).*

Proof: First note that $\pi_i(F_+)=\pi_i(X_5(2))=0$ for $i=3$ or 4. Let $x\in H^2(F_+)$(resp. $y\in H^2(X_5(2))$) be a specific generator and let $f:F_+\to\mathrm{CP}^\infty$ (resp. $g:X_5(2)\to\mathrm{CP}^\infty$) be the corresponding map. Then $f:F_+\to\mathrm{CP}^\infty$(resp. $g:X_5(2)\to CP^\infty$) is a 5-equivalence. Let H be the Hopf complex line bundle over CP^∞ and $\gamma(-)$ denote the normal bundle of $-$. It is easy to check that the bundle $(\gamma(F_+)-f^*(-3H))$ (resp. $(\gamma(X_5(2))-g^*(-3H))$ is a spin bundle with the first Pontryagin class zero, thereby its classifying map can be lifted to a map $\tilde{\nu}_f:F_+\to\mathrm{BO}\langle 8\rangle$ (resp.$\tilde{\nu}_g:X_5(2)\to\mathrm{BO}\langle 8\rangle$). Thus $f\times\tilde{\nu}_f:F_+\to\mathrm{CP}^\infty\times\mathrm{BO}\langle 8\rangle$ (resp. $g\times\tilde{\nu}_g:X_5(2)\to\mathrm{CP}^\infty\times\mathrm{BO}\langle 8\rangle$) is a normal 4-smoothing in ξ.

By Proposition 3.5, $(F_+,f\times\tilde{\nu}_f)$ and $(X_5(2),g\times\tilde{\nu}_g)\#\Sigma^{10}$ are ξ bordant where Σ^{10} is a homotopy sphere. By Theorem 3.3, F_+ and $X_5(2)\#\Sigma^{10}$ are stably diffeomorphic. Note that $H_5(X_5(2))=H_5(F_+)=0$. By the cancellation Theorem (cf. Netsvetaev [12]) it follows that $X_5(2)\#\Sigma^{10}$ and F_+ are diffeomorphic to each other. So our proof is completed by recalling the generalized Poincaré conjecture of dimension greater than 5. This completes the proof. ♠

4 Proof of Theorem 1.1

Let $M\subset S^7$ be a Dupin hypersurface, and let F be a focal manifolds. Different from the case of isoparametric hypersurfaces, the focal manifolds may not be taut submanifolds with Z_2-coefficients [16]. Note that F is nonorientable and M is a nonorientable circle bundle over F[7].

By [7] $\pi_1(M) \cong Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset S^3$, the fibre homotopy exact sequence argument shows that $\pi_1(F) \cong Z_2$. Let $\tilde{F}$ be the universal covers of F.

Proposition 4.1 *$\tilde{F}$ is diffeomorphic to $S^2 \times S^3$.*

Proof: Recall Smale's classification theorem for 5 manifolds asserts that the second homology group determines the diffeomorphism type of simply connected spin 5-manifold. As $F \subset S^7$ is a codimension 2 submanifold, it is well known that the second Stiefel-Whitney class $\bar{w}_2$ of its normal bundle vanishes. Note that $\bar{w}_2 = w_1^2 + w_2$. So $w_2(\tilde{F}) = 0$. By [7] the second homology group of F is isomorphic to Z. Thus the proof follows. ♠

Recall that $H_2(F) = 0$(compare [7]3.16). Combining the proposition above it is easy to check that the 2-type X of F is the associated RP^2 bundle over HP^∞ of the canonical bundle $BS^1 \to BS^3$. Let $\eta \to X$ be the nontrivial line bundle and $p : BSpin \to BO$ the natural projection. Consider the map $\xi = \eta \oplus p : X \times BSpin \to BO$. ξ is 3 coconnected. Let $f : F \to X$ be a map which induces isomorphisms on the first and second homotopy groups. Note that $-f^*(\eta + \tau(F)) : F \to BO$ is a spin bundle. Let $\nu_f : F \to BSpin$ denote a lifting. Thus $f \times \nu_f : F \to X \times BSpin$ is a 2-smoothing which represents a bordism class in the group $\Omega_5(\xi) \cong \Omega_6^{spin}(M\eta)$, where $M\eta$ denotes the Thom complex of η.

Lemma 4.2 $\Omega_6^{spin}(M\eta) \cong ker d_4$
where $d_4 : E_{5,1}^4 \cong H_5(M\eta; Z_2) \to E_{1,4}^4 \cong H_1(M\eta, Z) \cong Z_2$ is the differential in the Atiyah-Hirzebruch Spectral sequence.

Proof: As X has a cell-decomposition with 6-dimensional skeleton a manifold CP^3/Z_2 , which is a RP^2-bundle over S^4 associated to the S^2 bundle over S^4 with total space CP^3. By the Atiyah-Hirzebruch spectral sequence, the only nontrivial E^2-terms converging to $\Omega_6^{spin}(M\eta)$ is $E_{5,1}^2 \cong H_4(X, Z_2) \cong Z_2$.

The differential $d_2 : E_{7,0}^2 \to E_{5,1}^2$ and $d_2 : E_{5,1}^2 \to E_{3,2}^2$ are both trivial since they are dual to $\mu \circ Sq^2$ and Sq^2, where μ is the mod2 reduction. Therefore $E_{5,1}^\infty \cong ker d_4$ and the proof follows. ♠

Lemma 4.3 *$M\eta$ has a cell-decomposition with 6 dimensional skeleton $(RP^3 \cup_\alpha e^5) \cup_\beta e^6$, where $\alpha \in \pi_4(RP^3) \cong Z_2$ is the generator and $Sq^1(x) \neq 0$ for the generator $x \in H^5(M\eta, Z_2)$.*

Theorem 4.4 $\Omega_6^{spin}(M\eta) = 0$

Proof: By Lemma 4.2 we need only to show that d_4 is an isomorphism. Note that $\Omega_5^{spin}(M\eta)$ is a group of order 4 or 8 by the Atiyah-Hirzebruch spectral sequence, which is contributed by the $E^2_{5,0}$, $E^2_{3,2}$ and $E^2_{1,4}$. It is easy to see that the former two terms do survive in the E^∞ terms and $E^\infty_{1,4}=0$ if and only if $d_4: E^4_{5,1}\to E^4_{1,4}$ is an isomorphism. Thereby we are done if $\Omega_5^{spin}(M\eta)$ is of order 4.

Consider the exact sequence

$$\Omega_5^{spin}(RP^3)\to\Omega_5^{spin}(M\eta)\to\Omega_5^{spin}(S^5\cup_2 e^6\cup\cdots)\to\tilde{\Omega}_4^{spin}(RP^3)\to\tilde{\Omega}_4^{spin}(M\eta)\to 0$$

In the sequence above, $\Omega_4^{spin}(RP^3)$ is of order 4 and $\Omega_5^{spin}(RP^3)$ is of order at most 4 by the Atiyah-Hirzebruch spectral-sequence. Thus $\Omega_5^{spin}(M\eta)$ is of order 4 iff $\tilde{\Omega}_4^{spin}(M\eta)$ is of order 2.

As $\tilde{\Omega}_4^{spin}(M\eta)\cong\tilde{\Omega}_4^{spin}(RP^3\cup_\alpha e^5)$, where α is as in Lemma 4.3. By the exact sequence

$$\tilde{\Omega}_4^{spin}(S^4)\to\tilde{\Omega}_4^{spin}(RP^3)\to\tilde{\Omega}_4^{spin}(RP^3\cup_\alpha e^5)\to 0$$

the proof can be deduced if α_* is nonzero.

Note that $\Sigma RP^3\simeq\Sigma RP^2\vee S^4$, let $\beta\in\pi_3(RP^3)$ denotes the generator, i.e, the 2-fold covering map. We claim that $p\circ\Sigma\beta\in\pi_4(\Sigma RP^2\vee S^4)\in\pi_4(S^3)$ is a generator, where $p:\Sigma RP^2\vee S^4\to\Sigma RP^2\to S^3$ the collapse map. In fact, $\Sigma RP^4=\Sigma RP^3\cup_{\Sigma\beta}e^5=(\Sigma RP^2\vee S^4)\cup_{\Sigma\beta}e^5$, by collapsing $S^2\vee S^4$ we obtain a map $\pi:\Sigma RP^4\to S^3\cup_{p\circ\Sigma\beta}e^5$ such that

$$\pi^*: H^3(S^3\cup_{p\circ\Sigma\beta}e^5;Z_2)\to H^3(\Sigma RP^4;Z_2)\text{ and }\pi^*: H^5(S^3\cup_{p\circ\Sigma\beta}e^5;Z_2)\to H^5(\Sigma RP^4;Z_2)$$

are isomorphisms. Note that $\pi^*Sq^2(z)=Sq^2(\pi^*z)\neq 0$, here $z\in H^3(S^3\cup_{p\circ\Sigma\beta}e^5;Z_2)$ is the generator. Thus $p\circ\Sigma\beta$ is not null homotopy and hence $p\circ\Sigma\alpha\in\pi_5(S^3)$ is the generator too. This proves that $[\alpha]\in\pi_4^s(RP^3)\cong\pi_4^s(S^3)\oplus\pi_4^s(RP^2)$ is nontrivial and the image of the second component in $\pi_4^s(S^2)$ is nonzero too.

On the other hand, $\pi_4^s(RP^3)\to\Omega_4^{spin}(RP^3)$ is a well- defined homomorphism and the image of $[\alpha]$ under this homomorphism is nontrivial corresponding to the term $E^2_{2,2}$ in the AHSS. Therefore $\tilde{\Omega}_4^{spin}(RP^3\cup_\alpha e^5)$ is of order 2 and we are done. This completes the proof. ♠

Proof of lemma 4.3: First note that X has a cell-decomposition with 4-skeleton $RP^2\cup_\phi e^4$, where $\phi\in\pi_3(RP^2)\cong Z$ is a generator. Let $\eta\to X$ the nontrivial line bundle, then the Thom complex $M\eta$ has a cell-decomposition with 5-skeleton $RP^3\cup_\psi e^5$ for certain $\psi\in\pi_4(RP^3)\cong Z_2$. We want to show that ψ is not null homotopy. Otherwise, we obtain $H_5(M\eta;Z[Z_2])\cong Z_2[Z_2]$.

On the other hand, as the collapse of the disc bundle $D\eta \to M\eta$ induces an isomorphism on the fundmental groups. Thus the universal cover of $M\eta$ equals to $(\tilde{D\eta}/S_1)/S_2$, where $S_1 \cup (-S_2)$ is the boundary of $\tilde{D\eta}$. In fact $\tilde{D\eta} = CP^\infty \times [0,1]$, so $S_1 \cong S_2 \cong CP^\infty$. Thus we obtain that $H_5((\tilde{D\eta}/S_1)/S_2) \cong Z$.

This completes the proof. ♠

Now we are ready to prove our theorem advertised in the introduction.

Proof of Theorem 1.1: The dimension of a Dupin hypersurface M with six distinct principal curvatures are either 6 or 12 corresponding to the case of multiplicities $m = 1$ or 2 respectively. Any Dupin hypersurface is the normal bundle of each focal manifolds in the sphere. By Theorem 3.6, in the case of $m = 2$, one of the focal manifolds is homeomorphic to $X_5(2)$ and it turns out the hypersurface is homeomorphic to the normal sphere bundle of an embedding of $X_5(2)$ in S^{13} and we are done in this case.

For the case of $m = 1$, as we saw above, F has the normal 2-type $\xi = \eta \oplus p : X \times BSpin \to BO$ as that of $S^3 \times RP^2$. By Theorem 4.4, there is a 6-manifold (W, g) with boundary $(F, f \times \nu_f) \sqcup (S^3 \times RP^2, i)$, where $i : S^3 \times RP^2 \xrightarrow{proj} RP^2 \xrightarrow{inclu} X$. By Theorem 3.3 (c.f. also Kreck[9] Theorem 5.2 (b)), there is a unique obstruction $\theta \in L_6^{s,\tau}(Z_2, -, S(W))$ to make W to be a s-cobordism, where the obstruction monoid is a quotient of the group $L_6^{s,\tau}(Z_2, -)$.

By the exact sequence below Theorem 3.3(cf.also [9] lemma 4.1), the obstruction monoid $L_6^{s,\tau}(Z_2, -)$ is of order 2. Thus $L_6^{s,\tau}(Z_2, -, S(W))$ is of order at most 2. By [17] the homomorphism $L_6^s(1) \to L_6^s(Z_2, -)$ induced by the coefficients inclusion is an isomorphism, the generator of the former group is realized geometrically by a nontrivial framed $S^3 \times S^3$. If $\theta \neq 0$, then we can do the connected sum of nontrivial framed $S^3 \times S^3$ with W to cancel the obstruction. Thus F and $S^3 \times RP^2$ are s-cobordant. Applying the s-cobordism theorem it follows that F is diffeomorphic to $S^3 \times RP^2$.

As M is a nontrivial circle bundle over $F \cong S^3 \times RP^2$, which is determined by its twisted Euler class $e \in H^2(S^3 \times RP^2, Z_-) \cong Z_-$. $e(mod 2) = w_2 = 0$. Thus e is an even number. It is easy to verify that $\pi_1(M) \cong Q_{4|e|}$. Comparing this with the fact of $\pi_1(M) \cong Q_8$ we obtain that $e = \pm 2$ and M is diffeomorphic to $S^3 \times S^3/Q_8$. By the fact $SU(3)/T^2$ is a homogeneous example of isoparametric hypersurface with $g = 6$ and $m = 1$ we have that $SU(3)/T^2$ is diffeomorphic to $S^3 \times S^3/Q_8$. This completes the proof. ♠

5 Appendix: Smooth S^2-bundles over 10-complex

Let $\Pi : BSO(3) \to BSO$ be the inclusion. Up to homotopy, Π is a fibration with fibre the Stiefel manifold $SO/SO(3)$. We have seen in section 3 that the Dupin hypersurface is homeomorphic to a smooth S^2 bundle over $X_5(2)$. Motivated by this, in this section we study the number of stably equivalent smooth $SO(3)$ bundles over ten dimensional CW complex and when a stable bundle over a 11 dimensional complex has geometric dimension 3. The method to do this is to construct the Moore-Postnikov factorazition for the fibration $\Pi : BSO(3) \to BSO$ as far as we need and compute the indeterminacy of various obstructions along the line of Thomas [15].

For the sake of simplicity, throughout this section let w_i denote the i -th universal Stiefel-Whitney class, $K_i = K(Z_2, i)$ the Eilenberg-Maclane space and l_i for its basic class. Let p_i be the i-th universal Pontryagin class.

Theorem 5.1 *Let $\xi : X \to BSO$ be an oriented stable bundle ; for $dimX \leq 11$ the obstructions to represent ξ by a rank 3 bundle are as follows:*
$w_4(\xi) \in H^4(X; Z_2)$

$(\theta_6, p_2(\xi)) \in \frac{Ker\{(w_2(\xi)+Sq^2):H^6(X,Z_2)\to H^8(X,Z_2)\}}{Im\{(w_2(\xi)+Sq^2)Sq^1:H^3(X,Z_2)\to H^6(X,Z_2)\}} \oplus H^8(X, Z)$

$(\theta_7, \theta_9, \theta_{11}) \in \frac{H^7(X,Z_2)}{(Sq^2+w_2(\xi))H^5(X,Z_2)} \oplus \frac{\{ker(w_2(\xi)+Sq^2):H^9(X,Z_2)\to H^{11}(X,Z_2)\}}{(Sq^2+w_2(\xi))H^7(X,Z)} \oplus \frac{H^{11}(X,Z_3)}{(P_3^1+p_1(\xi))H^7(X,Z_3)}$

$\theta_8 \in \frac{\{KerSq^2:H^8(X,Z_4)\to H^{10}(X,Z_2)\}}{j\circ(w_2(\xi)+Sq^2)H^6(X,Z_2)}$

$\theta_9' \in \frac{\{KerSq^2:H^9(X,Z_2)\to H^{11}(X,Z_2)\}}{Im\{Sq^2:H^7(X,Z_4)\to H^9(X,Z_2)\}}$

$\theta_{10} \in \frac{H^{10}(X,Z_2)}{Im\{Sq^2:H^8(X,Z_2)\to H^{10}(X,Z_2)\}}$

$\beta_{10} \in \frac{H^{10}(X,Z_2)}{Im\{(w_2(\xi)+Sq^2):H^8(X,Z_2)\to H^{10}(X,Z_2)\}}$

where $(w_2(\xi) + Sq^2)\theta_7 \in Sq^1 H^8(X, Z_2)$, $(w_2(\xi) + Sq^2)\beta_{10} \in (w_3(\xi) + Sq^3)Sq^1 H^8(X, Z_2)$, $j : H^8(X, Z_2) \to H^8(X, Z_4)$ the $\times 2$ homomorphism and P_3^1 the first mod 3 p^{th} power operation.

$$
\begin{array}{ccccc}
K_9 \times K_9 & \xrightarrow{i_{11}} & E_{11} & & \\
 & & \downarrow \Pi_{11} & & \\
K_8 & \xrightarrow{i_{10}} & E_{10} & \xrightarrow{\theta_{10} \times \beta_{10}} & K_{10} \times K_{10} \\
 & & \downarrow \Pi_{10} & & \\
K(Z_4, 7) & \xrightarrow{i_9} & E_9 & \xrightarrow{\theta'_9} & K_9 \\
 & & \downarrow \Pi_9 & & \\
K_6 \times K_8 \times K(Z_3, 10) & \xrightarrow{i_8} & E_8 & \xrightarrow{\tilde{\theta}_8} & K(Z_4, 8) \\
 & & \downarrow \Pi_8 & & \\
K_5 \times K(Z, 7) & \xrightarrow{i_7} & E_7 & \xrightarrow{\theta_7 \times \theta_9 \times \theta_{11}} & K_7 \times K_9 \times K(Z_3, 11) \\
 & & \downarrow \Pi_7 & & \\
K_3 & \xrightarrow{i_5} & E_5 & \xrightarrow{\theta_6 \times p_2/2} & K_6 \times K(Z, 8) \\
 & & \downarrow \Pi_5 & & \\
 & & BSO & \xrightarrow{w_4} & K_4
\end{array}
$$

From the Moore-Postnikov factorization of $\Pi : BSO(3) \to BSO$ above one obtains immediately that

Corollary 5.2 $\pi_i(SO/SO(3)) \cong \begin{array}{cc} 0 & 1,2,4 \\ Z_2 & 3,5,6 \\ Z \oplus Z_4 & 7 \\ Z_2 \oplus Z_2 & 8,9 \\ Z_3 & 10 \end{array}$

Using the Postnikov tower we can also say something about the number of stably equivalent to ξ principal $SO(3)$ bundles. If η_1 and η_2 are both stable equivalent to ξ. Denote by $\delta_i(\eta_1, \eta_2)$ for the i-th obstruction. The result is as follows.

Theorem 5.3 *Let η_1 and η_2 be two stably equivalent to ξ principal $SO(3)$ bundles over X. For $dimX \leq 10$ the obstructions to finding an isomorphism between η_1 and η_2 are as follows:*
$\delta_3 \in \{Ker(w_2(\xi) + Sq^2)Sq^1 : H^3(X, Z_2) \to H^6(X, Z_2)$

$\delta_5 \in \frac{Ker\{(w_2(\xi)+Sq^2):H^5(X,Z_2)\to H^7(X,Z_2)\}}{Im\{(w_2(\xi)+Sq^2)Sq^1:H^2(X,Z_2)\to H^5(X,Z_2)\}}$

$\delta_7 \in Ker\{(w_2(\xi) + Sq^2) : H^7(X, Z) \to H^9(X, Z_2)\}$

$\delta_6 \in \frac{Ker\{j\circ(w_2(\xi)+Sq^2):H^6(X,Z_2)\to H^8(X,Z_2)\}}{Im\{(w_2(\xi)+Sq^2):H^4(X,Z_2)\to H^6(X,Z_2)\}}$

$\delta_8 \in \frac{Ker\{(w_2(\xi)+Sq^2):H^8(X,Z_2)\to H^{10}(X,Z_2)\}}{Im\{(w_2(\xi)+Sq^2):H^6(X,Z_2)\to H^8(X,Z_2)\}}$

$\delta_{10} \in \frac{H^{10}(X,Z_3)}{Im\{(p_1(\xi)+P_3^1):H^6(X,Z)\to H^{10}(X,Z_3)\}}$

$\delta_7' \in \frac{Ker\{Sq^2:H^7(X,Z_4)\to H^9(X,Z_2)\}}{Im\{j\circ Sq^2:H^5(X,Z_2)\to H^7(X,Z_4)\}}$

$\delta_8' \in \frac{Ker\{Sq^2:H^8(X,Z_2)\to H^{10}(X,Z_2)\}}{Im\{Sq^2:H^6(X,Z_4)\to H^8(X,Z_2)\}}$

$\delta_9 \in \frac{H^9(X,Z_2)}{Im\{Sq^2:H^7(X,Z_2)\to H^9(X,Z_2)\}}$

$\beta_9 \in H^9(X,Z_2)$

Proof of Theorem 1.2: By Theorem 1.1, any Dupin hypersurface with $g=6, m=2$ is homeomorphic to the normal sphere bundle of an embedded $X_5(2)$. Up to homeomorpism, it is a bundle stably equivalent to the stable normal bundle ξ of $X_5(2)$. We claim that set Δ of isomorphism class of $SO(3)$-bundles stably equivalent to ξ is of order at most 3. Let $z \in H^6(X_5(2),Z_2)$ be the generator. By Wu formula $Sq^2 : H^8(X_5(2),Z_2) \to H^{10}(X_5(2),Z_2)$ is an isomorphism since $w_2(X_5(2)) \neq 0$. The duality argument shows that $z \cup w_2^2 \neq 0$. Thus $Sq^2(z)\cup w_2 = Sq^2(z\cup w_2) + z\cup w_2^2 = 0$. By Theorem 4.2 the probably nontrivial obstructions to finding a homotopy between two elements in Δ are δ_6; δ_8; δ_8' and δ_{10} which lie in the following quotient groups respectively:
$\frac{Ker\{j\circ(w_2(\xi)+Sq^2):H^6(X_5(2),Z_2)\to H^8(X_5(2),Z_2)\}}{Im\{(w_2(\xi)+Sq^2):H^4(X_5(2),Z_2)\to H^6(X_5(2),Z_2)\}}$;
$\frac{Ker\{(w_2(\xi)+Sq^2):H^8(X_5(2),Z_2)\to H^{10}(X_5(2),Z_2)\}}{Im\{(w_2(\xi)+Sq^2):H^6(X_5(2),Z_2)\to H^8(X_5(2),Z_2)\}}$
$\frac{Ker\{Sq^2:H^8(X_5(2),Z_2)\to H^{10}(X_5(2),Z_2)\}}{Im\{Sq^2:H^6(X_5(2),Z_4)\to H^8(X_5(2),Z_2)\}}$ and
$\frac{H^{10}(X_5(2),Z_3)}{Im\{(p_1(\xi)+P_3^1):H^6(X_5(2),Z)\to H^{10}(X_5(2),Z_3)\}}$.

It is easy to check that the former three groups are zero and so there is no the corresponding obstructions. The last group is Z_3. Thus the order of Δ is at most 3. This completes the proof. ♠

For the same reason one can prove easily that

Corollary 5.4 *Let M be a homotopy CP^5. Then any two rank 3 vector bundles ξ and ξ' over M with $p_1(\xi) = p_1(\xi') \neq 0(mod3)$ are isomorphic.*

Proof of Theorem 5.1: As $\Pi^*(w_4) = 0$, we take the first stage of the Postnikov tower as follows

$$\begin{array}{ccccc} K_3 & \xrightarrow{i_5} & E_5 & & \\ & & \downarrow \Pi_5 & & \\ & & BSO & \xrightarrow{w_4} & K_4 \end{array}$$

Denote by $q_5 : BSO(3) \to E_5$ the lift of $\Pi : BSO(3) \to BSO$.

To compute the cohomology ring $H^*(E_5, Z_2)$, we make use of the Serre spectral sequence of the fibration $\Pi_5 : E_5 \to BSO$. By [13] the cohomology ring $H^*(K_3, Z_2)$ is a free polynomial algebra with generators $Sq^I(l_3) = Sq^{i_1}Sq^{i_2}\cdots Sq^{i_s}$ where $I = (i_1, i_2, \cdots, i_s)$ is admissible and the excess $e(I) < 3$. Let τ be the transgression, modulo decomposible terms we have

$$\tau(l_3) = w_4, \tau(Sq^1 l_3) = Sq^1 w_4 = w_5, \tau(Sq^2 l_3) = Sq^2 w_4 = w_6,$$
$$\tau(Sq^3 l_3) = w_7, \tau(Sq^2 Sq^1 l_3) = 0, \tau(Sq^4 Sq^2 l_3) = w_{10},$$
$$\tau(Sq^4 Sq^1 l_3) = w_9, \tau(Sq^5 Sq^2 l_3) = w_{11}.$$

Thus it is readily to see that

$$H^*(E_5, Z_2) \cong Z_2[w_2, w_3, w_8] \otimes [\theta_6, Sq^1\theta_6, Sq^4\theta_6, Sq^5\theta_6] \cdots (4.1)$$

up to dimension 11 where $\theta_6 \in H^6(E_5, Z_2)$ be the element charaterized by $q_5^*(\theta_6) = 0$ and $i_5^*\theta_6 = Sq^2Sq^1l_3$.

Let $\mu : K_3 \times E_5 \to E_5$ be the homotopy action of K_3 on E_5. By [15] one can readily see that

$$\mu^*(\theta_6) = 1 \times \theta_6 + Sq^1 l_3 \times w_2 + Sq^2Sq^1 l_3 \times 1$$

From this formula we can deduce that $\nu^*(Sq^2\theta_6 + w_2\theta_6) = 0$, where $\nu = \mu(id \times q_5) : K_3 \times BSO(3) \to E_5$. It is easy to check that ν^* is a monomorphism for $* \leq 9$ except in the dimension 8, $ker\nu^* = Z_2$ generated by w_8. Thus $Sq^2\theta_6 + w_2\theta_6 = 0$ or w_8. The latter case can not happen through checking the example QP^2, which is 7 connected with nonzero w_8. This shows that

$$Sq^2\theta_6 + w_2\theta_6 = 0 \cdots (4.2)$$

Similarly, $Sq^2Sq^1\theta_6 + w_3\theta_6 = 0 \cdots (4.3)$

To build the next stage, note first that $\Pi_5^*(p_2)(mod2) = \Pi_5^* {w_4}^2 = 0$, so there is an element in $H^8(E_5, Z)$ whose image under the homomorphism $H^8(E_5, Z) \xrightarrow{\times 2} H^8(E_5, Z)$ is $\Pi_5^*(p_2)$. We abbreviate this element by $p_2/2$. It is easy to verify that $p_2/2(\mathrm{mod}2) = w_8$ and $i_5^*(p_2/2) = 0$. The next stage of the Postnikov tower is as follows

$$\begin{array}{ccccc} K_5 \times K(Z,7) & \xrightarrow{i_7} & E_7 & & \\ & & \downarrow{\scriptstyle \Pi_7} & & \\ & & E_5 & \xrightarrow{\theta_6 \times p_2/2} & K_6 \times K(Z,8) \end{array}$$

Denote by $q_7 : BSO(3) \to E_7$ for the lift of $q_5 : BSO(3) \to E_5$.

Now let's compute the ring $H^*(E_7, Z_2)$. Modulo decomposible terms, the transgression satisfy the follows

$$\tau(l_5) = Sq^1\theta_6, \tau(l_7) = w_8, \tau(Sq^1 l_5) = Sq^1\theta_6,$$

$$\tau(Sq^4 l_5) = Sq^4\theta_6, \tau(Sq^5 l_5) = Sq^5\theta_6, \tau(Sq^4 l_7) = Sq^4 w_8 = w_{12}.$$

Thus these elements can not survive in their E^∞-terms. One can see further that

$$Sq^2 l_5, Sq^3 l_5, Sq^2 Sq^2 l_5 = Sq^3 Sq^1 l_5, Sq^4 Sq^2 l_5,$$

$$Sq^2 Sq^1 l_5; Sq^2 l_7 \text{ and } Sq^3 l_7$$

are all of the elements survive in E^∞-terms.

Thus one can obtain

$$H^*(E_7, Z_2) \cong Z_2[w_2, w_3] \otimes Z_2[\theta_7, \theta_8, \theta_9, , Sq^1\theta_7, Sq^2\theta_7, Sq^2 Sq^1\theta_7, Sq^4\theta_7; Sq^1\theta_9] \cdots (4.4)$$

up to dimension 11 where $\theta_i \in \mathrm{Ker} q_7^*$ satisfy

$$i_7^*(\theta_7) = Sq^2 l_5,$$

$$i_7^*(\theta_8) = Sq^2 Sq^1 l_5,$$

$$i_7^*(\theta_9) = Sq^2 l_7.$$

As above let $\mu : K_5 \times K(Z, 7) \times E_7 \to E_7$ be the homotopy action of $K_5 \times K(Z, 7)$ on E_7, $\gamma = \mu(id \times q_7) : K_5 \times K(Z, 7) \times BSO(3) \to E_7$. By [15] we have the following exact sequence

$$0 \to H^*(E_7, Z_2) \xrightarrow{\gamma^*} H^*(K_5 \times K(Z, 7) \times BSO(3)) \xrightarrow{\tau_1} H^{*+1}(E_6)$$

for $* \leq 16$. By the Künneth formula one can write

$$\mu^*(\theta_7) = 1 \times \theta_7 + (Sq^2 l_5 \times 1) + a(l_5 \times w_2)$$

for some $a \in Z_2$. From the exact sequence above one can check easily that $a = 1$ and so we get

$$\mu^*(\theta_7) = 1 \times \theta_7 + Sq^2 l_5 \times 1 + l_5 \times w_2.$$

Note that the elements θ_9 with the properties above may not be unique. With a suitable choice, we obtain

$$\mu^*(\theta_8) = 1 \times \theta_8 + Sq^2 Sq^1 l_5 \times 1 + l_5 \times w_3;$$

$$\mu^*(\theta_9) = 1 \times \theta_9 + l_7 \times w_2 + Sq^2 l_7 \times 1;$$

By the exact sequence we can get further the relations

$$Sq^2\theta_7 + w_2\theta_7 + Sq^1\theta_8 = 0; \cdots (4.5)$$

$$Sq^2\theta_8 + w_3\theta_7 = 0; \cdots (4.6)$$
$$Sq^2\theta_9 + w_2\theta_9 = 0. \cdots (4.7)$$

Now let's look at the cohomology ring of E_7 with Z_3-coefficient. Let $u_7 \in H^7(K(Z,7), Z_3)$ $\cong Z_3$ denote the generator, the transgression τ satisfies $\tau(u_7) = p_2/2(\mathrm{mod}3) \in H^8(E_5, Z_3) \cong H^8(BSO, Z_3)$; $\tau(P_3^1 u_7) = P_3^1(p_2/2) = -p_1p_2/2(\mathrm{mod}3)$ is decomposible. Let $\beta : H^i(-, Z_3)$ $\to H^{i+1}(X, Z_3)$ be the Bockstein operator associated to the exact sequence $0 \to Z_3 \to Z_9 \to Z_3 \to 0$. Then $\tau(\beta P_3^1(u_7)) = \beta(p_1p_2) = 0$. Thus $P_3^1 u_7$ and $\beta P_3^1(u_7)$ survive in E^∞ and it shows that

$$H^*(E_7, Z_3) \cong Z_3[p_1] \otimes Z_3[\theta_{11}, \beta\theta_{11}]$$

up to dimension 12 where θ_{11} lie in the kernel of q_7^* and $i_7^*(\theta_{11}) = P_3^1(u_7)$.

It is readily to see that

$$\mu^*(\theta_{11}) = 1 \times \theta_{11} + p_1 \times u_7 + P_3^1 u_7$$

Let $h : X \to E_7$ be a map. For any $x \in H^7(X, Z)$, x gives a map $f : X \to K(Z, 7)$. Denote by $g = \mu \circ (h \times f) : X \to E_7$. Then $g^*(\theta_{11}) - h^*(\theta_{11}) = (P_3^1 x + p_1 \cup x)$. This shows that the indeterminacy of the obstruction θ_{11} is $Coker\{(P_3^1 + p_1) : H^7(X, Z_3) \to H^{11}(X, Z_3)\}$.

The next stage of the Postnikov tower is as follows

$$\begin{array}{ccccc} K_6 \times K_8 \times K(Z_3, 10) & \xrightarrow{i_8} & E_8 & & \\ & & \downarrow \Pi_8 & & \\ & & E_7 & \xrightarrow{\theta_7 \times \theta_9 \times P_3^1} & K_7 \times K_9 \times K(Z_3, 11) \end{array}$$

Let $q_8 : BSO(3) \to E_8$ be a lift of $q_7 : BSO(3) \to E_7$.

Let $\alpha \in H^8(K_6, Z_4) \cong Z_2$ be the generator, α corresponds to the cohomology operator $j \circ Sq^2$ where $j : H^*(-, Z_2) \to H^*(-, Z_4)$ is the homomorphism of coefficients inclusion. Consider the Serre spetral sequence of $\Pi_8 : E_8 \to E_7$ with Z_4 coefficients. Note first, up to decomposible terms, $\tau(\alpha) = j \circ Sq^2\theta_7 = 0$ by the relation $Sq^2\theta_7 + Sq^1\theta_8 + w_2\theta_7 = 0$. Thus there exist an $\tilde{\theta}_8 \in H^8(E_8, Z_4)$ such that $i_8^*(\tilde{\theta}_8) = \alpha$.

On the other hand, the Z_2 coefficients spectral sequence shows that $i_8^* : H^8(E_8, Z_2) \to H^8(K_6 \times K_8 \times K(Z_3, 10), Z_2)$ is trivial and $H^8(E_8, Z_2)/(w_2, w_3) \cong Z_2$ generated by θ_8. The following commutative diagram of Bockstein sequences

$$\begin{array}{ccccccc} H^8(E_8, Z_2) & \xrightarrow{j} & H^8(E_8, Z_4) & \xrightarrow{\rho} & H^8(E_8, Z_2) & \to & H^9(E_8, Z_2) \\ \downarrow i_8^* & & \downarrow i_8^* & & \downarrow i_8^* & & \downarrow i_8^* \\ H^8(F, Z_2) & \xrightarrow{j} & H^8(F, Z_4) & \xrightarrow{\rho} & H^8(F, Z_2) & \to & H^9(F, Z_2) \end{array}$$

where $F = K_6 \times K_8 \times K(Z_3, 10)$, then shows that $\rho(\tilde{\theta}_8) \in H^8(E_8, Z_2)$ is nonzero and equal to θ_8 modulo decomposible term. Using the fact $Sq_4^1\alpha \in H^9(K_6, Z_2) = Sq^3$ we see $i_8^*(Sq_4^1\tilde{\theta}_8) = Sq^3 l_6$ where $Sq_4^1 \in H^9(K(Z_4, 8), Z_2)$ is the generator. Thus it is straightward to show that

$$H^*(E_8, Z_2) \cong Z_2[w_2, w_3] \otimes Z_2[Sq_4^1\tilde{\theta}_8, \theta_8, \beta_{10}]$$

$$H^*(E_8, Z_3) \cong Z_3[p_1]$$

up to dimension 11 where $Sq_4^1\tilde{\theta}_8$ and β_{10} both lie in the kernel of q_8^* and satisfy

$$i_8^*(Sq_4^1\tilde{\theta}_8,) = Sq^3 l_6;$$

$$i_8^*(\beta_{10}) = Sq^2 l_8.$$

$$\mu^*(\beta_{10}) = 1 \times \beta_{10} + l_8 \times w_2 + Sq^2 l_8 \times 1$$

This shows that as before the obstruction β_{10} satisfies that

$$(Sq^2 + w_2)\beta_{10} \in (Sq^3 Sq^1 + w_3 Sq^1)H^8(X, Z_2)$$

The homotopy action μ of K_6 on E_8 satisfies the relation

$$\mu^*(\tilde{\theta}_8) = j(Sq^2)l_6 \times 1 + j(w_2 \times l_6) + 1 \times \tilde{\theta}_8.$$

This shows the indeterminacy of the obstruction $\tilde{\theta}_8$ is $Coker\{j(Sq^2 + w_2) : H^6(X, Z_2) \to H^8(X, Z_4)\}$.

The next stage of the Postnikov tower is as follows

$$\begin{array}{ccccc} K(Z_4, 7) & \xrightarrow{i_9} & E_9 & & \\ & & \downarrow\Pi_9 & & \\ & & E_8 & \xrightarrow{\tilde{\theta}_8} & K(Z_4, 8) \end{array}$$

Let $q_8 : BSO(3) \to E_9$ be a lift of q_7.

It is very easy to see that $H^8(E_9, Z_2)/(w_2, w_3) = 0$, $H^9(E_9, Z_2)/(w_2, w_3) \cong Z_2$ generated by an θ_9' and $H^{10}(E_9, Z_2)/(w_2, w_3) \cong Z_2[Sq^1\theta_9', \beta_{10}]$, where θ_9' satisfies

$$i_9^*(\theta_9') = Sq^2 l_7',$$

where $l_7' \in H^7(K(Z_4, 7), Z_2)$ is the generator. By $Sq^2\tilde{\theta}_8 = 0$ it follows readliy that

$$Sq^2\theta_9' = 0 \cdots (4.8)$$

Now consider the principal fibration

$$\begin{array}{ccccc} K_8 & \xrightarrow{i_{10}} & E_{10} & & \\ & & \downarrow\Pi_{10} & & \\ & & E_9 & \xrightarrow{\tilde{\theta}_9'} & K_9 \end{array}$$

Let $q_{10} : BSO(3) \to E_{10}$ be a lift of q_9. It is readily to see that

$$H^*(E_{10}, Z_2) = Z_2[w_2, w_3] \otimes Z_2[\beta_{10}, \theta_{10}]$$

up to dimension 10 where θ_{10} satisfies

$$i_{10}^*(\theta_{10}) = Sq^2 l_8.$$

$$\mu^*(\theta_{10}) = 1 \times \theta_{10} + Sq^2 l_8 \times 1.$$

The next stage is as follows

$$\begin{array}{ccccc} K_9 \times K_9 & \xrightarrow{i_{11}} & E_{11} & & \\ & & \downarrow \Pi_{11} & & \\ & & E_{10} & \xrightarrow{\theta_{10} \times \beta_{10}} & K_{10} \times K_{10} \end{array}$$

It is easy to verify that $H^*(E_{11}, Z_2) \cong Z_2[w_2, w_3]$ and $H^*(E_{11}, Z_p) \cong Z_p[p_1]$

for odd prime p and $* \leq 11$.

Let $q_{11} : BSO(3) \longrightarrow E_{11}$ be a lift of q_{10}. Then q_{11}induces isomorphisms of the cohomology rings with Z_2 and Z_p coefficients with p odd prime up to dimension 11. Thus by Whitehead theorem it is readily to see that q_{11} is 11 connected and thereby the Postnikov tower of $\Pi : BSO(3) \to BSO$ is as advertised in the beginning of this section. This proves the theorem 5.1. ♠

By the general arguments of James-Thomas [8](or cf. Switzer [14]) the proof of Theorem 5.2 follows simply by looking up the Moore-Postnikov factorization of $\Pi : BSO(3) \to BSO$ as we showed above. ♠

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