Splitting Patterns and Trace Forms

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ABSTRACT. The splitting pattern of a quadratic form q over a field k consists of all distinct Witt indices that occur for q over extension fields of k. In small dimensions, the complete list of splitting patterns of quadratic forms is known. We show that *all* splitting patterns of quadratic forms of dimension at most nine can be realized by trace forms.

1. Constraints on splitting patterns

Let q be an anisotropic quadratic form of dimension $n \geq 2$ over a field k of characteristic $\neq 2$. Let

$$(i_0(q) = 0, i_1(q), i_2(q), \ldots, i_{h(q)}(q) = [n/2])$$

denote the *splitting pattern* of q; that is, $i_0(q) < i_1(q) < \cdots < i_{h(q)}(q)$ are all distinct Witt indices that occur for q over arbitrary field extensions of k. We refer to M. Knebusch's originating work on generic splitting of quadratic forms, [12], [13]. The index $i_j(q)$ occurs over the field K_j, $0 \leq j \leq h(q)$, in a *generic splitting tower* $K_0 = k, K_1 = k(q), \ldots, K_{h(q)}$ of q over k.

In general, it is still a wide open problem to determine all tuples of strictly increasing integers that are splitting patterns of quadratic forms. However, for an *excellent* quadratic form q over k, the splitting pattern is known in arbitrary dimension, [12, Section 7], [8, Cor. 2.8]. It depends only on the dimension n of q; we denote it by

$$(i_0(n) = 0, i_1(n), \ldots, i_{h(n)}(n) = [n/2])$$

and have for the smallest positive Witt index $i_1(n)$:

1.1 EXAMPLE. Let q be an anisotropic excellent quadratic form of dimension n over k. Write $n = 2^r + d$, where 2^r is the largest power of 2 less than n. Then the Witt index $i_1(n)$ of q over its function field $k(q)$ is given by

$$i_1(n) = d.$$

For arbitrary anisotropic forms we recall from [6], [9, Cor. 1.13]:

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1.2 Theorem. *Let q be an anisotropic quadratic form of dimension n over k. Then:*

$$i_1(q) \leq i_1(n).$$

In particular, we have by 1.1 and 1.2 for any field k:

1.3 Corollary. *Let q be an anisotropic quadratic form of dimension $n = 2^r + 1$ over k. Then:*

$$i_1(q) = 1.$$

It is natural to ask whether the inequality in 1.2 generalizes for higher Witt indices. Consider anisotropic quadratic forms q of dimension n with $h(n) \geq 2$. In other words, $n \neq 2^r, 2^r - 1$, and this implies $h(q) \geq 2$ by the characterization of all forms q of height $h(q) = 1$ in [12, Thm. 5.8], [18]. So, the indices $i_2(n)$ and $i_2(q)$ are defined, and we obtain:

1.4 Proposition. *Let q be an anisotropic quadratic form over k of dimension n with $h(n) \geq 2$. If $i_1(q) = i_1(n)$, then*

$$i_2(q) \leq i_2(n).$$

Proof. Let q_1 denote the anisotropic kernel of q over its function field $k(q)$. We have

$$i_2(q) = i_1(q) + i_1(q_1).$$

On the other hand,

$$i_2(n) = i_1(n) + i_1(n - 2i_1(n))$$

since $n - 2i_1(n)$ is the dimension of the anisotropic kernel of an n-dimensional anisotropic excellent quadratic form over its function field.

By assumption, $i_1(q) = i_1(n)$. Thus our claim $i_2(q) \leq i_2(n)$ amounts to showing $i_1(q_1) \leq i_1(n - 2i_1(n))$. Again by the assumption, the last inequality is equivalent to

$$i_1(q_1) \leq i_1(n - 2i_1(q));$$

that is,

$$i_1(q_1) \leq i_1(\dim q_1),$$

which is the result of Thm. 2.1 applied to the anisotropic from q_1 over $k(q)$. $\square$

By [9, Cor. 1.13], our assumption $i_1(q) = i_1(n)$ in Prop. 1.4 implies that q becomes an anisotropic Pfister neighbor over some field extension of k. The proof of the proposition carries over to the analogous result for higher Witt indices:

1.5 Corollary. *Let q be an anisotropic quadratic form over k of dimension n with min $\{h(n), h(q)\} \geq j + 1$. If $i_j(q) = i_j(n)$, then*

$$i_{j+1}(q) \leq i_{j+1}(n).$$

Proof. Replace q_1 in the proof of Prop. 1.4 by q_j, the anisotropic kernel of q over the field K_j in a generic splitting tower of q over k. We then have

$$i_{j+1}(q) = i_j(q) + i_1(q_j),$$
$$i_{j+1}(n) = i_j(n) + i_1(n - 2i_j(n))$$

and application of Thm. 2.1 to q_j yields the claim. □

Inspection of the list of splitting patterns in low dimensions, see 2.3 below with the patterns of excellent forms in each dimension given first, shows that the assumption $i_j(q) = i_j(n)$ in Cor. 1.5 is *not* necessary for anisotropic forms q of dimension $n \leq 10$.

1.6 Question. Can one drop in Cor. 1.5 the assumption that $i_j(q) = i_j(n)$?

2. Trace forms and splitting patterns

Let E be a commutative étale algebra of dimension n over a field k of characteristic $\neq 2$; so E is a product of separable extension fields of k. The *trace form* $\langle E \rangle$ of E, defined by

$$\langle E \rangle(x) = \mathrm{Tr}_{E/k}(x^2) \qquad \text{for } x \in E,$$

is a regular n-dimensional quadratic form over k; we just refer to [1], [2], [15].

So far, a basic question, which quadratic forms are (isometric to) trace forms, has been answered over arbitrary fields only in low dimensions. For a regular quadratic form $q = \langle a_1, \ldots, a_n \rangle$ over k let $w_m(q)$, $m = 0, 1, 2, \ldots$, denote the *Stiefel-Whitney invariants* of q, [3], given in terms of cup products as

$$w_m(q) = \sum_{i_1 < i_2 < \cdots < i_m} (a_{i_1}) \cup \cdots \cup (a_{i_m}) \in \mathrm{H}^m(k) = \mathrm{H}^m(\mathrm{Gal}\,(k_s/k), \mathbb{Z}/2\mathbb{Z}),$$

where k_s denotes a separable closure of k. The Galois cohomology groups $\mathrm{H}^0(k)$, $\mathrm{H}^1(k)$, and $\mathrm{H}^2(k)$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$, k^*/k^{*2}, and $Br_2(k)$; thus $w_0(q)$, $w_1(q)$, and $w_2(q)$ are the invariants given by the dimension mod 2, determinant $d = d(q)$, and Hasse-Witt invariant of q, respectively. We summarize the characterization due to M. Epkenhans, M. Krüskemper, and J.-P. Serre, compare [1, Thm. 11]:

2.1 Theorem. *A regular quadratic form q over k of dimension $n \leq 7$ is (isometric to) a trace form (of some étale k-algebra E) if and only if, over k,*
a. $n = 1$ *and* $q \cong \langle 1 \rangle$,
b. $n = 2$ *and* q *contains* $\langle 2 \rangle$,
c. $n = 3$ *and* q *contains* $\langle 1, 2 \rangle$,
d. $n = 4$, q *contains* $\langle 1 \rangle$, *and* $w_3(q) = 0$,
e. $n = 5$, q *contains* $\langle 1, 1 \rangle$, *and* $w_3(q) = 0$,
f. $n = 6$, q *contains* $\langle 1, 2 \rangle$, *and, over* $k(\sqrt{2d})$, q *contains* $\langle 1, 1, 2 \rangle$,
g. $n = 7$, q *contains* $\langle 1, 1, 2 \rangle$, *and, over* $k(\sqrt{2d})$, q *contains* $\langle 1, 1, 1, 2 \rangle$.

For $n \leq 5$ the results can be found in [5] and [16]; the $n = 6$ characterization from [17] was afterwards independently discovered and proved in [4]; the result for $n = 7$ then follows from a construction in [14].

By considering orthogonal sums of trace forms, we immediately obtain from 2.1 for arbitrary dimensions:

2.2 Lemma. *Let q be a regular quadratic form over k of dimension n. Let $m = [n/2]$ and put*

$$q_0 = \begin{cases} m \times \langle 1 \rangle & \text{if } n \equiv 0 \mod 4 \\ (m+1) \times \langle 1 \rangle & \text{if } n \equiv 1 \mod 4 \\ (m-1) \times \langle 1 \rangle \perp \langle 2 \rangle & \text{if } n \equiv 2 \mod 4 \\ m \times \langle 1 \rangle \perp \langle 2 \rangle & \text{if } n \equiv 3 \mod 4. \end{cases}$$

If q contains q_0, then q is a trace form.

Proof. By 2.1.d, every regular quadratic form q of dimension four that contains $\langle 1, 1 \rangle$ is a trace form. Thus every regular q of dimension $n \equiv 0 \bmod 4$ that contains $m \times \langle 1 \rangle$ is a trace form. The proof is analogous for $n \equiv 1, 2,$ *or* $3 \bmod 4$ based on 2.1 a – d. □

The following is a complete list of splitting patters in dimension at most ten.

2.3 Summary. The splitting patterns of anisotropic quadratic forms q of dimension $n \leq 10$ are given by:

$n = 2$ (0,1)
$n = 3$ (0,1)
$n = 4$ (0,2), (0,1,2)
$n = 5$ (0,1,2)
$n = 6$ (0,2,3), (0,1,3), (0,1,2,3)
$n = 7$ (0,3), (0,1,2,3)
$n = 8$ (0,4), (0,2,4), (0,1,2,4), (0,1,3,4), (0,1,2,3,4)
$n = 9$ (0,1,4), (0,1,2,3,4)
$n = 10$ (0,2,4,5), (0,2,3,5), (0,1,3,5), (0,2,3,4,5), (0,1,2,4,5), (0,1,2,3,5), (0,1,2,3,4,5).

Proof. By [11] or by [7], the tuples (0,2,3,4) and (0,1,5) fail to be splitting patterns in dimension eight and ten, respectively (see also [10]). Hence, for $n \leq 9$, the list follows from [9, Ex. 1.16]. For $n = 10$ the splitting pattern of excellent forms is (0,2,4,5), thus $i_1(10) = 2$ and we conclude from Thm. 1.2 that $i_1(q)$ is 1 or 2 for every anisotropic quadratic form of dimension ten. There are *at most four* splitting patterns in dimension ten with $i_1(q) = 1$ since there are exactly four splitting patterns in dimension eight, the ones different from (0,4), that would not give rise to the excluded tuple (0,1,5). There are *at most three* splitting patterns in dimension ten with $i_1(q) = 2$ since there are exactly three splitting patterns in dimension six. Thus, every splitting pattern in dimension ten is one of the seven tuples listed above. Moreover, all of the seven tuples occur as splitting patterns of quadratic forms in dimension 10; see Ex. 3.1 below. □

We have set up 2.1–2.3 in order to investigate which splitting patterns of quadratic forms are splitting patterns of trace forms. It might be a bit of a surprise to learn:

2.4 Theorem. *All splitting patterns of (anisotropic) quadratic forms of dimension $n \leq 9$ can be realized by trace forms.*

Proof. In any dimension n, the sum $n \times \langle 1 \rangle$ of n squares is a trace form; it is anisotropic for example over $\mathbb{Q}$. Hence we can realize in each dimension the splitting pattern of excellent forms, always listed first in 2.3, by a trace form over $\mathbb{Q}$. In particular, this settles the dimensions $n = 2$, 3, and 5.

For $n = 4$ the splitting pattern (0,1,2) can be realized by the anisotropic quadratic form $\langle 1, 1, 1, X \rangle$ over $\mathbb{Q}(X)$, say. By 2.2, this form is a trace form.

For $n = 6$ consider the quadratic forms $\langle 1, 1, 2, X, Y, -2XY \rangle$ and $\langle 1, 1, 1, 1, X, Y \rangle$ over $k = \mathbb{Q}(X, Y)$. Both of them are trace forms, by 2.2. The first one is isometric to $\langle 1, 1, 2, X \rangle \perp Y\langle 1, -2X \rangle$, hence it is anisotropic over $\mathbb{Q}(X, Y)$ since $\langle 1, 1, 2, X \rangle$ and $\langle 1, -2X \rangle$ are anisotropic over $\mathbb{Q}(X)$. Clearly, the second one is anisotropic over $\mathbb{Q}(X, Y)$, too.

The form $\langle 1, 1, 2, X, Y, -2XY \rangle$ is of (signed) discriminant $+1$; that is, it is an Albert form, and Witt index 2 does not occur in its splitting pattern. So, by 2.3, its splitting pattern is (0,1,3).

The form $\langle 1, 1, 1, 1, X, Y \rangle$ splits exactly one hyperbolic plane over $k(\sqrt{-XY})$ and since it is of discriminant $\neq 1$, Witt index 2 will occur, too. So its splitting pattern is (0,1,2,3).

For $n = 7$ the quadratic form $\langle 1, 1, 1, 1, 1, X, Y \rangle$ is a trace form, by 2.2. It is anisotropic over $k = \mathbb{Q}(X, Y)$ and splits exactly one hyperbolic plane over $k(\sqrt{-XY})$. Thus, by 2.3, its splitting pattern is $(0, 1, 2, 3)$.

For $n = 8$ we have to realize the four splitting patterns in 2.3 that are not patterns of excellent forms. Consider the quadratic forms $q_1 = 6 \times \langle 1 \rangle \perp \langle X, X \rangle, q_2 = 5 \times \langle 1 \rangle \perp \langle X, Y, XY \rangle, q_3 = 7 \times \langle 1 \rangle \perp \langle X \rangle, q_4 = 6 \times \langle 1 \rangle \perp \langle X, Y \rangle$ over $k = \mathbb{Q}(X, Y)$. All of them are anisotropic over $\mathbb{Q}(X, Y)$; we check it for q_2 : since $\langle 1, 1, 1, 1, 1, X \rangle$ and $\langle 1, X \rangle$ are anisotropic over $\mathbb{Q}(X)$, the form $q_2 = \langle 1, 1, 1, 1, 1, X \rangle \perp Y\langle 1, X \rangle$ is anisotropic over $\mathbb{Q}(X, Y)$.

The form $q_1 = 6 \times \langle 1 \rangle \perp \langle X, X \rangle$ is isometric to $\langle 1, 1, 1, X \rangle \otimes \langle 1, 1 \rangle$, thus $q_1 \cong q' \otimes \langle 1, 1 \rangle$ with $\dim q' = 4$. Moreover if, for some extension field K of $k = \mathbb{Q}(X, Y)$, the form q_K is isotropic we may choose q'_K to be isotropic. Thus q_1 has splitting pattern (0,2,4), compare [9, Rem.3.3].

The form $q_2 = 5 \times \langle 1 \rangle \perp \langle X, Y, XY \rangle$ splits exactly one hyperbolic plane over $K = \mathbb{Q}(\sqrt{-7})$. Namely, K has level 4 and, over K, we have $5 \times \langle 1 \rangle \cong \langle 1, -1 \rangle \perp \langle -1, -1, -1 \rangle$, so $q_{2,K} \cong H \perp \langle -1, -1, -1, X, Y, XY \rangle$ with a 6-dimensional anisotropic kernel $\langle -1, -1, -1, X, Y, XY \rangle$. The form q_2 splits exactly two hyperbolic planes over $K = k(\sqrt{-XY})$. Namely, over this extension field K of k, q_2 is isometric to $4 \times \langle 1 \rangle \perp \langle 1, X, -X, -1 \rangle \cong H \perp H \perp 4 \times \langle 1 \rangle$ with a 4-dimensional anisotropic kernel $4 \times \langle 1 \rangle$. Since q_2 has discriminant 1, over no extension field it will have Witt index 3, and hence q_2 has splitting pattern (0,1,2,4).

The form $q_3 = 7 \times \langle 1 \rangle \perp \langle X \rangle$ splits no or exactly one hyperbolic plane over every extension field K of k of level ≥ 8, since $7 \times \langle 1 \rangle$ stays anisotropic over such a field K. The form q_3 splits three or four hyperbolic planes over every extension field K of k of level ≤ 4, since $7 \times \langle 1 \rangle$ splits completely over such a field K. The Witt index of $q_{3,K}$ is 1 over $K = k(\sqrt{-X})$, and the Witt index of $q_{3,K}$ is 3 over $K = k(\sqrt{-7})$. Thus q_3 has splitting pattern $(0, 1, 3, 4)$.

The form $q_4 = 6 \times \langle 1 \rangle \perp \langle X, Y \rangle$ splits exactly one hyperbolic plane over $k(\sqrt{-XY})$ with anisotropic kernel $6 \times \langle 1 \rangle$, it splits exactly two hyperbolic planes over $k(\sqrt{-7})$ with anisotropic kernel $\langle -1, -1, X, Y \rangle$, and it splits exactly three hyperbolic planes over $k(\sqrt{-1})$ with anisotropic kernel $\langle X, Y\rangle$. Hence q_4 has splitting pattern (0,1,2,3,4).

This concludes the discussion of the 8-dimensional case.

For $n = 9$ it is only left, by 2.3, to realize the pattern (0,1,2,3,4) by a trace form. Consider the quadratic form $7 \times \langle 1\rangle \perp \langle X, Y\rangle$ over $\mathbb{Q}(X, Y)$. By 2.2, this form is a trace form. It is anisotropic over $\mathbb{Q}(X, Y)$ and splits exactly two hyperbolic planes over $\mathbb{Q}(\sqrt{-X}, \sqrt{-Y})$, hence Witt index 2 occurs in its pattern. Thus, by 2.3, its splitting pattern is (0,1,2,3,4). $\square$

The statement of Thm. 2.4 can be sharpened:

2.5 Corollary. *All splitting patterns of (anisotropic) quadratic forms of dimension $n \leq 9$ can be realized by trace forms of field extensions of $\mathbb{Q}(X, Y)$.*

Proof. The proof of Thm. 2.4 shows that for $n \leq 9$ all splitting patterns can be realized by trace forms of some étale algebra over $\mathbb{Q}(X, Y)$. By the main result in [5], over Hilbertian fields K of characteristic zero, every trace form of an étale K-algebra of dimension $n \geq 3$ is isometric to the trace form of some field extension of K. Thus it is only left to discuss the case $n = 2$. The trace form $\langle 2, 2d\rangle$ of a real quadratic extension $\mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$ remains anisotropic over $\mathbb{Q}(X, Y)$ and has splitting pattern (0,1). $\square$

3. Appendix

We illustrate how to realize *all* seven splitting patterns in dimension ten listed in 2.3. The following example can also be obtained from the classification of splitting patterns in [7].

3.1 Example. The form $10 \times \langle 1\rangle$ over $\mathbb{Q}$ has pattern (0,2,4,5), the form

$$\langle 1, X, Y, Z, T, XZ, XT, YZ, YT, XYZT\rangle$$

over $\mathbb{Q}(\sqrt{-1})(X, Y, Z, T)$ has pattern (0,2,3,5), the form $7 \times \langle 1\rangle \perp \langle X, Y, -XY\rangle$ over $\mathbb{Q}(X, Y)$ has pattern (0,1,3,5), the form $7 \times \langle 1\rangle \perp \langle X, X, X\rangle$ over $\mathbb{Q}(X)$ has pattern (0,2,3,4,5), the form $8 \times \langle 1\rangle \perp \langle X, Y\rangle$ over $\mathbb{Q}(X, Y)$ has a pattern (0,1,2,4,5), the form $5 \times \langle 1\rangle \perp \langle 2\rangle \perp \langle X, Y, Z, -2XYZ\rangle$ over $\mathbb{Q}(X, Y, Z)$ has pattern (0,1,2,3,5), and the form $6 \times \langle 1\rangle \perp \langle X, Y, Z, T\rangle$ over $\mathbb{Q}(X, Y, Z, T)$ has pattern (0,1,2,3,4,5).

It follows from 2.2 that, except for the second pattern (0,2,3,5), all patterns in 3.1 have been realized by *trace forms*. The form we have chosen to realize (0,2,3,5) is the neighbor of the Pfister form $\langle\langle X, Y, Z, T\rangle\rangle$ given by the complement of the Albert form $\langle XY, ZT, XYT, XYZ, XZT, YZT\rangle$ over $\mathbb{Q}(\sqrt{-1})(X, Y, Z, T)$. We do *not* know if also (0,2,3,5) is the splitting pattern of some 10-dimensional trace form.

The splitting pattern of an excellent quadratic form is an example of a pattern that can be realized in arbitrary dimension n by a trace form, namely by $n \times \langle 1\rangle$ over $\mathbb{Q}$, say. In this regard, we add:

3.2 Proposition. *For every dimension n, the splitting pattern*

$$(0, 1, 2, 3, 4, \ldots, [n/2])$$

can be realized by a trace form.

Proof. Given n, let $m = [n/2]$ and consider the quadratic form q_0 over $\mathbb{Q}$ defined in 2.2. The dimension of q_0 is $n-m$. Let φ be the quadratic form $\langle X_1, X_2, \ldots, X_m\rangle$ over $\mathbb{Q}(X_1, X_2, \ldots, X_m)$. Then

$$q \ := q_0 \perp \varphi$$

is an anisotropic, n-dimensional quadratic form over $k = \mathbb{Q}(X_1, X_2, \ldots, X_m)$, and q is a trace form by 2.2. Over $k(\sqrt{-X_1})$, q has Witt index 1; over $k(\sqrt{-X_1}, \sqrt{-X_2})$, q has Witt index 2 (if $n \geq 4$), and so on. Since q_0 contains $(m-1) \times \langle 1\rangle$, all consecutive integers $0, 1, 2, \ldots, m-1, m = [n/2]$ occur in the splitting pattern of q. $\square$

In view of the 2.4, 3.2, one might want to ask:

3.3 Question. Are all splitting patterns of (anisotropic) quadratic forms realizable by trace forms?

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