A COVERING TECHNIQUE FOR DERIVED EQUIVALENCES

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Throughout this paper k denotes an algebraically closed field. The word algebra always means a basic finite-dimensional k-algebra with identity, and all categories and functors are assumed to be k-linear.

Consider a Brauer tree with e edges whose exceptional vertex has multiplicity m. Then Rickard proved that the algebra defined by this Brauer tree is derived equivalent to the symmetric Nakayama algebra with e isoclasses of simples and of Loewy length $em+1$ ([17, Theorem 4.2]). In particular this solves the following problem posed by Broué [3] affirmatively in the cyclic defect case:

> For a finite group G, is a block of kG with abelian defect derived equivalent to its Brauer correspondent?

In this paper we develop a "covering technique" for derived equivalences to give a new approach to this problem (see Example 5.2), which extends Rickard's result above. In fact, as an application, we classify representation-finite selfinjective algebras of type A_n up to derived equivalences.

In the sequel, we consider covering functors to "algebras", and for an algebra A we deal with its repetition (= repetitive "algebra")$A^{\mathbb{Z}}$, which is no longer an algebra with identity but can be regarded as a category. Accordingly, we regard all algebras as categories and work over categories.

Our investigation was motivated by looking quiver presentations of the principal block Λ of $kSL(2,4)$ and its Brauer correspondent Γ when k is of characteristic 2. As we will see in section 5 we can find algebras A and B such that there are Galois covering functors $A^{\mathbb{Z}} \longrightarrow \Lambda$, $B^{\mathbb{Z}} \longrightarrow \Gamma$. Further it is not hard to see that A is derived equivalent to B. We want to induce a derived equivalence between Λ and Γ from one between A and B. This observation leads us the following problems, whose answers provide a desired tool.

(1) When does a derived equivalence between algebras A and B yield a derived equivalence between $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$?

(2) When does a derived equivalence between categories R and S yield a derived equivalence between quotient categories R/G and S/H by groups G and H of

automorphisms of R and S?

In section 1 we answer the problem (1) to show that if two algebras are derived equivalent, then their repetitions are derived equivalent. It immediately follows from this that for algebras A and B of finite global dimension, the repetitions $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$ are stably equivalent iff $A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$ are derived equivalent. Next in section 2 we show that a Galois covering functor $R \longrightarrow R/G$ induces a Galois covering functor from a full subcategory of bounded homotopy category of finitely generated projective R-modules consisting of a complete list of indecomposable objects to a dominant part of the corresponding full subcategory for R/G with the same group G. We expect that this enables us to calculate the derived category for R/G using that for R. In section 3 we give an answer to the problem (2) using the theorem in section 2. In section 4 we give some applications of our technique. The first one gives an infinite sequence of derived equivalent pairs of algebras from one such pair. Next we classify selfinjective algebras of type A_n under derived equivalences. This classification shows us that being stably equivalent is equivalent to being derived equivalent also for selfinjective algebras of type A_n. Notice that our technique may be applied to standard representation-finite selfinjective algebras. In fact, according to Hughes and Waschbüsch [10], a standard representation-finite selfinjective algebra has the form $A^{\mathbb{Z}}/G$ for some tilted algebra A and for some group G of automorphisms of $A^{\mathbb{Z}}$. In the last section we give an example to illustrate the proof in the previous section, and we explicitly present our solution of Broué's problem for the group $SL(2, 4)$ above.

Preliminaries

As stated in the introduction we regard every algebra as a category, more precisely as a finite spectroid. Recall from Gabriel-Roiter [6] that a category A is called *svelte* (= skeletally small) if the isoclasses of objects of A form a set; and a svelte category A is called a *spectroid* (= a *locally finite-dimensional category* in [4]) if the following three conditions are satisfied:

(i) A is *basic*, i.e., distinct objects of A are not isomorphic;
(ii) A is *semiperfect*, i.e., every object of A has a local endomorphism ring; and
(iii) A is *pointwise finite*, i.e., the space $A(x, y)$ is finite-dimensional for every $x, y \in A$.

For a spectroid A, A is called *finite* if A has only a finite number of objects; and A is called *locally bounded* if for every $x \in A$, there are only finitely many $y \in A$ such that $A(x, y) \neq 0$ or $A(y, x) \neq 0$.

For a spectroid A, we denote by $\operatorname{Mod} A$ the category of all (right) A-modules; by $\operatorname{mod} A$ the full subcategory of $\operatorname{Mod} A$ consisting of finitely presented objects; by $\operatorname{Pro} A$ the full subcategory of $\operatorname{Mod} A$ consisting of projective objects; and by $\operatorname{pro} A$ the full subcategory of $\operatorname{Pro} A$ consisting of finitely generated projective objects. In addition, $\underline{\mathrm{mod}}A$ denotes the (projectively) stable category of $\mathrm{mod}\, A$, and we set $D := \mathrm{Hom}_k(-, k)$.

For an additive category $\mathcal{A}$, we denote by $\mathcal{C}(\mathcal{A})$ (resp. $\mathcal{C}^b(\mathcal{A})$) the category of differential complexes (resp. of bounded differential complexes) in $\mathcal{A}$, and by $\mathcal{H}(\mathcal{A})$ (resp. $\mathcal{H}^b(\mathcal{A})$) the corresponding homotopy category. When $\mathcal{A}$ is an abelian category, we denote by $\mathcal{D}(\mathcal{A})$, $\mathcal{D}^b(\mathcal{A})$ the corresponding derived category. For a triangulated category $\mathcal{T}$, and a class of objects E in $\mathcal{T}$, $\mathrm{tri}(E)$ denotes the smallest full triangulated subcategory of $\mathcal{T}$ closed under isomorphisms that contains E. Further if $\mathcal{T}$ has infinite direct sums, by $\mathrm{Tri}(E)$ we denote the smallest full triangulated subcategory of $\mathcal{T}$ closed under isomorphisms and infinite direct sums that contains E. For a category A, $\mathcal{X}_A$ denotes the full subcategory of $\mathrm{pro}A$ consisting of representable functors $A(-, x)$ for some $x \in A$. We regard $\mathcal{X}_A \subseteq \mathcal{H}^b(\mathrm{pro}A)$ by considering each $A(-, x)$ as a complex concentrated in degree 0. By $\eta = \eta_A : A \longrightarrow \mathcal{X}_A$ we denote the Yoneda equivalence defined by $\eta(x) = A(-, x)$ for all $x \in A$. As in the notation $A(-, x)$, by (-) or (?) we denote a place for a valuable. When we want to stress the difference of valuables, we use different symbols like $A(-, ?)$. We refer the definitions of S-functors and S-equivalences to Keller-Vossieck [12].

1. Derived equivalences of Repetitions

We start with a categorical version of Rickard's theorem [16, Theorem 6.4].

Proposition 1.1. *Let A and B be categories. Then the following are equivalent:*

(1) *There is an S-equivalence $\mathcal{D}(\mathrm{Mod}\, A) \longrightarrow \mathcal{D}(\mathrm{Mod}\, B)$;*
(2) *There is a full subcategory E of $\mathcal{D}(\mathrm{Mod}\, A)$ such that the following are satisfied:*
   (a) *For all $T \in E$, $\mathcal{D}(\mathrm{Mod}\, A)(T, -)$ commutes with direct sums;*
   (b) *For all $T, U \in E$ and for all $n \neq 0$, $\mathcal{D}(\mathrm{Mod}\, A)(T, U[n]) = 0$;*
   (c) $\mathrm{Tri}(E) = \mathcal{D}(\mathrm{Mod}\, A)$*; and*
   (d) *E is equivalent to B.*
(3) *There is a full subcategory E of $\mathcal{H}^b(\mathrm{pro}A)$ such that the following are satisfied:*
   (a) *For all $T, U \in E$ and for all $n \neq 0$, $\mathcal{H}^b(\mathrm{pro}A)(T, U[n]) = 0$;*
   (b) $\mathrm{tri}(E)$ *contains $\mathcal{X}_A$; and*
   (c) *E is equivalent to B.*

Proof. Consider A and B as DG-categories concentrated in degree 0. Then the equivalence of (1) and (2) follows from Keller [11, Corollary 9.2]. Note that for each $T \in \mathcal{D}(\mathrm{Mod}\, A)$, $\mathcal{D}(\mathrm{Mod}\, A)(T, -)$ commutes with direct sums iff T is isomorphic to an object of $\mathcal{H}^b(\mathrm{pro}A)$. Then the equivalence of (2) and (3) follows. □

Definition 1.2. We say that categories A and B are *derived equivalent* if one of the equivalent conditions above holds. In this case the triple (A, E, B) in (3) is called a *tilting triple*. When both A and B are spectroids, E in (3) is called a *tilting spectroid* for A. The S-equivalence $\mathcal{D}(\mathrm{Mod}\, A) \longrightarrow \mathcal{D}(\mathrm{Mod}\, B)$ constructed by an equivalence

$\psi : E \longrightarrow B$ as in Keller [11] is denoted by φ_ψ, which restricts to the equivalence $\eta_B \psi : E \longrightarrow B \longrightarrow \mathcal{X}_B$.

Lemma 1.3. *Let A be a svelte pointwise finite category. Then for all $T, U \in \mathcal{H}^b(\mathrm{pro}A)$, we have*

$$\mathcal{H}(\mathrm{Mod}\, A)(T, U \otimes_A DA(?, -)) \cong D(\mathcal{H}(\mathrm{Mod}\, A)(U, T)).$$

Proof. For all $x, y \in A$, we have a canonical isomorphism

$$\mathrm{Hom}_A(A(-,x), A(-,y) \otimes_A DA(?, -)) \cong D\,\mathrm{Hom}_A(A(-,y), A(-,x)).$$

Hence for all $X, Y \in \mathrm{pro}A$, there is an isomorphism

$$\mathrm{Hom}_A(X, Y \otimes_A DA(?, -)) \cong D\,\mathrm{Hom}_A(Y, X)$$

which is natural in X and Y. Let $T = (T^i, d_T^i), U = (U^i, d_U^i) \in \mathcal{H}^b(\mathrm{pro}A)$, and put $\bar{U}^i = U^i \otimes_A DA(?, -), d_{\bar{U}}^i = d_U^i \otimes_A DA(?, -)$. Since $T \in \mathcal{H}^b(\mathrm{pro}A)$, there are $m < n$ in $\mathbb{Z}$ such that $T^i = 0$ whenever $i < m$ or $n < i$. Put

$$\begin{aligned}
\delta_+ : \bigoplus_{i=m}^{n} \mathrm{Hom}_A(T^i, \bar{U}^{i-1}) &\longrightarrow \bigoplus_{i=m}^{n} \mathrm{Hom}_A(T^i, \bar{U}^i)\\
(f^i)_i &\mapsto (f^{i+1} d_T^i + d_{\bar{U}}^{i-1} f^i)_i\\
\delta_- : \bigoplus_{i=m}^{n} \mathrm{Hom}_A(T^i, \bar{U}^i) &\longrightarrow \bigoplus_{i=m}^{n} \mathrm{Hom}_A(T^i, \bar{U}^{i+1})\\
(g^i)_i &\mapsto (g^{i+1} d_T^i - d_{\bar{U}}^i g^i)_i.
\end{aligned}$$

Then by definition, $\mathcal{H}(\mathrm{Mod}\, A)(T, U \otimes_A DA(?, -)) = \mathrm{Ker}\,\delta_- / \mathrm{Im}\,\delta_+$. Now δ_+, δ_- can be expressed as follows:

$$\delta_+ = \begin{pmatrix}
\mathrm{Hom}_A(T^m, d_{\bar{U}}^{m-1}) & \mathrm{Hom}_A(d_T^m, \bar{U}^m) & & \text{\Large 0}\\
 & & \ddots & \\
 & \ddots & & \mathrm{Hom}_A(d_T^{n-1}, \bar{U}^{n-1})\\
\text{\Large 0} & & & \mathrm{Hom}_A(T^n, d_{\bar{U}}^{n-1})
\end{pmatrix}$$

$$\delta_- = \begin{pmatrix}
\mathrm{Hom}_A(T^m, d_{\bar{U}}^{m}) & -\mathrm{Hom}_A(d_T^m, \bar{U}^{m+1}) & & \text{\Large 0}\\
 & & \ddots & \\
 & \ddots & & -\mathrm{Hom}_A(d_T^{n-1}, \bar{U}^{n})\\
\text{\Large 0} & & & \mathrm{Hom}_A(T^n, d_{\bar{U}}^{n})
\end{pmatrix}.$$

Define $\tilde{\delta}_+, \tilde{\delta}_-$ as follows:

$$\tilde{\delta}_+ = \begin{pmatrix} D\operatorname{Hom}_A(d_U^{m-1}, T^m) & D\operatorname{Hom}_A(U^m, d_T^m) & & 0 \\ & & \ddots & \\ & \ddots & & D\operatorname{Hom}_A(U^{n-1}, d_T^{n-1}) \\ 0 & & & D\operatorname{Hom}_A(d_U^{n-1}, T^n) \end{pmatrix}$$

$$\tilde{\delta}_- = \begin{pmatrix} D\operatorname{Hom}_A(d_U^{m}, T^m) & -D\operatorname{Hom}_A(U^{m+1}, d_T^m) & & 0 \\ & & \ddots & \\ & \ddots & & -D\operatorname{Hom}_A(U^{n}, d_T^{n-1}) \\ 0 & & & D\operatorname{Hom}_A(d_U^{n}, T^n) \end{pmatrix}.$$

Then δ_+, δ_- are isomorphic to $\tilde{\delta}_+, \tilde{\delta}_-$, respectively, and

$$\operatorname{Ker}\delta_- / \operatorname{Im}\delta_+ \cong \operatorname{Ker}\tilde{\delta}_- / \operatorname{Im}\tilde{\delta}_+.$$

Since A is pointwise finite, $\tilde{\delta}_+, \tilde{\delta}_-$ are in $\operatorname{mod} k$. Hence

$$\operatorname{Ker}\tilde{\delta}_- / \operatorname{Im}\tilde{\delta}_+ \cong D(\operatorname{Ker}(D\tilde{\delta}_+)/\operatorname{Im}(D\tilde{\delta}_-)).$$

Now by the matrix expressions of $\tilde{\delta}_+$ and $\tilde{\delta}_-$ we have

$$\begin{aligned} D\tilde{\delta}_- : \bigoplus_{i=m+1}^{n+1} \operatorname{Hom}_A(U^i, T^{i-1}) &\longrightarrow \bigoplus_{i=m}^{n} \operatorname{Hom}_A(U^i, T^i) \\ (g^i)_i &\mapsto ((-d_T^{i-1})g^i + g^{i+1}d_U^i)_i \\ D\tilde{\delta}_+ : \bigoplus_{i=m}^{n} \operatorname{Hom}_A(U^i, T^i) &\longrightarrow \bigoplus_{i=m-1}^{n-1} \operatorname{Hom}_A(U^i, T^{i+1}) \\ (f^i)_i &\mapsto (-(-d_T^i)f^i + f^{i+1}d_U^i)_i. \end{aligned}$$

Thus

$$\operatorname{Ker}(D\tilde{\delta}_+)/\operatorname{Im}(D\tilde{\delta}_-) \cong \mathcal{H}(\operatorname{Mod} A)(U, T'),$$

where $T' := (T^i, -d_T^i)$. But obviously $T \cong T'$ in $\mathcal{C}(\operatorname{Mod} A)$. As a consequence, we have $\mathcal{H}(\operatorname{Mod} A)(T, U \otimes_A DA(?, -)) \cong D\mathcal{H}(\operatorname{Mod} A)(U, T)$. $\square$

We recall the definition of the repetition of a spectroid (Cf. [6]). Note that we slightly changed the definition.

Definition 1.4. Let A be a spectroid. We define a spectroid, the *repetition* $A^{\mathbb{Z}}$ of A as follows. Objects are the pairs $x^{[n]} := (x, n)$ with $x \in A$ and $n \in \mathbb{Z}$.

$$A^{\mathbb{Z}}(x^{[n]}, y^{[m]}) := \begin{cases} \{f^{[n]} := (f, n) | f \in A(x, y)\} & \text{if } m = n; \\ \{\varphi^{[n]} := (\varphi, n) | \varphi \in DA(y, x)\} & \text{if } m = n+1; \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The composition

$$A^{\mathbb{Z}}(y^{[m]}, z^{[l]}) \times A^{\mathbb{Z}}(x^{[n]}, y^{[m]}) \longrightarrow A^{\mathbb{Z}}(x^{[n]}, z^{[l]})$$

is given as follows:

(i) If $m = n, l = m$, then this is the composition of A:

$$A(y, z) \times A(x, y) \longrightarrow A(x, z).$$

(ii) If $m = n, l = m + 1$, then this is given by the right A-module structure of $DA(-, ?)$:

$$DA(z, y) \times A(x, y) \longrightarrow DA(z, x).$$

(iii) If $m = n + 1, l = m$, then this is given by the left A-module structure of $DA(-, ?)$:

$$A(y, z) \times DA(y, x) \longrightarrow DA(z, x).$$

(iv) Otherwise the composition is zero.

For every $n \in \mathbb{Z}$, we denote by $A^{[n]}$ the full subcategory of $A^{\mathbb{Z}}$ formed by $x^{[n]}$ with $x \in A$, and by $1^{[n]} : A \xrightarrow{\sim} A^{[n]} \hookrightarrow A^{\mathbb{Z}}, x \mapsto x^{[n]}$, the embedding functor.

Note that if a spectroid A is locally bounded, then so is $A^{\mathbb{Z}}$.

Theorem 1.5. *Let A and B be spectroids. If A and B are derived equivalent, then their repetitions are derived equivalent.*

Proof. Let (A, E, B) be a tilting triple. For every $n \in \mathbb{Z}$, $A^{\mathbb{Z}}(-, 1^{[n]}?)$ is an A-$A^{\mathbb{Z}}$-bimodule. For every $x \in A$, we have $A(-, x) \otimes_A A^{\mathbb{Z}}(-, 1^{[n]}?) \cong A^{\mathbb{Z}}(-, x^{[n]})$. Hence for every $P \in \mathcal{H}^b(\text{pro}A)$, we see $P^{[n]} := P \otimes_A A^{\mathbb{Z}}(-, 1^{[n]}?) \in \mathcal{H}^b(\text{pro}A^{\mathbb{Z}})$. Since E is a subcategory of $\mathcal{H}^b(\text{pro}A)$, we can define a full subcategory $\hat{E}$ of $\mathcal{H}^b(\text{pro}A^{\mathbb{Z}})$ consisting of $T^{[n]}$ for $T \in E$ and $n \in \mathbb{Z}$. We have only to show that $(A^{\mathbb{Z}}, \hat{E}, B^{\mathbb{Z}})$ is a tilting triple.

(a) For all $T^{[i]}, U^{[j]} \in \hat{E}$ and for all $n \neq 0$, $\mathcal{H}^b(\text{pro}A^{\mathbb{Z}})(T^{[i]}, U^{[j]}[n]) = 0$.

In fact, the left hand side is equal to

$$\begin{aligned}
&\mathcal{H}^b(\text{pro}A^{\mathbb{Z}})(T \otimes_A A^{\mathbb{Z}}(-, 1^{[i]}?), (U \otimes_A A^{\mathbb{Z}}(--, 1^{[j]}??))[n]) \\
\cong\ &\mathcal{H}^b(\text{pro}A^{\mathbb{Z}})(T \otimes_A A^{\mathbb{Z}}(-, 1^{[i]}?), U[n] \otimes_A A^{\mathbb{Z}}(--, 1^{[j]}??)) \\
\cong\ &\mathcal{H}(\text{Mod}\, A^{\mathbb{Z}})(T \otimes_A A^{\mathbb{Z}}(-, 1^{[i]}?), U[n] \otimes_A A^{\mathbb{Z}}(--, 1^{[j]}??)) \\
\cong\ &\mathcal{H}(\text{Mod}\, A)(T, \text{Hom}_{A^{\mathbb{Z}}}(A^{\mathbb{Z}}(-, 1^{[i]}?), U[n] \otimes_A A^{\mathbb{Z}}(--, 1^{[j]}??))) \\
\cong\ &\mathcal{H}(\text{Mod}\, A)(T, U[n] \otimes_A A^{\mathbb{Z}}(1^{[i]}?, 1^{[j]}??)) \\
\cong\ &\begin{cases} \mathcal{H}(\text{Mod}\, A)(T, U[n] \otimes_A A(?, ??)) \cong \mathcal{H}^b(\text{pro}A)(T, U[n]) = 0 & \text{if } j = i; \\ \mathcal{H}(\text{Mod}\, A)(T, U[n] \otimes_A DA(??, ?)) & \text{if } j = i + 1; \textit{and} \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Now in case $j = i+1$, we have by Lemma 1.3, $\mathcal{H}(\mathrm{Mod}\,A)(T, U[n] \otimes_A DA(??,?)) \cong D\mathcal{H}(\mathrm{Mod}\,A)(U[n], T) \cong D\mathcal{H}^b(\mathrm{pro}A)(U, T[-n]) = 0$.

(b) $\mathrm{tri}(\hat{E}) \supseteq \mathcal{X}_{A^{\mathbb{Z}}}$.

In fact, let $\mathcal{E}$ be the full subcategory of $\mathcal{H}^b(\mathrm{pro}A)$ formed by the objects P such that $P \otimes_A A^{\mathbb{Z}}(-, 1^{[i]}?) \in \mathrm{tri}(\hat{E})$ for all $i \in \mathbb{Z}$. Then $\mathcal{E} = \mathrm{tri}(\mathcal{E})$ because the functors $\mathcal{H}^b(\mathrm{pro}A) \longrightarrow \mathcal{H}^b(\mathrm{pro}A^{\mathbb{Z}}), P \mapsto P \otimes_A A^{\mathbb{Z}}(-, 1^{[i]}?)$ are S-functors. Since for all $i \in \mathbb{Z}$, and for all $T \in E$, $T \otimes_A A^{\mathbb{Z}}(-, 1^{[i]}?) = T^{[i]} \in \hat{E} \subseteq \mathrm{tri}(\hat{E})$, we have $E \subseteq \mathcal{E}$. Hence $\mathcal{E} \supseteq \mathrm{tri}(E) \supseteq \mathcal{X}_A$. Therefore, for all $x \in A$, and for all $i \in \mathbb{Z}$, we have $\mathrm{tri}(\hat{E}) \ni A(-, x) \otimes A^{\mathbb{Z}}(-, 1^{[i]}?) \cong A^{\mathbb{Z}}(-, x^{[i]})$. Thus $\mathrm{tri}(\hat{E}) \supseteq \mathcal{X}_{A^{\mathbb{Z}}}$.

(c) $\hat{E}$ is equivalent to $B^{\mathbb{Z}}$.

Let $\psi : E \longrightarrow B$ be an equivalence. For $T, U \in E$ put $\psi_U^T : E(T, U) \longrightarrow B(\psi T, \psi U)$ to be the isomorphism given by ψ. We construct an equivalence $\psi^{\mathbb{Z}} : \hat{E} \longrightarrow B^{\mathbb{Z}}$. For $T^{[n]} \in \hat{E}$, we define $\psi^{\mathbb{Z}}(T^{[n]}) := (\psi T)^{[n]}$. Then this gives a bijection on objects. Define the value of $\psi^{\mathbb{Z}}$ on morphisms as follows:

$$
\begin{aligned}
&\hat{E}(T^{[i]}, U^{[j]}) \\
\xrightarrow{\sim}\ &\mathcal{H}(\mathrm{Mod}\,A^{\mathbb{Z}})(T \otimes_A A^{\mathbb{Z}}(-, 1^{[i]}?), U \otimes_A A^{\mathbb{Z}}(--, 1^{[j]}??)) \\
\xrightarrow{\sim}\ &\mathcal{H}(\mathrm{Mod}\,A)(T, U \otimes_A A^{\mathbb{Z}}(1^{[i]}?, 1^{[j]}??)) \\
\xrightarrow{\sim}\ &\begin{cases} \mathcal{H}(\mathrm{Mod}\,A)(T, U) = E(T, U) \xrightarrow{\psi_U^T} B(\psi T, \psi U) & \text{if } j = i; \\ D\mathcal{H}(\mathrm{Mod}\,A)(U, T) = DE(U, T) \xrightarrow{(D\psi_T^U)^{-1}} DB(\psi U, \psi T) & \text{if } j = i+1; \textit{and} \\ 0 & \text{otherwise} \end{cases} \\
\xrightarrow{\sim}\ &B^{\mathbb{Z}}((\psi T)^{[i]}, (\psi U)^{[j]})) = B^{\mathbb{Z}}(\psi^{\mathbb{Z}} T^{[i]}, \psi^{\mathbb{Z}} U^{[j]})
\end{aligned}
$$

for $T^{[i]}, U^{[j]} \in \hat{E}$. If we could show that the defined correspondence $\psi^{\mathbb{Z}}$ is a k-functor, then by construction $\psi^{\mathbb{Z}}$ turns out to be an equivalence. It is obvious that $\psi^{\mathbb{Z}}$ preserves identities and that $\psi^{\mathbb{Z}}$ is k-linear. Hence it only remains to show that $\psi^{\mathbb{Z}}$ commutes with composition. Let $T^{[i]}, U^{[j]}, V^{[l]} \in \hat{E}$. We have to show the commutativity of the following diagram $(*)$:

$$
\begin{array}{ccc}
\hat{E}(U^{[j]}, V^{[l]}) \times \hat{E}(T^{[i]}, U^{[j]}) & \longrightarrow & \hat{E}(T^{[i]}, V^{[l]}) \\
{\scriptstyle (\psi^{\mathbb{Z}}, \psi^{\mathbb{Z}})}\downarrow & & \downarrow{\scriptstyle \psi^{\mathbb{Z}}} \\
B^{\mathbb{Z}}((\psi U)^{[j]}, (\psi V)^{[l]}) \times B^{\mathbb{Z}}((\psi T)^{[i]}, (\psi U)^{[j]}) & \longrightarrow & B^{\mathbb{Z}}((\psi T)^{[i]}, (\psi V)^{[l]})
\end{array}
$$

If the upper left corner or the lower right corner is zero, then $(*)$ is obviously commutative. Hence we may assume that $j \in \{i, i+1\}$, $l \in \{j, j+1\}$ and $l \in \{i, i+1\}$.

(i) Case $j = i, l = j$. In this case $i = j = l$ and $(*)$ is isomorphic to the following commutative diagram:

$$\begin{array}{ccc} E(U,V)\times E(T,U) & \longrightarrow & E(T,V) \\ {\scriptstyle (\psi^U_V,\psi^T_U)}\Big\downarrow & & \Big\downarrow{\scriptstyle \psi^T_V} \\ B(\psi U,\psi V)\times B(\psi T,\psi U) & \longrightarrow & B(\psi T,\psi V). \end{array}$$

(ii) Case $j = i, l = j+1$. In this case $l = i+1$ and $(*)$ is isomorphic to the following commutative diagram:

$$\begin{array}{ccc} DE(V,U)\times E(T,U) & \longrightarrow & DE(V,T) \\ {\scriptstyle ((D\psi^V_U)^{-1},\psi^T_U)}\Big\downarrow & & \Big\downarrow{\scriptstyle (D\psi^V_T)^{-1}} \\ DB(\psi V,\psi U)\times B(\psi T,\psi U) & \longrightarrow & DB(\psi V,\psi T). \end{array}$$

(iii) Case $j = i+1, l = j$. In this case $l = i+1$ and $(*)$ is isomorphic to the following commutative diagram:

$$\begin{array}{ccc} E(U,V)\times DE(U,T) & \longrightarrow & DE(V,T) \\ {\scriptstyle (\psi^U_V,(D\psi^U_T)^{-1})}\Big\downarrow & & \Big\downarrow{\scriptstyle (D\psi^V_T)^{-1}} \\ B(\psi U,\psi V)\times DB(\psi U,\psi T) & \longrightarrow & DB(\psi V,\psi T). \end{array}$$

(iv) Case $j = i+1, l = j+1$. In this case $l = i+2 \notin \{i, i+1\}$. Hence $(*)$ is commutative.

In any case the diagram $(*)$ is commutative. Thus $\psi^{\mathbb{Z}} : \hat{E} \longrightarrow B^{\mathbb{Z}}$ is a functor that is fully faithful and bijective on objects. As a consequence, $(A^{\mathbb{Z}}, \hat{E}, B^{\mathbb{Z}})$ is a tilting triple. □

Remark 1.6. Let A and B be locally bounded spectroids. Then as in Rickard [16] the following are equivalent:

(1) There is an S-equivalence $\mathcal{D}(\mathrm{Mod}\, A) \longrightarrow \mathcal{D}(\mathrm{Mod}\, B)$; and
(2) There is an S-equivalence $\mathcal{D}^b(\mathrm{mod}\, A) \longrightarrow \mathcal{D}^b(\mathrm{mod}\, B)$.

Corollary 1.7. *Let A and B be locally bounded spectroids of finite global dimension. Then the following are equivalent:*

(1) *There is an S-equivalence $\underline{\mathrm{mod}} A^{\mathbb{Z}} \longrightarrow \underline{\mathrm{mod}} B^{\mathbb{Z}}$; and*
(2) *$A^{\mathbb{Z}}$ and $B^{\mathbb{Z}}$ are derived equivalent.*

Proof. (2) ⇒(1). This follows from a categorical version of [17, Corollary 2.2] or [12, 2.3 Exemple].

(1) ⇒(2). By assumption using a theorem of Happel [8] or of Keller-Vossieck [12], the statement (1) implies that A and B are derived equivalent, which implies the statement (2) by Theorem 1.5. □

2. Coverings of spectroids of homotopy categories

Throughout this section R is a *locally bounded* spectroid, G a group of automorphisms of R and $F : R \longrightarrow R/G$ is the canonical Galois covering functor. Note in this case that an R-module is finitely presented iff it is finitely generated iff it is finite dimensional. Recall that a category is called an *aggregate* (= a Krull-Schmidt category) in case it is additive, svelte, pointwise finite and each object is a finite direct sum of indecomposables with local endomorphism algebras ([6, 3.5]). Note also that both $\mathcal{H}^b(\text{pro}R)$ and $\mathcal{H}^b(\text{pro}R/G)$ are aggregates because both $\mathcal{C}^b(\text{pro}R)$ and $\mathcal{C}^b(\text{pro}R/G)$ are. In this section we will show that under a suitable condition on G, the covering functor F induces a Galois covering with the group G from a spectroid of $\mathcal{H}^b(\text{pro}R)$ to a full subcategory of a spectroid of $\mathcal{H}^b(\text{pro}R/G)$. The following example of a Galois covering will be used later.

Example 2.1. Let A be a locally bounded spectroid and $\nu : A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$ an automorphism (so-called Nakayama automorphism) of $A^{\mathbb{Z}}$ given by $x^{[n]} \mapsto x^{[n+1]}$. Then $A^{\mathbb{Z}}/\langle \nu \rangle$ is isomorphic to the trivial extension $T(A) := A \ltimes DA$. Thus we have a Galois covering $A^{\mathbb{Z}} \longrightarrow T(A)$ with group $\langle \nu \rangle$.

Recall that the group G acts on $\text{Mod}\, R$: For each $g \in G$, and for each $M \in \text{Mod}\, R$,

$$^{g}M := M \circ g^{-1} : R^{\text{op}} \xrightarrow{g^{-1}} R^{\text{op}} \xrightarrow{M} \text{Mod}\, k.$$

Since for every $g \in G$, $^{g}(-) : \text{pro}R \longrightarrow \text{pro}R$ is an equivalence, it induces an S-equivalence $\mathcal{H}^b(\text{pro}R) \longrightarrow \mathcal{H}^b(\text{pro}R)$, which is denoted also by $^{g}(-)$.

Lemma 2.2. *The action of G on $\mathcal{H}^b(\text{pro}R)$ is* locally bounded*, i.e., for every pair (X, Y) of objects of $\mathcal{H}^b(\text{pro}R)$, there are only a finite number of $g \in G$ such that $\mathcal{H}^b(\text{pro}R)(X, {}^{g}Y) \cong \mathcal{H}^b(\text{pro}R)({}^{g^{-1}}X, Y) \neq 0$.*

Proof. Straightforward. See also Remark 2.4. $\square$

Let $F^{\cdot} : \text{Mod}\, R/G \longrightarrow \text{Mod}\, R$, $F^{\cdot}(M) := M \circ F^{\text{op}}$ be the *pull-up* functor, and $F_{\cdot} : \text{Mod}\, R \longrightarrow \text{Mod}\, R/G$, $(F_{\cdot}(M))(a) := \bigoplus_{x \in a} M(x)$ the *push-down* functor. Then $F_{\cdot}$ is the left adjoint of $F^{\cdot}$, and both $F_{\cdot}$ and $F^{\cdot}$ are exact functors. Hence $F^{\cdot}, F_{\cdot}$ induce S-functors, which we denote also by $F^{\cdot}, F_{\cdot}$, respectively:

$$\mathcal{H}(\text{Mod}\, R) \underset{F^{\cdot}}{\overset{F_{\cdot}}{\rightleftarrows}} \mathcal{H}(\text{Mod}\, R/G).$$

Then again $F_{\cdot}$ is the left adjoint of $F^{\cdot}$. Since $F_{\cdot}R(-, x) \cong R/G(-, Fx)$ for all $x \in R$, $F_{\cdot}$ induces a functor

$$F_{\cdot} : \mathcal{H}^b(\text{pro}R) \longrightarrow \mathcal{H}^b(\text{pro}R/G).$$

Lemma 2.3. *Let $X \in \mathcal{H}^b(\text{pro}R)$. Then the following hold.*

(1) *For all $g \in G$, $F_{\cdot}{}^{g}X \cong F_{\cdot}X$ in $\mathcal{H}^b(\text{pro}R/G)$;*

(2) *$F^{\cdot}F.X \cong \bigoplus_{g\in G} {}^gX$ in $\mathcal{H}(\mathrm{Mod}\, R)$, canonically; and*
(3) *For all $Y \in \mathcal{H}^b(\mathrm{pro}R)$, $F.$ induces an isomorphism*

$$\bigoplus_{g\in G} \mathcal{H}^b(\mathrm{pro}R)(X, {}^gY) \xrightarrow{\sim} \mathcal{H}^b(\mathrm{pro}R/G)(F.X, F.Y).$$

Proof. (1) and (2). Both isomorphisms are given by Gabriel [4, Lemma 3.2] in the category of differential complexes, and hence in the homotopy category.
(3) Since $\mathcal{H}^b(\mathrm{Pro}R)(X, -)$ commutes with direct sums, we have

$$\begin{aligned}\mathcal{H}^b(\mathrm{pro}R/G)(F.X, F.Y) \cong& \mathcal{H}(\mathrm{Mod}\, R/G)(F.X, F.Y)\\ \cong& \mathcal{H}(\mathrm{Mod}\, R)(X, F^{\cdot}F.Y)\\ \cong& \mathcal{H}^b(\mathrm{Pro}R)(X, \bigoplus_{g\in G} {}^gY)\\ \cong& \bigoplus_{g\in G} \mathcal{H}^b(\mathrm{pro}R)(X, {}^gY). \quad \square\end{aligned}$$

Remark 2.4. In (3) above, since $\mathcal{H}^b(\mathrm{pro}R/G)(F.X, F.Y)$ is finite-dimensional, the subset $G_{X,Y} := \{g \in G | \mathcal{H}^b(\mathrm{pro}R)(X, {}^gY) \neq 0\}$ of G is finite. Thus Lemma 2.2 also follows from (3), and we have an isomorphism

$$\mathcal{H}^b(\mathrm{pro}R/G)(F.X, F.Y) \cong \mathcal{H}^b(\mathrm{pro}R)(X, \bigoplus_{g\in G_{X,Y}} {}^gY).$$

Lemma 2.5. *Let X be an indecomposable object of $\mathcal{H}^b(\mathrm{pro}R)$. Then*
(1) *$F.X$ is indecomposable iff ${}^gX \ncong X$ for all $1 \neq g \in G$.*
(2) *Assume that ${}^gX \ncong X$ for all $1 \neq g \in G$ and let $Y \in \mathcal{H}^b(\mathrm{pro}R)$. Then $F.Y \cong F.X$ implies that $Y \cong {}^gX$ for some $g \in G$.*

Proof. (1) Put $H := \{g \in G | X \cong {}^gX\} \ni 1$. Then $H \subseteq G_{X,X}$. By Remark 2.4 we have

$$\mathcal{H}^b(\mathrm{pro}R/G)(F.X, F.X) \cong \mathcal{H}^b(\mathrm{pro}R)(X, \bigoplus_{g\in G_{X,X}} {}^gX).$$

Since $\mathrm{rad}\, \mathcal{H}^b(\mathrm{pro}R)(X, {}^gX) = \mathcal{H}^b(\mathrm{pro}R)(X, {}^gX)$ for all $g \in G\backslash H$, we have

$$\begin{aligned}&\mathcal{H}^b(\mathrm{pro}R/G)(F.X, F.X)/\,\mathrm{rad}\, \mathcal{H}^b(\mathrm{pro}R/G)(F.X, F.X)\\ \cong& \bigoplus_{g\in H} \mathcal{H}^b(\mathrm{pro}R)(X, {}^gX)/\,\mathrm{rad}\, \mathcal{H}^b(\mathrm{pro}R)(X, {}^gX)\\ \cong& \bigoplus^{H} k.\end{aligned}$$

Hence $F.X$ is indecomposable iff $H = \{1\}$.

(2) Assume that $F.Y \cong F.X$. Then by Lemma 2.3 we have $\bigoplus_{g\in G} {}^gY \cong \bigoplus_{g\in G} {}^gX$, in particular Y is a direct summand of $\bigoplus_{g\in G} {}^gX$. We first show that $Y \cong \bigoplus_{g\in H} {}^gX$

for some subset H of G. Let $Y \xrightarrow{\sigma} \bigoplus_{g\in G} {}^gX \xrightarrow{\pi} Y$ be morphisms in $\mathcal{H}^b(\mathrm{Pro}R)$ such that $\pi\sigma = 1_Y$. Then since $Y \in \mathcal{H}^b(\mathrm{pro}R)$, we have

$$\begin{aligned}
&\mathcal{H}^b(\mathrm{Pro}R)(Y, \bigoplus_{g\in G} {}^gX)\\
\cong &\bigoplus_{g\in G} \mathcal{H}^b(\mathrm{pro}R)(Y, {}^gX)\\
\cong &\bigoplus_{g\in N} \mathcal{H}^b(\mathrm{pro}R)(Y, {}^gX)\\
\cong &\mathcal{H}^b(\mathrm{pro}R)(Y, \bigoplus_{g\in N} {}^gX),
\end{aligned}$$

where $N := G_{Y,X}$ (2.4) is a finite set. Take $\sigma' \in \mathcal{H}^b(\mathrm{pro}R)(Y, \bigoplus_{g\in N} {}^gX)$ that corresponds to σ. Then we have $\sigma = \tau\sigma'$, where $\tau : \bigoplus_{g\in N} {}^gX \longrightarrow \bigoplus_{g\in G} {}^gX$ is the inclusion. Since $(\pi\tau)\sigma' = \pi\sigma = 1_Y$, σ' is a section, thus Y is a direct summand of $\bigoplus_{g\in N} {}^gX \in \mathcal{H}^b(\mathrm{pro}R)$. Therefore $Y \cong \bigoplus_{g\in H} {}^gX$ for some $H \subseteq N$ because $\mathcal{H}^b(\mathrm{pro}R)$ is an aggregate.

Now since $F.X \cong F.Y \cong \bigoplus_{g\in H} F.{}^gX$ is indecomposable by the statement (1) above, H must be a one element set, say $H = \{g\}$ with $g \in G$. As a consequence, $Y \cong {}^gX$. $\square$

Definition 2.6. (1) Let $I \subseteq S$ be sets or classes and assume that S has a G-action. Then I is called G-*stable* if $gX \in I$ for each $X \in I$ and for each $g \in G$.

(2) Suppose that the group G acts on a category $\mathcal{A}$. Then we say that the action of G is *free* (or that G acts *freely*) on isoclasses of indecomposable objects of $\mathcal{A}$ if $gX \not\cong X$ for each $1 \neq g \in G$ and for each indecomposable object $X \in \mathcal{A}$.

(3) For each spectroid S, we choose a spectroid $\mathrm{ind}\,\mathcal{H}^b(\mathrm{pro}S)$ of $\mathcal{H}^b(\mathrm{pro}S)$ such that it contains $\mathcal{X}_S$ and for each object (X^i, d^i) the X^i are direct sums of representables.

(4) Assume that G acts freely on isoclasses of indecomposable objects of $\mathcal{H}^b(\mathrm{pro}R)$. Then first we choose a spectroid $\mathcal{I}$ of $\mathcal{H}^b(\mathrm{pro}R)$ so that the object set forms a G-stable complete set of representatives of isoclasses of indecomposables. Namely the object set is a union of some G-orbits of indecomposables. Then the action of G is *free* on $\mathcal{I}$. Next we define a G-action on $\mathrm{ind}\,\mathcal{H}^b(\mathrm{pro}R)$ by identifying this with $\mathcal{I}$ as follows. For each $X \in \mathcal{H}^b(\mathrm{pro}R)$, fix an isomorphism $v(X) : X \xrightarrow{\sim} i(X) \in \mathcal{I}$. Then for each $g \in G$, the new action $X \mapsto {}^g|X$ is defined by $i({}^g|X) := {}^gi(X)$. For $\alpha : X \longrightarrow Y$ in $\mathrm{ind}\,\mathcal{H}^b(\mathrm{pro}R)$, ${}^g|\alpha$ is defined by $i({}^g|\alpha) := {}^gi(\alpha)$, where $i(\alpha) := v(Y)\alpha v(X)^{-1}$. Put $w(X) := v({}^g|X)^{-1}({}^gv(X))$. Then we have a commutative diagram:

$$\begin{array}{ccc}
{}^gX & \xrightarrow{{}^g\alpha} & {}^gY \\
{\scriptstyle w(X)}\big\downarrow & & \big\downarrow{\scriptstyle w(Y)} \\
{}^g|X & \xrightarrow{{}^g|\alpha} & {}^g|Y.
\end{array}$$

Since ${}^gR(-,x) = R({}^g(-),x) \cong R(-,gx)$, we have ${}^g|R(-,x) = R(-,gx)$. We simply write ${}^gX$ for ${}^g|X$ if this makes no confusion. We also say that G acts *freely* on $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$ instead of saying that G acts freely on isoclasses of indecomposable objects of $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$.

Now assume that G acts *freely* on $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$. Then by Lemma 2.5(1), $F.:\mathcal{H}^b(\mathrm{pro}R) \longrightarrow \mathcal{H}^b(\mathrm{pro}R/G)$ preserves indecomposability. As in Gabriel [4, 3.6], we can construct a functor $\bar{F}:\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R) \longrightarrow \operatorname{ind}\mathcal{H}^b(\mathrm{pro}R/G)$ such that the following diagram of functors is commutative up to isomorphisms:

$$\begin{array}{ccc}
\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R) & \xrightarrow{\bar{F}} & \operatorname{ind}\mathcal{H}^b(\mathrm{pro}R/G) \\
\big\downarrow & & \big\downarrow \\
\mathcal{H}^b(\mathrm{pro}R) & \xrightarrow{F.} & \mathcal{H}^b(\mathrm{pro}R/G),
\end{array}$$

where the vertical arrows are the inclusion functors. Then by Lemma 2.3(1) $\bar{F}g = \bar{F}$ for all $g \in G$. We set $\operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G)$ to be the full subcategory of $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R/G)$ consisting of $\bar{F}X$ for some $X \in \operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$.

Theorem 2.7. *Assume that G acts freely on* $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$. *Then*

$$\bar{F}:\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R) \longrightarrow \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G)$$

is a Galois covering with group G.

Proof. Recall that $\bar{F}$ is a Galois covering with group G iff

(1) $\bar{F}$ is a covering functor;
(2) $\bar{F}g = \bar{F}$ for all $g \in G$;
(3) $\bar{F}:\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R) \longrightarrow \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G)$ is surjective on objects; and
(4) G acts transitively on $\bar{F}^{-1}(X)$ for all $X \in \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G)$.

The statement (2) follows by construction. The statement (3) is trivial. The statement (4) follows from Lemma 2.5(2). Hence it only remains to show the statement (1). Let $X \in \operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$ and $Y \in \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G)$. Then it follows from Lemma 2.3(3) that $\bar{F}$ induces an isomorphism

$$\bigoplus_{\bar{F}Z=Y} \operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)(X,Z) \longrightarrow \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G)(\bar{F}X,Y).$$

Next we show that $\bar{F}$ induces an isomorphism

$$\bigoplus_{\bar{F}Z=Y} \operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)(Z,X) \longrightarrow \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G)(Y,\bar{F}X).$$

Note that we have the right adjoint F_ρ of the pull-up functor $F^{\cdot}$:

$$\operatorname{Mod} R \underset{F_\rho}{\overset{F^{\cdot}}{\rightleftarrows}} \operatorname{Mod} R/G,$$

defined as $F_\rho(M)(a) := \prod_{x\in a} M(x)$. These also induce S-functors

$$\mathcal{H}(\operatorname{Mod} R) \underset{F_\rho}{\overset{F^{\cdot}}{\rightleftarrows}} \mathcal{H}(\operatorname{Mod} R/G),$$

where again F_ρ is the right adjoint of $F^{\cdot}$. Note that for $X \in \mathcal{H}^b(\mathrm{pro}R)$, we have $F.X = F_\rho X$. By definition of $\operatorname{ind}^1 \mathcal{H}^b(\mathrm{pro}R/G)$, there is some $Z \in \operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$ such that $\bar{F}Z = Y$. Then

$$\begin{aligned}
\mathcal{H}^b(\mathrm{pro}R/G)(Y, F.X) &\cong \mathcal{H}(\operatorname{Mod} R/G)(F.Z, F_\rho X)\\
&\cong \mathcal{H}(\operatorname{Mod} R)(F^{\cdot}F.Z, X)\\
&\cong \mathcal{H}(\operatorname{Mod} R)(\bigoplus_{g\in G} {}^gZ, X)\\
&\cong \prod_{g\in G} \mathcal{H}^b(\mathrm{pro}R)({}^gZ, X)\\
&\cong \bigoplus_{g\in G} \mathcal{H}^b(\mathrm{pro}R)({}^gZ, X).
\end{aligned}$$

The last isomorphism follows from the fact that the action of G on $\mathcal{H}^b(\mathrm{pro}R)$ is locally bounded. This gives the desired isomorphism. $\square$

Remark 2.8. Since the action of G on $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$ is locally bounded by Lemma 2.2, we can form the quotient $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)/G$ and $\bar{F}$ induces an isomorphism

$$\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)/G \longrightarrow \operatorname{ind}^1 \mathcal{H}^b(\mathrm{pro}R/G).$$

The following is useful in computation of examples.

Lemma 2.9. *If G acts freely on R and G is torsion-free, then the action of G on $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$ is free.*

Proof. (1) G acts freely on isoclasses of indecomposables in $\operatorname{mod} R$.

Let $g \in G$ and X be an indecomposable object in $\operatorname{mod} R$. Put $\operatorname{supp} X := \{x \in R | X(x) \neq 0\}$. Assume that ${}^gX \cong X$. Then $\operatorname{supp} {}^gX = \operatorname{supp} X$. Thus g induces a permutation on the finite set $\operatorname{supp} X$. Therefore there is some $n \in \mathbb{N}$ such that g^n fixes some object of $\operatorname{supp} X$. Since G acts freely on R, $g^n = 1$. Thus g has a finite order. Hence $g = 1$ because G is torsion-free.

(2) G acts freely on $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$.

Let $g \in G$ and X be an indecomposable object in $\mathcal{H}^b(\mathrm{pro}R)$. Since $X \neq 0$ in $\mathcal{H}^b(\mathrm{pro}R)$, $H^n(X) \neq 0$ for some $n \in \mathbb{N}$. Assume that ${}^gX \cong X$. Then we have a quasi-isomorphism $X \longrightarrow {}^gX$, which induces an isomorphism $H^n(X) \xrightarrow{\sim} H^n({}^gX) \cong {}^gH^n(X)$. Let $H^n(X) \cong \bigoplus_{i=1}^s Y_i$ for some indecomposables Y_i in $\mathrm{mod}\, R$. Then since $\bigoplus_{i=1}^s {}^gY_i \cong \bigoplus_{i=1}^s Y_i$, there is a permutation π on the set $\{1, \dots, s\}$ such that for all i, we have ${}^gY_i \cong Y_{\pi(i)}$. Hence for some $a \in \mathbb{N}$ and for some i, ${}^{g^a}Y_i \cong Y_i$. Then by (1) above, we have $g^a = 1$. Hence $g = 1$. □

3. Pushdown of a derived equivalence

In this section we investigate how to get derived equivalences from those between coverings. Throughout the section R, S are *locally bounded* spectroids and (R, E, S) is a tilting triple with the S-equivalence $\varphi = \varphi_\psi : \mathcal{D}(\mathrm{Mod}\, R) \longrightarrow \mathcal{D}(\mathrm{Mod}\, S)$ defined by an isomorphism $\psi : E \longrightarrow S$ (Definition 1.2). The S-equivalence $\mathcal{H}^b(\mathrm{pro}R) \longrightarrow \mathcal{H}^b(\mathrm{pro}S)$ induced from φ is denoted also by φ. Further G is a group of automorphisms of R and $F : R \longrightarrow R/G$ is the canonical Galois covering. We always assume that G acts *freely* on $\mathrm{ind}\,\mathcal{H}^b(\mathrm{pro}R)$.

We start with the fundamental fact.

Lemma 3.1. *Let $\rho : M \longrightarrow N$ be an isomorphism of spectroids and H a group of automorphisms of M. Put ${}^\rho H := \{\rho h \rho^{-1} | h \in H\}$ to be a group of automorphisms of N. Then ρ induces an isomorphism $\rho/H : M/H \longrightarrow N/{}^\rho H$.*

Proof. Let $L : N \longrightarrow N/{}^\rho H$ be the canonical Galois covering. Then it is enough to show the three facts: (1) $(L\rho)h = L\rho$ for all $h \in H$; (2) $L\rho$ is surjective on objects; and (3) H acts transitively on $(L\rho)^{-1}(a)$ for all $a \in N/{}^\rho H$. These are easy to check and are left to the reader. □

Proposition 3.2. *The S-equivalence $\varphi : \mathcal{H}^b(\mathrm{pro}R) \longrightarrow \mathcal{H}^b(\mathrm{pro}S)$ induces an isomorphism*

$$\varphi/G : \mathrm{ind}\,\mathcal{H}^b(\mathrm{pro}R)/G \longrightarrow \mathrm{ind}\,\mathcal{H}^b(\mathrm{pro}S)/{}^\varphi G.$$

Proof. This is immediate from the lemma above. □

It is interesting to know when the group ${}^\varphi G$ is induced from a group H of automorphisms of S. For, if this is the case, we have $\mathrm{ind}\,\mathcal{H}^b(\mathrm{pro}S)/{}^\varphi G \cong \mathrm{ind}^1\,\mathcal{H}^b(\mathrm{pro}S/H)$. This makes it possible to relate R/G to S/H directly as we have an isomorphism $\mathrm{ind}^1\,\mathcal{H}^b(\mathrm{pro}R/G) \cong \mathrm{ind}^1\,\mathcal{H}^b(\mathrm{pro}S/H)$ by Theorem 2.7.

Definition 3.3. Let H be a group of automorphisms of S. Then we set $\mathcal{H}H := \{{}^h(-) | h \in H\}$ to be a group of automorphisms of $\mathrm{ind}\,\mathcal{H}^b(\mathrm{pro}S)$.

Proposition 3.4. *If the tilting spectroid E is G-stable (2.6), then there is a group H of automorphisms of S such that ${}^\varphi G = \mathcal{H}H$.*

Proof. Recall that $\varphi|_E = \eta\psi$, where $\eta = \eta_S : S \longrightarrow \mathcal{X}_S$ is the Yoneda isomorphism. Since E is G-stable, we can define a group $H(\cong G)$ of automorphisms of S as $H := \{\psi^g(-)\psi^{-1} | g \in G\}$. Let $S(-,\alpha) : S(-,x) \longrightarrow S(-,y)$ be in $\mathcal{X}_S$, and $g \in G$. Put $g' := \varphi^g(-)\varphi^{-1} \in {}^{\varphi}G$ and $h := \psi^g(-)\psi^{-1} \in H$. Note that $h = \eta^{-1}g'\eta$. Then it is enough to show that $g'S(-,x) = {}^hS(-,x)$ and $g'S(-,\alpha) = {}^hS(-,\alpha)$. Now since $hx = \eta^{-1}g'\eta x = \eta^{-1}(g'S(-,x))$, we have $g'S(-,x) = \eta(hx) = S(-,hx) = {}^hS(-,x)$. The rest is shown similarly. □

Definition 3.5. Let H be a group of automorphisms of S. Then the tilting spectroid E is called (G,H)-*compatible* (with respect to ψ) if E is G-stable and ${}^{\varphi}G = \mathcal{H}H$.

Theorem 3.6. *Let (R,E,S) be a tilting triple of locally bounded spectroids and G, H groups of automorphisms of R, S, respectively. Assume that the action of G on* $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$ *is free and that E is (G,H)-compatible. Then the action of H on* $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}S)$ *is free, and R/G and S/H are derived equivalent.*

Proof. Since G acts freely on $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}R)$, so does the group ${}^{\varphi}G$ on $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}S)$, thus so does the group H. By Theorem 2.7 and Proposition 3.2, we have an isomorphism $\varphi/G : \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G) \xrightarrow{\sim} \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}S/H)$. Note that $\operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}S/H) \supseteq \mathcal{X}_{S/H}$. Here for simplicity, we denote the object class of a category by the same notation as the category itself if this makes no confusion. For a subcategory A of the category $\mathcal{H}^b(\mathrm{pro}R/G)$, we denote by $\langle A\rangle$ the smallest subcategory of $\mathcal{H}^b(\mathrm{pro}R/G)$ containing A and closed under isomorphisms. Let E' be the full subcategory $(\varphi/G)^{-1}(\mathcal{X}_{S/H})$ of $\operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G)$, which coincides with $\bar{F}(\varphi^{-1}(\mathcal{X}_S)) = \bar{F}(E)$. It is enough to show that $(R/G, E', S/H)$ is a tilting triple. Now it is obvious that $E' \subseteq \mathcal{H}^b(\mathrm{pro}R/G)$.

(a) For all $T, U \in E'$ and for all $n \neq 0$, $\mathcal{H}^b(\mathrm{pro}R/G)(T, U[n]) = 0$.

In fact, for $P = R/G$ or S/H, let $\bar{\Sigma}_P$ be the automorphism of $\operatorname{ind}\mathcal{H}^b(\mathrm{pro}P)$ defined by $\bar{\Sigma}_P X \cong X[1]$ for all $X \in \operatorname{ind}\mathcal{H}^b(\mathrm{pro}P)$. Then the left hand side is isomorphic to

$$\begin{aligned}\operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}R/G)(T, \bar{\Sigma}^n_{R/G}U) &\cong \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}S/H)((\varphi/G)(T), (\varphi/G)(\bar{\Sigma}^n_{R/G}U)) \\ &\cong \operatorname{ind}^1\mathcal{H}^b(\mathrm{pro}S/H)((\varphi/G)(T), \bar{\Sigma}^n_{S/H}(\varphi/G)(U)) \\ &\cong \mathcal{H}^b(\mathrm{pro}S/H)((\varphi/G)(T), (\varphi/G)(U)[n]) \\ &= 0\end{aligned}$$

because $(\varphi/G)(T), (\varphi/G)(U) \in \mathcal{X}_{S/H}$.

(b) $\mathrm{tri}(E') \supseteq \mathcal{X}_{R/G}$.

In fact, since $F.$ is an S-functor, we have

$$\mathcal{X}_{R/G} \subseteq \langle F.(\mathcal{X}_R)\rangle \subseteq \langle F.(\mathrm{tri}(E))\rangle \subseteq \mathrm{tri}(F.(E)) = \mathrm{tri}(E').$$

(c) φ/G induces an isomorphism $E' \xrightarrow{\sim} \mathcal{X}_{S/H} \cong S/H$. Hence $E' \cong S/H$.

By (a), (b), (c) above, $(R/G, E', S/H)$ is a tilting triple. □

4. Applications

In this section we give some applications of Theorems 1.5 and 3.6. In particular, we classify representation-finite selfinjective algebras of Type A_n under derived equivalences. As the first application, we obtain the following.

Proposition 4.1. *Let (A, E, B) be a tilting triple of locally bounded spectroids with an isomorphism $\psi : E \longrightarrow B$ and ν_A, ν_B the Nakayama automorphisms of $A^{\mathbb{Z}}, B^{\mathbb{Z}}$, respectively. Then we have*

(1) *The tilting spectroid $\hat{E}$ for $A^{\mathbb{Z}}$ (Theorem 1.5)is $(\langle \nu_A^n \rangle, \langle \nu_B^n \rangle)$-compatible (with respect to $\psi^{\mathbb{Z}}$) for all $n \in \mathbb{Z}$; and*

(2) *$A^{\mathbb{Z}}/\langle \nu_A^n \rangle$ and $B^{\mathbb{Z}}/\langle \nu_B^n \rangle$ are derived equivalent. In particular, when $n = 1$, trivial extensions $T(A)$ and $T(B)$ are derived equivalent.*

Proof. (1) (a) $\hat{E}$ is $\langle \nu_A^n \rangle$-stable for all $n \in \mathbb{Z}$.

It is enough to show the assertion for $n = 1$. Let $T = (T^i, d^i) \in E$ and $m \in \mathbb{Z}$. Then

$$\begin{aligned} {}^{\nu_A}(T^i \otimes_A A^{\mathbb{Z}}(-, 1^{[m]}?)) &= T^i \otimes_A A^{\mathbb{Z}}({\nu_A}^{-1}(-), 1^{[m]}?) \\ &\cong T^i \otimes_A A^{\mathbb{Z}}(-, \nu_A 1^{[m]}?) \\ &= T^i \otimes_A A^{\mathbb{Z}}(-, 1^{[m+1]}?). \end{aligned}$$

Thus ${}^{\nu_A}(T^{[m]}) = T^{[m+1]}$. Hence $\hat{E}$ is $\langle \nu_A \rangle$-stable.

(b) ${}^{\varphi}\langle \nu_A^n \rangle = \mathcal{H}\langle \nu_B^n \rangle$ for all $n \in \mathbb{Z}$, where $\varphi = \varphi_\psi$.

It is enough to show the assertion for $n = 1$. Let $T^{[m]} \in \hat{E}$. Then

$$\begin{aligned} \varphi_{\psi^{\mathbb{Z}}}{}^{\nu_A}(T^{[m]}) &= \varphi_{\psi^{\mathbb{Z}}}(T^{[m+1]}) = (\eta\psi^{\mathbb{Z}})(T^{[m+1]}) \\ &= B^{\mathbb{Z}}(-, \psi^{\mathbb{Z}}(T^{[m+1]})) = B^{\mathbb{Z}}(-, (\psi T)^{[m+1]}) \\ &= {}^{\nu_B}B^{\mathbb{Z}}(-, (\psi T)^{[m]}) = {}^{\nu_B}(\varphi_{\psi^{\mathbb{Z}}}(T^{[m]})). \end{aligned}$$

Thus the assertion holds.

(2) This follows from (1) by Theorem 3.6 because $\langle \nu_A^n \rangle$ is torsion-free. □

Let Λ be a representation-finite selfinjective algebra of type A_n. This type of algebras was completely classified by Riedtmann [19], to which we refer terminologies and notation used here. We exclude the trivial case that $\Lambda = k$. Let $\mathcal{C}$ be a *configuration* of $\mathbb{Z}A_n$ and G an *admissible* group of automorphisms of $\mathbb{Z}A_n$ stabilizing $\mathcal{C}$ such that $(\mathbb{Z}A_n)_{\mathcal{C}}/G$ is isomorphic to the Auslander-Reiten quiver of Λ. Define an automorphism τ of $\mathbb{Z}A_n$ by $\tau(p, q) := (p-1, q)$. When n is odd, define an automorphism ϕ of $\mathbb{Z}A_n$ by $\phi(p, q) := (p + q - \frac{n+1}{2}, n + 1 - q)$, the reflection at the central line of $\mathbb{Z}A_n$. We set m to be the number of isoclasses of simple Λ-modules and e the *period* of $\mathcal{C}$. Then e divides both n and m: $n = er$, $m = es$. Set $g = \tau^{-e}$. Then g stabilizes $\mathcal{C}$ and $\langle g^s \rangle = \langle (\tau^{-1})^{es} \rangle = \langle \tau^m \rangle$. By [19, 3.3] the group G may be written as $G = \langle g^s \rangle$

or $G = \langle g^s\phi\rangle$. By definition, Λ is called a *wreath-like* algebra in the former case, and is called a *Möbius* algebra in the latter case. In the latter case n must be odd and $e = n$, thus $m = ns$. As in [19, 6.2], $\mathcal{C}$ defines a quiver $\tilde{Q}_{\mathcal{C}}$ which has the integers as vertices and whose arrows are divided into two groups: α-arrows and β-arrows. Further any automorphism h of $\mathbb{Z}A_n$ stabilizing $\mathcal{C}$ induces an automorphism of $\tilde{Q}_{\mathcal{C}}$, which we denote also by h. Then G can be regarded as a group of automorphisms of $\tilde{Q}_{\mathcal{C}}$, and the quiver $\tilde{Q}_{\mathcal{C}}/\langle\tau^n\rangle$ is identified with the *Brauer quiver* $Q_{\mathcal{C}}$ corresponding to $\mathcal{C}$. Let I be the ideal of $k\tilde{Q}_{\mathcal{C}}$ defined by all possible zero relations $\alpha\beta = 0 = \beta\alpha$ and commutativity relations $\alpha^{a(x)} = \beta^{b(x)}$ for all vertices x in $\tilde{Q}_{\mathcal{C}}$, where $\alpha^{a(x)}$ (resp. $\beta^{b(x)}$) is the unique path consisting of α-arrows (resp. β-arrows) from x to $x+n$ in $\tilde{Q}_{\mathcal{C}}$. Set $R := k\tilde{Q}_{\mathcal{C}}/I$. Then again G can be regarded as a group of automorphisms of R. By [19, Theorem 6.2], we have a Galois covering functor $R \longrightarrow \Lambda$ with group G.

First we investigate the wreath-like case.

Definition 4.2. Let $m, n \geq 1$. By $N_{m,n}$ we denote the (basic) connected selfinjective Nakayama algebra with m isoclasses of simple modules and Loewy length $n+1$.

Remark 4.3. Let $\vec{A}_n$ be the quiver

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n.$$

Then $N_{m,n} \cong B^{\mathbb{Z}}/\langle\gamma^m\rangle$, where $B = k\vec{A}_n$ and γ is an automorphism of $B^{\mathbb{Z}}$ defined by

$$\gamma(p^{[i]}) := \begin{cases} 1^{[i+1]} & \text{if } p = n; \\ (p+1)^{[i]} & \text{otherwise.} \end{cases}$$

The next proposition generalizes a Rickard's theorem on derived equivalences of Brauer tree algebras [17, Theorem 4.2] (cf. a theorem of Gabriel-Riedtmann [5, 1.8 Theorem 1]).

Proposition 4.4. *For a selfinjective algebra Λ the following are equivalent:*

(1) *Λ is derived equivalent to a selfinjective Nakayama algebra $N_{m,n}$; and*
(2) *Λ is a wreath-like algebra of type A_n with m isoclasses of simple modules.*

Proof. By Rickard [17, Corollary 2.2], (1) implies (2).

(2) $\Rightarrow$ (1). Assume that Λ is a wreath-like algebra of type A_n with m isoclasses of simple modules. We have to show that Λ is derived equivalent to $N_{m,n}$. Cyclic paths in $Q_{\mathcal{C}}$ consisting of $\bar{\alpha}$-arrows (resp. $\bar{\beta}$- arrows) are called $\bar{\alpha}$-orbits (resp. $\bar{\beta}$-orbits). Then by Riedtmann [19, Proposition 3.5], if $e \neq n$, then $Q_{\mathcal{C}}$ has exactly one *exceptional* orbit, i.e., either an $\bar{\alpha}$-orbit or a $\bar{\beta}$-orbit which is stable under $\bar{\tau}^e$, which may be assumed to be an $\bar{\alpha}$-orbit. If $e = n$, then choose one $\bar{\alpha}$-orbit arbitrarily and call it exceptional. By deleting one arrow from each cycle in $Q_{\mathcal{C}}$ we obtain a quiver Q of the form in Figure 1 satisfying the following:

- Q has n vertices;

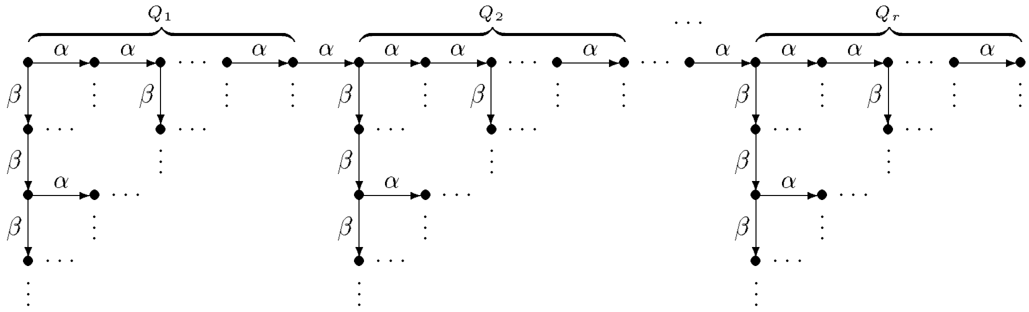

FIGURE 1

- The first horizontal full α-path L from the above corresponds to the exceptional $\bar{\alpha}$-orbit;
- The number of vertices of the subquiver Q_i is equal to e for all $i = 1, \dots, r$;
- Q_i and Q_j are isomorphic for all $i, j = 1, \dots, r$; and
- Each Q_i is a full subquiver of the quiver in Figure 2.

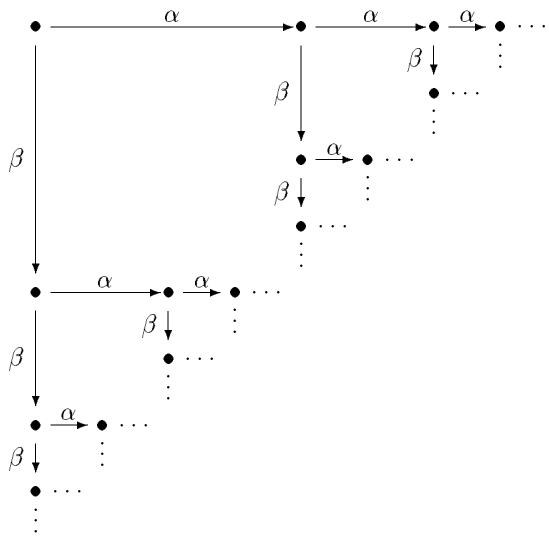

FIGURE 2

Each horizontal or vertical full α- or β-path is called a *branch* of Q. We call a vertex x in Q a *junction* if x belongs to two branches. Let A be the spectroid defined by the quiver Q with all possible relations $\alpha\beta = \beta\alpha = 0$. Then by the definition of repetitions, we can construct a display-functor ([6, p. 74]) $\Phi : k\tilde{Q}_{\mathcal{C}} \longrightarrow A^{\mathbb{Z}}$ with $\operatorname{Ker}\Phi = I$. Thus Φ gives identifications $A^{\mathbb{Z}} = R$ and $\nu_A = g^r$.

We now construct a nice tilting spectroid E for A. We write $x\hat{} := A(-,x)$ for all $x \in A$ for short. Let $x \in A$. There is a unique shortest path from a vertex $x_0 \in L$ to x in Q:

$$x_0 \overbrace{\xrightarrow{\beta}\xrightarrow{\beta}\cdots\xrightarrow{\beta}}^{a_0} x_1 \overbrace{\xrightarrow{\alpha}\xrightarrow{\alpha}\cdots\xrightarrow{\alpha}}^{a_1} x_2 \xrightarrow{\beta} \cdots \to \cdots \to x_t = x,$$

where $x_0, x_1, \dots, x_{t-1}$ are junctions if $t > 0$. We then put

$$T_x := (\cdots \to 0 \to x_0\hat{} \xrightarrow{(\beta^{a_0})\hat{}} x_1\hat{} \xrightarrow{(\alpha^{a_1})\hat{}} x_2\hat{} \to \cdots \to x_t\hat{} \to 0 \to \cdots),$$

where $x_0\hat{}$ is in degree zero. Let E be the full subcategory of $\mathcal{H}^b(\mathrm{pro}A)$ defined by $\{T_x | x \in A\}$. We show that (A, E, B) is a tilting triple.

(a) $\mathrm{tri}(E) \supseteq \mathcal{X}_A$.

In fact by construction, we have a triangle of the form $x\hat{}[-t] \to T_x \to T_{x_{t-1}} \to x\hat{}[1-t]$. Hence (a) follows.

(b) $\mathcal{H}^b(\mathrm{pro}A)(T_x, T_y[i]) = 0$ for all $x, y \in A$ and all $i \neq 0$.

First remark by construction that if $A(x,y) \cong \mathrm{Hom}_A(x\hat{}, y\hat{}) \neq 0$ for some $x, y \in A$, then x and y are in the same branch of Q and x is a *predecessor* of y in Q (i.e., there is a path from x to y in Q). Further the k-dimension of $\mathrm{Hom}_A(x\hat{}, y\hat{})$ is at most 1.

By this remark, the assertion is clear if $|i| \geq 2$.

Case $i = -1$.

Assume that $\mathcal{H}^b(\mathrm{pro}A)(T_x, T_y[-1]) \neq 0$. Let $0 \neq \bar{f} \in \mathcal{H}^b(\mathrm{pro}A)(T_x, T_y[-1])$, $f = (f_j)$, $f_j : x_{j+1}\hat{} \to y_j\hat{}$. Put $p := \min\{j \geq 0 | f_j \neq 0\}$. Then by the remark above, $x_p = y_p$ and this is a proper predecessor of x_{p+1}. Thus $f_p \in \mathrm{Hom}_A(x_{p+1}\hat{}, x_p\hat{}) = 0$, a contradiction.

Case $i = 1$.

Assume that $0 \neq \bar{f} \in \mathcal{H}^b(\mathrm{pro}A)(T_x, T_y[1])$, $f = (f_j)$, $f_j : x_j\hat{} \to y_{j+1}\hat{}$. Put $p := p(f) := \max\{j \geq 0 | f_j \neq 0\}$, and put $f' := (f'_j)$, where

$$f'_j = \begin{cases} f_j & \text{if } j < p \\ 0 & \text{if } j \geq p. \end{cases}$$

By the remark above, $x_p = y_p$ and f_p factors through $y_p\hat{} \to y_{p+1}\hat{}$. This shows that f and f' are homotopic. Since $p(f') < p(f)$, induction shows that $\bar{f} = 0$, a contradiction.

(c) $E \cong B$.

Define a linear order $\leq$ on the set of vertices of Q lexicographically as follows: Let $x, y \in Q$ and let

$$T_x = (\cdots \to 0 \to x_0\hat{} \xrightarrow{(\beta^{a_0})\hat{}} x_1\hat{} \xrightarrow{(\alpha^{a_1})\hat{}} x_2\hat{} \to \cdots \to x_t\hat{} = x\hat{} \to 0 \to \cdots)$$

$$T_y = (\cdots \to 0 \to y_0\hat{} \xrightarrow{(\beta^{b_0})\hat{}} y_1\hat{} \xrightarrow{(\alpha^{b_1})\hat{}} y_2\hat{} \to \cdots \to y_u\hat{} = y\hat{} \to 0 \to \cdots).$$

(i) If x_0 is a predecessor of y_0, then we define $x \leq y$.

(ii) Assume that $x_0 = y_0, \dots, x_i = y_i$ for $i \geq 0$. If $y_i = y$ (i.e., if $T_y^{i+1} = 0$), then define $x \leq y$. In case $x_i \neq x$ and $y_i \neq y$, if x_{i+1} is a predecessor of y_{i+1}, then define $x \leq y$.

Then in particular for $x, y \in L$, $x \leq y$ iff there is a path from x to y in L; for other branches $z_0 \longrightarrow z_1 \longrightarrow \cdots \longrightarrow z_p$, we have $z_1 < z_2 < \cdots < z_p < z_0$.

Claim 1. $x > y$ implies $\mathcal{H}^b(\mathrm{pro}A)(T_x, T_y) = 0$.

In fact, assume that $x > y$ and that $0 \neq \bar{f} \in \mathcal{H}^b(\mathrm{pro}A)(T_x, T_y)$, where $f = (f^i) \in \mathcal{C}^b(\mathrm{pro}A)$, $f^i : x_i\hat{} \longrightarrow y_i\hat{}$. Put $p := \max\{i | f_i \neq 0\}$. If $x_p \neq y_p$, then there is a path from x_p to y_p because $f_p \neq 0$, which means that $x \leq y$, a contradiction. Hence $x_p = y_p$. It then follows from $x > y$ that $y_p \neq y$. Here note that $z\hat{}$ are uniserial for all $z \in Q$, and the dimension vector of $y_i\hat{}$ is as follows for all $1 \leq i \leq u$:

$$(\underline{\dim}\ y_i\hat{})_z = \begin{cases} 1 & \text{if } y_{i-1}, z, y_i \text{ are in the same branchand } y_{i-1} \leq z \leq y_i, \\ 0 & \text{otherwise} \end{cases}$$

for all $z \in Q$. Put h to be the map $y_p\hat{} \longrightarrow y_{p+1}\hat{}$ in T_y. Since $(f^i) \in \mathcal{C}^b(\mathrm{pro}A)$ and $f_{p+1} = 0$, we must have $hf_p = 0$. Considering the dimension vectors of $y_p\hat{}$ and $y_{p+1}\hat{}$, this forces $f_p = 0$, a contradiction.

Claim 2. $x \leq y$ implies $\dim \mathcal{H}^b(\mathrm{pro}A)(T_x, T_y) = 1$.

In fact, assume that $x \leq y$ and put $p = \max\{i | x_i = y_i\}$. From the assumption, $x_p = x$ implies $y_p = y$. Thus we have the following three cases:

(i) $x_p = x$ and $y_p = y$; (ii) $x_p \neq x$ and $y_p = y$; or (iii) $x_p \neq x$ and $y_p \neq y$.

In each case we easily calculate the $\mathcal{C}^b(\mathrm{pro}A)(T_x, T_y)$ as follows:

Cases (i) and (ii). $\mathcal{C}^b(\mathrm{pro}A)(T_x, T_y) = kf$, where $f = (\dots, 0, 1_{x_0\hat{}}, \dots, 1_{x_p\hat{}}, 0, \dots)$ with $1_{x_0\hat{}}$ in degree zero.

Case (iii). The map $x_p\hat{} \longrightarrow x_{p+1}\hat{}$ in T_x has the form $(\gamma^a)\hat{}$, and the map $x_p\hat{} \longrightarrow y_{p+1}\hat{}$ in T_y has the form $(\gamma^{a+b})\hat{}$ for $\gamma = \alpha$ or β and for some $a, b \in \mathbb{N}$. Then $\mathcal{C}^b(\mathrm{pro}A)(T_x, T_y) = kf$, where $f = (\dots, 0, 1_{x_0\hat{}}, \dots, 1_{x_p\hat{}}, (\gamma^b)\hat{}, 0, \dots)$ with $1_{x_0\hat{}}$ in degree zero. Therefore in any case we have $\dim \mathcal{H}^b(\mathrm{pro}A)(T_x, T_y) \leq 1$. Now the morphism f constructed above is not homotopic to zero because $(\beta^{a_0})\hat{}$ is not a section. Hence the claim follows.

The two claims above show that $E \cong B$.

By (a), (b) and (c) above, (A, E, B) is a tilting triple. Then by Theorem 1.5, $(A^{\mathbb{Z}}, \hat{E}, B^{\mathbb{Z}})$ is a tilting triple, where $\hat{E}$ is the full subcategory of $\mathcal{H}^b(\mathrm{pro}A^{\mathbb{Z}})$ with object set $\{T^{[n]} := T \otimes_A A^{\mathbb{Z}}(-, 1^{[n]}?) | T \in E, n \in \mathbb{Z}\}$.

Next we show that $\hat{E}$ is $\langle g \rangle$-stable. Name the vertices of Q as $v(0), \dots, v(n-1)$ so that $v(p) < v(q)$ iff $p < q$. Then by the argument in (c) above E is presented by the quiver $T_{v(0)} \longrightarrow T_{v(1)} \longrightarrow \cdots \longrightarrow T_{v(n-1)}$ with no relations. By construction for each

$p \in \{0, \dots, n-1\}$ we have

$$\begin{aligned} {}^g(A(-, v(p)) \otimes_A A^{\mathbb{Z}}(-, 1^{[i]})) &= {}^g A^{\mathbb{Z}}(-, v(p)^{[i]}) \\ &= A^{\mathbb{Z}}(-, g(v(p)^{[i]})) \\ &= \begin{cases} A^{\mathbb{Z}}(-, v(p+e)^{[i]}) & \text{if } p+e < n; \\ A^{\mathbb{Z}}(-, v(p+e-n)^{[i+1]}) & \text{if } p+e \geq n. \end{cases} \end{aligned}$$

Let $q \in \mathbb{Z}$. Then we have $q = ni + p$ for a unique $p \in \{0, \dots, n-1\}$ and for a unique $i \in \mathbb{Z}$. Introduce the notation $\hat{v}(q) := v(p)^{[i]}$ and $T_{\hat{v}(q)} := T^{[i]}_{v(p)}$. Then the above equality is written as ${}^g A^{\mathbb{Z}}(-, \hat{v}(q)) = A^{\mathbb{Z}}(-, \hat{v}(q+e))$ for all $q \in \mathbb{Z}$. Therefore g acts as

$$ {}^g T_{\hat{v}(q)} = T_{\hat{v}(q+e)} \qquad \text{for all } q \in \mathbb{Z}. $$

Hence $\hat{E}$ is $\langle g \rangle$-stable.

Put $\{w(0), \dots, w(n-1)\}$ to be the vertices of B so that $B(-, w(p)) := \varphi_E(T_{v(p)})$ for all $p \in \{0, \dots, n-1\}$. Put similarly $\hat{w}(q) := w(p)^{[i]}$ for all $q \in \mathbb{Z}$ as above. Define an automorphism h of $B^{\mathbb{Z}}$, by $h(\hat{w}(q)) := \hat{w}(q+e)$. Then by the formula of g-action above, $\hat{E}$ is $(\langle g \rangle, \langle h \rangle)$-compatible. Put $H := \langle h^s \rangle$. Then $\hat{E}$ is (G, H)-compatible. Hence by Theorem 3.6, $\Lambda \cong A^{\mathbb{Z}}/G$ is derived equivalent to $B^{\mathbb{Z}}/H \cong N_{m,n}$ because G is torsion-free. $\square$

Remark 4.5. (1) If n divides m, say $m = ns$ (e.g., this occurs if $e = n$), then the implication (2) $\Rightarrow$ (1) follows from Proposition 4.1 because Λ has the form $A^{\mathbb{Z}}/\langle \nu_A^s \rangle$ for some algebra A which is tilted from the algebra $B = k\vec{A}_n$, and $B^{\mathbb{Z}}/\langle \nu_B^s \rangle \cong N_{m,n}$.

(2) In the proof of (c) above, note that A is derived equivalent to the algebra $B := k\vec{A}_n$ from the beginning (for instance by Assem-Happel [1, Theorem]). Therefore $\mathcal{H}^b(\mathrm{pro}A) \simeq \mathcal{H}^b(\mathrm{pro}B)$, and by Happel [8] $\mathcal{H}^b(\mathrm{pro}B)$ is equivalent to the stable category $\underline{\mathrm{mod}}B^{\mathbb{Z}}$, which has no oriented cycle. Further the k-dimension of $(\underline{\mathrm{mod}}B^{\mathbb{Z}})(X, Y)$ is at most 1 for all indecomposables X, Y in $\underline{\mathrm{mod}}B^{\mathbb{Z}}$. Using these it is enough for (c) to verify that $\mathcal{H}^b(\mathrm{pro}A)(T_x, T_y) \neq 0$ for all $x, y \in Q$ with $x \leq y$.

Next we classify Möbius algebras under derived equivalences. Before doing so we have to give "standard" forms of Möbius algebras (cf. [19, Fig. 36] and [20, Figure 2]).

Definition 4.6. An algebra is called a *standard Möbius* algebra if it is isomorphic to the algebra $M_{p,s}$ for some $p, s \in \mathbb{N}$ defined by the quiver in Figure 3 with relations

(i) Paths of length $p+2$ are equal to 0;

(ii) $\begin{cases} \beta_0^{[i+1]}\alpha_p^{[i]} = 0, \\ \alpha_0^{[i+1]}\beta_p^{[i]} = 0 \end{cases}$ for all $i = 0, \dots, s-2$, and $\begin{cases} \alpha_0^{[0]}\alpha_p^{[s-1]} = 0, \\ \beta_0^{[0]}\beta_p^{[s-1]} = 0 \end{cases}$; and

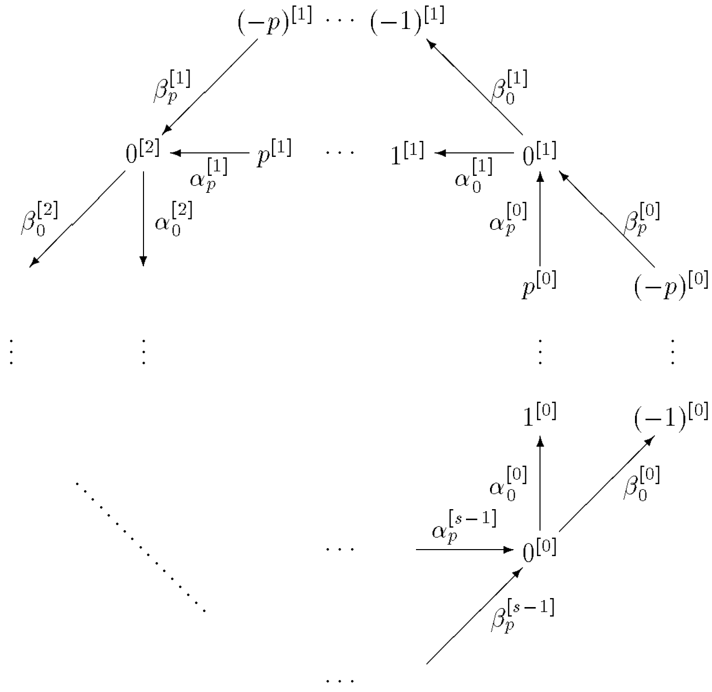

FIGURE 3

(iii) $\alpha_p^{[i]} \cdots \alpha_0^{[i]} = \beta_p^{[i]} \cdots \beta_0^{[i]}$ for all $i = 0, \ldots, s-1$.

Remark 4.7. Let $\vec{A}_p^p$ be the quiver

$$(-p) \xleftarrow{\beta_{p-1}} \cdots \xleftarrow{\beta_2} (-2) \xleftarrow{\beta_1} (-1) \xleftarrow{\beta_0} 0 \xrightarrow{\alpha_0} 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{p-1}} p.$$

Then $M_{p,s} \cong B^{\mathbb{Z}}/\langle \nu_B^s \phi_B \rangle$, where $B := k\vec{A}_p^p$, and ϕ_B is an automorphism of $B^{\mathbb{Z}}$ defined by $\phi_B(q^{[i]}) := (-q)^{[i]}$.

Proposition 4.8. *Let Λ be a Möbius algebra of type A_n with m isoclasses of simple modules. Then Λ is derived equivalent to the standard Möbius algebra $M_{p,s}$ with $n = 2p+1$, $m = sn$ (thus $p = (n-1)/2$, $s = m/n$).*

Proof. Since Λ is a Möbius algebra, the automorphism ϕ of $\mathbb{Z}A_n$ stabilizes $\mathcal{C}$. Thus ϕ induces an automorphism (again denoted by ϕ) of $Q_{\mathcal{C}}$ exchanging $\bar{\alpha}$-arrows and $\bar{\beta}$-arrows. It is easy to see that there is a unique vertex $x \in Q_{\mathcal{C}}$ satisfying $\phi(x) = x$. Put $v(0) := x$. By deleting one arrow from each cycle in $Q_{\mathcal{C}}$ we obtain a subquiver Q of $Q_{\mathcal{C}}$ of the form in Figure 4 satisfying the following.

- Q has n vertices;
- $\phi(v(0)) = v(0)$ and ϕ induces an isomorphism from Q_- to Q_+ exchanging α-arrows and β-arrows;
- both Q_- and Q_+ are full subquivers of the quiver in Figure 2.

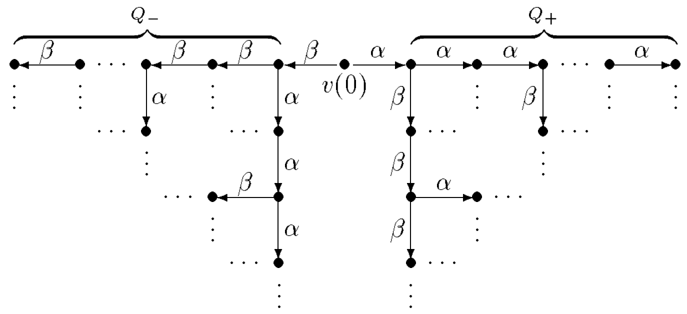

FIGURE 4

Let A be the spectroid defined by the quiver Q with all possible relations $\alpha\beta = \beta\alpha = 0$, and let ϕ_A the automorphism of $A^{\mathbb{Z}}$ defined by $\phi(q^{[i]}) := (-q)^{[i]}$. Put L to be the full subquiver of Q defined by the α-branch and the β-branch connected to the vertex $v(0)$. The rest is quite similar as in the proof of Theorem 4.4, and the details are left to the reader. Defining $T_x \in \mathcal{H}^b(\text{pro}A)$ for all $x \in Q$ in the same way, (A, E, B) is shown to be a tilting triple, where $B = k\vec{A}_p^p$, and E is the full subcategory of $\mathcal{H}^b(\text{pro}A)$ defined by $\{T_x | x \in Q\}$. Further the tilting spectroid $\hat{E}$ of the induced tilting triple $(A^{\mathbb{Z}}, \hat{E}, B^{\mathbb{Z}})$ is shown to be (G, H)-compatible, where $H := \langle \nu_B^s \phi_B \rangle$. Hence $\Lambda \cong A^{\mathbb{Z}}/G$ is derived equivalent to $M_{p,s} \cong B^{\mathbb{Z}}/H$. $\square$

Summarizing the two propositions above we obtain the following.

Theorem 4.9. *Let Λ be a representation-finite selfinjective algebra of type A_n with m isoclasses of simple modules.*

(1) *If Λ is a wreath-like algebra, then it is derived equivalent to $N_{m,n}$;*
(2) *If Λ is a Möbius algebra, then it is derived equivalent to $M_{\frac{n-1}{2}, \frac{m}{n}}$; and*
(3) *A selfinjective algebra Γ is stably equivalent to the algebra Λ iff Γ is derived equivalent to the algebra Λ.*

5. EXAMPLES

In this section we present an example illustrating a course of the proof of Proposition 4.4, and give our solution of Broué's problem in the introduction for the principal block of $kSL(2, 4)$ in characteristic 2. This is an alternative approach to the same solution as given in Rickard [18]. Our technique may be used to solve the problem for the groups $SL(2, p^n)$ with p prime.

Example 5.1. Let Λ be a non-symmetric wreath-like algebra defined by the quiver in Figure 5 with relations $\alpha^3 - \beta^2 = \alpha\beta = \beta\alpha = 0$.
Then $n = 6, e = 2, r = 3, m = 4, s = 2$. The Brauer quiver $Q_{\mathcal{C}}$ of Λ is as in Figure 6, where we presented α-arrows by solid arrows and β-arrows by dotted arrows.

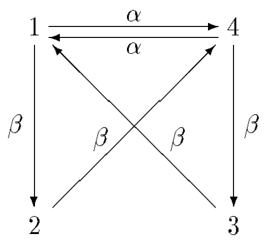

FIGURE 5

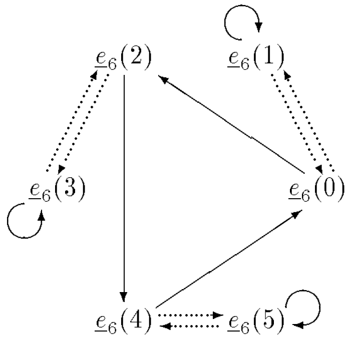

FIGURE 6

The algebra A defined in the proof of Theorem 4.9 looks as Figure 7.

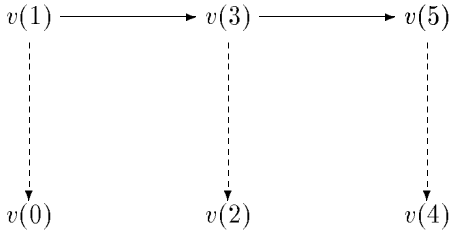

FIGURE 7

The tilting spectroids E and $\hat{E}$ constructed in the proof of the theorem are as in Figure 8, where E has no relations and $\hat{E}$ has relations that all paths of length 7 are zero, and g^s acts as ${}^{g^s}T\hat{v}(q) = T\hat{v}(q+4)$.

Thus $E \cong k\vec{A}_6$ and by Theorem 4.9, Λ is derived equivalent to the selfinjective Nakayama algebra defined in Figure 9.

Let F be a unique square-free tilting A-module. Consider F as a tilting spectroid by identifying it with the full subcategory of $\mathcal{H}^b(\text{pro}A) \simeq \mathcal{D}^b(\text{mod}\,A)$ consisting of indecomposable direct summands of F. Then note that the tilting spectroid $\hat{F}$ is not G-stable, and hence is not (G, H)-compatible.

$$E = \begin{pmatrix} v(1)^\wedge \\ \downarrow \\ v(0)^\wedge \end{pmatrix} \xrightarrow{1^\wedge} v(1)^\wedge \xrightarrow{\alpha^\wedge} \begin{pmatrix} v(3)^\wedge \\ \downarrow \\ v(2)^\wedge \end{pmatrix} \xrightarrow{1^\wedge} v(3)^\wedge \xrightarrow{\alpha^\wedge} \begin{pmatrix} v(5)^\wedge \\ \downarrow \\ v(4)^\wedge \end{pmatrix} \xrightarrow{1^\wedge} v(5)^\wedge$$

$$\hat{E} = \cdots \longrightarrow T_{\hat{v}(-1)} \longrightarrow T_{\hat{v}(0)} \longrightarrow T_{\hat{v}(1)} \longrightarrow T_{\hat{v}(2)} \longrightarrow T_{\hat{v}(3)} \longrightarrow T_{\hat{v}(4)} \longrightarrow T_{\hat{v}(5)} \longrightarrow T_{\hat{v}(6)} \longrightarrow \cdots$$

FIGURE 8

$$\begin{tikzcd} 1 \arrow[d, "\alpha"'] & 4 \arrow[l, "\alpha"'] \\ 2 \arrow[r, "\alpha"'] & 3 \arrow[u, "\alpha"'] \end{tikzcd} ; \quad \alpha^7 = 0.$$

FIGURE 9

Example 5.2. Assume that the characteristic of k is 2. Then the principal block Λ of $kSL(2,4)$ is derived equivalent to its Brauer correspondent Γ.

Proof. The algebra Λ is given by the left quiver in Figure 10 with commutativity relations $\beta_2\beta_1\alpha_2\alpha_1 = \alpha_2\alpha_1\beta_2\beta_1$, and zero relations $\alpha_1\alpha_2 = 0 = \beta_1\beta_2$. The algebra Γ is given by the right quiver in Figure 10 with commutativity relations $\alpha_2\alpha_1 = \gamma_1\gamma_3$, $\beta_2\beta_1 = \alpha_1\alpha_2$, $\gamma_2\gamma_1 = \beta_1\beta_2$, and zero relations $\beta_1\alpha_1 = 0 = \alpha_2\beta_2$, $\gamma_1\beta_1 = 0 = \beta_2\gamma_2$, $\alpha_1\gamma_1 = 0 = \gamma_2\alpha_2$ (see Rickard [18], Okuyama [14] or Koshita [13]).
Let A be the spectroid defined by the left quiver in Figure 11 with relations $\alpha\gamma = 0 = \beta\delta$, and let B the spectroid defined by the right quiver in Figure 11 with relations $\alpha\gamma = 0 = \beta\delta$.
Then $\Lambda \cong A^{\mathbb{Z}}/G$ and $\Gamma \cong B^{\mathbb{Z}}/H$, where $G = \langle g \rangle, H = \langle h \rangle$ and both g and h are defined by $p^{[i]} \mapsto \begin{cases} (p+3)^{[i]} & \text{for } p = 1,2,3 \\ (p-3)^{[i+1]} & \text{for } p = 4,5,6 \end{cases}$. Define a full subcategory E of $\mathcal{H}^b(\mathrm{pro}A^{\mathbb{Z}})$ by the following 6 objects: $(\underline{2^\wedge}), (\underline{3^\wedge}), (\underline{5^\wedge}), (\underline{6^\wedge})$, $(\underline{2^\wedge \oplus 3^\wedge} \xrightarrow{(\alpha^\wedge, \beta^\wedge)} 1^\wedge)$ and $(\underline{5^\wedge \oplus 6^\wedge} \xrightarrow{(\varepsilon^\wedge, \zeta^\wedge)} 4^\wedge)$, where the underline stands for the place of degree zero. Then it is easy to verify that (A, E, B) is a tilting triple and the tilting spectroid $\hat{E}$ is (G, H)-compatible. Hence $\Lambda \cong A^{\mathbb{Z}}/G$ is derived equivalent to $\Gamma \cong B^{\mathbb{Z}}/H$. The obtained

$$\begin{tikzcd} 2 \arrow[r, shift right, "\alpha_2"'] & 1 \arrow[l, shift right, "\alpha_1"'] \arrow[r, shift left, "\beta_1"] & 3 \arrow[l, shift left, "\beta_2"] \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[dl, shift right, "\alpha_1"'] \arrow[dr, shift left, "\gamma_2"] & \\ 2 \arrow[ur, shift right, "\alpha_2"'] \arrow[rr, shift right, "\beta_1"'] & & 3 \arrow[ul, shift left, "\gamma_1"] \arrow[ll, shift right, "\beta_2"'] \end{tikzcd}$$

FIGURE 10

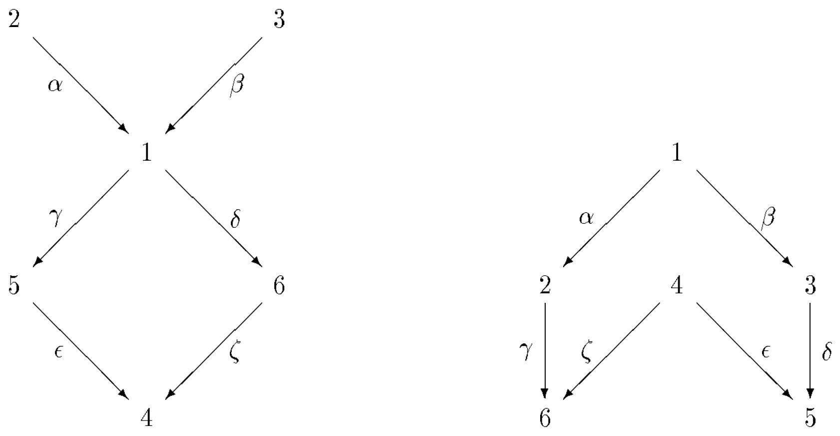

FIGURE 11

tilting spectroid E' for Λ is the same as in Rickard [18]. $\square$

Remark 5.3. Put $T^n(A) := A^{\mathbb{Z}}/\langle \nu_A^n \rangle$ for a locally bounded spectroid A. Consider the setting in the example above. Then we can call the Λ and Γ as $T^{\frac{1}{2}}(A)$ and $T^{\frac{1}{2}}(B)$, respectively because $g^2 = \nu_A$ and $h^2 = \nu_B$. Instead of taking $A^{\mathbb{Z}}$, $B^{\mathbb{Z}}$, we can use $T(A)$, $T(B)$ as double coverings of Λ, Γ, respectively. Then all computations are executed within the limits of algebras.

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