Schurian vector space categories of polynomial growth

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Abstract

The purpose of this work is to prove criteria of polynomial growth and domestic representation type of schurian vector space categories over an algebraically closed field. A list of all infinitely sincere schurian vector space categories of polynomial growth is obtained.

Introduction

Throughout the paper k is an algebraically closed field. A vector space category $\mathcal{K} = (\mathcal{K}, |-|)$ is a Krull-Scmidt k-category (see [Ri2]) together with a k-linear faithful functor $|-| : \mathcal{K} \to \operatorname{mod} k$.

The factor space category $\mathcal{V}(\mathcal{K})$ of $\mathcal{K}$ is defined as follows. The objects of $\mathcal{V}(\mathcal{K})$ are triples (X, V, φ), where $X \in \operatorname{Ob} \mathcal{K}$, V is a finite-dimensional k-vector space, and $\varphi : |X| \to V$ is a k-linear map. A morphism $(X, V, \varphi) \to (X', V', \varphi')$ is a pair (f, g), $f \in \mathcal{K}(X, X')$, $g \in \operatorname{Hom}_k(V, V')$ such that $\varphi'|f| = g\varphi$.

The concept of a factor space category was introduced by Nazarova and Roiter in [NR1]. It plays an important role in the representation theory of finite-dimensional associative algebras (see, for example, [NR1],[Ri1],[Ri2],[Si],[B1],[B2]).

We shall assume that $\mathcal{K}$ has a finite number of pairwise nonisomorphic indecomposable objects. We fix a set of representatives of the isomorphism classes of indecomposable objects in $\mathcal{K}$ and denote it by $\operatorname{ind} \mathcal{K}$. Denote by $\operatorname{ind}_m \mathcal{K}$ the set of objects which have the dimension m. Set $S\mathcal{K} = \operatorname{ind}_1 \mathcal{K}$, $L\mathcal{K} = \operatorname{ind}_2 \mathcal{K}$.

The vector space category $\mathcal{K}$ is called *schurian* (resp., *linear*) if $\operatorname{End}_{\mathcal{K}}(X) \cong k$ (resp., $\dim_k |X| = 1$) for any indecomposable object X of $\mathcal{K}$.

Let $\leq$ be the relation on $\operatorname{ind} \mathcal{K}$ satisfying $X \leq Y$ iff $\mathcal{K}(Y, X) \neq 0$. It should be noted that the relation $\leq$ is not transitive. The restriction of $\leq$ to $S\mathcal{K}$ is a partial order, i.e. $S\mathcal{K}$ is a poset (i.e. partially ordered set). We shall write $X < Y$ if $X \leq Y$ and $X \neq Y$.

For $(X, V, \varphi) \in \operatorname{Ob} \mathcal{V}(\mathcal{K})$, we define its dimension vector $\vec{d} = \vec{d}\,((X, V, \varphi))$ by setting $\vec{d} = (d_0; (d_Y)_{Y \in \operatorname{ind} \mathcal{K}})$, where $d_0 := \dim_k V$ and, for all $Y \in \operatorname{ind} \mathcal{K}$, d_Y is the multiplicity of Y as the summand in a decomposition of X into indecomposable direct summands. We set $d = d\,((X, V, \varphi)) = d_0 + \sum_{Y \in \operatorname{ind} \mathcal{K}} d_Y$.

The criterion of tameness for linear schurian vector space categories (= posets) was obtained by Nazarova in [N]. The criteria of polynomial growth and domestic representation type in this case were obtained in [ZN1] and [BZN]. In the case of nonlinear vector space categories, a problem of a characterization of tame, polynomial growth and domestic categories remains an open problem. The important classes of such categories are successfully studied in [Ri2],[NBR].

Let us remark that a tame criterion for schurian vector space categories was announced by Nazarova (see, for example, [NBR], [Ro2]). A finite growth criterion for schurian vector space categories was announced by Bondarenko, Nazarova and Roiter (see, for example, [NBR],[Ro2]).

A factor space (X, V, φ) of dimension vector $\vec{d}$ is called *sincere* if $d_0 \neq 0$ and $d_Y \neq 0$ for all $Y \in \operatorname{ind}\mathcal{K}$. The vector space category $\mathcal{K}$ is called *sincere* if there exists a sincere indecomposable factor space of $\mathcal{K}$. The vector space category $\mathcal{K}$ is called *infinitely sincere* if there exist infinitely many indecomposable pairwise nonisomorphic sincere factor spaces in some dimension $\vec{d}$.

For a given object $X \in L\mathcal{K}$, we shall identify the set of all one-dimensional subspaces of $|X|$ with the points of the one-dimensional projective space and denote it by $\mathbf{P}^1(X)$. For $X \in L\mathcal{K}$ and $V \in \mathbf{P}^1(X)$, we denote by $S_{X,V} = S\mathcal{K}_{X,V}$ (resp., $S^{X,V} = S\mathcal{K}^{X,V}$) the set of all $Z \in S\mathcal{K}$ such that $\mathcal{K}(X, Z) \neq 0$ (resp., $\mathcal{K}(Z, X) \neq 0$) and $V = \operatorname{Ker}|\varphi|$ (resp., $V = \operatorname{Im}|\varphi|$) for any $0 \neq \varphi \in \mathcal{K}(X, Z)$ (resp., $0 \neq \varphi \in \mathcal{K}(Z, X)$). Since $S_{X,V} \subseteq S\mathcal{K}$ (resp., $S^{X,V} \subseteq S\mathcal{K}$), the elements of $S_{X,V}$ (resp., $S^{X,V}$) form a poset.

For an object $X \in L\mathcal{K}$, we denote by $\Omega_X = \Omega\mathcal{K}_X$ (resp., $\Omega^X = \Omega\mathcal{K}^X$) the set of all $V \in \mathbf{P}^1(X)$ such that $S_{X,V} \neq \emptyset$ (resp., $S^{X,V} \neq \emptyset$). We set $S\mathcal{K}^X = \cup_{V \in \Omega^X} S\mathcal{K}^{X,V}$, $S\mathcal{K}_X = \cup_{V \in \Omega_X} S\mathcal{K}_{X,V}$.

The aim of the present paper is to prove the following seven theorems.

Theorem 1. *The schurian vector space category $\mathcal{K}$ is of polynomial growth if and only if the following conditions are satisfied:*

(a) $\dim_k|X| \leq 2$ *for all* $X \in \operatorname{ind}\mathcal{K}$*;*

(b) $\dim_k\mathcal{K}(X, Y) + \dim_k\mathcal{K}(Y, X) > 0$ *for all* $X \in L\mathcal{K}$, $Y \in S\mathcal{K}$*;*

(c) $\dim_k\mathcal{K}(X, Z) + \dim_k\mathcal{K}(Z, X) \geq 3$ *for all* $X, Z \in L\mathcal{K}, X \neq Z$ *;*

(d) $S_{X,V} \cap S^{Y,U} = \emptyset$ *for all* $X, Y \in L\mathcal{K}$, $V \in \mathbf{P}^1(X)$, $U \in \mathbf{P}^1(Y)$ *such that* $|\mathcal{K}(X, Y)|(V) = U$*;*

(e) $S_{X,V}$ *and* $S^{X,V}$ *do not contain the posets* $(1, 1, 1)$ *and* $(1, 2)$ *for any* $X \in L\mathcal{K}$ *and any* $V \in \mathbf{P}^1(X)$*;*

(f) if $1 < \operatorname{ord}\Omega_X < 4$ *(resp.,* $1 < \operatorname{ord}\Omega^X < 4$*), then* $S\mathcal{K}_X$ *(resp.,* $S\mathcal{K}^X$*) does not contain the poset* $(1, 1, 1, 1)$;

(g) S^{X_1,V_1} *or* S_{X_n,V_n} *does not contain the poset* $(1, 1)$ *for any* $X_1, \ldots, X_n \in L\mathcal{K}$, $V_i \in \mathbf{P}^1(X_i)$, $1 \leq i \leq n$, $n \in \mathbf{N}$, *such that* $|\mathcal{K}(X_i, X_{i+1})|(V_i) = V_{i+1}$ *for* $1 \leq i < n$.

(h) $S\mathcal{K}$ *does not contain the following posets:* $(1, 1, 1, 1, 1)$, $(1, 1, 1, 2)$, $(2, 2, 3)$, $(1, 3, 4)$, $(1, 2, 6)$, $(N, 5)$ *and* $\{A < B > C < D > A, S, T\}$.

Let $\mu_\mathcal{K}(\vec{d})$ be the number of series of the given dimension $\vec{d}$ (see Subsection 1.2).

Theorem 2. *Let $\mathcal{K}$ be a schurian vector space category of polynomial growth. Then* $\mu_\mathcal{K}(\vec{d}) \leq 1$ *for any* $\vec{d}$.

Theorem 3. *Let $\mathcal{K}$ be a schurian vector space category. The following conditions are equivalent:*

(a) $\mathcal{K}$ is of polynomial growth;

(b) $\mathcal{K}$ is of linear growth;

(c) $\mathcal{K}$ is of finite growth.

Theorem 4. *Let $\mathcal{K}$ be a nonlinear infinitely sincere schurian vector space category of polynomial growth. Then $\mathcal{K}$ is isomorphic or antiisomorphic to one of the following categories: $\mathcal{A}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda$, where $\lambda \in k \setminus \{0, 1\}$ (see Subsection* 1.4 *).*

Theorem 5. *A schurian vector space category of polynomial growth $\mathcal{K}$ is of domestic representation type if and only if $S\mathcal{K}$ is domestic and $\mathcal{K}$ does not contain a full subcategory which is isomorphic or antiisomorphic to one of the categories $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda$ (see* (1.4)*).*

Let $\chi_{\mathcal{K}}(\vec{d}) = d_0^2 + \sum_{X,Y \in \operatorname{ind}\mathcal{K}} d_X d_Y \dim_k \mathcal{K}(X, Y) - d_0 \sum_{X \in \operatorname{ind}\mathcal{K}} d_X \dim_k |X|$ be a quadratic form of the vector space category $\mathcal{K}$.

Theorem 6. *Let $\mathcal{K}$ be a schurian vector space category of polynomial growth. There are infinitely many indecomposable pairwise nonisomorphic factor spaces of the dimension $\vec{d}$ if and only if $\chi_{\mathcal{K}}(\vec{d}) = 0$.*

Let $n(\mathcal{K})$ be the number of critical (see Subsection 1.1) subsets in $S\mathcal{K}$.

Theorem 7. *Let $\mathcal{K}$ be a schurian vector space category of domestic representation type. Then $\mathcal{K}$ is of m-parameter representation type if and only if $n(\mathcal{K}) + \operatorname{ord} L\mathcal{K} = m$.*

Remark. In the proof of Theorems 1-7 the assumption that k is algebraically closed is not used. In fact, these Theorems hold for the vector space categories (over an arbitrary field k) which satisfy the following condition: $\operatorname{End}_{\mathcal{K}}(X) \cong k$ for all $X \in \operatorname{ind}\mathcal{K}$.

For the case of the linearly ordered sets $S_{X,V}$ and $S^{X,V}$ for all $X \in L\mathcal{K}$ and $V \in \mathbf{P}^1(X)$, the criteria of polynomial growth and domestic representation type are obtained in [B3].

The plan of the paper is as follows. In Section 1 all definitions and examples of schurian vector space categories are given. The differentiation algorithm of schurian vector space categories of polynomial growth is described in Section 2. In Section 3 Theorems 1-7 are proved.

1 Preliminaries

1.1

Let S be a poset. Recall that the width $w(S)$ is the maximal number of pairwise incomparable elements of S. If $A, B \in S$ are incomparable, we write $A - B$. Given subsets $S_1, S_2 \subset S$, the note $S_1 - S_2$ (resp., $S_1 < S_2$) means that $A - B$ (resp., $A < B$) for any $A \in S_1$, $B \in S_2$. We denote by $\max S$ (resp., $\min S$) the set of all maximal (resp., minimal) elements of S, and by S^{op}, the set which is antiisomorphic to S. A linearly ordered set is called a *chain*. Throughout instead of one-point sets we shall write simply the corresponding elements.

We say that the poset S is a *sum* of its subposets $S_1, \ldots, S_n$ and write $S = S_1 + \ldots + S_n$ if $S = S_1 \cup \ldots \cup S_n$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. The sum $S = S_1 + \ldots + S_n$ is *cardinal* if the elements of distinct subsets S_i and S_j are incomparable. The cardinal sum of chains consisting of $n_1, \ldots, n_m$ elements is denoted by $(n_1, \ldots, n_m)$. We denote by (N, l) the cardinal sum of the four-point set $N = \{A < B > C < D\}$ and the chain of length l. A *semichain* is a poset S of width ≤ 2 which does not contain the subset $(1, 2)$. The poset $(1, 1)$ (resp., $(1, 1, 1)$, $(1, 1, 1, 1)$) will be called a *pair* (resp., *triple*, *quadruple*).

The posets

$$K_1 = (1,1,1,1), K_2 = (2,2,2), K_3 = (1,3,3), K_4 = (N,4), K_5 = (1,2,5)$$

are called *critical sets*. The posets $\psi_1 - \psi_{20}$ and the posets $\psi_1^{op} - \psi_{20}^{op}$ [Z] (see also [Si;Ch.15]) are called *defining sets*. We set

$$W_1 = (1,1,1,1,1), W_2 = (1,1,1,2), W_3 = (2,2,3), W_4 = (1,3,4),$$

$$W_5 = (N,5), W_6 = (1,2,6), G = \{A < B > C < D > A, S, T\}.$$

We say that an element A of the poset S satisfies the condition $(*)$ if $A \in (1,2,2)$ or $A \in (1,1,1,1)$. It is easy to see that the elements of sets $W_1 - W_6$, G, $K_1 - K_5$, $\psi_1 - \psi_{20}$, $\psi_1^{op} - \psi_{20}^{op}$ satisfy the condition $(*)$.

1.2

Denote by $\mathcal{K}^{|-|}$ the vector space category $(\mathrm{Im}\,|-|, |-|')$ with the identity functor $|-|'$. It follows immediately from definitions that $\mathcal{V}(\mathcal{K})$ is equivalent to $\mathcal{V}(\mathcal{K}^{|-|})$. We denote by ρ the functor-equivalence from $\mathcal{V}(\mathcal{K}^{|-|})$ to $\mathcal{V}(\mathcal{K})$. Let $F = k < x, y >$ or $F = k[x]$. By an *F-representation* of $\mathcal{K}^{|-|}$ we mean a triple (Y, U, ψ), where $Y \in \mathrm{Ob}\,\mathcal{K}^{|-|}$, U is a free F-module of finite rank, and $\psi : F \otimes_k Y \to U$ is an F- linear map. Given an F-representation (Y, U, ψ) of $\mathcal{K}^{|-|}$, we consider a functor $G_{(Y,U,\psi)} = \rho(- \otimes_F (Y, U, \psi))$ from the category of finite-dimensional F-modules to $\mathcal{V}(\mathcal{K})$.

We fix a set of representatives of the isomorphism classes of indecomposable objects of $\mathcal{V}(\mathcal{K})$ and denote it by $\mathrm{ind}\,\mathcal{V}(\mathcal{K})$. Let $S(k[x])$ be the simple $k[x]$-modules.

Following [D], $\mathcal{K}$ is called *wild* if there exists an $k < x, y >$-representation (Y, U, ψ) of $\mathcal{K}^{|-|}$ such that the functor $G_{(Y,U,\psi)}$ preserves indecomposability and isomorphism classes.

A category $\mathcal{K}$ is called *tame* if for any dimension $\vec{d}$ there exist finitely many $k[x]$-representations (Y_i, U_i, ψ_i), $1 \le i \le n_{\vec{d}}$, such that the following satisfies: all but a finite number (up to isomorphism) of indecomposable objects of $\mathcal{V}(\mathcal{K})$ of dimension $\vec{d}$ are isomorphic to $G_{(Y_i,U_i,\psi_i)}(S)$ for some $S \in S(k[x])$. The subset of $\{G_{(Y_i,U_i,\psi_i)}(S) \mid S \in S(k[x])\}$ consisting of all factor spaces of the given dimension $\vec{d}$ will be called a *series*.

Let $\mu_{\mathcal{K}}(\vec{d})$ be the minimal number of $k[x]$-representations (Y_i, U_i, ψ_i) satisfying the above condition. $\mathcal{K}$ is called of *finite growth* if there is a natural number m such that $\mu_{\mathcal{K}}(\vec{d}) \le m$ for all $\vec{d}$.

We can give the equivalent definition of tameness replacing $\vec{d}$ by d. Then let $\mu_{\mathcal{K}}(d)$ be the minimal number of the corresponding $k[x]$-representations. $\mathcal{K}$ is called of *polynomial growth* (resp., *linear growth*) [Sk2] if there is a natural number m such that $\mu_{\mathcal{K}}(d) \le d^m$ (resp., $\mu_{\mathcal{K}}(d) \le md$) for all $d \ge 2$.

Following [Ri1], $\mathcal{K}$ is *domestic* if there exist finitely many $k[x]$-representations (Y_i, U_i, ψ_i), $1 \le i \le n$, such that for each dimension $\vec{d}$, all but a finite number (up to isomorphism) of indecomposable objects of $\mathcal{V}(\mathcal{K})$ of dimension $\vec{d}$ are isomorphic to $G_{(Y_i,U_i,\psi_i)}(M)$ for some i and some indecomposable $k[x]$-module M. Let $\nu(\mathcal{K})$ be the minimal number of $k[x]$-representations (Y_i, U_i, ψ_i) satisfying the above condition. $\mathcal{K}$ is called of *m-parameter* representation type if $\nu(\mathcal{K}) = m$.

Sometimes we shall consider the vector space categories $\mathcal{K}$ with infinite set $\mathrm{ind}\,\mathcal{K}$. In this case, a vector space category $\mathcal{K}$ is called tame, of polynomial growth etc. if any full subcategory with only finitely many indecomposable objects is tame, of polynomial growth etc., respectively.

For a given object X and a natural number n we denote by X^n a direct sum $X^n = \oplus_{i=1}^{n} X$. For $T \subset \mathrm{ind}\,\mathcal{K}$ denote by $\mathcal{K}(T)$ the full subcategory with the set of objects T.

We say that a basis $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$ is fixed in a vector space category $\mathcal{K}$ if a basis $\mathcal{B}_0(X) = \{x^i, 1 \le i \le \dim_k|X|\}$ of the space $|X|$ is fixed for every object $X \in \mathrm{ind}\,\mathcal{K}$, and a basis $\mathcal{B}_1(X,Y) = \{b^i_{X,Y}, 1 \le i \le \dim_k|\mathcal{K}(X,Y)|\}$ of the space $|\mathcal{K}(X,Y)|$ is fixed for every two objects $X \ne Y$ in $\mathrm{ind}\,\mathcal{K}$. Then $\mathcal{B}_0 = \cup_{X\in \mathrm{ind}\,\mathcal{K}} \mathcal{B}_0(X)$, $\mathcal{B}_1 = \cup_{X\ne Y \in \mathrm{ind}\,\mathcal{K}} \mathcal{B}_1(X,Y)$. For the sake of simplicity we write x (resp., $b_{X,Y}$) instead of x^1 (resp., $b^1_{X,Y}$) if $\dim_k|X| = 1$ (resp., $\dim_k \mathcal{K}(X,Y) = 1$). Let $Mb^i_{X,Y}$ be the matrix of the mapping $b^i_{X,Y}$ in the basis $\mathcal{B}_0$.

Fix a basis $\mathcal{B}$ in $\mathcal{K}$ and $(X, V, \varphi) \in \mathcal{V}(\mathcal{K})$. We fix a decomposition $X \cong X_1^{d_1} \oplus \ldots \oplus X_n^{d_n}$, where $X_i \in \mathrm{ind}\,\mathcal{K}$ and the linear isomorphism $V \cong k^{d_0}$. Then $|X_i^{d_i}| = \oplus_{j=1}^{\dim_k|X_i|} (kx_i^j)^{d_i}$ and the matrix $M\varphi$ of the mapping φ has the block form

$$M\varphi = \left(\begin{array}{ccccccc} M\varphi_{x_1^1} & \cdots & M\varphi_{x_1^{m_1}} & \cdots & M\varphi_{x_n^1} & \cdots & M\varphi_{x_n^{m_n}} \end{array} \right),$$

where $m_j = \dim_k|X_j|$ and $\varphi_{x_l^j}$ is the restriction of the mapping φ to the subspace $(kx_l^j)^{d_l}$.

We say that the block matrix $M\varphi$ is a *matrix representation* of $\mathcal{K}$. It is easy to see that any isomorphism $(f, g) : (X, V, \varphi) \to (X', V', \varphi')$ in $\mathcal{V}(\mathcal{K})$ induces a pair of matrices $M_f \in \mathrm{Gl}(d_1m_1 + \ldots + d_nm_n, k)$, $M_g \in \mathrm{Gl}(d_0, k)$ such that $M_gM\varphi M_f = M\varphi'$. Then we define a matrix problem

$$(Mat_{\mathcal{K}}, G_{\mathcal{K}})$$

in the sense of [Si], which is equivalent to the classification of indecomposable objects of $\mathcal{V}(\mathcal{K})$ (see [Si] for details).

Denote by $p_{x_l^j}$ (resp., $i_{x_l^j}$) the natural projection (resp., embedding) of the space $|X_l|$ (resp., $(kx_l^j)^{d_l}$) onto (resp., into) the space $(kx_l^j)^{d_l}$ (resp., $|X_l|$). If T is a subset of $\mathcal{B}_0$, then we denote by $M\varphi_T$ the block matrix $(M\varphi_z)_{z\in T}$.

Suppose that $P \in S\mathcal{K}$. A basis $\mathcal{B}$ is called a P-basis if the following conditions hold:

1. If $X \in L\mathcal{K}$ and $\dim_k \mathcal{K}(P,X) = 1$, then $x^1 = b^1_{P,X}(p)$;
2. If $X \in L\mathcal{K}$ and $\dim_k \mathcal{K}(X,P) = 1$, then $x^2 \in \mathrm{Ker} b^1_{X,P}$, $p = b^1_{X,P}(x^1)$.

For an object $A \in \mathrm{ind}\,\mathcal{K}$, we set $A^- = \{B \in \mathrm{ind}\,\mathcal{K} \mid \mathcal{K}(B,A) \ne 0, B \ne A\}$, $A^+ = \{B \in \mathrm{ind}\,\mathcal{K} \mid \mathcal{K}(A,B) \ne 0, B \ne A\}$. For a given object $P \in S\mathcal{K}$, we set $\mathcal{B}_0^{P,+} = \{t \in \mathcal{B}_0 \mid t = |\alpha|(p) \text{ for some } \alpha \in \mathrm{Mor}\,\mathcal{K}\}$, $\mathcal{B}_0^{P,-} = \{t \in \mathcal{B}_0 \mid t \notin \mathcal{B}_0^{P,+}, t \notin |A| \text{ for any } A \in P^-\}$, $L_1 = \{|X| \mid X \in \mathrm{ind}\,\mathcal{K}, \mathcal{K}(P,X) = 0\}$, $L_2 = \{kx^2 \mid X \in L\mathcal{K}, \dim_k \mathcal{K}(P,X) = 1\}$. Consider the category L with $\mathrm{Ob}L = L_1 \cup L_2$ and with the following spaces of morphisms:

$L(|X|, |Y|) = |\mathcal{K}(X,Y)|$, $\quad L(kx^2, ky^2) = \sum_{\varphi\in\mathcal{K}(X,Y)} p_{y^2}|\varphi| i_{x^2}$,

$L(|X|, ky^2) = \sum_{\varphi\in\mathcal{K}(X,Y)} p_{y^2}|\varphi|$, $\quad L(kx^2, |Y|) = \sum_{\varphi\in\mathcal{K}(X,Y)} |\varphi| i_{x^2}$.

Denote by $\mathcal{PL}$ the additive closure of the category L with the identity functor $|-|$. We denote by $\hat{\mathcal{V}}(\mathcal{K})$ the full subcategory of $\mathcal{V}(\mathcal{K})$ consisting of the objects (X, V, φ) such that $M\varphi_{\mathcal{B}_0^{P,+}} = 0$ for a fixed P-basis. The following lemma follows immediately from definitions.

Lemma 1. $\hat{\mathcal{V}}(\mathcal{K}) \cong \mathcal{V}(\mathcal{PL})$. ▪

Following Ringel [Ri1] the one-dimensional objects of $\mathcal{K}$ will be pictured by points • and the two-dimensional objects by squares with two points. Edges will represent non-zero homomorphisms.

For a given objects A, B, the note $A \subset B$ means that A is isomorphic to a direct summand of B. If $A \subset B$ or $A = B$, we write $A \subseteq B$.

Let S be a poset. We denote by $\mathcal{K}(S)$ a linear schurian vector space category with the set of indecomposable objects $\operatorname{ind}\mathcal{K}(S) = \{kA \mid A \in S\}$, with the following spaces of morphisms

$$\mathcal{K}(S)(kA, kB) = \begin{cases} k & \text{, if } B \leq A; \\ 0 & \text{, otherwise,} \end{cases}$$

for any $A, B \in S$ and with the identity functor $|-|$. Obviously, $\mathcal{V}(\mathcal{K}(S)) \cong R(S)$, where $R(S)$ is the category of representations of the poset S.

Let (S, σ) be a poset with involution satisfying the following condition: if $\sigma(A) \neq A$, then $\sigma(A) - A$ for any $A \in S$ (see [NBR]). For any $A \in S$ we set

$$A_\sigma = \begin{cases} kA & \text{, if } \sigma(A) = A; \\ kA \oplus k\sigma(A) & \text{, otherwise.} \end{cases}$$

We denote by $\mathcal{K}(S, \sigma)$ a schurian vector space category with the set of indecomposable objects $\operatorname{ind}\mathcal{K}(S, \sigma) = \{X_\sigma \mid X \in S\}$, with the following spaces of morphisms

$$\mathcal{K}(S, \sigma)(X_\sigma, Y_\sigma) = \begin{cases} k & \text{, if } X_\sigma = Y_\sigma; \\ \mathcal{K}(S)(X_\sigma, Y_\sigma) & \text{, otherwise} \end{cases}$$

for any $A, B \in S$ and with the identity functor $|-|$. Obviously, $\mathcal{V}(\mathcal{K}(S, \sigma)) \cong R(S, \sigma)$, where $R(S, \sigma)$ is the category of representations of the poset with involution (S, σ) (see [NBR]).

We denote by $J_s(\lambda)$ the $s \times s$ Jordan cell with eigenvalue $\lambda \in k$. We denote by E_s the $s \times s$ identity matrix. If $s = 0$, then E_s is the "empty" matrix, i.e. matrix with zero number of rows and columns. Consider the following well-known indecomposable factor spaces.

1. $\mathcal{K}_0 = (0, k, 0)$;
2. $\mathcal{K}_{A,0} = (A, 0, 0)$ for every $A \in \operatorname{ind}\mathcal{K}$;
3. $\mathcal{K}_A = (A, k, \varphi)$ for every $A \in S\mathcal{K}$, where $M\varphi = (1)$;
4. $\mathcal{K}_{\{A,B\}} = (A \oplus B, k, \varphi)$ for every pair (A, B) in $S\mathcal{K}$, where $M\varphi = (11)$;
5. $\mathcal{K}_{\{A,B,C\}} = (A \oplus B \oplus C, k^2, \varphi)$ for every triple (A, B, C) in $S\mathcal{K}$, where

$$M\varphi = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array};$$

6. $\mathcal{K}_{\{A,B,C,\diamond\}} = (A \oplus B \oplus C, k, \varphi)$ for every triple (A, B, C) in $S\mathcal{K}$, where $M\varphi = (111)$;
7. $\mathcal{K}_{\{A,B,C<D\}} = (A \oplus B \oplus C \oplus D, k^2, \varphi)$ for every subset $\{A, B, C < D\} \subseteq S\mathcal{K}$, where

$$M\varphi = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array};$$

8. $\mathcal{K}_{A,s,0,-} = (A^s, k^s, \varphi)$ for every $A \in L\mathcal{K}$, where $s \in \mathbf{N}$ and

$$M\varphi = \begin{array}{c} \begin{array}{cc} a^1 & a^2 \end{array} \\ \begin{pmatrix} E_s & J_s(0) \end{pmatrix} \end{array};$$

9. $\mathcal{K}_{A,s,0,+} = (A^s, k^s, \varphi)$ for every $A \in L\mathcal{K}$, where $s \in \mathbf{N}$ and

$$M\varphi = \begin{array}{cc} a^1 & a^2 \\ \multicolumn{2}{c}{\begin{pmatrix} J_s(0) & E_s \end{pmatrix}} \end{array};$$

10. $\mathcal{K}_{A,s,-} = (A^{s+1}, k^s, \varphi)$ for every $A \in L\mathcal{K}$, where $s \in \mathbf{N}$ and

$$M\varphi_{a^1} = \begin{pmatrix} E_s & 0 \end{pmatrix}, \qquad M\varphi_{a^2} = \begin{pmatrix} 0 & E_s \end{pmatrix};$$

11. $\mathcal{K}_{A,s,+} = (A^s, k^{s+1}, \varphi)$ for every $A \in L\mathcal{K}$, where $s \in \mathbf{N}$ and

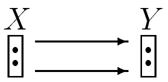
$$M\varphi_{a^1} = \begin{pmatrix} 0 \\ E_s \end{pmatrix}, \qquad M\varphi_{a^2} = \begin{pmatrix} E_s \\ 0 \end{pmatrix};$$

12. $\mathcal{K}_{A,s,\lambda} = (A^s, k^s, \varphi)$ for every $A \in L\mathcal{K}$, where $s \in \mathbf{N}, \lambda \in k \setminus \{0\}$ and

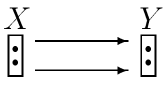
$$M\varphi = \begin{array}{cc} a^1 & a^2 \\ \multicolumn{2}{c}{\begin{pmatrix} E_s & J_s(\lambda) \end{pmatrix}} \end{array}.$$

1.3

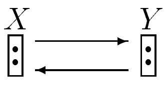
We list below the non-polynomial growth and wild categories which are used in the proof of Theorem 1.

A : 

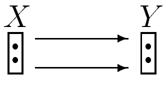
$$Mb^1_{X,Y} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Mb^2_{X,Y} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

B : 

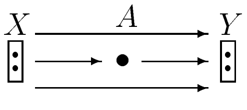
$$Mb^1_{X,Y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Mb^2_{X,Y} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

C : 

$$Mb^1_{X,Y} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Mb^1_{Y,X} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix};$$

D : 

$$Mb^1_{X,Y} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Mb^2_{X,Y} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

E : 

$$Mb^1_{X,Y} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Mb^2_{X,Y} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$Mb^1_{X,A} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad Mb^3_{X,Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad Mb^1_{A,Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\mathbf{F}: \begin{tikzcd}[row sep=tiny] & \overset{A}{\bullet} \\ \overset{Z}{\boxed{\vdots}} \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \overset{B}{\bullet} \\ & \overset{C}{\bullet} \\ & \overset{D}{\bullet} \end{tikzcd} \qquad \begin{array}{l} Mb^1_{Z,A} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \\ Mb^1_{Z,B} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \\ Mb^1_{Z,C} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \\ Mb^1_{Z,D} = \begin{pmatrix} 1 & 1 \end{pmatrix}. \end{array} \qquad \mathbf{G}: \begin{tikzcd}[row sep=tiny] & \overset{A}{\bullet} \\ \overset{Z}{\boxed{\vdots}} \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & \overset{B}{\bullet} \\ & \overset{C}{\bullet} \\ & \overset{D}{\bullet} \end{tikzcd} \qquad \begin{array}{l} Mb^1_{Z,A} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \\ Mb^1_{Z,B} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \\ Mb^1_{Z,C} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \\ Mb^1_{Z,D} = \begin{pmatrix} 0 & 1 \end{pmatrix}. \end{array}$$

The infinite growth of the categories $\mathbf{A}$ and $\mathbf{C}$ follows from [NBR] (see p.20, where $\mathbf{A}$ corresponds to the antichain 1) and $\mathbf{C}$ corresponds to the antichain 2) from Lemma 17). Let us prove that the categories $\mathbf{A}$ and $\mathbf{C}$ are not of polynomial growth. Suppose that $\mathcal{K} = \mathbf{A}$ or $\mathcal{K} = \mathbf{C}$. For any set $\{r_i, s_i \mid r_i, s_i \in \mathbf{N},\ 1 \le i \le t,\ t \in \mathbf{N}\}$ we consider the following $k[x]$-representation $R(r_1, \dots, r_t, s_1, \dots, s_t) = (Z, U, \psi)$ of $\mathcal{K}^{|-|}$. Let $Z := |X|^{d_X} \oplus |Y|^{d_Y}$, $U := F^{d_0}$, where $F = k[x]$, $d_X = \sum_{i=1}^t r_i$, $d_Y = \sum_{i=1}^t s_i$, $d_0 = d_X + d_Y$, and u_i, $1 \le i \le d_0$, are the natural free generators of U, $\{x_i^k\}$, $1 \le k \le 2$, $1 \le i \le d_X$, is a basis of $|X|^{d_X}$, $\{y_i^k\}$, $1 \le k \le 2$, $1 \le i \le d_Y$, is a basis of $|Y|^{d_Y}$. For given $1 \le m \le t-1$ we set $\mu_m := \sum_{i=1}^m r_i$, $\nu_m := \sum_{i=1}^m s_i$. We consider a new basis $\{z_i^k \mid 1 \le k \le 2,\ 1 \le i \le d_0\}$ of Z, where $z_1^1 := x_1^1$, $z_1^2 := x_1^2$,

$$z_i^k := \begin{cases} x_{j+1}^k & \text{, if } z_{i-1}^k = x_j^k \text{ and } j \ne \mu_m \text{ for } 1 \le m \le t; \\ y_{j+1}^k & \text{, if } z_{i-1}^k = y_j^k \text{ and } j \ne \nu_m \text{ for any } 1 \le m \le t; \\ y_{\mu_m+1}^k & \text{, if } z_{i-1}^k = x_{\mu_m}^k \text{ for some } 1 \le m \le t; \\ x_{\nu_m+1}^k & \text{, if } z_{i-1}^k = y_{\nu_m}^k \text{ for some } 1 \le m < t \end{cases}$$

for $1 < i \le d_0$, $1 \le k \le 2$. Then we define the map ψ as follows:

$$\psi(z_i^k \otimes 1) := \begin{cases} u_i & \text{, if } k = 1; \\ u_{i+1} & \text{, if } k = 2 \text{ and } i \ne d_0; \\ x u_1 & \text{, if } k = 2 \text{ and } i = d_0. \end{cases}$$

It is not difficult to verify that $G_{R(r_1,\dots,r_t,s_1,\dots,s_t)}(S)$ is the indecomposable object of $\mathcal{V}(\mathcal{K})$ of dimension $d = 2d_0$ for any $S \in S(k[x])$ and $G_{R(r_1,\dots,r_t,s_1,\dots,s_t)}(S_1) \cong G_{R(r'_1,\dots,r'_t,s'_1,\dots,s'_t)}(S_2)$ if and only if $S_1 \cong S_2$ and either $r_i = r'_i, s_i = s'_i$ for all i or $(r'_1, \dots, r'_t, s'_1, \dots, s'_t) = (r_{l+1}, \dots, r_t, r_1, \dots, r_l, s_{l+1}, \dots, s_t, s_1, \dots, s_l)$ for some $1 \le l < t$. From the above information it follows as in [Sk1;p.504] that the category $\mathcal{K}$ is not of polynomial growth.

The wildness of the categories $\mathbf{D}, \mathbf{E}$ follows from [NBR] (see p.18, where $\mathbf{D}$ corresponds to the set $\mathcal{N}_{7'}$ and $\mathbf{E}$ corresponds to the set $\bar{\mathcal{N}}_{71}$).

Consider the following $k < x, y >$-representation (Z, U, ψ) of $\mathbf{B}^{|-|}$. Set $Z = |X|^6 \oplus |Y|^7$, $U = F^{12}$, where $F = k < x, y >$, and $u_i, 1 \le i \le 12$, are natural free generators of U, $\{x_j^k\}, 1 \le k \le 2, 1 \le j \le 6$, is a basis of $|X|^6$, $\{y_l^s\}, 1 \le s \le 2, 1 \le l \le 7$, is a basis of $|Y|^7$. Then we define the map ψ as follows: $\psi(x_1^1 \otimes 1) = u_7$, $\psi(x_2^1 \otimes 1) = y u_7$, $\psi(x_3^1 \otimes 1) = u_1$, $\psi(x_4^1 \otimes 1) = u_2$, $\psi(x_5^1 \otimes 1) = u_3$, $\psi(x_6^1 \otimes 1) = u_6$, $\psi(x_1^2 \otimes 1) = u_1 + u_3 + u_4$, $\psi(x_2^2 \otimes 1) = u_2 + u_3 + x u_4$, $\psi(x_3^2 \otimes 1) = 0$, $\psi(x_4^2 \otimes 1) = u_6$, $\psi(x_5^2 \otimes 1) = u_5$, $\psi(x_6^2 \otimes 1) = 0$, $\psi(y_1^1 \otimes 1) = u_{10}$, $\psi(y_2^1 \otimes 1) = u_9$, $\psi(y_3^1 \otimes 1) = u_8$, $\psi(y_4^1 \otimes 1) = 0$, $\psi(y_5^1 \otimes 1) = u_{12}$, $\psi(y_6^1 \otimes 1) = u_{11}$, $\psi(y_7^1 \otimes 1) = 0$, $\psi(y_1^2 \otimes 1) = u_2$, $\psi(y_2^2 \otimes 1) = u_3$, $\psi(y_3^2 \otimes 1) = u_4$, $\psi(y_4^2 \otimes 1) = u_7$, $\psi(y_5^2 \otimes 1) = u_8$, $\psi(y_6^2 \otimes 1) = u_9$, $\psi(y_7^2 \otimes 1) = u_{12}$.

It is not difficult to verify that the functor $G_{(Z,U,\psi)}$ which acts from the category of finite-dimensional $k < x, y >$-modules to $\mathcal{V}(\mathcal{K})$ preserves indecomposability and isomorphism classes. So, $\mathbf{B}$ is wild.

For the proof of the non-polynomial growth of the category $\mathbf{F}$ we define an embedding $\mu : \operatorname{Ob}\mathcal{V}(\mathcal{K}(G)) \to \operatorname{Ob}\mathcal{V}(\mathbf{F})$, where $G = \{A < B > C < D > A, S, T\}$. Let (Y, W, ψ) be an arbitrary object of the category $\mathcal{V}(\mathcal{K}(G))$. We fix a decomposition $Y \cong \oplus_{F \in \operatorname{ind}\mathcal{K}(G)} F^{d_F}$. Consider $(X, V, \varphi) \in \mathcal{V}(\mathbf{F})$ such that $X = Z^{d_B} \oplus A^{d_S} \oplus B^{d_T} \oplus C^{(d_D+d_C)} \oplus D^{(d_D+d_A)}$, $V = W \oplus k^{d_D}$,

$$M\varphi_{z^1} = 0, M\varphi_{z^2} = \begin{pmatrix} M\psi_b \\ 0 \end{pmatrix}, M\varphi_a = \begin{pmatrix} M\psi_s \\ 0 \end{pmatrix}, M\varphi_b = \begin{pmatrix} M\psi_t \\ 0 \end{pmatrix},$$

$$M\varphi_c = \begin{pmatrix} 0 & M\psi_c \\ E & 0 \end{pmatrix}, M\varphi_d = \begin{pmatrix} M\psi_d & M\psi_a \\ E & 0 \end{pmatrix}.$$

We set $\mu((Y, W, \psi)) = (X, V, \varphi)$. It is easy to see that μ preserves indecomposability and isomorphism classes. Moreover, if $(Y, W, \psi), (Y', W', \psi') \in \operatorname{Ob}\mathcal{V}(\mathcal{K}(G))$ and $\vec{d}((Y, W, \psi)) = \vec{d}((Y', W', \psi'))$, then $\vec{d}(\mu(Y, W, \psi)) = \vec{d}(\mu(Y', W', \psi'))$. Therefore the non-polynomial growth of the category $\mathbf{F}$ follows from the non-polynomial growth of the category $\mathcal{K}(G)$ (see [Sk1;p.507]).

For the proof of the non-polynomial growth of the category $\mathbf{G}$ we define an embedding $\mu : \operatorname{Ob}\mathcal{V}(\mathcal{K}(G)) \to \operatorname{Ob}\mathcal{V}(\mathbf{G})$, where $G = \{A < B > C < D > A, S, T\}$. Let (Y, W, ψ) be an arbitrary object of the category $\mathcal{V}(\mathcal{K}(G))$. We fix a decomposition $Y \cong \oplus_{F \in \operatorname{ind}\mathcal{K}(G)} F^{d_F}$. Consider $(X, V, \varphi) \in \mathcal{V}(\mathbf{G})$ such that $X = Z^{(d_B+d_D)} \oplus A^{d_S} \oplus B^{d_T} \oplus C^{(d_B+d_C)} \oplus D^{(d_D+d_A)}$, $V = W \oplus k^{(d_B+d_D)}$,

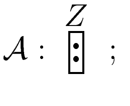
$$M\varphi_{z^1} = \begin{pmatrix} 0 & 0 \\ E & 0 \\ 0 & E \end{pmatrix}, M\varphi_{z^2} = \begin{pmatrix} M\psi_b & M\psi_d \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, M\varphi_a = \begin{pmatrix} M\psi_s \\ 0 \\ 0 \end{pmatrix},$$

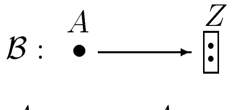
$$M\varphi_b = \begin{pmatrix} M\psi_t \\ 0 \\ 0 \end{pmatrix}, M\varphi_c = \begin{pmatrix} 0 & M\psi_c \\ E & 0 \\ 0 & 0 \end{pmatrix}, M\varphi_d = \begin{pmatrix} 0 & M\psi_a \\ 0 & 0 \\ E & 0 \end{pmatrix}.$$

We set $\mu((Y, W, \psi)) = (X, V, \varphi)$. It is easy to see that μ preserves indecomposability and isomorphism classes. Moreover, if $(Y, W, \psi), (Y', W', \psi') \in \operatorname{Ob}\mathcal{V}(\mathcal{K}(G))$ and $\vec{d}((Y, W, \psi)) = \vec{d}((Y', W', \psi'))$, then $\vec{d}(\mu(Y, W, \psi)) = \vec{d}(\mu(Y', W', \psi'))$. Therefore the non-polynomial growth of the category $\mathbf{F}$ follows from the non-polynomial growth of the category $\mathcal{K}(G)$ (see [Sk1;p.507]).

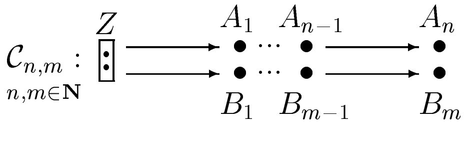
1.4

We list below the schurian vector space categories of polynomial growth which are used in Sections 2,3.



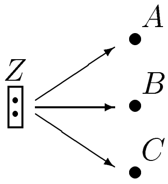


$$Mb_{A,Z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix};$$



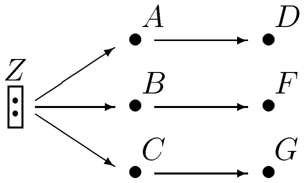
$$Mb_{Z,A_1} = \begin{pmatrix} 1 & 0 \end{pmatrix},$$
$$Mb_{Z,B_1} = \begin{pmatrix} 0 & 1 \end{pmatrix};$$

$\mathcal{D}$:



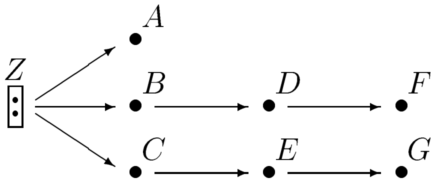
$$Mb_{Z,A} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad Mb_{Z,B} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad Mb_{Z,C} = \begin{pmatrix} 0 & 1 \end{pmatrix};$$

$\mathcal{E}$:



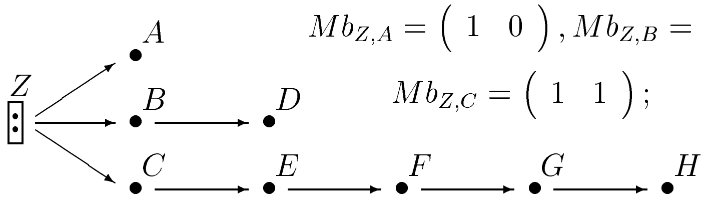
$$Mb_{Z,A} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad Mb_{Z,B} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad Mb_{Z,C} = \begin{pmatrix} 1 & 1 \end{pmatrix};$$

$\mathcal{F}$:



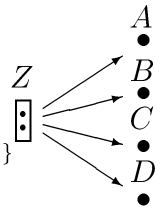
$$Mb_{Z,A} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad Mb_{Z,B} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad Mb_{Z,C} = \begin{pmatrix} 1 & 1 \end{pmatrix};$$

$\mathcal{G}$:



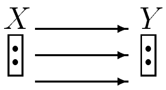
$$Mb_{Z,A} = \begin{pmatrix} 1 & 0 \end{pmatrix}, Mb_{Z,B} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad Mb_{Z,C} = \begin{pmatrix} 1 & 1 \end{pmatrix};$$

$\mathcal{H}_\lambda$:
$\lambda \in k \backslash \{0,1\}$



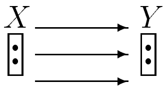
$$Mb_{Z,A} = \begin{pmatrix} 1 & 1 \end{pmatrix}, \quad Mb_{Z,B} = \begin{pmatrix} 1 & \lambda \end{pmatrix}, \quad Mb_{Z,C} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad Mb_{Z,D} = \begin{pmatrix} 1 & 0 \end{pmatrix};$$

$\mathcal{I}$:



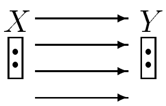
$$Mb^1_{X,Y} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Mb^2_{X,Y} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Mb^3_{X,Y} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix};$$

$\mathcal{J}$:

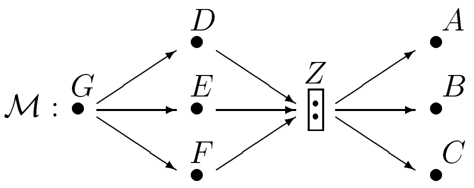


$$Mb^1_{X,Y} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Mb^2_{X,Y} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Mb^3_{X,Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$\mathcal{L}$:

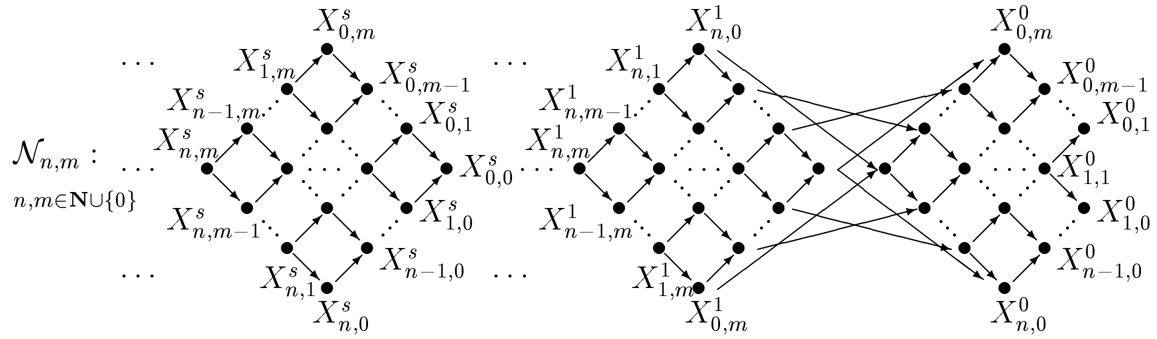


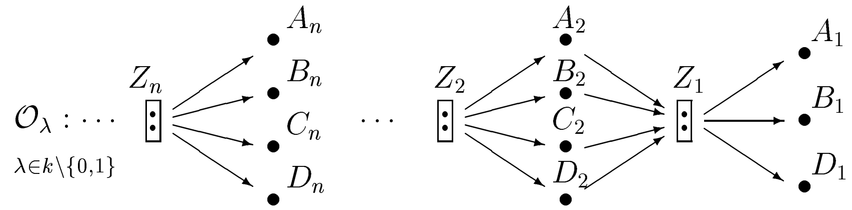
$$Mb^1_{X,Y} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Mb^2_{X,Y} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$Mb^3_{X,Y} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Mb^4_{X,Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix};$$



$$Mb_{D,Z} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Mb_{E,Z} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad Mb_{F,Z} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$Mb_{Z,A} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad Mb_{Z,B} = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad Mb_{Z,C} = \begin{pmatrix} 1 & 1 \end{pmatrix};$$





$$Mb_{Z_s,A_s} = \begin{pmatrix} 1 & 0 \end{pmatrix}, Mb_{Z_s,B_s} = \begin{pmatrix} 0 & 1 \end{pmatrix}, Mb_{Z_t,C_t} = \begin{pmatrix} \lambda & 1 \end{pmatrix},$$

$$Mb_{Z_s,D_s} = \begin{pmatrix} 1 & 1 \end{pmatrix}, Mb_{A_{s+1},Z_s} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, Mb_{B_{s+1},Z_s} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$Mb_{C_{s+1},Z_s} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, Mb_{D_{s+1},Z_s} = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}, s,t \in \mathbf{N}, t > 1.$$

Remark. The spaces of morphisms in the category $\mathcal{N}_{n,m}$ are defined by the following conditions.

1. $X^k_{i,j} < X^k_{i,j+1}$, where $0 \le i \le n$, $0 \le j < m$,$k \in \mathbf{N} \cup \{0\}$,$k+i+j \ne 0$;
2. $X^k_{i,j} < X^k_{i+1,j}$, where $0 \le i < n$, $0 \le j \le m$,$k \in \mathbf{N} \cup \{0\}$,$k+i+j \ne 0$;
3. $X^k_{i,m} < X^{k+1}_{i,0}$, where $0 \le i \le n$, $k \in \mathbf{N} \cup \{0\}$, $k+i+m \ne 0$;
4. $X^k_{n,j} < X^{k+1}_{0,j}$, where $0 \le j \le m$, $k \in \mathbf{N} \cup \{0\}$, $k+j+n \ne 0$.

1.5

Lemma 2. *Let $\mathcal{K}$ be a schurian vector space category which satisfies all the conditions of Theorem 1, $X, Y \in \operatorname{ind} \mathcal{K}$, $X \ne Y$. If $\mathcal{K}(X,Y) \ne 0$, then $\mathcal{K}(Y,X) = 0$.*

Proof. If $X \in S\mathcal{K}$ or $Y \in S\mathcal{K}$, then proof is evident. Assume that $X, Y \in L\mathcal{K}$, $X \neq Y$, $0 \neq \varphi \in \mathcal{K}(X,Y)$, $0 \neq \psi \in \mathcal{K}(Y,X)$. Since $\mathcal{K}$ is schurian, we get $\varphi\psi = \psi\varphi = 0$ and hence $\operatorname{Ker}|\varphi| \in \mathbf{P}^1(X)$, $\operatorname{Im}|\varphi| \in \mathbf{P}^1(Y)$. Therefore there exists a basis $\mathcal{B}$ of the category $\mathcal{K}$ such that $M|\varphi| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Suppose $M|\psi| = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\varphi\psi = \psi\varphi = 0$, we get $a = b = c = 0$. Consequently, $\dim_k \mathcal{K}(Y,X) = 1$. Similarly, we have $\dim_k \mathcal{K}(X,Y) = 1$. This contradicts $\dim_k \mathcal{K}(X,Y) + \dim_k \mathcal{K}(Y,X) \geq 3$. ▪

Given $\Phi \subseteq S\mathcal{K}, X \in L\mathcal{K}$, the note $\Phi < X$ (resp., $\Phi > X$) means that $A < X$ (resp., $A > X$) for any $A \in \Phi$. We set $\max \operatorname{ind} \mathcal{K} = \{X \in \operatorname{ind} \mathcal{K} \mid \mathcal{K}(Y,X) = 0$ for any $Y \in \operatorname{ind} \mathcal{K}, Y \neq X\}$, $\min \operatorname{ind} \mathcal{K} = \{X \in \operatorname{ind} \mathcal{K} \mid \mathcal{K}(X,Y) = 0$ for any $Y \in \operatorname{ind} \mathcal{K}, Y \neq X\}$.

Lemma 3. *If $U, V \in \mathbf{P}^1(X)$ and $U \neq V$, then $S\mathcal{K}_{X,U} < S\mathcal{K}^{X,V}$.*

Proof. Suppose $A \in S\mathcal{K}_{X,U}$, $B \in S\mathcal{K}^{X,V}$. Then there exist $\varphi \in \mathcal{K}(X,A)$, $\psi \in \mathcal{K}(B,X)$ such that $U = \operatorname{Ker}|\varphi|$, $V = \operatorname{Im}|\psi|$. Since $V \not\subseteq U$, we have $\varphi\psi \neq 0$ and hence $A < B$. ▪

Lemma 4. *Suppose that $X \in L\mathcal{K}, A, B \in S\mathcal{K}$ and $A < B$. Then*
1. If $X < A$ and $B \in S\mathcal{K}^{X,V}$, then $A \in S\mathcal{K}^{X,V}$;
2. If $X > B$ and $A \in S\mathcal{K}_{X,V}$, then $B \in S\mathcal{K}_{X,V}$.

Proof. 1. From $X < A$ we conclude that $\mathcal{K}(A,X) \neq 0$. Since $\dim_k |A| = \dim_k |B| = 1$ and $A < B$, then there exists $\delta \in \mathcal{K}(B,A)$ such that $|\delta|$ is an isomorphism. Therefore we have $\operatorname{Im}|\varphi| = \operatorname{Im}|\varphi\delta| = V$ for any $0 \neq \varphi \in \mathcal{K}(A,X)$ and hence $A \in S\mathcal{K}^{X,V}$.
The statement 2 is dual to the statement 1. ▪

We denote by $\mathcal{K}^{op}$ the opposite vector space category to $\mathcal{K}$ (see [Si] for details). We shall identify the sets $\operatorname{ind} \mathcal{K}^{op}$ and $\operatorname{ind} \mathcal{K}$.

Lemma 5. *Suppose that $X \in L\mathcal{K}, A, B \in S\mathcal{K}$ and $A - B$. If $A > X$ and $X > B$, then there exists $V \in \mathbf{P}^1(X)$ such that $A \in S\mathcal{K}^{X,V}, B \in S\mathcal{K}_{X,V}$.*

Proof. Since $A > X$ (resp., $X > B$), then there exists $0 \neq \varphi \in \mathcal{K}(A,X)$ (resp., $0 \neq \psi \in \mathcal{K}(X,B)$). From $A - B$ we conclude that $\psi\varphi = 0$ and hence $\operatorname{Im}|\varphi| = \operatorname{Ker}|\psi| \in \mathbf{P}^1(X)$ for all $0 \neq \varphi \in \mathcal{K}(A,X)$,$0 \neq \psi \in \mathcal{K}(X,B)$. ▪

Lemma 6. *Let $\mathcal{K}$ be a schurian vector space category, $X, Y \in L\mathcal{K}$. Then*
1. If $\dim_k \mathcal{K}(X,Y) = 4$, then $\mathcal{K}(\{X,Y\}) \cong \mathcal{L}$;
2. If $\dim_k \mathcal{K}(X,Y) = 3$, then $\mathcal{K}(\{X,Y\})$ is equivalent to one of the categories $\mathcal{I}, \mathcal{J}$.
3. If $\dim_k \mathcal{K}(X,Y) + \dim_k \mathcal{K}(Y,X) = 2$, then $\mathcal{K}(\{X,Y\})$ is equivalent to one of the categories $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{D}^{op}$.

Proof. 3. If $\dim_k \mathcal{K}(X,Y) = \dim_k \mathcal{K}(Y,X) = 1$, then from the proof of Lemma 2 it follows that $\mathcal{K} \cong \mathbf{C}$. Assume that $\dim_k \mathcal{K}(X,Y) = 2$. Let $\{\varphi, \psi\}$ be the basis of $|\mathcal{K}(X,Y)|$. From the classification of indecomposable representations of a pencil, solved by Kronecker, it follows that there exist the basis of $|X|$ and $|Y|$ such that one of the following situations may take place:

a). $M|\varphi| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $M|\psi| = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in k$. It is clear that in this case $\mathcal{K} \cong \mathbf{B}$.

b). $M|\varphi| = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$, $M|\psi| = \begin{pmatrix} \mu & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda\mu \neq 1$. In this case $\mathcal{K} \cong \mathbf{A}$.

c). $M|\varphi| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M|\psi| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. In this case $\mathcal{K} \cong \mathbf{D}^{op}$.

d). $M|\varphi| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M|\psi| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case $\mathcal{K} \cong \mathbf{D}$.

2. Let $\{\varphi, \psi, \delta\}$ be the basis of $|\mathcal{K}(X,Y)|$. From proof of the statement 3 it follows that there exist the basis of $|X|$ and $|Y|$ such that one of the following situations may take place:

a). $M|\varphi| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $M|\psi| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then we may assume that $M|\delta| = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$. If $b \neq 0$, then $\mathcal{K} \cong \mathcal{J}$. If $b = 0$, then $\mathcal{K} \cong \mathcal{I}$.

b). $M|\varphi| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M|\psi| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then we may assume that $M|\delta| = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$. If $a = 0$ or $b = 0$, then $\mathcal{K} \cong \mathcal{I}$. If $a \neq 0$ and $b \neq 0$, then $\mathcal{K} \cong \mathcal{J}$.

c). $M|\varphi| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M|\psi| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then we may assume that $M|\delta| = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$. In this case $\mathcal{K} \cong \mathcal{I}$.

d). $M|\varphi| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M|\psi| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then we may assume that $M|\delta| = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$. In this case $\mathcal{K} \cong \mathcal{I}$.

1. It follows from the statement 2. ∎

Lemma 7. *Let $\mathcal{K}$ be a schurian vector space category, $X \in L\mathcal{K}$, $A_i \in S_{X,V_i}$ (resp., $A_i \in S^{X,V_i}$), $1 \leq i \leq 3$, where $V_i \neq V_j$ for $i \neq j$. Then $\mathcal{K}(\{X, A_1, A_2, A_3\})$ is equivalent to the category $\mathcal{D}$ (resp., $\mathcal{D}^{\mathrm{op}}$).*

Proof is evident. ∎

Lemma 8. *Let $\mathcal{K}$ be a schurian vector space category, $X \in L\mathcal{K}$, $A_i \in S_{X,V_i}$ (resp., $A_i \in S^{X,V_i}$), $1 \leq i \leq 4$, where $V_i \neq V_j$ for $i \neq j$. Then $\mathcal{K}(\{X, A_1, \ldots, A_4\})$ is equivalent to one of the categories $\mathcal{H}_\lambda$ (resp., $\mathcal{H}_\lambda^{\mathrm{op}}$), $\lambda \in k \backslash \{0,1\}$.*

Proof is evident. ∎

Lemma 9. *Let $\mathcal{K}$ be a schurian vector space category satisfying the condition (c) of Theorem 1, $X, Y \in L\mathcal{K}$, $X > Y$. The following conditions are equivalent:*

1. $\mathcal{K}(\{X,Y\}) \cong \mathcal{I}$;

2. There exist $U \in \mathbf{P}^1(X), V \in \mathbf{P}^1(Y)$ such that $|\mathcal{K}(X,Y)|(U) = V$.

Proof follows from Lemma 6. ∎

Lemma 10. *Let $\mathcal{K}$ be a schurian vector space category satisfying the condition (c) of Theorem 1, $X, Y \in L\mathcal{K}$, $X > Y$. Assume that there exist $U_1, U_2 \in \mathbf{P}^1(X)$, $V_1, V_2 \in \mathbf{P}^1(Y)$ such that $|\mathcal{K}(X,Y)|(U_i) = V_i$ for $1 \leq i \leq 2$. Then $U_1 = U_2$, $V_1 = V_2$.*

Proof follows from Lemmas 9,6. ■

Lemma 11. *Let $\mathcal{K}$ be a schurian vector space category satisfying all the conditions of Theorem 1, $X, Y \in L\mathcal{K}$, $X > Y$, $\mathcal{K}(\{X, Y\}) \cong \mathcal{I}$, $\Phi \subseteq S\mathcal{K}$, $\Phi < X$ and $\Phi > Y$. Then $\Phi = \Phi_1 \cup \Phi_2$, $\Phi_1 \cap \Phi_2 = \emptyset$, Φ_1, Φ_2 do not contain the subsets* $(1,1,1)$ *and* $(1,2)$.

Proof. From Lemma 9 it follows that there exist $U \in \mathbf{P}^1(X)$, $V \in \mathbf{P}^1(Y)$ such that $|\mathcal{K}(X, Y)|(U) = V$. Put $\Phi_1 = S\mathcal{K}_{X,U}$, $\Phi_2 = S\mathcal{K}^{Y,V}$. From the condition (d) of Theorem 1 it follows that $\Phi_1 \cap \Phi_2 = \emptyset$. The condition (e) of Theorem 1 conclude that Φ_1, Φ_2 do not contain the subsets $(1,1,1)$ and $(1,2)$.

Suppose $A \in \Phi$, $A \notin S\mathcal{K}_{X,U}$. Then there exists $0 \neq \varphi \in \mathcal{K}(X, A)$ such that $\operatorname{Ker}|\varphi| \neq U$. Then $\operatorname{Im}|\psi| = |\psi\varphi|(U) = V$ for any $0 \neq \psi \in \mathcal{K}(A, Y)$ and hence $A \in S\mathcal{K}^{Y,V}$.

Suppose $A \in \Phi$, $A \notin S\mathcal{K}^{Y,V}$. Then there exists $0 \neq \psi \in \mathcal{K}(A, Y)$ such that $\operatorname{Im}|\psi| \neq V$. Then if $|\varphi|(U) \neq 0$ for some $\varphi \in \mathcal{K}(X, A)$, then $|\psi\varphi|(U) \nsubseteq V$, a contradiction. Therefore $|\varphi|(U) = 0$ for any $\varphi \in \mathcal{K}(X, A)$ and hence $A \in S\mathcal{K}_{X,U}$. ■

Lemma 12. *Let $\mathcal{K}$ be a schurian vector space category satisfying the conditions (b),(e) and (g) of Theorem 1. Assume that $\Phi \subseteq S\mathcal{K}$, $\Phi = (1,1,1,1)$ or $\Phi = (1,2,2)$. Then for any $Z \in L\mathcal{K}$ we have $Z < \Phi$ or $\Phi < Z$.*

Proof. Since $\mathcal{K}$ satisfies the condition (b) of Theorem 1, we have $A < Z$ or $A > Z$ for any $Z \in L\mathcal{K}$ and $A \in \Phi$. Suppose the assertion of the Lemma is false. Then one of the following situations occur:

1). $\Phi = \{A_1, A_2, A_3, A_4\}, A_1 < Z, \{A_2, A_3, A_4\} > Z$. From Lemma 5 it follows that there exists $V \in \mathbf{P}^1(Z)$ such that $A_1 \in S\mathcal{K}_{Z,V}$, $\{A_2, A_3, A_4\} \subseteq S\mathcal{K}^{Z,V}$, a contradiction to the condition (e) of Theorem 1.

2). $\Phi = \{A_1, A_2, A_3, A_4\}, \{A_1, A_2\} < Z, \{A_3, A_4\} > Z$. From Lemma 5 it follows that there exists $V \in \mathbf{P}^1(Z)$ such that $\{A_1, A_2\} \subseteq S\mathcal{K}_{Z,V}$, $\{A_3, A_4\} \subseteq S\mathcal{K}^{Z,V}$, a contradiction to the condition (g) of Theorem 1.

3). $\Phi = \{A, B < C, D < E\}, A < Z, \{B < C, D < E\} > Z$. From Lemma 5 it follows that there exists $V \in \mathbf{P}^1(Z)$ such that $A \in S\mathcal{K}_{Z,V}$, $\{B < C, D < E\} \subseteq S\mathcal{K}^{Z,V}$, a contradiction to the condition (e) of Theorem 1.

4). $\Phi = \{A, B < C, D < E\}, A > Z, \{B < C, D < E\} < Z$. This case is dual to case 3).

5). $\Phi = \{A, B < C, D < E\}, A < Z, B < Z, E > Z$. From Lemma 5 it follows that there exists $V \in \mathbf{P}^1(Z)$ such that $A, B \in S\mathcal{K}_{Z,V}$, $E \in S\mathcal{K}^{Z,V}$. If $C < Z$, then $C \in S\mathcal{K}_{Z,V}$ because $C - E$. Hence $\{A, B < C\} \subseteq S\mathcal{K}_{Z,V}$, a contradiction to the condition (e) of Theorem 1. If $C > Z$, then $C \in S\mathcal{K}^{Z,V}$ because $C - A$, a contradiction to the condition (g) of Theorem 1.

6). $\Phi = \{A, B < C, D < E\}, A > Z, B < Z, E > Z$. This case is dual to case 5). ■

Lemma 13. *Let $\mathcal{K}$ be a schurian vector space category satisfying the conditions (b),(e),(g) of Theorem 1. Let $\Phi \subseteq S\mathcal{K}$ be one of the following sets $K_1, \dots, K_5, G$. Then any object $Z \in L\mathcal{K}$ satisfies $\Phi > Z$ or $\Phi < Z$.*

Proof. Conversely, suppose that there are $B, C \in \Phi$ such that $B < Z$ and $C > Z$. Since Φ satisfies the condition $(*)$, Φ contains one of the subsets $\{C, C_1, C_2, C_3\}$,$\{C, C_1 < C_2, C_3 < C_4\}$,$\{C_1, C < C_2, C_3 < C_4\}$,$\{C_1, C_2 < C, C_3 < C_4\}$. Lemma 12 implies that $C_i > Z$ for $1 \leq i \leq 4$. By the same reason, Φ contains one of the subsets $\{B, B_1, B_2, B_3\}$,

$\{B, B_1 < B_2, B_3 < B_4\}$,$\{B_1, B < B_2, B_3 < B_4\}$,$\{B_1, B_2 < B, B_3 < B_4\}$ and $B_i < Z$ for $1 \leq i \leq 4$. We get a contradiction to the form of the sets $K_1, \ldots, K_5, G$. ■

2 Differentiation algorithm

In this section we construct an algorithm of differentiation of schurian vector space categories satisfying all the conditions of Theorem 1, which generalizes the Nazarova–Roiter's differentiation algorithm for posets (see [NR2]). Throughout the section $\mathcal{K}$ stands for the schurian vector space category satisfying all the conditions of Theorem 1.

Since $\mathcal{V}(\mathcal{K}) \cong \mathcal{V}(\mathcal{K}^{|-|})$, from now on we will assume that $\mathcal{K} \equiv \mathcal{K}^{|-|}$. Therefore we write X (resp., $\mathcal{K}(X,Y)$) instead $|X|$ (resp., $|\mathcal{K}(X,Y)|$), where $X, Y \in \operatorname{ind}\mathcal{K}$.

2.1 Categories $\mathcal{C}_{n,m}$.

Suppose that $\mathcal{K} = \mathcal{C}_{n,m}$ for some $n, m \in \mathbf{N}$. Consider the following indecomposable objects of $\mathcal{V}(\mathcal{K})$:

$\mathcal{K}_{Z,A_i,s} = (Z^s \oplus A_i, k^{s+1}, \varphi)$, where $s \in \mathbf{N}$ and

$$M\varphi_{z^1} = \begin{pmatrix} 0 & 0 \\ E_{s-1} & 0 \\ 0 & 1 \end{pmatrix}, M\varphi_{z^2} = \begin{pmatrix} 1 & 0 \\ 0 & E_{s-1} \\ 0 & 0 \end{pmatrix}, M\varphi_{a_i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix};$$

$\mathcal{K}_{Z,B_j,s} = (Z^s \oplus B_j, k^{s+1}, \varphi)$, where $s \in \mathbf{N}$ and

$$M\varphi_{z^1} = \begin{pmatrix} 0 & 0 \\ E_{s-1} & 0 \\ 0 & 1 \end{pmatrix}, M\varphi_{z^2} = \begin{pmatrix} 1 & 0 \\ 0 & E_{s-1} \\ 0 & 0 \end{pmatrix}, M\varphi_{b_j} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

$\mathcal{K}_{Z,A_i,s,0} = (Z^s \oplus A_i, k^s, \varphi)$, where $s \in \mathbf{N}$ and

$$M\varphi_{z^1} = \begin{pmatrix} 0 & E_{s-1} \\ 0 & 0 \end{pmatrix}, M\varphi_{z^2} = \begin{pmatrix} E_{s-1} & 0 \\ 0 & 1 \end{pmatrix}, M\varphi_{a_i} = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$\mathcal{K}_{Z,B_j,s,0} = (Z^s \oplus B_j, k^s, \varphi)$, where $s \in \mathbf{N}$ and

$$M\varphi_{z^1} = \begin{pmatrix} E_{s-1} & 0 \\ 0 & 1 \end{pmatrix}, M\varphi_{z^2} = \begin{pmatrix} 0 & E_{s-1} \\ 0 & 0 \end{pmatrix}, M\varphi_{b_j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$\mathcal{K}_{\{Z,A_i,B_j,s\}} = (Z^s \oplus A_i \oplus B_j, k^{s+1}, \varphi)$, where $s \in \mathbf{N}$ and

$$M\varphi_{z^1} = \begin{pmatrix} 0 & 0 \\ E_{s-1} & 0 \\ 0 & 1 \end{pmatrix}, M\varphi_{z^2} = \begin{pmatrix} 1 & 0 \\ 0 & E_{s-1} \\ 0 & 0 \end{pmatrix}, M\varphi_{\{a_i,b_j\}} = \begin{array}{c} \begin{array}{cc} a_i & b_j \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}.$$

The proof of this result will be given in (2.6).

Proposition 1. Suppose that $\mathcal{K} = \mathcal{C}_{n,m}$ for some $n, m \in \mathbf{N}$. Then $\operatorname{ind}\mathcal{V}(\mathcal{K}) = \{\mathcal{K}_0$, $\mathcal{K}_{A_i}$, $\mathcal{K}_{B_j}$, $\mathcal{K}_{Z,0}$, $\mathcal{K}_{A_i,0}$, $\mathcal{K}_{B_j,0}$, $\mathcal{K}_{\{A_i,B_j\}}$, $\mathcal{K}_{Z,s,-}$, $\mathcal{K}_{Z,s,+}$, $\mathcal{K}_{Z,s,0,-}$, $\mathcal{K}_{Z,s,0,+}$, $\mathcal{K}_{Z,s,\lambda}$, $\mathcal{K}_{Z,A_i,s}$, $\mathcal{K}_{Z,B_j,s}$, $\mathcal{K}_{Z,A_i,s,0}$, $\mathcal{K}_{Z,B_j,s,0}$, $\mathcal{K}_{\{Z,A_i,B_j,s\}}$ $|s \in \mathbf{N}$, $\lambda \in k$, $\lambda \neq 0$, $1 \leq i \leq n$, $1 \leq j \leq m\}$.

2.2 Category $\mathcal{D}$.

Consider the following indecomposable objects of $\mathcal{V}(\mathcal{D})$:

$\mathcal{D}_{\{Z,A,B,C\}} = (Z \oplus A \oplus B \oplus C, k^2, \varphi)$, where:

$$M\varphi = \begin{array}{c} \begin{array}{ccccc} z^1 & z^2 & a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array};$$

$\mathcal{D}_{\{Z,A,B,C,s,\diamond\}} = (Z^s \oplus A \oplus B \oplus C, k^{s+2}, \varphi)$, where $s \in \mathbf{N}$ and

$$M\varphi_{z^1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ E_{s-1} & 0 \\ 0 & 1 \end{pmatrix}, M\varphi_{z^2} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & E_{s-1} \\ 0 & 0 \end{pmatrix}, M\varphi_{\{a,b,c\}} = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array};$$

$\mathcal{D}_{\{Z,A,B,C,s,-\}} = (Z^{2s} \oplus A \oplus B \oplus C, k^{2s+1}, \varphi)$, where $s \in \mathbf{N}$ and

$$M\varphi_{z^1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{s-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & E_{s-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}, M\varphi_{z^2} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & E_{s-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_{s-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M\varphi_{\{a,b,c\}} = \begin{array}{c} \\ 1 \\ \vdots \\ s+1 \\ \vdots \\ 2s+1 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{array};$$

$\mathcal{D}_{\{Z,A,B,C,s,+\}} = (Z^{2s+1} \oplus A \oplus B \oplus C, k^{2s+2}, \varphi)$, where $s \in \mathbf{N}$ and

$$M\varphi_{z^1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ E_{s-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & E_s & 0 \end{pmatrix}, M\varphi_{z^2} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & E_{s-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & E_s \end{pmatrix},$$

$$M\varphi_{\{a,b,c\}} = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ s+2 \\ \vdots \\ 2s+2 \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}.$$

The proof of this result will be given in (2.6).

Proposition 2. $\operatorname{ind}\mathcal{V}(\mathcal{D}) = \{\mathcal{D}_0,\ \mathcal{D}_A,\ \mathcal{D}_B,\ \mathcal{D}_C,\ \mathcal{D}_{Z,0},\ \mathcal{D}_{A,0},\ \mathcal{D}_{B,0},\ \mathcal{D}_{C,0},\ \mathcal{D}_{\{A,B\}},$ $\mathcal{D}_{\{A,C\}}, \mathcal{D}_{\{B,C\}}, \mathcal{D}_{Z,s,-}, \mathcal{D}_{Z,s,+}, \mathcal{D}_{Z,s,0,-}, \mathcal{D}_{Z,s,0,+},\ \mathcal{D}_{Z,s,\lambda},\ \mathcal{D}_{Z,A,s},\ \mathcal{D}_{Z,B,s},\ \mathcal{D}_{Z,C,s},\ \mathcal{D}_{Z,A,s,0},$ $\mathcal{D}_{Z,B,s,0},\ \mathcal{D}_{Z,C,s,0},\ \mathcal{D}_{\{Z,A,B,s\}},\ \mathcal{D}_{\{Z,A,C,s\}},\ \mathcal{D}_{\{Z,B,C,s\}},\ \mathcal{D}_{\{A,B,C\}},\ \mathcal{D}_{\{A,B,C,\diamond\}},\ \mathcal{D}_{\{Z,A,B,C\}},$ $\mathcal{D}_{\{Z,A,B,C,s,\diamond\}},\ \ \mathcal{D}_{\{Z,A,B,C,s,-\}},\ \mathcal{D}_{\{Z,A,B,C,s,+\}},\ \mid\ s \in \mathbf{N},\quad \lambda \in k, \lambda \neq 0\}$.

2.3

Lemma 14. *Suppose that $L\mathcal{K} = \max \operatorname{ind}\mathcal{K} = \{Z\}$, $S\mathcal{K}_{Z,V}$ does not contain the subset $(1,1)$ for any $V \in \mathbf{P}^1(Z)$ and $S\mathcal{K}$ does not contain the subsets $(1,1,1,1)$ and $(1,2,2)$.*

If $Z \subseteq X$, then $A \nsubseteq X$ for any $(X,V,\varphi) \in \operatorname{ind}\mathcal{V}(\mathcal{K})$ and any $A \in S\mathcal{K} \setminus S\mathcal{K}_Z$.

Proof. Having fixed an arbitrary basis $\mathcal{B}$ in $\mathcal{K}$ we will consider the decomposition $X = Z^s \oplus Y$, where $Z \nsubseteq Y$. Let φ_Y denote the restriction of φ to the subspace Y. Then (Y,V,φ_Y) is the object of the category $\mathcal{V}(\mathcal{K}(T))$, where $T = \operatorname{ind}\mathcal{K}\backslash\{Z\}$. Let $(Y,V,\varphi_Y) = \oplus_{i=1}^m (Y_i,V_i,\varphi_{Y,i})$ be a decomposition of (Y,V,φ_Y) into the direct sum of indecomposable objects. We fix an arbitrary basis in the spaces V_i, Y_i and Z. Then the matrix $M\varphi$ can be written in the form:

$$M\varphi = \begin{array}{c} \\ 1 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cc} M\varphi_{z^1} & M\varphi_{z^2} \end{array} \\ \begin{pmatrix} M_{11} & M_{21} & M_{31} & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 \\ M_{1m} & M_{2m} & 0 & 0 & M_{3m} \end{pmatrix} \end{array}.$$

Since $\hat{\mathcal{K}} = \mathcal{K}(T)$ is the linear schurian category and $S\hat{\mathcal{K}}$ does not contain $(1,1,1,1)$ and $(1,2,2)$ as subsets, from [K] it follows that $\operatorname{ind}\mathcal{V}(\hat{\mathcal{K}}) = \{\hat{\mathcal{K}}_0,\ \hat{\mathcal{K}}_B,\ \hat{\mathcal{K}}_{\{C_1,C_2\}},\ \hat{\mathcal{K}}_{\{D_1,D_2,D_3\}},$ $\hat{\mathcal{K}}_{\{D_1,D_2,D_3,\diamond\}}, \hat{\mathcal{K}}_{B,0}, \hat{\mathcal{K}}_{\{E,F,G<H\}}\ \mid\ B \in S\mathcal{K}, (C_1,C_2), (D_1,D_2,D_3), \{E,F,G<H\} \subseteq S\mathcal{K}\}$. Let $A \in S\mathcal{K}, A \notin S\mathcal{K}_Z$ and $A \subseteq Y_i$ for a certain $1 \leq i \leq m$. It is easily seen that the matrices M_{1i}, M_{2i} can be turned into zero matrices by adding the columns of the matrix M_{3i}. Since $Z \subseteq X$, we get $(Y_i,V_i,\varphi_{Y,i}) \neq (X,V,\varphi)$ and hence $(Y_i,V_i,\varphi_{Y,i})$ can be distinguished by a direct summand in (X,V,φ), which contradicts indecomposability of (X,V,φ). ▪

2.4

An object $P \in S\mathcal{K}$ is called *admissible* if the following conditions hold:

1. $P \in \max S\mathcal{K}$;
2. If $P \in S\mathcal{K}^{X,V}$ for some $V \in \mathbf{P}^1(X)$, then $P \geq A$ for any $A \in S\mathcal{K}^{X,V}$;
3. If $Z \in \max\operatorname{ind}\mathcal{K}, Z \in L\mathcal{K}$, then $P \in \max(\operatorname{ind}\mathcal{K}\backslash\{Z\})$;
4. If $Z \in \max\operatorname{ind}\mathcal{K}$,$Z \in L\mathcal{K}$ and $w(S\mathcal{K}_{Z,V}) = 2$ for some $V \in \mathbf{P}^1(Z)$, then $P \in S\mathcal{K}_{Z,V}$;
5. If $Z \in \max\operatorname{ind}\mathcal{K}, Z \in L\mathcal{K}$ and $P \in S\mathcal{K}^{Y,V}$ for some $V \in \mathbf{P}^1(Y)$, then $\mathcal{K}(Z,Y)(U) \neq V$ for any $U \in \mathbf{P}^1(Z)$.

For given $X, Y \in \operatorname{ind}\mathcal{K}$ we denote by $\operatorname{Mat}_{\mathcal{B}}(X,Y)$ the vector space with the basis $\{Mb^i_{X,Y}\ \mid\ 1 \leq i \leq \dim_k\mathcal{K}(X,Y)\}$. For a matrix M we denote by ψ_M the corresponding map. Let $P \in S\mathcal{K}$ be an admissible object. If $P \notin \max\operatorname{ind}\mathcal{K}$, then we denote by Z an element of $\max\operatorname{ind}\mathcal{K}$.

We define a new vector space category $\partial_P\mathcal{K}$ in the following manner. In order to obtain the set $\operatorname{ind}\partial_P\mathcal{K}$, the objects P and Z are removed from the set $\operatorname{ind}\mathcal{K}$ and the following indecomposable objects are added:

1. an one-dimensional object $S_{\{A,B,C\}} := k$ and a two-dimensional object $L_{\{A,B,C\}} := k^2$ for every quadruple (P, A, B, C) in $S\mathcal{K}$;
2. an one-dimensional object $S_{\{A,B\}} := k$ for every triple (P, A, B) in $S\mathcal{K}$;
3. an one-dimensional object $S_Y := k$ for all $Y \in L\mathcal{K}$ such that $P \in S\mathcal{K}^Y$;
4. the one-dimensional objects $S_{Z,0} := k, S_{Z,s,0} := k, S_{Z,s,-} := k, s \in \mathbf{N}$ if $P \in S\mathcal{K}_Z$;
5. the one-dimensional objects $S_{Z,A,s,0} := k,\ s \in \mathbf{N}$ for every pair (P, A) in $S\mathcal{K}_Z$;
6. the one-dimensional objects $S_{\{Z,A,B,s\}} := k,\ s \in \mathbf{N}$ for every triple (P, A, B) in $S\mathcal{K}_Z$;
7. the one-dimensional objects $S_{\{Z,A,B,C,s\}} := k,\ s \in \mathbf{N}$ and the two-dimensional objects $L_{\{Z,A,B,C\}} := k^2,\ L_{\{Z,A,B,C,s,-\}} := k^2,\ L_{\{Z,A,B,C,s,+\}} := k^2,\ s \in \mathbf{N}$ for every quadruple (P, A, B, C) in $S\mathcal{K}_Z$;
8. an one-dimensional object $S_{\{Z,A,B,C\}} := k$ for every quadruple (P, A, B, C) in $S\mathcal{K}$ if $\dim_k\mathcal{K}(Z, P) = 2$ and $(A, B, C) \subseteq S\mathcal{K}_Z$.

The subset of $\operatorname{ind}\partial_P\mathcal{K}$ consisting of new (resp., old) objects is denoted by $N\partial_P\mathcal{K}$ (resp., $O\partial_P\mathcal{K}$). For each subset $\Phi \subset \operatorname{ind}\partial_P\mathcal{K}$, we set $N\Phi := \Phi \cap N\partial_P\mathcal{K}$, $O\Phi := \Phi \cap O\partial_P\mathcal{K}$.

We define a map ψ from $\operatorname{ind}\partial_P\mathcal{K}$ to the set of all subsets of $\operatorname{ind}\mathcal{K}$ by the rule $\psi(S_{\{A,B,C\}}) = \{A, B, C\}$, $\psi(S_{\{A,B\}}) = \{A, B\}$, $\psi(S_Y) = \{Y\}$, $\psi(L_{\{A,B,C\}}) = \{A, B, C\}$, $\psi(S_{Z,0}) = \{Z\}$, $\psi(S_{Z,s,0}) = \{Z\}$, $\psi(S_{Z,s,-}) = \{Z\}$, $\psi(S_{Z,A,s,0}) = \{Z, A\}$, $\psi(S_{\{Z,A,B,s\}}) = \{Z, A, B\}$, $\psi(S_{\{Z,A,B,C,s\}}) = \{Z, A, B, C\}$, $\psi(L_{\{Z,A,B,C\}}) = \{Z, A, B, C\}$, $\psi(L_{\{Z,A,B,C,s,-\}}) = \{Z, A, B, C\}$, $\psi(L_{\{Z,A,B,C,s,+\}}) = \{Z, A, B, C\}$, $\psi(\ S_{\{Z,A,B,C\}}\ ) = \{Z, A, B, C\}$ and $\psi(X) = X$ for $X \in O\partial_P\mathcal{K}$.

We denote by ψ_Z the set of objects $Y \in N\partial_P\mathcal{K}$ such that $Z \in \psi(Y)$.

We set $O = \{Y \in \operatorname{ind}\partial_P\mathcal{K} \mid \psi(Y) \subseteq S\mathcal{K}_Z \cup \{Z\}, Y \neq S_{Z,i,0}\}$.

For an arbitrary subcategory M in $\partial_P\mathcal{K}$, let us denote by $\psi(M)$ the full subcategory in $\mathcal{K}$ whose indecomposable objects are P and the elements of the set $\{\psi(Y) \mid Y \in \operatorname{ind}M\}$.

For an arbitrary subset T of $\operatorname{ind}\partial_P\mathcal{K}$ the set $\{\psi(Y) \mid Y \in T\}$ is denoted by $\psi(T)$.

Let an arbitrary P-basis $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$ be fixed in the category $\mathcal{K}$. The spaces of morphisms in the category $\partial_P\mathcal{K}$ are defined by the following conditions.

1. $\partial_P\mathcal{K}(A, A) := k$ for any $A \in \operatorname{ind}\partial_P\mathcal{K}$;
2. $\partial_P\mathcal{K}(A, B) := \mathcal{K}(A, B)$ for any $A, B \in O\partial_P\mathcal{K}$;
3. If (P, A, B, C) is a quadruple in $S\mathcal{K}$, then $\partial_P\mathcal{K}(\{Y \in \operatorname{ind}\partial_P\mathcal{K} \mid \psi(Y) \subseteq \{A, B, C, P\}\}) \cong \mathcal{M}$, where $A \to A$, $B \to B$, $C \to C$, $L_{\{A,B,C\}} \to Z$, $S_{\{A,B,C\}} \to G$, $S_{\{A,B\}} \to F$, $S_{\{B,C\}} \to E$, $S_{\{A,C\}} \to D$;
4. $\partial_P\mathcal{K}(S_{\{A,B\}}, A) := k$, $\partial_P\mathcal{K}(S_{\{A,B\}}, B) := k$;
5. If $Y \in L\mathcal{K}$ and $\dim_k\mathcal{K}(P, Y) = 1$, then $\partial_P\mathcal{K}(\{Y, S_Y\}) \cong \mathcal{B}$, where $Y \to Z, S_Y \to A$;
6. If $\operatorname{ord}\Omega\mathcal{K}_Z = 4$ and $S\mathcal{K}_Z = \{A, B, C, P\}$, $\mathcal{K}(\{Z, A, B, C, P\}) \cong \mathcal{H}_\lambda$, where $Z \to Z, P \to D, A \to A, B \to B, C \to C$,then $\partial_P\mathcal{K}(O) \cong \mathcal{O}_\lambda$, where $A \to A_1$, $B \to B_1$, $C \to D_1$, $L_{\{A,B,C\}} \to Z_1$, $S_{\{A,B\}} \to C_2$, $S_{\{A,C\}} \to A_2$, $S_{\{B,C\}} \to B_2$, $S_{Z,0} \to D_2$, $L_{\{Z,A,B,C\}} \to Z_2$, $S_{Z,A,s,0} \to A_{2s+1}$, $S_{Z,B,s,0} \to B_{2s+1}$, $S_{Z,C,s,0} \to D_{2s+1}$, $S_{\{A,B,C\}} \to C_3$, $L_{\{Z,A,B,C,s,-\}} \to Z_{2s+1}$, $S_{\{Z,A,B,s\}} \to C_{2s+2}$, $S_{\{Z,A,C,s\}} \to A_{2s+2}$, $S_{\{Z,B,C,s\}} \to B_{2s+2}$, $S_{Z,s,-} \to D_{2s+2}$, $L_{\{Z,A,B,C,s,+\}} \to Z_{2s+2}$, $S_{\{Z,A,B,C,s\}} \to C_{2s+3}$;
7. If $\operatorname{ord}\Omega\mathcal{K}_Z \leq 3, S\mathcal{K}_{Z,U} = \{A_1 < \ldots < A_n\}, S\mathcal{K}_{Z,W} = \{B_1 < \ldots < B_m\}, P \in S\mathcal{K}_{Z,V}$, where V, U, W are pairwise different proper subspace,$n, m \in \mathbf{N} \cup \{0\}$, then

$\partial_P\mathcal{K}\,(O) \cong \mathcal{N}_{n,m}$, where $A_i \to X^0_{i,0}$,$B_j \to X^0_{0,j}$,$S_{\{A_i,B_j\}} \to X^0_{i,j}$, $S_{Z,A_i,s,0} \to X^s_{i,0}$,$S_{Z,B_j,s,0} \to X^s_{0,j}$,$S_{\{Z,A_i,B_j,s\}} \to X^s_{i,j}$,$S_{Z,0} \to X^1_{0,0}$,$S_{Z,s,-} \to X^{s+1}_{0,0}$;

8. If $X \in \operatorname{ind}\partial_P\mathcal{K} \setminus \psi_Z$, $Y \in \{S_{\{B,C\}}, S_{\{A,B,C\}}\}$ for some $A, B, C \in S\mathcal{K}$, then $\partial_P\mathcal{K}\,(X,Y) := \{\psi_F \mid F \in \cap_{T\in\psi(Y)}\mathrm{Mat}_{\mathcal{B}}\,(X,T)\}$;

9. If $X \in \operatorname{ind}\partial_P\mathcal{K} \setminus \psi_Z$, then $\partial_P\mathcal{K}\,(X, L_{\{A,B,C\}}) := \{\begin{pmatrix} \varphi_{AC} + \varphi_{AB} \\ \varphi_{BC} - \varphi_{AB} \end{pmatrix} \mid \varphi_{LN} \in \mathrm{Mat}_{\mathcal{B}}(X,L) \cap \mathrm{Mat}_{\mathcal{B}}(X,N)\}$.

10. If $X \in \operatorname{ind}\partial_P\mathcal{K}, X \neq S_{Z,0}$, then $\partial_P\mathcal{K}\,(X, S_Y) := \{\psi_F \mid \begin{pmatrix} F \\ 0 \end{pmatrix} \in \mathrm{Mat}_{\mathcal{B}}\,(X,Y)\}$;

11. If $\mathcal{K}(\{Z,X\}) \cong \mathcal{L}$, then $\partial_P\mathcal{K}(S_{Z,0}, S_X) := k$;

12. $\partial_P\mathcal{K}(S_{\{Z,A,B,C\}}, S_{\{A,B,C\}}) := k$;

13. If $X \in O\partial_P\mathcal{K}$ and $\mathcal{K}(\{Z,X\}) \ncong \mathcal{I}$, then $\partial_P\mathcal{K}(S_{Z,0}, X) := k^2$;

14. If $\mathcal{K}(\{Z,X\}) \cong \mathcal{I}$, then $\partial_P\mathcal{K}(S_{Z,0}, X) := k\alpha$, where $M\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

15. If $X \in SO\partial_P\mathcal{K}, X \notin S\mathcal{K}_{Z,Z_P}$, then $\partial_P\mathcal{K}(S_{Z,0}, X) := k$;

16. If $S_{\{A,B\}} \notin O$, then $\partial_P\mathcal{K}(S_{Z,0}, S_{\{A,B\}}) := k$;

17. If $S_{\{A,B\}} \notin O$, $S_{\{B,C\}} \notin O$ and $S_{\{A,C\}} \notin O$, then $\partial_P\mathcal{K}(S_{Z,0}, S_{\{A,B,C\}}) := k$;

18. $\partial_P\mathcal{K}(\{S_{Z,s,0} \mid s \in \mathbf{N}\}) \cong \mathcal{N}^{op}_{0,0}$, where $S_{Z,s,0} \to X^s_{0,0}$, and $\partial_P\mathcal{K}(S_{Z,s,0}, T) := k$ for any $T \in O$;

19. If $S_{\{A,B\}} \notin O$, $S_{\{B,C\}} \notin O$, $S_{\{A,C\}} \in O$, then $\partial_P\mathcal{K}(X, S_{\{A,B,C\}}) := k$ for any $X \in \psi_Z$, $X \neq S_{Z,0}$;

20. $\varphi \in \operatorname{Mor}\partial_P\mathcal{K}$ if and only if $\varphi = \sum_{j=1}^k \varphi_{j1}\dots\varphi_{jn(j)}$, where φ_{js} are defined by the conditions 1-19.

We define a functor $|-| : \partial_P\mathcal{K} \to \operatorname{mod} k$ by the rule $|X| := X$, $|\partial_P\mathcal{K}\,(X,Y)| := \partial_P\mathcal{K}\,(X,Y)$ for any $X, Y \in \operatorname{ind}\partial_P\mathcal{K}$.

For any $X \in L\mathcal{K}$ and $A \in S^{X,V}$ (resp., $A \in S_{X,V}$), let us denote by X^A (resp., X_A) the vector space $V \in \mathbf{P^1}\,(V)$ such that $A \in S\mathcal{K}^{X,V}$ (resp., $A \in S\mathcal{K}_{X,V}$).

2.5

In this section we deal with the construction of mappings $\mathbf{d}_P : \operatorname{Ob}\mathcal{V}(\mathcal{K}) \to \operatorname{Ob}\mathcal{V}(\partial_P\mathcal{K})$ and $\mathbf{i}_P : \operatorname{Ob}\mathcal{V}(\partial_P\mathcal{K}) \to \operatorname{Ob}\,\mathcal{V}(\mathcal{K})$. We assume that an arbitrary P-basis is fixed in the category $\mathcal{K}$ and consider an arbitrary object (X, V, φ) of the category $\mathcal{V}(\mathcal{K})$. We fix a decomposition $X \cong X_1^{d_1} \oplus \dots \oplus X_n^{d_n}$, where $X_i \in \operatorname{ind}\mathcal{K}$, and a linear isomorphism $V \cong k^{d_0}$. According to Sec. 1, the problem concerning the classification of indecomposable objects of the category $\mathcal{V}(\mathcal{K})$ is equivalent to the matrix problem ($\mathrm{Mat}_{\mathcal{K}}, G_{\mathcal{K}}$). By using the elementary transformations of rows, the matrix $M\varphi$ can be transformed to the following form

$$M\varphi = \begin{array}{c} \\ V_1 \\ V_2 \end{array} \begin{array}{c} \begin{array}{cccc} M\varphi_{z^1} & M\varphi_{z^2} & M\varphi_{\mathcal{B}_0^{P,+}} & M\varphi_{\mathcal{B}_0^{P,-}} \end{array} \\ \begin{pmatrix} M_1 & M_2 & M_3 & M_4 \\ M_5 & M_6 & 0 & M_7 \end{pmatrix} \end{array},$$

where the rows of the matrix M_3 are linearly independent. We get the decomposition $V = V_1 \oplus V_2$. Adding columns of the matrix $M\varphi_{\mathcal{B}_0^{P,+}}$ to the matrix $M\varphi_{z^1}$, we can turn the block M_1 into zero. Consider the object (X, V_2, φ'), where

$$M\varphi' = \begin{array}{c} \\ V_2 \end{array} \begin{array}{c} \begin{array}{cccc} M\varphi_{z^1} & M\varphi_{z^2} & M\varphi_{\mathcal{B}_0^{P,+}} & M\varphi_{\mathcal{B}_0^{P,-}} \end{array} \\ \begin{pmatrix} M_5 & M_6 & 0 & M_7 \end{pmatrix} \end{array}.$$

If we decompose (X, V_2, φ') into the following direct sum $(X, V_2, \varphi') = \oplus_{i=1}^{m}(X_i, V_{2i}, \varphi_i')$, where $(X_i, V_{2i}, \varphi_i') = \oplus_{j=1}^{t_i} (Y_i, W_i, \varphi_i'')$, $(Y_i, W_i, \varphi_i'') \in \operatorname{ind}\mathcal{V}(\mathcal{K})$ and $(Y_k, W_k, \varphi_k'') \ncong (Y_s, W_s, \varphi_s'')$ for $k \neq s$, then the matrix $M\varphi$ takes the form

$$M\varphi = \begin{array}{c} \\ 1 \\ 21 \\ \vdots \\ 2n \end{array} \begin{array}{c} \begin{array}{cccccccccc} 11 & \cdots & 1n & 21 & \cdots & 2n & 3 & 41 & \cdots & 4n \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & M_{21} & \cdots & M_{2n} & M_3 & M_{41} & \cdots & M_{4n} \\ M_{51} & 0 & 0 & M_{61} & 0 & 0 & 0 & M_{71} & 0 & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & M_{5n} & 0 & 0 & M_{6n} & 0 & 0 & 0 & M_{7n} \end{pmatrix} \end{array},$$

where $M\varphi_i' = \begin{array}{c} \\ V_{2i} \end{array} \begin{array}{c} \begin{array}{cccc} M\varphi_{z^1} & M\varphi_{z^2} & M\varphi_{\mathcal{B}_0^{P,+}} & M\varphi_{\mathcal{B}_0^{P,-}} \end{array} \\ \begin{pmatrix} M_{5i} & M_{6i} & 0 & M_{7i} \end{pmatrix} \end{array}$.

According to Lemma 1, the classification of indecomposable objects of the category $\hat{\mathcal{V}}(\mathcal{K})$ is equivalent to the problem of classification of indecomposable objects of the category $\mathcal{PL}$. Consider the following sets:

1. $R_1 = \{\mathcal{PL}_0,\, \mathcal{PL}_{A,0},\, \mathcal{PL}_A \mid A \in S\mathcal{PL}\}$;
2. $R_2 = \{\mathcal{PL}_{\{A,B\}} \mid (A, B) \subseteq S\mathcal{PL}\}$;
3. $R_3 = \{\mathcal{PL}_{\{A,B,C\}}, \mathcal{PL}_{\{A,B,C,\diamond\}} \mid (A, B, C) \subseteq S\mathcal{PL}\}$;
4. $R_4 = \{\mathcal{PL}_{Z,0},\, \mathcal{PL}_{Z,s,-},\, \mathcal{PL}_{Z,s,+},\, \mathcal{PL}_{Z,s,0,-},\, \mathcal{PL}_{Z,s,0,+},\, \mathcal{PL}_{Z,s,\lambda},\, \lambda \neq 0\}$;
5. $R_5 = \{\mathcal{PL}_{Z,A,s},\, \mathcal{PL}_{Z,A,s,0} \mid A \in S\mathcal{PL}_Z\}$;
6. $R_6 = \{\mathcal{PL}_{\{Z,A,B,s\}} \mid (A, B) \subseteq S\mathcal{PL}_Z\}$;
7. $R_7 = \{\mathcal{PL}_{\{Z,A,B,C\}},\, \mathcal{PL}_{\{Z,A,B,C,s,\diamond\}},\, \mathcal{PL}_{\{Z,A,B,C,s,-\}},\, \mathcal{PL}_{\{Z,A,B,C,s,+\}} \mid (A, B, C) \subseteq S\mathcal{PL}_Z\}$.

Since $\mathcal{K}$ satisfies all the conditions of Theorem 1 and P is admissible, then the category $\mathcal{PL}$ satisfies the conditions of Lemma 14 and $S\mathcal{PL}$ does not contain the subsets $(1, 1, 1, 1)$ and $(1, 1, 2)$.

Therefore from Lemma 14, Propositions 1,2 and [K] it follows that $\operatorname{ind}\mathcal{V}(\mathcal{PL}) = \cup_{i=1}^{7} R_i$. Hence $\operatorname{ind}\hat{\mathcal{V}}(\mathcal{K}) = \cup_{i=1}^{7} R_i'$, where

1. $R_1' = \{\mathcal{K}_0,\, \mathcal{K}_{X,1,0,+},\, \mathcal{K}_{A,0},\, \mathcal{K}_A \mid (P, A) \subseteq S\mathcal{K}, X \in L\mathcal{K}, \dim_k \mathcal{K}(P, X) = 1\}$;
2. $R_2' = \{\mathcal{K}_{\{A,B\}} \mid (P, A, B) \subseteq S\mathcal{K}\}$;
3. $R_3' = \{\mathcal{K}_{\{A,B,C\}},\, \mathcal{K}_{\{A,B,C,\diamond\}} \mid (P, A, B, C) \subseteq S\mathcal{K}\}$;
4. $R_4' = \{\mathcal{K}_{Z,0},\, \mathcal{K}_{Z,s,-},\, \mathcal{K}_{Z,s,+},\, \mathcal{K}_{Z,s,0,-},\, \mathcal{K}_{Z,s,0,+},\, \mathcal{K}_{Z,s,\lambda} \mid \lambda \in k, \lambda \neq 0\}$;
5. $R_5' = \{\mathcal{K}_{Z,A,s},\, \mathcal{K}_{Z,A,s,0} \mid A \in S\mathcal{K}_Z, P - A\}$;
6. $R_6' = \{\mathcal{K}_{\{Z,A,B,s\}} \mid (A, B) \subseteq S\mathcal{K}_Z, P - \{A, B\}\}$;
7. $R_7' = \{\, \mathcal{K}_{\{Z,A,B,C\}},\, \mathcal{K}_{\{Z,A,B,C,s,\diamond\}},\, \mathcal{K}_{\{Z,A,B,C,s,-\}},\, \mathcal{K}_{\{Z,A,B,C,s,+\}} \mid (A, B, C) \subseteq S\mathcal{K}_Z,\ P - \{A, B, C\}\}$.

Note that the matrices $M\varphi$ of indecomposable objects of the categories $\mathcal{C}_{n,m}$ and $\mathcal{D}$ in (2.1), (2.2) are given in the basis, which is not P-basis.

If we have one of the cases

1. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_A$; 2. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{X,1,0,+}$;
3. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{A,B\}}$; 4. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{A,B,C\}}$;
5. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{A,B,C,\diamond\}}$; 6. $(Y_j, W_j, \varphi''_j) \in \{\mathcal{K}_{Z,s,-}, \mathcal{K}_{Z,s,0,-}\}$;
7. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{Z,A,s}$; 8. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,s\}}$;
9. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,C,s,\diamond\}}$; 10. $(Y_j, W_j, \varphi''_j) \in \{\mathcal{K}_{\{Z,A,B,C,s,-\}}, \mathcal{K}_{\{Z,A,B,C,s,+\}}\}$;
11. $(Y_j, W_j, \varphi''_j) \in \{\mathcal{K}_{Z,A,s,0}, \mathcal{K}_{Z,s,+}, \mathcal{K}_{Z,s,0,+}, \mathcal{K}_{Z,s,\lambda}, \lambda \in k, \lambda \neq 0\}$;
12. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,C\}}$ and $\dim_k \mathcal{K}(Z,P) = 1$;
13. $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,C\}}$ and $\dim_k \mathcal{K}(Z,P) = 2$,

then, adding rows of the horizontal band $2j$ to the horizontal band 1, we can turn into zero the following blocks of the horizontal band 1:

1. $M_{4j}(a)$; 2. $M_{4j}(x^2)$;
3. $M_{4j}(b)$; 4. $M_{4j}(a)$ and $M_{4j}(b)$;
5. $M_{4j}(c)$; 6. M_{2j} except the column 1;
7. M_{2j} and $M_{4j}(a)$; 8. M_{2j} and $M_{4j}(b)$;
9. M_{2j}, $M_{4j}(a)$ and $M_{4j}(b)$; 10. M_{2j} and $M_{4j}(c)$;
11. M_{2j}; 12. M_{2j} and $M_{4j}(c)$;
13. $M_{4j}(a)$ and $M_{4j}(b)$,

respectively, where $M_t(y^i)$ means the vertical band of the matrix M_t corresponding to the basis vector y^i of the object Y.

Moreover, if $\dim_k(Z,P) = 2$, then adding columns of the matrix $M\varphi_{\mathcal{B}_0^{P,+}}$ to the matrix $M\varphi_{z^2}$, we can turn the block M_2 into zero.

Let M' denote the matrix consisting of nonzero blocks of the band 1 in the matrix $M\varphi$ excepting the block $M_3(p)$. Let G' denote the subset in G consisting of elementary transformations which do not change the horizontal band 2 and zero blocks of the band 1 in the matrix $M\varphi$. By an immediate checking one can see that $(\mathrm{Mat}_{\mathcal{K}}, G') = (\mathrm{Mat}_{\partial_P\mathcal{K}}, G_{\partial_P\mathcal{K}})$.

Furthermore, the block $M_{4j}(c)$ corresponds to the object $S_{\{A,B,C\}}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{A,B,C\}}$, the block $M_{4j}(a)$ to $S_{\{A,B\}}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{A,B\}}$, the blocks $M_{4j}(a)$ and $M_{4j}(b)$ to $L_{\{A,B,C\}}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{A,B,C,\diamond\}}$, the block M_{3j} to S_X, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{X,1,0,+}$, the block M_{3j} to $A \in O\partial_P\mathcal{K}$, where $(W_j, Y_j, \varphi''_j) = \mathcal{K}_{A,0}$, the blocks $M_{4j}(a)$ and $M_{4j}(b)$ to $L_{\{Z,A,B,C\}}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,C\}}$, the blocks $M_{4j}(a)$ and $M_{4j}(b)$ to $L_{\{Z,A,B,C,s,-\}}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,C,s,-\}}$, the blocks $M_{4j}(a)$ and $M_{4j}(b)$ to $L_{\{Z,A,B,C,s,+\}}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,C,s,+\}}$, the block $M_{4j}(c)$ to $S_{\{Z,A,B,C,s\}}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,C,s,\diamond\}}$, the block $M_{4j}(a)$ to $S_{\{Z,A,B,s\}}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,s\}}$, the block $M_{4j}(a)$ to $S_{Z,A,s,0}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{Z,A,s,0}$, the column 1 of the block M_{2j} to $S_{Z,s,-}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{Z,s,-}$, the column 1 of the block M_{2j} to $S_{Z,s,0}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{Z,s,0,-}$, the column 1 of the block M_{2j} to $S_{Z,0}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{Z,0}$, the block $M_{4j}(c)$ to $S_{\{Z,A,B,C\}}$, where $(Y_j, W_j, \varphi''_j) = \mathcal{K}_{\{Z,A,B,C\}}$.

Let $\mathbf{d}_P(X,V,\varphi)$ denote the object $\mathcal{V}(\partial_P\mathcal{K})$ corresponding to the matrix M'. Thus, we have constructed the mapping $\mathbf{d}_P : \mathrm{Ob}\,\mathcal{V}(\mathcal{K}) \to \mathrm{Ob}\,\mathcal{V}(\partial_P\mathcal{K})$. Obviously, the matrix $M\varphi$ can be uniquely restored up to the block $M\varphi_p$ by the matrix M'. In order to restore the block $M\varphi_p$, we transform the matrix M' to the following form

$$M' = \begin{array}{c c} & \begin{array}{cccc} M\varphi_{z^2} & M\varphi_{z^1} & M\varphi_{\mathcal{B}_0^{P,+}} & M\varphi_{\mathcal{B}_0^{P,-}} \end{array} \\ \begin{array}{c} 11 \\ 12 \end{array} & \begin{pmatrix} M'_{11} & M'_{21} & M'_{31} & M'_{41} \\ M'_{12} & M'_{22} & 0 & M'_{42} \end{pmatrix} \end{array},$$

where the rows M'_{31} are linearly independent. Then we set $M\varphi_p = \begin{array}{c} 11 \\ 12 \\ 2 \end{array}\begin{pmatrix} 0 \\ E \\ 0 \end{pmatrix}$. Thus, we have obtained the mapping $\mathbf{i}_P : \mathrm{Ob}\,\mathcal{V}(\partial_P\mathcal{K}) \to \mathrm{Ob}\,\mathcal{V}(\mathcal{K})$. The following proposition follows immediately from definitions of $\mathbf{d}_P, \mathbf{i}_P$.

Proposition 3.

1. The mappings $\mathbf{d}_P$ and $\mathbf{i}_P$ carry indecomposable objects to indecomposable ones or to zero;

2. If $(X,V,\varphi) \cong (X',V',\varphi')$, then $\mathbf{d}_P((X,V,\varphi) \cong \mathbf{d}_P((X',V',\varphi'))$;

3. Let (X, V, φ), (X', V', φ') be indecomposable and $\mathbf{d}_P((X, V, \varphi)) \neq 0$, $\mathbf{d}_P((X',V',\varphi')) \neq 0$. Then $\mathbf{d}_P((X,V,\varphi))$ and $\mathbf{d}_P((X',V',\varphi'))$ are indecomposable and $\mathbf{d}_P((X,V,\varphi)) \cong \mathbf{d}_P((X',V',\varphi'))$ implies $(X,V,\varphi) \cong (X',V',\varphi')$;

4. $\mathbf{d}_P$ carries to zero only a finite number of objects from $\mathrm{ind}\,\mathcal{V}(\mathcal{K})$ of fixed dimension $\vec{d}$ which are different from $\mathcal{K}_{Z,s,\lambda}, \lambda \neq 0$;

5. $d(\mathbf{d}_P((X,V,\varphi))) \leq d((X,V,\varphi))$ and if (X,V,φ) is sincere, then the inequality is strict. ▪

For each object $(X,V,\varphi) \in \mathcal{V}(\mathcal{K})$ we denote by $\mathrm{supp}\,(X,V,\varphi)$ the set $\{Y \in \mathrm{ind}\,\mathcal{K} \mid Y \subseteq X\}$.

Remark. From the definition of $\mathbf{d}_P$ it follows that

1. If $P \in \max\,\mathrm{ind}\mathcal{K}$, then $\mathrm{Ker}\,\mathbf{d}_P = \{\mathcal{K}_0, \mathcal{K}_{P,0}, \mathcal{K}_A \mid A \in S\mathcal{K}, A - P\}$;

2. If $Z = \max\,\mathrm{ind}\mathcal{K}$, $Z \in L\mathcal{K}$, $\dim_k\mathcal{K}(Z,P) = 2$, then $\mathrm{Ker}\,\mathbf{d}_P = \{\mathcal{K}_0, \mathcal{K}_{P,0}, \mathcal{K}_A, T \mid A \in S\mathcal{K}, A - P, Z \in \mathrm{supp}\,T\}$;

3. If $Z = \max\,\mathrm{ind}\mathcal{K}$, $Z \in L\mathcal{K}$, $P \in S\mathcal{K}_Z$, then $\mathrm{Ker}\,\mathbf{d}_P = \{\mathcal{K}_0, \mathcal{K}_{P,0}, \mathcal{K}_A, \mathcal{K}_{Z,s,0,+}, \mathcal{K}_{Z,s,+}, \mathcal{K}_{Z,s,\lambda}, \mathcal{K}_{Z,B,s} \mid A \in S\mathcal{K}, A - P, B \in S\mathcal{K}_Z, B - P, \lambda \in k, \lambda \neq 0, s \in \mathbf{N}\}$.

The following proposition can be proved in the same way as the Statement 1 in [ZN1] (see also Chapter 15 in [Si]).

Proposition 4. *Under the differentiation $\mathbf{d}_P$, a series turns into a series except the series $\mathcal{K}_{Z,s,\lambda}, \lambda \neq 0$, i.e. almost all representations of a given dimension generated by one $k[x]$-representation turns into representations again generated by a one $k[x]$-representation. Besides, every series of the initial set is obtained from the series of the derived set by the integration $\mathbf{i}_P$ except the series $\mathcal{K}_{Z,s,\lambda}, \lambda \neq 0$.* ▪

2.6

Proof of Proposition 1. Consider several cases:

Case 1. $n = m = 0$. It is well known that $\mathrm{ind}\,\mathcal{V}(\mathcal{C}_{0,0}) = \{\mathcal{K}_0, \mathcal{K}_{Z,0}, \mathcal{K}_{Z,s,0,-}, \mathcal{K}_{Z,s,0,+}, \mathcal{K}_{Z,s,-}, \mathcal{K}_{Z,s,+}, \mathcal{K}_{Z,s,\lambda} \mid s \in \mathbf{N}, \lambda \in k \setminus \{0\}\}$.

Case 2. $n > 0, m = 0$. Consider $\partial_{A_1}\mathcal{K}$ and denote it by $\mathcal{K}'$. From the case 1 and (2.4) it follows that the category $\mathcal{K}'$ is linear and $\mathrm{ind}\,\mathcal{K}' = \{A_2, \ldots, A_n, S_{Z,0}, S_{Z,s,-}, S_{Z,s,0} \mid s \in \mathbf{N}\}$. Since $w(S\mathcal{K}') = 2$, then from [K] it follows that $\mathrm{ind}\,\mathcal{V}(\mathcal{K}') = \{\mathcal{K}'_0, \mathcal{K}'_{X,0}, \mathcal{K}'_X, \mathcal{K}'_{\{C,D\}} \mid X \in \mathrm{ind}\,\mathcal{K}', (C,D) \in S\mathcal{K}'\}$.

It is easily seen that $\mathbf{i}_{A_1}(\mathcal{K}'_0) \cong \mathcal{K}_{A_1}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{A_i,0}) \cong \mathcal{K}_{A_i,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{A_i}) \cong \mathcal{K}_{A_i}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,0,0}}) \cong \mathcal{K}_{Z,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,0}}) \cong \mathcal{K}_{Z,A_1,1,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,s,-},0}) \cong \mathcal{K}_{Z,s,-}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,s,-}}) \cong \mathcal{K}_{Z,A_1,s+1,0}$,

$\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,s,0},0}) \cong \mathcal{K}_{Z,s,0,-}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,s,0}}) \cong \mathcal{K}_{Z,A_1,s}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{\{A_i,S_{Z,0}\}}) \cong \mathcal{K}_{Z,A_i,1,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{\{A_i,S_{Z,s,-}\}}) \cong \mathcal{K}_{Z,A_i,s+1,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{\{A_i,S_{Z,s,0}\}}) \cong \mathcal{K}_{Z,A_i,s}$, for $1 < i \leq n$, $s \in \mathbf{N}$. Hence in this case Proposition 1 follows from Proposition 3 and Remark to Proposition 3.

Case 3. $n > 0, m > 0$. Consider $\partial_{A_1}\mathcal{K}$ and denote it by $\mathcal{K}'$. From the case 2 and (2.4) it follows that the category $\mathcal{K}'$ is linear and $\operatorname{ind}\mathcal{K}' = \{A_2, \dots, A_n, B_j,\ S_{Z,0},\ S_{Z,s,-},\ S_{Z,s,0},\ S_{Z,B_j,s,0} \mid s \in \mathbf{N},\ 1 \leq j \leq m\}$. Since $w(S\mathcal{K}') = 2$, then from [K] it follows that $\operatorname{ind}\mathcal{V}(\mathcal{K}') = \mathcal{K}'_0,\ \mathcal{K}'_{X,0},\ \mathcal{K}'_X,\ \mathcal{K}'_{\{C,D\}} \mid X \in \operatorname{ind}\ \mathcal{K}',\ (C, D) \in S\mathcal{K}'\}$.

It is easily seen that $\mathbf{i}_{A_1}(\mathcal{K}'_0) \cong \mathcal{K}_{A_1}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{A_i,0}) \cong \mathcal{K}_{A_i,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{A_i}) \cong \mathcal{K}_{A_i}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{B_j,0}) \cong \mathcal{K}_{B_j,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{B_j}) \cong \mathcal{K}_{\{A_1,B_j\}}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,0},0}) \cong \mathcal{K}_{Z,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,0}}) \cong \mathcal{K}_{Z,A_1,1,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,s,-},0}) \cong \mathcal{K}_{Z,s,-}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,s,-}}) \cong \mathcal{K}_{Z,A_1,s+1,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,s,0},0}) \cong \mathcal{K}_{Z,s,0,-}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,s,0}}) \cong \mathcal{K}_{Z,A_1,s}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,B_j,s,0},0}) \cong \mathcal{K}_{Z,B_j,s,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{S_{Z,B_j,s,0}}) \cong \mathcal{K}_{\{Z,B_j,A_1,s\}}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{\{A_i,B_j\}}) \cong \mathcal{K}_{\{A_i,B_j\}}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{\{A_i,S_{Z,0}\}}) \cong \mathcal{K}_{Z,A_i,1,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{\{A_i,S_{Z,s,-}\}}) \cong \mathcal{K}_{Z,A_i,s+1,0}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{\{A_i,S_{Z,s,0}\}}) \cong \mathcal{K}_{Z,A_i,s}$, $\mathbf{i}_{A_1}(\mathcal{K}'_{\{S_{Z,B_j,s,0},A_i\}}) \cong \mathcal{K}_{\{Z,B_j,A_i,s\}}$, for $1 < i \leq n$, $1 \leq j \leq m$, $s \in \mathbf{N}$. Hence in this case Proposition 1 follows from Proposition 3 and Remark to Proposition 3. ∎

Proof of Proposition 2. Consider $\partial_A\mathcal{K}$ and denote it by $\mathcal{K}'$. From (2.4) it follows that the category $\mathcal{K}'$ is linear and $\operatorname{ind}\mathcal{K}' = \{B, C, S_{\{B,C\}},\ S_{\{Z,B,C,s\}},\ S_{Z,0},\ S_{Z,s,-},\ S_{Z,s,0},\ S_{Z,B,s,0},\ S_{Z,C,s,0} \mid s \in \mathbf{N}\}$. Since $w(S\mathcal{K}') = 2$, then from [K] it follows that $\operatorname{ind}\mathcal{V}(\mathcal{K}') = \{\mathcal{K}'_0,\ \mathcal{K}'_{X,0},\ \mathcal{K}'_X,\ \mathcal{K}'_{\{L,M\}} \mid X \in \operatorname{ind}\mathcal{K}',\ (L, M) \in S\mathcal{K}'\}$.

It is easily seen that $\mathbf{i}_A(\mathcal{K}'_0) \cong \mathcal{K}_A$, $\mathbf{i}_A(\mathcal{K}'_{B,0} \cong \mathcal{K}_{B,0}$, $\mathbf{i}_A(\mathcal{K}'_B) \cong \mathcal{K}_{\{A,B\}}$, $\mathbf{i}_A(\mathcal{K}'_{C,0} \cong \mathcal{K}_{C,0}$, $\mathbf{i}_A(\mathcal{K}'_C) \cong \mathcal{K}_{\{A,C\}}$, $\mathbf{i}_A(\mathcal{K}'_{S_{\{B,C\}},0}) \cong \mathcal{K}_{\{B,C\}}$, $\mathbf{i}_A(\mathcal{K}'_{S_{\{B,C\}}}) \cong \mathcal{K}_{\{A,B,C\}}$, $\mathbf{i}_A(\mathcal{K}'_{S_{Z,0},0}) \cong \mathcal{K}_{Z,0}$, $\mathbf{i}_A(\mathcal{K}'_{S_{Z,0}}) \cong \mathcal{K}_{Z,A,1,0}$, $\mathbf{i}_A(\mathcal{K}'_{S_{Z,s,-},0}) \cong \mathcal{K}_{Z,s,-}$, $\mathbf{i}_A(\mathcal{K}'_{S_{Z,s,-}}) \cong \mathcal{K}_{Z,A,s+1,0}$, $\mathbf{i}_A(\mathcal{K}'_{S_{Z,s,0},0}) \cong \mathcal{K}_{Z,s,0,-}$, $\mathbf{i}_A(\mathcal{K}'_{S_{Z,s,0}}) \cong \mathcal{K}_{Z,A,s}$ $\mathbf{i}_A(\mathcal{K}'_{S_{Z,B,s,0},0}) \cong \mathcal{K}_{Z,B,s,0}$, $\mathbf{i}_A(\mathcal{K}'_{S_{Z,B,s,0}}) \cong \mathcal{K}_{\{Z,B,A,s\}}$, $\mathbf{i}_A(\mathcal{K}'_{S_{Z,C,s,0},0}) \cong \mathcal{K}_{Z,C,s,0}$, $\mathbf{i}_A(\mathcal{K}'_{S_{Z,C,s,0}}) \cong \mathcal{K}_{\{Z,C,A,s\}}$, $\mathbf{i}_A(\mathcal{K}'_{S_{\{Z,B,C,s\}},0}) \cong \mathcal{K}_{\{Z,B,C,s\}}$, $\mathbf{i}_A(\mathcal{K}'_{S_{\{Z,B,C,s\}}}) \cong \mathcal{K}_{\{Z,A,B,C,s,\diamond\}}$, $\mathbf{i}_A(\mathcal{K}'_{\{B,C\}} \cong \mathcal{K}_{\{A,B,C,\diamond\}}$, $\mathbf{i}_A(\mathcal{K}'_{\{S_{\{B,C\}},S_{Z,0}\}}) \cong \mathcal{K}_{\{Z,A,B,C\}}$, $\mathbf{i}_A(\mathcal{K}'_{\{S_{Z,B,s,0},S_{Z,C,s,0}\}}) \cong \mathcal{K}_{\{Z,A,B,C,s,-\}}$, $\mathbf{i}_A(\mathcal{K}'_{\{S_{\{Z,B,C,s\}},S_{Z,s,-}\}}) \cong \mathcal{K}_{\{Z,A,B,C,s,+\}}$, for $s \in \mathbf{N}$. Hence Proposition 2 follows from Proposition 3 and Remark to Proposition 3. ∎

2.7

Let P be an admissible object.

Lemma 15. *Suppose that $X \in SO\partial_P\mathcal{K}$. The following statements hold:*

1. $X < S_{\{A,B\}} \Leftrightarrow X \leq A$ or $X \leq B$;

2. $X < S_{\{A,B,C\}} \Leftrightarrow X \leq A$ or $X \leq B$ or $X \leq C$;

3. $X > S_{\{A,B\}} \Leftrightarrow X > A$ and $X > B$;

4. $X > S_{\{A,B,C\}} \Leftrightarrow X > A$ and $X > B$ and $X > C$;

5. $S_{\{A,B\}} \geq S_{\{C,D\}} \Leftrightarrow (A \geq C$ or $B \geq C)$ and $(A \geq D$ or $B \geq D)$;

6. $S_{\{A,B,C\}} \geq S_{\{D,F\}} \Leftrightarrow (A \geq D$ or $B \geq D$ or $C \geq D)$ and $(A \geq F$ or $B \geq F$ or $C \geq F)$;

7. $S_{\{A,B\}} \geq S_{\{D,F,G\}} \Leftrightarrow (A \geq D$ or $B \geq D)$ and $(A \geq F$ or $B \geq F)$ and $(A \geq G$ or $B \geq G)$;

8. $S_{\{A,B,C\}} \geq S_{\{D,F,G\}} \Leftrightarrow (A \geq D$ or $B \geq D$ or $C \geq D)$ and $(A \geq F$ or $B \geq F$ or $C \geq F)$ and $(A \geq G$ or $B \geq G$ or $C \geq G)$.

Proof follows from the conditions 3,4,8 and 20 of the definition of $\operatorname{Mor}\partial_P\mathcal{K}$. ∎

Lemma 16. *If* $X, Y \in L\mathcal{K}$, $S_Y \in NS\partial_P\mathcal{K}$, $S_Y < X$, *then* $\mathcal{K}(\{X,Y\}) \cong \mathcal{I}$ *and* $\dim_k \partial_P\mathcal{K}(X, S_Y) = 2$.

Proof. Put $U = X^P, V = Y^P$. From the condition (c) of Theorem 1 it follows that $\mathcal{K}(X,Y)(U) \neq 0$. If $\mathcal{K}(X,Y)(U) = Y$, then $\dim_k \mathcal{K}(P,Y) = 2$, a contradiction. Therefore $\mathcal{K}(X,Y)(U) = V$ and hence $\mathcal{K}(\{X,Y\}) \cong \mathcal{I}$ by Lemma 9. Then from the condition 10 of the definition of $\operatorname{Mor}\partial_P\mathcal{K}$ it follows that $\dim_k \mathcal{K}(X, S_Y) = 2$. ∎

Lemma 17. *Suppose that* $X = L_{\{X_1,X_2,X_3\}}, \Phi \subseteq S\partial_P\mathcal{K}$, $S_{Z,0} \notin \Phi$. *Then* $\Phi \subseteq S\partial_P\mathcal{K}^{X,V}$ *if and only if there exists* $s \in \{1,2,3\}$ *such that* $\Phi - X_s$ *and* $\Phi > X_i$ *for* $i \neq s$.

Proof follows from the definition of $\operatorname{Mor}\partial_P\mathcal{K}$. ∎

Lemma 18. *If* $S_{\{A,B\}} \in S\partial_P\mathcal{K}^{X,V}$, *then* $A, B \in S\partial_P\mathcal{K}^{X,V}$ *and* $S_{\{A,B\}}$ *is comparable with all the elements of* $S\partial_P\mathcal{K}^{X,V}$, *which are different from* $S_{Z,0}$.

Proof. Since $B < S_{\{A,B\}}$, $A < S_{\{A,B\}}$, we obtain $A, B \in S\partial_P\mathcal{K}^{X,V}$ by Lemma 4.

Suppose $X = L_{\{X_1,X_2,X_3\}}$. Then from Lemma 17 it follows that there exists $s \in \{1,2,3\}$ such that $\{A,B\} - X_s$, $\{A,B\} > X_i$ for $i \neq s$. Hence $S\mathcal{K}$ contains the subset of non-polynomial growth $G = \psi(X) \cup \{A,B,P\}$, a contradiction. Therefore we may assume that $X \in OL\partial_P\mathcal{K}$.

Suppose $Y \in S\partial_P\mathcal{K}^{X,V}$, $Y \neq S_{Z,0}$, $Y - S_{\{A,B\}}$. It is clear that $Y \neq S_T$ for any $T \in L\mathcal{K}$. If $Y \in O\partial_P\mathcal{K}$, then Lemma 15 implies that $Y - A$ or $Y - B$, a contradiction to the condition (e) of Theorem 1. If $Y = S_{\{C,D\}}$ or $Y = S_{\{C,D,E\}}$, then from Lemma 15 it follows that there exists $T \in \{C,D,E\}$ such that $T - A$ or $T - B$. We get a contradiction to the condition (e) of Theorem 1 because $T \in S\mathcal{K}^{X,V}$ on account of Lemma 4.

From the definition of $\operatorname{Mor}\partial_P\mathcal{K}$ it follows that if $Y \in \psi_Z$, then there exists $T \in \psi(Y)$ such that $T \in S\mathcal{K}^{X,V}$, $T - A$ or $T - B$, which is impossible because of the condition (e) of Theorem 1. ∎

Lemma 19. *Suppose that* $P \in \max\operatorname{ind}\mathcal{K}$, $Y \in L\mathcal{K}, S_Y \in S\partial_P\mathcal{K}, A \in S\partial_P\mathcal{K}$. *Then* $S_Y - A$ *if and only if either* $A \in S\partial_P\mathcal{K}_{Y,Y^P}$ *or* $B \in S\mathcal{K}^{Y,V}$ *for any* $B \in \psi(A)$, *where* $V \neq Y^P$.

Proof. If $A < Y$, then Lemma 5 implies that $A \in S\partial_P\mathcal{K}_{Y,Y^P}$. If $A > Y$, then from the condition 10 of the definition of $\operatorname{Mor}\partial_P\mathcal{K}$ we conclude that $A \in S\mathcal{K}^{Y,V}$, where $V \neq Y^P$. Then Lemma 4 implies that $B \in S\mathcal{K}^{Y,V}$ for any $B \in \psi(A)$. ∎

Lemma 20. *Suppose that* $Y \in L\mathcal{K}, S_Y \in S\partial_P\mathcal{K}$. *Then*

1. $S_Y \in S\partial_P\mathcal{K}^{Y,Y^P}$*;*

2. $S_Y \leq A$ *for any* $A \in S\partial_P\mathcal{K}^{Y,Y^P}$*;*

3. If $S_Y \in S\partial_P\mathcal{K}^{X,V}$ *for some* $X \in L\partial_P\mathcal{K}$, *then* S_Y *is comparable with all the elements of* $S\partial_P\mathcal{K}^{X,V}$.

Proof. 1. Evident.

2. It immediately follows from the condition 10 of the definition of $\operatorname{Mor}\partial_P\mathcal{K}$.

3. It is clear that $X \in OL\partial_P\mathcal{K}$. Indeed, if $X = L_{\{X_1,X_2,X_3\}}$, then $\{X_1,X_2,X_3\} < Y$ and hence from Lemma 5 it follows that $\{X_1,X_2,X_3\} \in S\mathcal{K}_{Y,Y^P}$, a contradiction to the condition (e) of Theorem 1.

Suppose $A < Y$, $A \in S\partial_P\mathcal{K}^{X,V}$, $A - S_Y$. Since $S_Y \in S\partial_P\mathcal{K}^{X,V}$, we get $\mathcal{K}(\{X,Y\}) \cong \mathcal{I}$ and $\mathcal{K}(Y,X)(U) = V$, where $U = Y^P$. From Lemma 5 it follows that $\psi(A) \in S\mathcal{K}_{Y,U}$, a contradiction to the condition (d) of Theorem 1.

If $A > Y$, $A \in S\partial_P\mathcal{K}^{X,V}$, then from Lemma 10 it follows that $A \in S\partial_P\mathcal{K}^{Y,Y^P}$. Therefore $A \neq S_{Z,0}$ and hence A is comparable with S_Y because of the condition 10 of the definition of $\mathrm{Mor}\,\partial_P\mathcal{K}$. ∎

Lemma 21. *Suppose that $A = S_{\{A_1,A_2,A_3\}}$. Then*

1. If $X \in L\partial_P\mathcal{K}, X \neq L_{\{Z,A_1,A_2,A_3\}}$ and $\partial_P\mathcal{K}\,(A,X) \neq 0$, then $\dim_k\partial_P\mathcal{K}\,(A,X) = 2$;

2. If $X \in OL\partial_P\mathcal{K}$, then $\partial_P\mathcal{K}\,(X,A) = 0$;

3. If $X = L_{\{B_1,B_2,B_3\}}$, $B_i \neq A_j$ for $i,j \in \{1,2,3\}$ and $\partial_P\mathcal{K}\,(X,A) \neq 0$, then $\dim_k\partial_P\mathcal{K}\,(X,A) = 2$.

Proof. 1. Suppose $X \in O\partial_P\mathcal{K}$. If $\dim_k\partial_P\mathcal{K}(A,X) = 1$, then there exists $V \in \mathbf{P}^1(X)$ such that $A \in S\partial_P\mathcal{K}^{X,V}$. Then Lemma 4 implies that $A_1, A_2, A_3 \in S\mathcal{K}^{X,V}$, a contradiction to the condition (e) of Theorem 1. Therefore we may assume that $X = L_{\{X_1,X_2,X_3\}}$ for some $X_1, X_2, X_3 \in S\mathcal{K}$. If $\dim_k\partial_P\mathcal{K}(A,X) = 1$, then there exists $V \in \mathbf{P}^1(X)$ such that $A \in S\partial_P\mathcal{K}^{X,V}$. From Lemma 4 it follows that $\psi(A) \subseteq S\partial_P\mathcal{K}^{X,V}$. Then Lemma 17 implies that there exists $s \in \{1,2,3\}$ such that $\psi(A) - X_s$ and hence $S\mathcal{K}$ contains the wild subset $\{P, X_s, A_1, A_2, A_3\}$, a contradiction.

2. Suppose, contrary to our claim, that $\partial_P\mathcal{K}(X,A) \neq 0$. Then from Lemma 5 it follows that $A_1, A_2, A_3 \in S\mathcal{K}_{X,X^P}$, a contradiction to the condition (e) of Theorem 1.

3. Suppose, contrary to our claim, that $\dim_k\partial_P\mathcal{K}(X,A) = 1$. Then we may assume that $A < X_1$, $A - X_2$, $A - X_3$. Hence $A_i \not\geq X_j$ for any $i,j \in \{1,2,3\}$. Since $S\mathcal{K}$ is the poset of polynomial growth, we obtain that any A_i is comparable with at least two elements of $\psi(X)$. Therefore $\dim_k\partial_P\mathcal{K}(X,A_i) = 2$ for all i and hence $\dim_k\partial_P\mathcal{K}(X,A) = 2$ because of the condition 8 of the definition of $\mathrm{Mor}\,\partial_P\mathcal{K}$. ∎

Lemma 22. *If $S_{\{A,B\}} \in S\partial_P\mathcal{K}_{X,V}, X \in OL\partial_P\mathcal{K}$, then $A, B \in S\mathcal{K}_{X,V}, V = X^P$ and $S_{\{A,B\}}$ is comparable with all the elements of $S\partial_P\mathcal{K}_{X,V}$.*

Proof. From Lemma 5 it follows that $A, B \in S\mathcal{K}_{X,X^P}$. Lemma 21 implies that $S_{\{C,D,E\}} \notin S\partial_P\mathcal{K}_{X,X^P}$ for any $C, D, E \in S\mathcal{K}$. From Lemma 16 we conclude that $S_Y \notin S\partial_P\mathcal{K}_{X,X^P}$ for any $Y \in L\mathcal{K}$.

Suppose $C \in OS\partial_P\mathcal{K}_{X,X^P}$, $C - S_{\{A,B\}}$. Then from Lemma 15 it follows, that $C - A$ or $C - B$, a contradiction to the condition (e) of Theorem 1.

Suppose $E = S_{\{C,D\}}$, $E - S_{\{A,B\}}$, $E \in S\partial_P\mathcal{K}_{X,X^P}$. From Lemma 5 it follows that $C, D \in S\mathcal{K}_{X,X^P}$. Since $S\mathcal{K}_{X,X^P}$ is a semichain, we get $\{C,D\} > \{A,B\}$ or $\{C,D\} < \{A,B\}$, a contradiction to the fact that $S_{\{A,B\}} - E$. ∎

Lemma 23. *Suppose that $Y \in L\mathcal{K}$, $X \in L\partial_P\mathcal{K}$, $S_Y \in S\partial_P\mathcal{K}_{X,V}$ for some $V \in \mathbf{P}^1(X)$. Then $X = L_{\{X_1,X_2,X_3\}}$ for some $X_1, X_2, X_3 \in S\mathcal{K}$ and S_Y is comparable with all the elements of $S\partial_P\mathcal{K}_{X,V}$.*

Proof. From Lemma 16 and the definition of $\mathrm{Mor}\,\partial_P\mathcal{K}$ it follows that $X = L_{\{X_1,X_2,X_3\}}$ for some $X_i \in S\mathcal{K}$, $1 \leq i \leq 3$. Let $\mathcal{B}$ be a P-basis as in (2.5). Then $Mb_{X,X_1} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $Mb_{X,X_2} = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $Mb_{X,X_3} = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

It is clear that $\psi(X) > Y$. We distinguish several cases:

(a). $X_i \in S\mathcal{K}^{Y,U_i}$ for all $1 \le i \le 3$, $U_i \ne U_j$ for $i \ne j$. By Lemma 8, we can choose a P-basis as in (2.5) for which $Mb_{X_1,Y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Mb_{X_2,Y} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix}$, $Mb_{X_3,Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since $\mathcal{K}$ satisfies all the conditions of Theorem 1, we have $\lambda \in k \setminus \{0,1\}$. Then from the condition 10 of the definition of $\operatorname{Mor} \partial_P \mathcal{K}$ it follows that $\operatorname{Mat}_{\mathcal{B}}(X, S_Y) = k \begin{pmatrix} \lambda & 1 \end{pmatrix}$. Let $V \in \mathbf{P}^1(X)$, $V = kv$, where $v = x^1 - \lambda x^2$. Then $S_Y \in S\mathcal{K}_{X,V}$.

Suppose $T \in S\partial_P\mathcal{K}$, $T > Y$, $T \ne X_i$ for all i, $T \ne S_Y$. Since $S\mathcal{K}$ is a poset of polynomial growth and $P, X_i \in S\mathcal{K}^Y$, we obtain $T > X$.

It is clear that if $T < Y$, then $T \notin S\mathcal{K}_{X,V}$. Hence $S\mathcal{K}_{X,V} = \{S_Y\}$.

(b). $X_1, X_2 \in S\mathcal{K}^{Y,U}$ for some $U \in \mathbf{P}^1(Y)$ and $\dim_k \mathcal{K}(X_3, Y) = 2$. It is easily seen that we can choose a P-basis as in (2.5) for which $Mb_{X_1,Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $Mb_{X_2,Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $Mb^1_{X_3,Y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Mb^2_{X_3,Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then from the condition 10 of the definition of $\operatorname{Mor} \partial_P \mathcal{K}$ it follows that $\operatorname{Mat}_{\mathcal{B}}(X, S_Y) = k \begin{pmatrix} 1 & 1 \end{pmatrix}$. Let $V \in \mathbf{P}^1(X)$, $V = k(x^1 - x^2)$. Then $S_Y, X_3 \in S\partial_P\mathcal{K}_{X,V}$.

From Lemma 5 it follows that if $T \in S\partial_P\mathcal{K}$, $T < Y$ and $T - S_Y$, then $\dim_k \partial_P\mathcal{K}(X,T) = 2$. Therefore if $T \in S\mathcal{K}_{X,V}$ and $T < Y$, then $T < S_Y$.

Suppose $A \in S\mathcal{K}$, $A > Y$, $A \ne X_i$ for all i, $A \ne P$. Since $S\mathcal{K}$ is the poset of polynomial growth, we see that A is comparable with at least two elements of $\{X_1, X_2, X_3, P\}$. Hence we obtain that if $A \not\ge B$ for any $B \in \{X_1, X_2, X_3, P\}$, then either $A \in S\mathcal{K}^{Y,U}$ or $A \in S\mathcal{K}^{Y,Y^P}$. This implies that if $T \in S\partial_P\mathcal{K}$, $T > Y$, $T \ne X_i$ for all i, $T \ne S_Y$, $T < X$, then one of the following situations may take place:

(α) $T \in O\partial_P\mathcal{K}$, $T \in S\mathcal{K}^{Y,U}$. Then $T < X_1$, $T < X_2$ and hence $\dim_k \partial_P\mathcal{K}(X,T) = 2$;

(β) $T \in O\partial_P\mathcal{K}$, $T \in S\mathcal{K}^{Y,Y^P}$. Then $T < P$, $T < X_3$. Hence $T \in S\partial_P\mathcal{K}_{X,V}$ and $S_Y < T < X_3$;

(γ) $T = S_{\{T_1,T_2\}}$, $T_1, T_2 \in S\mathcal{K}^{Y,U}$.

Then $\{T_1, T_2\} < \{X_1, X_2\}$ and hence $\dim_k \partial_P\mathcal{K}(X,T) = 2$;

(δ) $T = S_{\{X_3,A\}}$ for some $A \in S\mathcal{K}^{Y,U}$. Then $A < X_1$, $A < X_2$ and A is comparable with all elements of $S\partial_P\mathcal{K}^{Y,U}$. It is easily seen that in this case $T \in S\mathcal{K}_{X,V}$ and $S_Y < X_3 < T$.

(c). $X_1 \in S\mathcal{K}^{Y,U_1}$, $X_2 \in S\mathcal{K}^{Y,U_2}$ for some $U_1, U_2 \in \mathbf{P}^1(Y)$, $U_1 \ne U_2$, $\dim_k \mathcal{K}(X_3, Y) = 2$. By Lemma 7, we can choose a P-basis as in (2.5) for which $Mb_{X_1,Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $Mb_{X_2,Y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $Mb^1_{X_3,Y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $Mb^2_{X_3,Y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then from the condition 10 of the definition of $\operatorname{Mor} \partial_P \mathcal{K}$ it follows that $\dim_k \partial_P\mathcal{K}(X, S_Y) = 2$. Therefore $S_Y \notin S\partial_P\mathcal{K}_X$.

(d). $\dim_k \mathcal{K}(X_1, Y) = \dim_k \mathcal{K}(X_2, Y) = 2$. Similar to the case **(c)** we obtain that $\dim_k \partial_P\mathcal{K}(X, S_Y) = 2$. Therefore $S_Y \notin S\partial_P\mathcal{K}_X$. ∎

Lemma 24. *If* $X \in NL\partial_P\mathcal{K}, Y \in L\partial_P\mathcal{K}$, *then* $\dim_k \partial_P\mathcal{K}(X,Y) + \dim_k \partial_P\mathcal{K}(Y,X) \ge 3$.

Proof. Suppose $X = L_{\{X_1,X_2,X_3\}}$, $\psi(X) > Y$. Let $\mathcal{B}$ be a P-basis as in (2.5). Then $Mb_{X,X_1} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $Mb_{X,X_2} = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $Mb_{X,X_3} = \begin{pmatrix} 1 & 1 \end{pmatrix}$.

Assume that $0 \ne a_i \in \mathcal{K}(X_i, Y)$, $1 \le i \le 3$. Then $\sum_{i=1}^3 t_i Ma_i Mb_{X,X_i} = \begin{pmatrix} A & B \end{pmatrix}$, where $A = t_1 Ma_1 + t_3 Ma_3$, $B = t_2 Ma_2 + t_3 Ma_3$. Therefore $a_i b_{X,X_i}$, $1 \le i \le 3$, are linearly dependent if and only if $(X_1, X_2, X_3) \subseteq S\mathcal{K}^{Y,V}$ for some $V \in \mathbf{P}^1(Y)$.

Suppose $Y \in OL\partial_P\mathcal{K}$. Since $\mathcal{K}$ satisfies the condition (e) of Theorem 1, we obtain $(X_1, X_2, X_3) \not\subseteq S\mathcal{K}^{Y,V}$ for any $V \in \mathbf{P}^1(Y)$ and hence $\dim_k \partial_P\mathcal{K}(X,Y) \geq 3$.

Suppose $Y = L_{\{Y_1,Y_2,Y_3\}}$ for some $Y_1, Y_2, Y_3 \in S\mathcal{K}$. Then Lemma 17 implies that if $(X_1, X_2, X_3) \subseteq S\mathcal{K}^{Y,V}$ for some $V \in \mathbf{P}^1(Y)$, then there exists $s \in \{1,2,3\}$ such that $(X_1, X_2, X_3) - Y_s$. Therefore $S\mathcal{K}$ contains the wild subset (X_1, X_2, X_3, Y_s, P), a contradiction. Hence $\dim_k \partial_P\mathcal{K}(X,Y) \geq 3$.

If $Y \in OL\partial_P\mathcal{K}$, then from Lemma 27 it follows that $\psi(X) > Y$.

Since $S\mathcal{K}$ is the poset of polynomial growth, we obtain that if $X = L_{\{X_1,X_2,X_3\}}$, $Y = L_{\{Y_1,Y_2,Y_3\}}$ and $X > Y$, then $X_i \not\leq Y_j$ for all $i, j \in \{1,2,3\}$.

It is easily seen that if $X, Y, Z \in L\mathcal{K}$, $\dim_k\mathcal{K}(X,Y) \geq 3$ and $\dim_k\mathcal{K}(Y,Z) \geq 3$, then $\dim_k\mathcal{K}(X,Z) \geq 3$ for any schurian vector space category $\mathcal{K}$. Hence proof follows from the above informations and the fact that the statement of lemma is true for the categories $\mathcal{O}_\lambda$, $\lambda \in k \setminus \{0,1\}$. ∎

Lemma 25. *Let $\mathcal{K}$ satisfies all the conditions of Theorem 1 and $\mathcal{K}'$ be a full subcategory of $\partial_P\mathcal{K}$ such that* $\operatorname{ind}\mathcal{K}'$ *does not contain S_Y for all $Y \in L\mathcal{K}$ and* $\operatorname{ind}\mathcal{K}' \cap LO\partial_P\mathcal{K} = \emptyset$. *Assume that $P \in \max\operatorname{ind}\mathcal{K}$. Then $\mathcal{K}'$ also satisfies all the conditions of Theorem 1.*

Proof. Suppose the category $\mathcal{K}'$ does not satisfy the conditions of Theorem 1. Then from (3.1) it follows that $\mathcal{K}'$ is wild. Therefore $\psi(\mathcal{K}')$ is also wild because of Proposition 4. Since the category $\psi(\mathcal{K}')$ is linear, we obtain that the poset $S\mathcal{K}$ is wild, a contradiction. ∎

Lemma 26. *Assume that $Y \in L\mathcal{K}$, $S_Y \in SN\partial_P\mathcal{K}$. Then*
1. *$\partial_P\mathcal{K}(S_Y, X) = 0$ for any $X \in LN\partial_P\mathcal{K}$;*
2. *$\partial_P\mathcal{K}(X, S_Y) > 0$ for any $X \in LN\partial_P\mathcal{K}$;*
3. *If $X \in OL\partial_P\mathcal{K}$ and $Y < X$, then $\dim_k\partial_P\mathcal{K}(X, S_Y) = 2$;*
4. *If $X \in OL\partial_P\mathcal{K}$ and $Y > X$, then $\dim_k\partial_P\mathcal{K}(S_Y, X) \geq 1$;*
5. *If $X \in OL\partial_P\mathcal{K}$, $X \neq Y$ and $\dim_k\partial_P\mathcal{K}(S_Y, X) = 1$, then $\mathcal{K}(\{X,Y\}) \cong \mathcal{I}$.*

Proof. 1. It follows from Lemma 27.

2. It follows from Lemmas 24,6 and the condition 10 of the definition of the spaces of morphisms in the category $\partial_P\mathcal{K}$.

3. It follows from Lemma 16.

4. It follows from Lemma 6.

5. Since $\dim_k\partial_P\mathcal{K}(S_Y, X) = 1$, $\mathcal{K}(Y,X)(Y^P) \in \mathbf{P}^1(X)$ and hence $\mathcal{K}(\{X,Y\}) \cong \mathcal{I}$ because of Lemma 9. ∎

Lemma 27. *Suppose $X \in OL\partial_P\mathcal{K}$, $Y \in \{S_{\{A,B\}}, S_{\{A,B,C\}}, L_{\{A,B,C\}}\}$ for some $A, B, C \in S\mathcal{K}$. Then either $\psi(Y) < X$ or $\psi(Y) > X$. If $\psi(Y) < X$, then $Y \neq S_{\{A,B,C\}}$, $Y \neq L_{\{A,B,C\}}$. If $\psi(Y) < X$ and $Y = S_{\{A,B\}}$, then $P \in S\mathcal{K}^{X,V}$, $A, B \in S\mathcal{K}_{X,V}$ for some $V \subseteq \mathbf{P}^1(X)$.*

Proof. Suppose $N, M \in \psi(Y)$, $N > X$, $M < X$. Then from Lemma 5 it follows that $N \in S\mathcal{K}^{X,V}$, $P \in S\mathcal{K}^{X,V}$, $M \in S\mathcal{K}_{X,V}$ and we get a contradiction to the fact that P is admissible. Hence $\psi(Y) < X$ or $\psi(Y) > X$.

Assume that $\psi(Y) < X$, $\psi(Y) = \{A, B, C\}$. Then $\{A, B, C\} \subseteq S\mathcal{K}_{X,X^P}$, a contradiction to the condition (e) of Theorem 1.

If $\psi(Y) < X$ and $Y = S_{\{A,B\}}$, then $A, B \in S\mathcal{K}_{X,X^P}$. ∎

3 Proof of the theorems

3.1

We claim that the conditions (a)–(h) of Theorem 1 are necessary.

Indeed, the necessity of the conditions (a) and (b) follows from Lemma 2.4 in [Ri1].

The necessity of the condition (h) follows from the polynomial growth criterion for posets [ZN1].

It is easily seen that if the condition (d) is not fulfilled, then $\mathcal{K}$ contains a full subcategory isomorphic to $\mathbf{E}$. Therefore, the necessity of the condition (d) follows from (1.3).

Suppose the condition (c) is not fulfilled. If $\dim_k\mathcal{K}(X,Y)+\dim_k\mathcal{K}(Y,X)=0$ for $X,Y\in L\mathcal{K}$, then Lemma 2.4 in [Ri1] shows that $\mathcal{K}$ is wild.

Assume that $\dim_k\mathcal{K}(X,Y)+\dim_k\mathcal{K}(Y,X)=1, 0\neq\varphi\in\mathcal{K}(X,Y)$, where $X,Y\in L\mathcal{K}$. If φ is the isomorphism, Lemma 3.6 in [Ri1] allows us to conclude that the category $\mathcal{K}$ is wild. If φ is not the isomorphism, then we can choose a basis for which

$$M\varphi=\begin{pmatrix}1&0\\0&0\end{pmatrix}.$$

In this case $\mathcal{K}(\{X,Y\})\cong\mathcal{K}(S,\sigma)$, where $S=\{A,B,C<D\},\sigma(A)=C,\sigma(B)=D$. In [NBR] it was proved that the poset with involution (S,σ) is wild and hence $\mathcal{K}$ is wild too. If $\dim_k\mathcal{K}(X,Y)+\dim_k\mathcal{K}(Y,X)=2$ for $X,Y\in L\mathcal{K}$, then from Lemma 6 it follows that $\mathcal{K}$ contains a full subcategory isomorphic to one of the categories $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D},\mathbf{D}^{op}$. But in (1.3) it was proved that the categories $\mathbf{A},\mathbf{C}$ are not of polynomial growth and the categories $\mathbf{B},\mathbf{D},\mathbf{D}^{op}$ are wild.

Suppose the condition (e) is not fulfilled. If $S\mathcal{K}_{X,V}$ (resp., $S\mathcal{K}^{X,V}$) contains the subset (A_1,A_2,A_3), then $\mathcal{K}(\Phi)\cong\mathcal{K}(S,\sigma)$, where $\Phi=\{X,A_1,A_2,A_3\},S=\{A_1<X_2,A_2<X_2,A_3<X_2,X_1\}$ (resp., $S=\{X_2<A_1,X_2<A_2,X_2<A_3,X_1\}$),$\sigma(X_1)=X_2,\sigma(A_i)=A_i, 1\leq i\leq 3$. From [NBR] it follows that the poset with involution (S,σ) is wild and hence $\mathcal{K}$ is wild too. If $S\mathcal{K}_{X,V}$ (resp., $S\mathcal{K}^{X,V}$) contains the subset $(A_1<A_2,A_3)$, then $\mathcal{K}(\Phi)\cong\mathcal{K}(S,\sigma)$, where $\Phi=\{X,A_1,A_2,A_3\},S=\{A_1<A_2<X_2,A_3<X_2,X_1\}$ (resp., $S=\{X_2<A_1<A_2,X_2<A_3,X_1\}$),$\sigma(X_1)=X_2,\sigma(A_i)=A_i,1\leq i\leq 3$. From [NBR] it follows that the poset with involution (S,σ) is wild and hence $\mathcal{K}$ is wild too.

Suppose the condition (f) is not fulfilled and $\mathcal{K}$ satisfies the conditions (e) and (h). If $\operatorname{ord}\Omega\mathcal{K}_X=3$ (resp., $\operatorname{ord}\Omega\mathcal{K}^X=3$) and $S\mathcal{K}_X$ (resp., $S\mathcal{K}^X$) contains the subset (A,B,C,D), then $\mathcal{K}(\Phi)\cong\mathbf{F}$ (resp., $\mathcal{K}(\Phi)\cong\mathbf{F}^{op}$), where $\Phi=\{X,A,B,C,D\}$, therefore the non-polynomial growth of the category $\mathcal{K}$ follows from (1.3). If $\operatorname{ord}\Omega\mathcal{K}_X=2$ (resp., $\operatorname{ord}\Omega\mathcal{K}^X=2$) and $S\mathcal{K}_X$ (resp., $S\mathcal{K}^X$) contains the subset (A,B,C,D), then $\mathcal{K}(\Phi)\cong\mathbf{G}$ (resp., $\mathcal{K}(\Phi)\cong\mathbf{G}^{op}$), where $\Phi=\{X,A,B,C,D\}$, therefore the non-polynomial growth of the category $\mathcal{K}$ follows from (1.3).

Suppose the condition (g) is not fulfilled and $\mathcal{K}$ satisfies the condition (c). Then there exist $X_1,\dots,X_n\in L\mathcal{K},F,H,P,Q\in S\mathcal{K}$, $V_i\in\mathbf{P}^1(X_i)$, $1\leq i\leq n$, $n\in\mathbf{N}$, such that $\mathcal{K}\left(X_i,X_{i+1}\right)(V_i)=V_{i+1}$, $\dim_k\mathcal{K}\left(X_i,X_{i+1}\right)=3$ for $1\leq i<n$,$F-H$,$P-Q$ and $(F,H)\subseteq S\mathcal{K}^{X_1,V_1}$, $(P,Q)\subseteq S\mathcal{K}_{X_n,V_n}$. We fix a basis $\mathcal{B}\left(\mathcal{K}\right)$ such that $x_i^1\in V_i$ for all i. For the proof of the non-polynomial growth of the category $\mathcal{K}$ we define an embedding $\mu:\operatorname{Ob}\mathcal{V}(\mathcal{K}(G))\to\operatorname{Ob}\mathcal{V}(\mathcal{K})$, where $G=\{A<B>C<D>A,S,T\}$. Let (Y,W,ψ) be an arbitrary object of the category $\mathcal{V}(\mathcal{K}(G))$. We fix a decomposition $Y\cong\oplus_{R\in\operatorname{ind}\mathcal{K}(G)}R^{d_R}$. Consider $(X,V,\varphi)\in\mathcal{V}(\mathcal{K})$ such that $X=(\oplus_{i=1}^{n}X_i^{(d_B+d_D)})\oplus F^{d_S}\oplus H^{d_T}\oplus P^{(d_B+d_C)}\oplus$

$Q^{(d_D+d_A)}$, $V = W \oplus (\oplus_{i=1}^n (M_i \oplus N_i))$, where $M_i = k^{d_B}$,$N_i = k^{d_D}$,$1 \le i \le n$,

$$M\varphi_f = \begin{array}{c} W \\ \vdots \end{array} \begin{pmatrix} M\psi_s & \\ 0 & \end{pmatrix}, M\varphi_h = \begin{array}{c} W \\ \vdots \end{array} \begin{pmatrix} M\psi_t & \\ 0 & \end{pmatrix}, M\varphi_{x_1^2} = \begin{array}{c} W \\ \vdots \end{array} \begin{pmatrix} M\psi_b & M\psi_d \\ 0 & 0 \end{pmatrix},$$

$$M\varphi_p = \begin{array}{c} W \\ \vdots \\ M_n \\ N_n \end{array} \begin{pmatrix} 0 & M\psi_c \\ 0 & 0 \\ E & 0 \\ 0 & 0 \end{pmatrix}, M\varphi_q = \begin{array}{c} W \\ \vdots \\ M_n \\ N_n \end{array} \begin{pmatrix} 0 & M\psi_a \\ 0 & 0 \\ 0 & 0 \\ E & 0 \end{pmatrix},$$

$$M\varphi_{x_1^1} = \begin{array}{c} W \\ M_1 \\ N_1 \\ \vdots \end{array} \begin{pmatrix} 0 & 0 \\ E & 0 \\ 0 & E \\ 0 & 0 \end{pmatrix}, M\varphi_{x_i^j} = \begin{array}{c} \vdots \\ M_{i-j+1} \\ N_{i-j+1} \\ \vdots \end{array} \begin{pmatrix} 0 & 0 \\ E & 0 \\ 0 & E \\ 0 & 0 \end{pmatrix},$$

for $1 < i \le n$,$1 \le j \le 2$. We set $\mu((Y, W, \psi)) = (X, V, \varphi)$. It is easy to see that μ preserves indecomposability and isomorphism classes. Moreover, if $(Y, W, \psi), (Y', W', \psi') \in \operatorname{Ob} \mathcal{V}(\mathcal{K}(G))$ and $\vec{d}((Y, W, \psi)) = \vec{d}((Y', W', \psi'))$, then $\vec{d}(\mu(Y, W, \psi)) = \vec{d}(\mu(Y', W', \psi'))$. Therefore the non-polynomial growth of the category $\mathcal{K}$ follows from the non-polynomial growth of the category $\mathcal{K}(G)$ (see [Sk1;p.507]).

3.2

Lemma 28. *Let $\mathcal{K}$ be a schurian vector space category satisfying all the conditions of Theorem 1. Suppose that there exist $P \in S\mathcal{K}$, $Z \in L\mathcal{K}$ such that*

1. $Z = \min LP^-$;

2. P is an admissible object of the category $\mathcal{K}(\Phi)$, where $\Phi = \{Z\} \cup Z^+$;

3. If $P \in S\mathcal{K}_{Z,V}$, then $S\mathcal{K}^{Z,V}$ does not contain the subset $(1,1)$ and $\mathcal{K}(X, Z)(U) \ne V$ for any $X \in L\mathcal{K}, X \ne Z$ and any $U \in \mathbf{P}^1(X)$.

Assume that $A \in P^+ \cup \{P\}$, $B \in Z^- \setminus S\mathcal{K}^{Z,V}$ in case $P \in S\mathcal{K}_{Z,V}$ for some $V \in \mathbf{P}^1(Z)$ and $B \in Z^-$ otherwise.

If $A \subseteq Y$, then $B \not\subseteq Y$ for any $(Y, T, \varphi) \in \operatorname{ind} \mathcal{V}(\mathcal{K})$.

Proof. Set $F = S\mathcal{K}^{Z,Z_P}$ in case $P \in S\mathcal{K}_Z$ and $F = \emptyset$ otherwise. Denote $Z^- \setminus F$ by G. Fix an arbitrary P-basis $\mathcal{B}$, a decomposition $Y \cong Y_1^{d_1} \oplus \cdots \oplus Y_n^{d_n}$, where $Y_i \in \operatorname{ind} \mathcal{K}$, and a linear isomorphism $T \cong k^{d_0}$. By using the elementary transformations of rows, the matrix $M\varphi$ can be transformed to the following form

$$M\varphi = \begin{array}{c} \\ T_1 \\ T_2 \end{array} \begin{array}{c} \begin{array}{cccccc} M\varphi_G & M\varphi_F & M\varphi_{z^1} & M\varphi_{z^2} & M\varphi_{\mathcal{B}_0^{P,+}} & M\varphi_{\mathcal{B}_0^{P,-}} \end{array} \\ \begin{pmatrix} M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \\ N_1 & N_2 & N_3 & N_4 & 0 & N_6 \end{pmatrix} \end{array},$$

where the rows of the matrix M_5 are linearly independent. We get the decomposition $T = T_1 \oplus T_2$. Adding columns of the matrix $M\varphi_{\mathcal{B}_0^{P,+}}$ to the matrices $M\varphi_{z^1}$ and $M\varphi_G$, we can turn the blocks M_1 and M_3 into zero. Consider the object (Y, T_2, ψ), where

$$M\psi = T_2 \begin{array}{c} \\ \end{array}\begin{array}{cccccc} M\varphi_G & M\varphi_F & M\varphi_{z^1} & M\varphi_{z^2} & M\varphi_{\mathcal{B}_0^{P,+}} & M\varphi_{\mathcal{B}_0^{P,-}} \\ \end{array}$$
$$\begin{pmatrix} N_1 & N_2 & N_3 & N_4 & 0 & N_6 \end{pmatrix}.$$

Consider the decomposition $Y = H \oplus S \oplus Q$ such that $H_i \in G$, $S_j \in F$, $Q_k \notin G \cup F$ for any indecomposable $H_i \subseteq H$, $S_j \subseteq S$, $Q_k \subseteq Q$. Denote by δ (resp., μ) the restriction of ψ to the subspace Q (resp., $S \oplus Q$). Then (Q, T_2, δ) (resp., $(S \oplus Q, T_2, \mu)$) is the object of the category $\mathcal{V}(\mathcal{K}_1)$ (resp., $\mathcal{V}(\mathcal{K}_2)$), where $\mathcal{K}_1 = \mathcal{K}(\Phi)$ (resp., $\mathcal{K}_2 = \mathcal{K}(\Phi \cup F)$). Let $(Q, T_2, \delta) = \oplus_{i=1}^m (Q_i, T_{2i}, \delta_i)$ be a decomposition of (Q, T_2, δ) into the direct sum of indecomposable objects. Then the matrix $M\psi$ can be written in the form:

$$M\psi = \begin{array}{c} \\ 1 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 1 & 2 & 31 & \cdots & 3m & 41 & \cdots & 4m & 5 & 61 & \cdots & 6m \end{array} \\ \begin{pmatrix} N_{11} & N_{21} & N_{31} & 0 & 0 & N_{41} & 0 & 0 & 0 & N_{61} & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ N_{1m} & N_{2m} & 0 & 0 & N_{3m} & 0 & 0 & N_{4m} & 0 & 0 & 0 & N_{6m} \end{pmatrix} \end{array}.$$

From (2.5) it follows that $(Q_i, T_{2i}, \delta_i) \in \cup_{j=1}^7 R'_j$, where R'_j are as in (2.5). It is easily seen that if $C \in Z^+, C \subseteq Q_i$ for a certain $1 \le i \le m$, then the matrix N_{2i} can be turned into zero matrix by adding the columns of the matrices N_{3i}, N_{4i} and N_{6i}. Let $(S \oplus Q, T_2, \mu) = \oplus_{i=1}^l (S_i \oplus Q_i, T_{2i}, \mu_i)$ be a decomposition of $(S \oplus Q, T_2, \mu)$ into the direct sum of indecomposable objects.

Then the matrix $M\psi$ can be written in the form:

$$\begin{array}{c} \\ 1 \\ \vdots \\ l \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} 1 & 21 & \cdots & 2l & 31 & \cdots & 3l & 41 & \cdots & 4l & 5 & 61 & \cdots & 6l \end{array} \\ \begin{pmatrix} N_{11} & N_{21} & 0 & 0 & N_{31} & 0 & 0 & N_{41} & 0 & 0 & 0 & N_{61} & 0 & 0 \\ \vdots & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 \\ N_{1l} & 0 & 0 & N_{2l} & 0 & 0 & N_{3l} & 0 & 0 & N_{4l} & 0 & 0 & 0 & N_{6l} \end{pmatrix} \end{array}.$$

From the above information it follows that if $C \in Z^+$, $C \subseteq S_i \oplus Q_i$, $D \in F$, then $D \nsubseteq S_i \oplus Q_i$. Hence $(S_i \oplus Q_i, T_{2i}, \mu_i) \in R' = \cup_{j=1}^8 R'_j$, where $R'_8 = \{\mathcal{K}_{Z,E,s,0}, \mathcal{K}_{Z,E,s}, \mathcal{K}_{E,0}, \mathcal{K}_E \mid E \in F, s \in \mathbf{N}\}$. Set $R1 = \{\mathcal{K}_0, \mathcal{K}_{A_1}, \mathcal{K}_{Z,B_1,s}, \mathcal{K}_{Z,s,+}, \mathcal{K}_{Z,s,0,+} \mid (P, A_1) \subseteq S\mathcal{K}, (B_1, P) \subseteq S\mathcal{K}_Z \cup S\mathcal{K}^Z, s \in \mathbf{N}\}$, $R2 = \{\mathcal{K}_{Z,s,\lambda} \mid \lambda \in k, \lambda \neq 0, s \in \mathbf{N}\}$, $R = R' \setminus R1$. It is easily seen that if $(S_i \oplus Q_i, T_{2i}, \mu_i) \in R$, then the matrix N_{1i} can be turned into zero matrix by adding the columns of the matrices N_{2i}, N_{3i}, N_{4i} and N_{6i}. Therefore if $(S_i \oplus Q_i, T_{2i}, \mu_i) \in R$, then $(S_i \oplus Q_i, T_{2i}, \mu_i)$ can be distinguished by a direct summand in (Y, T_2, ψ).

If $(S_i \oplus Q_i, T_{2i}, \mu_i) \in R1 \cup R2$, then the matrices M_{6i}, M_{4i} and M_{2i} can be turned into zero matrix by adding the rows of the horizontal band i.

Let $(Y, T_2, \psi) = \oplus_{i=1}^t (Y_i, T_{2i}, \psi_i)$ be a decomposition of (Y, T_2, ψ) into the direct sum of indecomposable objects. From the above information it follows that if $(Y_i, T_{2i}, \psi_i) \notin R$ and $(Y_i, T_{2i}, \psi_i) \neq (Y, T, \varphi)$, then (Y_i, T_{2i}, ψ_i) can be distinguished by a direct summand in (Y, T, φ), which contradicts indecomposability of (Y, T, φ). Since $A \subseteq Y$, we obtain $(Y, T, \varphi) \ncong (Y, T_2, \psi)$. Therefore $(Y_i, T_{2i}, \psi_i) \in R$ for all i and hence $B \nsubseteq Y$. ∎

3.3

We claim that if $\mathcal{K}$ is a sincere vector space category satisfying all the conditions of Theorem 1 and $\mathcal{K} \ncong \mathcal{A}$, then the differentiation algorithm presented in (2.4) can be applied to $\mathcal{K}[P]$, where $\mathcal{K}[P] = \mathcal{K}$ or $\mathcal{K}[P] = \mathcal{K}^{op}$ or $\mathcal{K}[P](\operatorname{ind}\mathcal{K}[P] \setminus \{P\}) \cong \mathcal{K}$, or $\mathcal{K}[P](\operatorname{ind}\mathcal{K}[P] \setminus \{P\}) \cong \mathcal{K}^{op}$, $P \in S\mathcal{K}[P]$, P is an admissible object of $\mathcal{K}[P]$ and $\mathcal{K}[P]$ also satisfies all the conditions of Theorem 1.

An object $Z \in L\mathcal{K}$ will be called *singular* if there exist $X_1, \ldots, X_n \in L\mathcal{K}$, $V_i \in \mathbf{P}^1(X_i)$, $1 \leq i \leq n$, $n \in \mathbf{N}$, such that $X_n = Z$, $\mathcal{K}(X_i, X_{i+1})(V_i) = V_{i+1}$, for $1 \leq i < n$, $X_1^- = S\mathcal{K}^{X_1,V_1}$, ord max $S\mathcal{K}^{X_1,V_1} = 2$ and $L\mathcal{K} \cap (Z^- \cup \{Z\}) = \{X_1, \ldots, X_n\}$.

Denote by $Sg\mathcal{K}$ the set of singular objects. For given $X, Y \in L\mathcal{K}$ such that $X < Y$ we set $S\mathcal{K}[X,Y] = \{A \in S\mathcal{K} \mid X < A < Y\}$.

One of the following situations may take place:

a). $Sg\mathcal{K} = \emptyset$. If $L\mathcal{K} = \emptyset$, then any object $P \in \max S\mathcal{K}$ is admissible. In this case we set $\mathcal{K}[P] := \mathcal{K}$. Put $Z = \max L\mathcal{K}$. It is clear that if $Z^- \neq \emptyset$, then there exists an admissible object $P \in \max S\mathcal{K}$, because $Z \notin Sg\mathcal{K}$. Then we set $\mathcal{K}[P] := \mathcal{K}$. Suppose $Z^- = \emptyset$.

aa). $L\mathcal{K} = \{Z\}$. Since $\mathcal{K} \ncong \mathcal{A}$, we get $Z^+ \neq \emptyset$.

Assume that there exists $V \in \mathbf{P}^1(Z)$ such that $\operatorname{ord} S\mathcal{K}_{Z,V} = 2$. If ord max$S\mathcal{K}_{Z,V} = 1$ and $P = \max S\mathcal{K}_{Z,V}$, then P is admissible. In this case we set $\mathcal{K}[P] := \mathcal{K}$. If ord max$S\mathcal{K}_{Z,V} = 2$ and $\max S\mathcal{K}_{Z,V} = \{P_1, P_2\}$, then the category $\mathcal{K}$ can be supplemented with a new one-dimensional indecomposable object $P := k$ such that $\mathcal{K}(P,P) := k$, $\mathcal{K}(P,P_1) := k$, $\mathcal{K}(P,P_2) := k$, $\mathcal{K}(P,X) = \mathcal{K}(P_1,X)\mathcal{K}(P,P_1) + \mathcal{K}(P_2,X)\mathcal{K}(P,P_2)$, $\mathcal{K}(Y,P) := \{\psi_F \mid F \in \operatorname{Mat}_{\mathcal{B}}(Y,P_1) \cap \operatorname{Mat}_{\mathcal{B}}(Y,P_2)\}$ for any $Y \in \operatorname{ind}\mathcal{K}$, where $\mathcal{B}$ is an arbitrary basis in $\mathcal{K}$. Let $\mathcal{K}[P]$ denote the additive closure of the obtained category. It is clear that $\mathcal{K}[P]$ satisfies all the conditions of Theorem 1 and P is admissible.

If $\operatorname{ord} S\mathcal{K}_{Z,V} \leq 1$ for any $V \in \mathbf{P}^1(Z)$, then any object $P \in \max S\mathcal{K}$ is admissible. Then we set $\mathcal{K}[P] := \mathcal{K}$.

ab). $L\mathcal{K} \neq \{Z\}$. Set $X = \max(L\mathcal{K} \setminus \{Z\})$. We distinguish two cases.

Case 1. $\mathcal{K}(\{X,Z\}) \cong \mathcal{I}$. Then there exist $U \in \mathbf{P}^1(Z)$ and $V \in \mathbf{P}^1(X)$ such that $\mathcal{K}(Z,X)(U) = V$.

If ord max$S\mathcal{K}_{Z,U} \cap S\mathcal{K}[X,Z] = 1$ and $P = \max S\mathcal{K}_{Z,U}$, then P is admissible because $Sg\mathcal{K} = \emptyset$. Then we set $\mathcal{K}[P] := \mathcal{K}$.

If ord max$S\mathcal{K}_{Z,U} = 2$ and $\max S\mathcal{K}_{Z,U} = \{P_1, P_2\} \subseteq S\mathcal{K}[X,Z]$, then the category $\mathcal{K}$ can be supplemented with a new one-dimensional indecomposable object $P := k$ such that $\mathcal{K}(P,P) := k$, $\mathcal{K}(P,P_1) := k$, $\mathcal{K}(P,P_2) := k$, $\mathcal{K}(P,Y) := \mathcal{K}(P_1,Y)\mathcal{K}(P,P_1) + \mathcal{K}(P_2,Y)\mathcal{K}(P,P_2)$, $\mathcal{K}(Y,P) := \{\psi_F \mid F \in \operatorname{Mat}_{\mathcal{B}}(Y,P_1) \cap \operatorname{Mat}_{\mathcal{B}}(Y,P_2)\}$ for any $Y \in \operatorname{ind}\mathcal{K}$, where $\mathcal{B}$ is an arbitrary basis in $\mathcal{K}$. Let $\mathcal{K}[P]$ denote the additive closure of the obtained category. From Lemma 11 it follows that $\mathcal{K}[P]$ satisfies all the conditions of Theorem 1. Since $Sg\mathcal{K} = \emptyset$, P is admissible.

Assume that $S\mathcal{K}_{Z,U} \cap S\mathcal{K}[X,Z] = \emptyset$. Then the category $\mathcal{K}$ can be supplemented with a new one-dimensional indecomposable object $P := k$ such that $\mathcal{K}(P,P) := k$, $P \in S\mathcal{K}_{Z,U}$, $\dim_k \mathcal{K}(P,X) = 2$. Let $\mathcal{K}[P]$ denote the additive closure of the obtained category. From Lemma 11 it follows that $\mathcal{K}[P]$ satisfies all the conditions of Theorem 1. Since $Sg\mathcal{K} = \emptyset$, P is admissible.

Case 2. $\mathcal{K}(\{X,Z\}) \ncong \mathcal{I}$. From Lemma 6 it follows that in this case $S\mathcal{K}_{Z,V} \cap S\mathcal{K}[X,Z] = S\mathcal{K}_{Z,V}$, $S\mathcal{K}^{X,U} \cap S\mathcal{K}[X,Z] = S\mathcal{K}^{X,U}$ for any $V \in \mathbf{P}^1(Z)$, $U \in \mathbf{P}^1(X)$.

Assume that there exists $V \in \mathbf{P}^1(Z)$ such that $\operatorname{ord} S\mathcal{K}_{Z,V} = 2$. If $\operatorname{ord} \max S\mathcal{K}_{Z,V} = 1$ and $P = \max S\mathcal{K}_{Z,V}$, then P is admissible. Then we set $\mathcal{K}[P] := \mathcal{K}$. If $\operatorname{ord} \max S\mathcal{K}_{Z,V} = 2$ and $\max S\mathcal{K}_{Z,V} = \{P_1, P_2\}$, then the category $\mathcal{K}$ can be supplemented with a new one-dimensional indecomposable object $P := k$ such that $\mathcal{K}(P,P) := k$, $\mathcal{K}(P,P_1) := k$, $\mathcal{K}(P,P_2) := k$, $\mathcal{K}(P,Y) = \mathcal{K}(P_1,Y)\mathcal{K}(P,P_1) + \mathcal{K}(P_2,Y)\mathcal{K}(P,P_2)$, $\mathcal{K}(Y,P) := \{\psi_F \mid F \in \operatorname{Mat}_{\mathcal{B}}(Y,P_1) \cap \operatorname{Mat}_{\mathcal{B}}(Y,P_2)\}$ for any $Y \in \operatorname{ind}\mathcal{K}$, where $\mathcal{B}$ is an arbitrary basis in $\mathcal{K}$. Let $\mathcal{K}[P]$ denote the additive closure of the obtained category. It is clear that $\mathcal{K}[P]$ satisfies all the conditions of Theorem 1 and P is admissible.

Therefore we may assume that $\operatorname{ord} S\mathcal{K}_{Z,V} \leq 1$ for any $V \in \mathbf{P}^1(Z)$. Suppose also that $S\mathcal{K}[X,Z] \neq \emptyset$.

If $w(S\mathcal{K}^{X,V}) \leq 1$ for any $V \in \mathbf{P}^1(X)$, then any $P \in \max S\mathcal{K}[X,Z]$ is admissible. In this case we set $\mathcal{K}[P] := \mathcal{K}$. Assume that there exists $V \in \mathbf{P}^1(X)$ such that $w(S\mathcal{K}^{X,V}) = 2$. We may assume that $S\mathcal{K}[X,Z] = S\mathcal{K}^{X,V}$. Indeed, if $S\mathcal{K}[X,Z] \neq S\mathcal{K}^{X,V}$, then any $P \in \max(S\mathcal{K}[X,Z] \setminus S\mathcal{K}^{X,V})$ is admissible. Then we set $\mathcal{K}[P] := \mathcal{K}$. It is clear that there exists $\varphi \in \mathcal{K}(Z,X)$ such that $\operatorname{Im}\varphi \neq V$, $\operatorname{Im}\varphi = W \in \mathbf{P}^1(X)$, $\operatorname{Ker}\varphi = U \in \mathbf{P}^1(Z)$ and $S\mathcal{K}_{Z,U} \cap S\mathcal{K}^{X,V} = \emptyset$. Then the category $\mathcal{K}$ can be supplemented with a new one-dimensional indecomposable object $P := k$ such that $\mathcal{K}(P,P) := k$, $P \in S\mathcal{K}_{Z,U} \cap S\mathcal{K}^{X,W}$. Let $\mathcal{K}[P]$ denote the additive closure of the obtained category. It is clear that $\mathcal{K}[P]$ satisfies all the conditions of Theorem 1 and P is admissible.

Suppose $S\mathcal{K}[X,Z] = \emptyset$. Since $\mathcal{K}$ satisfies all the conditions of Theorem 1, we have $\dim_k \mathcal{K}(Z,X) \geq 3$ and hence $\mathcal{K}(\{X,Z\})$ is isomorphic to one of the categories $\mathcal{J}, \mathcal{L}$. Hence it follows that there exists a basis $\mathcal{B}$ of the category $\mathcal{K}$ and a basis morphism $b_\varphi \in \mathcal{K}(Z,X)$ such that $Mb_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Therefore, the category $\mathcal{K}$ can be supplemented with a new one-dimensional indecomposable object $P := k$ such that $\mathcal{K}(P,P) := k$, $\mathcal{K}(Z,P) := k\alpha$, $\mathcal{K}(P,X) := k\beta$, $\mathcal{K}(P,Y) := \mathcal{K}(X,Y)\mathcal{K}(P,X)$ for all $Y \in \operatorname{ind}\mathcal{K}, Y \neq X, Y \neq Z$, $\mathcal{K}(Y,P) := 0$ for all $Y \in \operatorname{ind}\mathcal{K}$, $Y \neq Z$, where $M\alpha = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $M\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Let $\mathcal{K}[P]$ denote the additive closure of the obtained category. Evidently, $\mathcal{K}[P]$ satisfies all the conditions of Theorem 1 and we can apply the differentiation algorithm with respect to the point P to this category.

b). $Sg\mathcal{K} \neq \emptyset$. Put $Z = \min Sg\mathcal{K}$. We distinguish several cases.

ba). $L\mathcal{K} = Sg\mathcal{K} = \{Z\}$. Then there exists $V \in \mathbf{P}^1(Z)$ such that $Z^- = S\mathcal{K}^{Z,V}$, $w(S\mathcal{K}^{Z,V}) = 2$. If $S\mathcal{K}_{Z,V} \neq \emptyset$, then $w(S\mathcal{K}_{Z,V}) = 1$, because $\mathcal{K}$ satisfies the condition (g) of Theorem 1. If $\min S\mathcal{K}_{Z,V} \not\subseteq \min S\mathcal{K}$, then there exists $P \in \min S\mathcal{K}$ such that $P \notin S\mathcal{K}_Z$. Therefore P is an admissible object of the category $\mathcal{K}[P] := \mathcal{K}^{op}$. If $P = \min S\mathcal{K}_{Z,V} \in \min S\mathcal{K}$, then P is an admissible object of the category $\mathcal{K}[P] := \mathcal{K}^{op}$. Therefore we may assume that $S\mathcal{K}_{Z,V} = \emptyset$.

Suppose that $Z^+ \neq \emptyset$. Set $\mathcal{K}' = \mathcal{K}(Z^+ \cup \{Z\})$. If $\mathcal{K}'$ contains an admissible object P, then $\mathcal{K}$, P and Z satisfy the conditions of Lemma 28. Hence $\mathcal{K}$ is not sincere, a contradiction. If $\mathcal{K}'$ does not contain an admissible object, then similarly to the case **aa)** the category $\mathcal{K}$ can be supplemented with a new one-dimensional object P, which is admissible of the category $\mathcal{K}_1 := \mathcal{K}(Z^+ \cup \{Z\})[P]$. Since $\mathcal{K}_1$, P and Z satisfy the conditions of Lemma 28 and $Z^+ \neq \emptyset$, then $\mathcal{K}$ is not sincere, a contradiction. Therefore we may assume that $Z^+ = \emptyset$.

If $\operatorname{ord} \min S\mathcal{K}^{Z,V} = 1$ and $\min S\mathcal{K}^{Z,V} = P$, then P is an admissible object of the category $\mathcal{K}[P] := \mathcal{K}^{op}$.

If $\min S\mathcal{K}^{Z,V} = \{P_1, P_2\}$, then the category $\mathcal{K}$ can be supplemented with a new one-dimensional indecomposable object P such that $\mathcal{K}(P,P) := k$, $P \in S\mathcal{K}^{Z,V}$, $P < P_1$, $P < P_2$. Let $\mathcal{K}_1$ denote the additive closure of the obtained category. It is clear that $\mathcal{K}_1^{op}$ satisfies all the conditions of Theorem 1 and P is an admissible object of the category $\mathcal{K}[P] := \mathcal{K}_1^{op}$.

bb). $L\mathcal{K} = Sg\mathcal{K} \neq \{Z\}$. Then there exist $X_1, \ldots, X_n \in L\mathcal{K}$, $V_i \in \mathbf{P}^1(X_i)$, $1 \leq i \leq n$, $n \in \mathbf{N}$,$n > 1$, such that $X_n = Z$, $\mathcal{K}(X_i, X_{i+1})(V_i) = V_{i+1}$, for $1 \leq i < n$, $X_1^- = S\mathcal{K}^{X_1,V_1}$, $\operatorname{ord}\max S\mathcal{K}^{X_1,V_1} = 2$ and $L\mathcal{K} = \{X_1, \ldots, X_n\}$. Analogously to the case **ba)** we may assume that $Z^+ = \emptyset$.

If $\operatorname{ord}\min S\mathcal{K}^{Z,V_n} \cap S\mathcal{K}[Z, X_{n-1}] = 1$ and $P = \min S\mathcal{K}^{Z,V_n}$, then P is an admissible object of the category $\mathcal{K}[P] := \mathcal{K}^{op}$.

If $\operatorname{ord}\min S\mathcal{K}^{Z,V_n} = 2$ and $\min S\mathcal{K}^{Z,V_n} = \{P_1, P_2\} \subseteq S\mathcal{K}[Z, X_{n-1}]$, then the category $\mathcal{K}$ can be supplemented with a new one-dimensional indecomposable object $P := k$ such that $\mathcal{K}(P,P) := k$, $P \in S\mathcal{K}^{Z,V_n}$, $P < P_1$, $P < P_2$, $\dim_k \mathcal{K}(X, P) = 2$. Let $\mathcal{K}_1$ denote the additive closure of the obtained category. From Lemma 11 it follows that $\mathcal{K}_1$ satisfies all the conditions of Theorem 1. It is clear that P is an admissible object of the category $\mathcal{K}[P] := \mathcal{K}_1^{op}$.

Assume that $S\mathcal{K}^{Z,V_n} \cap S\mathcal{K}[Z, X_{n-1}] = \emptyset$. Then the category $\mathcal{K}$ can be supplemented with a new one-dimensional indecomposable object $P := k$ such that $\mathcal{K}(P,P) := k$, $P \in S\mathcal{K}^{Z,V_n}$, $\dim_k \mathcal{K}(X_{n-1}, P) = 2$. Let $\mathcal{K}_1$ denote the additive closure of the obtained category. From Lemma 11 it follows that $\mathcal{K}_1$ satisfies all the conditions of Theorem 1. It is clear that P is an admissible object of the category $\mathcal{K}[P] := \mathcal{K}_1^{op}$.

bc). $L\mathcal{K} \neq Sg\mathcal{K}$. Then there exist $X_1, \ldots, X_n \in L\mathcal{K}$, $V_i \in \mathbf{P}^1(X_i)$, $1 \leq i \leq n$, $n \in \mathbf{N}$, such that $X_n = Z$, $\mathcal{K}(X_i, X_{i+1})(V_i) = V_{i+1}$, for $1 \leq i < n$, $X_1^- = S\mathcal{K}^{X_1,V_1}$, $\operatorname{ord}\max S\mathcal{K}^{X_1,V_1} = 2$ and $L\mathcal{K} \cap (Z^- \cup \{Z\}) = \{X_1, \ldots, X_n\}$. Since $L\mathcal{K} \neq Sg\mathcal{K}$, we have $Z^+ \neq S\mathcal{K}_{Z,V_n}$. Hence, analogously to the case **ba)**, we obtain that $\mathcal{K}$ is not sincere, a contradiction.

Remark. It is easily seen that if $\mathcal{K}$ is sincere, $\mathcal{K}$ does not contain the admissible objects, $\mathcal{K}[P]$ is one of the above categories, P is an admissible object of $\mathcal{K}[P]$, then $\max \operatorname{ind} \mathcal{K}[P] \subseteq L\mathcal{K}[P]$. Hence $d(\mathbf{d}_P(X, V, \varphi)) < d(X, V, \varphi)$ for any $(X, V, \varphi) \in \operatorname{ind} \mathcal{V}(\mathcal{K}[P])$ such that $Y \subseteq X$ for all $Y \in \operatorname{ind} \mathcal{K}[P]$, $Y \neq P$.

3.4

Suppose $\mathcal{K}$ satisfies all the conditions of Theorem 1. Let P be an admissible object and let $P \in \max \operatorname{ind} \mathcal{K}$. We will show that $\partial_P \mathcal{K}$ also satisfies all the conditions of Theorem 1.

A. From the definition of the category $\partial_P \mathcal{K}$ it immediately follows that the category $\partial_P \mathcal{K}$ satisfies the condition (a) of Theorem 1.

B. We will be proving that the category $\partial_P \mathcal{K}$ satisfies the condition (b) of Theorem 1. Suppose, contrary to our claim, that $\dim_k \partial_P \mathcal{K}(X, Y) + \dim_k \partial_P \mathcal{K}(Y, X) = 0$ for certain $X \in L\partial_P \mathcal{K}, Y \in S\partial_P \mathcal{K}$. If $X \in NL\partial_P \mathcal{K}$, then Lemma 25 implies that $Y = S_D$ for certain $D \in L\mathcal{K}$, which contradicts Lemma 26. Therefore we may assume that $X \in OL\partial_P \mathcal{K}$. Then we have $Y \in N\partial_P \mathcal{K}$. From Lemma 26 we conclude that $Y \neq S_D$ for any $D \in L\mathcal{K}$. Hence $Y = S_{\{A,B\}}$ or $Y = S_{\{A,B,C\}}$ for certain $A, B, C \in S\mathcal{K}$. Lemma 27 now implies that either $\psi(Y) > X$ or $\psi(Y) < X$. Suppose $\psi(Y) > X$. Then we have $\mathcal{K}(A, X) \neq 0$. Since $\mathcal{K}(Y, A) \neq 0$ and $A \in S\mathcal{K}$, we get $\mathcal{K}(Y, X) \neq 0$, which contradicts the assumption.

If $\psi(Y) < X$, then Lemma 27 shows that $Y = S_{\{A,B\}}$ and $A, B \in S\mathcal{K}_{X,V}$ for some $V \in \mathbf{P}^1(X)$. But the definition of $\partial_P\mathcal{K}$ then gives $Y \in S\partial_P\mathcal{K}_{X,V}$, a contradiction.

C. Lemmas 25,24 imply that the category $\partial_P\mathcal{K}$ satisfies the condition (c) of Theorem 1.

D. We will be proving that the category $\partial_P\mathcal{K}$ satisfies the condition (d) of Theorem 1. On the contrary, suppose that there exist $X, Y \in L\partial_P\mathcal{K}, A \in S\partial_P\mathcal{K}$ and $V \in \mathbf{P}^1(X), U \in \mathbf{P}^1(Y)$ such that $\partial_P\mathcal{K}(X,Y)(V) = U$ and $A \in S\partial_P\mathcal{K}_{X,V} \cap S\partial_P\mathcal{K}^{Y,U}$. Consider all possible cases.

(a) Suppose that $A = S_Z$ for a certain $Z \in L\mathcal{K}$. We set $U' = \partial_P\mathcal{K}(X,Z)(V)$. Since $\partial_P\mathcal{K}(Z,Y)\partial_P\mathcal{K}(X,Z) \subseteq \partial_P\mathcal{K}(X,Y)$ and $\dim_k\partial_P\mathcal{K}(Z,Y) \geq 3$, we get $U' \in \mathbf{P}^1(Z)$. And since $A \in S\partial_P\mathcal{K}^{Y,U}$, we obtain $\partial_P\mathcal{K}(Z,Y)(Z^A) = U$. Lemma 10 now yields $Z^A = U'$ and hence $A \in S\partial_P\mathcal{K}^{Z,U} \cap S\partial_P\mathcal{K}_{X,V}$. We may assume therefore that $Z = Y$. Lemma 26 shows that $X \in NL\mathcal{K}$, i. e. $X = L_{\{X_1,X_2,X_3\}}$ for certain $X_1, X_2, X_3 \in S\mathcal{K}$. Since $A \in S\partial_P\mathcal{K}_{X,V}$ and $\partial_P\mathcal{K}(\{X,Y\}) \cong \mathcal{I}$, proof of Lemma 23 shows that there exists $k \in \{1,2,3\}$ such that $\dim_k\mathcal{K}(X_k,Y) = 2, X_t \in S\mathcal{K}^Y$ for $t \neq k$ and $Z^{X_t} \neq Z^{X_s}$ for $t \neq k, s \neq k, t, s \in \{1,2,3\}$. It can be easily seen that $V = X_{X_k}, U = Y^{X_t}$, where $t \neq k$. But then we have $A \in S\partial_P\mathcal{K}^{Y,X_t}$, which is impossible because of Lemma 20.

(b) Suppose that $A \neq S_Z$ for any $Z \in L\mathcal{K}$. Lemma 25 then gives $X \in OL\partial_P\mathcal{K}$ or $Y \in OL\partial_P\mathcal{K}$. From Lemma 27 it follows that if $Y \in NL\partial_P\mathcal{K}$, then $X \in NL\partial_P\mathcal{K}$ too. We therefore may assume that $Y \in OL\partial_P\mathcal{K}$. Suppose $X \in OL\partial_P\mathcal{K}$. From Lemma 21 we see that $A \neq S_{\{A_1,A_2,A_3\}}$ for any $A_1, A_2, A_3 \in S\mathcal{K}$. If $A = S_{\{A_1,A_2\}}$ for certain $A_1, A_2 \in S\mathcal{K}$, then Lemmas 18,22 imply $A_1 \in S\mathcal{K}_{X,V} \cap S\mathcal{K}^{Y,U}$ and hence the category $\mathcal{K}$ does not satisfy the condition (d) of Theorem 1, a contradiction. Therefore we may assume that $X \in NL\partial_P\mathcal{K}$, i. e. $X = L_{\{X_1,X_2,X_3\}}$ for certain $X_1, X_2, X_3 \in S\mathcal{K}$. Since $\partial_P\mathcal{K}(\{X,Y\}) \cong \mathcal{I}$, proof of Lemma 23 shows that there exists $k \in \{1,2,3\}$ such that $\dim_k\mathcal{K}(X_k,Y) = 2, X_t \in S\mathcal{K}^Y$ for $t \neq k$ and $Z^{X_t} = Z^{X_s}$ for $t \neq k, s \neq k, t, s \in \{1,2,3\}$. From Lemma 21 we conclude that $A \neq S_{\{A_1,A_2,A_3\}}$ for any $A_1, A_2, A_3 \in S\mathcal{K}$. It is easily seen that $V = X_{X_k}, U = Y^{X_t}$, where $t \neq k$. Let $A \in OS\partial_P\mathcal{K}$. Since $A \in S\mathcal{K}_{X,X_k}$, we get $A \neq X_t$ for $t \neq k$. Since $S\partial_P\mathcal{K}^{Y,U}$ is the semichain and $A < X$, we obtain $A < X_t$ for $t \neq k$. Then we have $\dim_k\partial_P\mathcal{K}(X,A) = 2$, which is impossible. Finally, let $A = S_{\{A_1,A_2\}}$ for certain $A_1, A_2 \in S\mathcal{K}$. Since $S_{\{A_1,A_2\}} < X$, we get $\{A_1, A_2\} \neq \{X_t, X_s\}$, where $t, s \in \{1,2,3\}, t \neq k, s \neq k$. From Lemma 18 we conclude that $A_1, A_2 \in S\mathcal{K}^{Y,U}$. It follows that $\{A_1, A_2\} < \{X_t, X_s\}$, where $t \neq k, s \neq k, t, s \in \{1,2,3\}$. But then we have $A < X_s, A < X_t$ and hence $\dim_k\partial_P\mathcal{K}(X,A) = 2$, which is impossible.

E. We will prove that the category $\partial_P\mathcal{K}$ satisfies the condition (e) of Theorem 1. Suppose, contrary to our claim, that $S\partial_P\mathcal{K}_{X,V}$ or $S\partial_P\mathcal{K}^{X,V}$ contains the subset $(1,1,1)$ or $(1,2)$ for certain $X \in L\partial_P\mathcal{K}$ and $V \in \mathbf{P}^1(X)$. Consider each case separately.

(a) Suppose $\Phi \subseteq S\partial_P\mathcal{K}^{X,V}$ and $\Phi = (1,1,1)$ or $\Phi = (1,2)$. Lemmas 18,20,21 show that if $A \in \Phi$, then $A \neq S_{\{A_1,A_2\}}, A \neq S_Z, A \neq S_{\{A_1,A_2,A_3\}}$ for any $Z \in L\mathcal{K}, A_1, A_2, A_3 \in S\mathcal{K}$. Hence $\Phi \subseteq OS\partial_P\mathcal{K}$. Since $\mathcal{K}$ satisfies the condition (e) of Theorem 1, we have $X \in NL\partial_P\mathcal{K}$, which contradicts Lemma 25.

(b) Suppose $\Phi \subseteq S\partial_P\mathcal{K}_{X,V}$ and $\Phi = (1,1,1)$ or $(1,2)$. If $X \in NL\partial_P\mathcal{K}$, then from Lemma 25 we conclude that $S_Y \in \Phi$ for a certain $Y \in L\mathcal{K}$, which is impossible (see proof of Lemma 23). We therefore may assume that $X \in OL\partial_P\mathcal{K}$. From Lemmas 21,22,23 we see that if $A \in \Phi$, then $A \neq S_{\{A_1,A_2,A_3\}}, A \neq S_{\{A_1,A_2\}}, A \neq S_Z$ for any $Z \in L\mathcal{K}, A_1, A_2, A_3 \in S\mathcal{K}$. Therefore we have $\Phi \subseteq OS\partial_P\mathcal{K}$ and hence $\mathcal{K}$ does not satisfy the condition (e) of Theorem 1, a contradiction.

F. We will prove that the category $\partial_P\mathcal{K}$ satisfies the condition (f) of Theorem 1. Suppose this statement is false and consider the following possible cases.

(a) Let $\operatorname{ord}\Omega\partial_P\mathcal{K}^X = 2$ and let $S\partial_P\mathcal{K}^X$ contains the subset $(1,1,1,1)$ for a certain $X \in L\partial_P\mathcal{K}$. Since $\partial_P\mathcal{K}$ has already been proved to satisfy the condition (e) of Theorem 1, there exist $V_1, V_2 \in \mathbf{P}^1(X)$ such that $V_1 \neq V_2$, $\Phi_i \subseteq S\partial_P\mathcal{K}^{X,V_i}, \Phi_i = (1,1), 1 \leq i \leq 2$. Lemmas 18,20,21 now show that if $A \in \Phi_i$ for a certain i, then $A \neq S_{\{A_1,A_2\}}, A \neq S_Z, A \neq S_{\{A_1,A_2,A_3\}}$ for any $Z \in L\mathcal{K}, A_1, A_2, A_3 \in S\mathcal{K}$. Therefore we have $\Phi_1, \Phi_2 \subseteq OS\partial_P\mathcal{K}$. Since $\mathcal{K}$ satisfies the condition (f) of Theorem 1, we obtain $X \in NL\partial_P\mathcal{K}$, which contradicts Lemma 25.

(b) Let $\operatorname{ord}\Omega\partial_P\mathcal{K}_X = 2$ and let $S\partial_P\mathcal{K}_X$ contains the subset $(1,1,1,1)$ for a certain $X \in L\partial_P\mathcal{K}$. Since we have already proved that $\partial_P\mathcal{K}$ satisfies the condition (e) of Theorem 1, there exist $V_1, V_2 \in \mathbf{P}^1(X)$ such that $V_1 \neq V_2$, $\Phi_i \subseteq S\partial_P\mathcal{K}_{X,V_i}, \Phi_i = (1,1), 1 \leq i \leq 2$. Since $\operatorname{ord}\Omega\partial_P\mathcal{K}_X = 2$, we obtain $X \in OL\partial_P\mathcal{K}$. Lemmas 21,22,23 now show that if $A \in \Phi_i$ for a certain i, then $A \neq S_{\{A_1,A_2,A_3\}}, A \neq S_{\{A_1,A_2\}}, A \neq S_Z$ for any $A_1, A_2, A_3 \in S\mathcal{K}, Z \in L\mathcal{K}$. Therefore we have $\Phi_i \subseteq OS\partial_P\mathcal{K}, 1 \leq i \leq 2$ and hence $\mathcal{K}$ does not satisfy the condition (f) of Theorem 1, a contradiction.

(c) Let $\operatorname{ord}\Omega\partial_P\mathcal{K}^X = 3$ and let $S\partial_P\mathcal{K}^X$ contains the subset $(1,1,1,1)$ for a certain $X \in L\partial_P\mathcal{K}$. Since $\partial_P\mathcal{K}$ has already been proved to satisfy the condition (e) of Theorem 1, there exist pairwise distinct $V_i \in \mathbf{P}^1(X), 1 \leq i \leq 3$ such that $A \in S\partial_P\mathcal{K}^{X,V_1}, B \in S\partial_P\mathcal{K}^{X,V_2}, \Phi = (1,1) \subseteq S\partial_P\mathcal{K}^{X,V_3}$. Similar to the case (a), Lemmas 18,20,21 now imply $\Phi \subseteq OS\partial_P\mathcal{K}$. From Lemma 21 we see that $A \neq S_{\{A_1,A_2,A_3\}}, B \neq S_{\{B_1,B_2,B_3\}}$ for any $A_i, B_j \in S\mathcal{K}$. If $A = S_{\{A_1,A_2\}}$, then Lemma 18 shows that $A_1, A_2 \in S\mathcal{K}^{X,V_1}$ and hence $\mathcal{K}$ does not satisfy the condition (f) of Theorem 1, which contradicts the assumption. Therefore we have $A \neq S_{\{A_1,A_2\}}, B \neq S_{\{B_1,B_2\}}$ for any $A_1, A_2, B_1, B_2 \in S\mathcal{K}$. Suppose $A = S_Y$ for a certain $Y \in L\mathcal{K}$. From Lemma 20 we conclude that $B \neq S_Z$ for any $Z \in L\mathcal{K}$. Therefore we have $B \in OS\partial_P\mathcal{K}$ and hence $S\mathcal{K}^X$ contains the subset $(1,1,1,1) = \Phi \cup \{P, B\}$ and $\operatorname{ord}\Omega\mathcal{K}^X = 3$. Thus $\mathcal{K}$ does not satisfy the condition (f) of Theorem 1, a contradiction. We therefore may assume that $A \neq S_Y, B \neq S_Y$ for any $Y \in L\mathcal{K}$ and hence that $A, B \in OS\partial_P\mathcal{K}$. But Lemma 25 now gives $X \in OL\partial_P\mathcal{K}$ and hence $\mathcal{K}$ does not satisfy the condition (f) of Theorem 1, a contradiction.

(d) Let $\operatorname{ord}\Omega\partial_P\mathcal{K}_X = 3$ and let $S\partial_P\mathcal{K}_X$ contains the subset $(1,1,1,1)$ for a certain $X \in L\partial_P\mathcal{K}$. Since $\partial_P\mathcal{K}$ has been proved to satisfy the condition (e) of Theorem 1, there exist pairwise distinct $V_i \in \mathbf{P}^1(X), 1 \leq i \leq 3$ such that $A \in S\partial_P\mathcal{K}_{X,V_1}, B \in S\partial_P\mathcal{K}_{X,V_2}, \Phi = (1,1) \subseteq S\partial_P\mathcal{K}_{X,V_3}$. Similar to the case (b), Lemmas 21,22,23 now imply $\Phi \subseteq OS\partial_P\mathcal{K}$. Assume $X \in NL\partial_P\mathcal{K}$, i. e. $X = L_{\{X_1,X_2,X_3\}}$ for certain $X_i \in S\mathcal{K}$. Let $\Phi = \{C, D\}$. If $\Phi < P$, then we have $\Phi - X_i$ for any i and hence $S\mathcal{K}$ contains the wild subset $\{X_1, X_2, X_3, C, D\}$, which is impossible by the assumption. If $\Phi - P$, then Lemma 15 yields $\Phi < X_k$ for a certain $k \in \{1,2,3\}$ and $\Phi - X_t$ for $t \neq k$, because $\Phi \subseteq S\mathcal{K}_{X,V_1}$. But then $S\mathcal{K}$ contains the wild subset $\{P, X_t, X_s, C, D\}$, where $t \neq k, s \neq k, s \neq t$, which contradicts the assumption. We therefore may assume that $X \in OL\partial_P\mathcal{K}$. Lemma 23 then gives $A \neq S_Z, B \neq S_Z$ for any $Z \in L\mathcal{K}$. Lemma 21 now gives $A \neq S_{\{A_1,A_2,A_3\}}, B \neq S_{\{B_1,B_2,B_3\}}$ for any $A_i, B_j \in S\mathcal{K}$. If $A = S_{\{A_1,A_2\}}$ for certain $A_1, A_2 \in S\mathcal{K}$, then Lemma 22 shows that $A_1, A_2 \in S\mathcal{K}_{X,V_1}$ and hence $\mathcal{K}$ does not satisfy the condition (f) of Theorem 1, which is impossible. Therefore we may assume that $A, B \in SO\partial_P\mathcal{K}$, which contradicts the assumption that $\mathcal{K}$ satisfies the condition (f) of Theorem 1.

G. We will prove that the category $\partial_P\mathcal{K}$ satisfies the condition (g) of Theorem 1. On the contrary, suppose that $\Phi_1 = (1,1) \subseteq S\mathcal{K}^{X_1,V_1}, \Phi_2 = (1,1) \subseteq S\mathcal{K}_{X_n,V_n}$ for certain

$X_1, \ldots, X_n \in L\partial_P\mathcal{K}, V_i \in \mathbf{P}^1(X_i), 1 \leq i \leq n$, such that $\partial_P\mathcal{K}(X_i, X_{i+1})(V_i) = V_{i+1}$ for $1 \leq i < n$. Lemmas 18,20,21 show that if $A \in \Phi_1$, then $A \neq S_{\{A_1,A_2\}}, A \neq S_Z, A \neq S_{\{A_1,A_2,A_3\}}$ for any $A_i \in S\mathcal{K}, Z \in L\mathcal{K}$. Therefore we have $\Phi_1 \subseteq OS\partial_P\mathcal{K}$. From Lemmas 21,22,23 we see that if $A \in \Phi_2$, then $A \neq S_{\{A_1,A_2,A_3\}}, A \neq S_{\{A_1,A_2\}}, A \neq S_Z$ for any $A_i \in S\mathcal{K}, Z \in L\mathcal{K}$. Let $X_1 \in NL\partial_P\mathcal{K}$, i. e. $X_1 = L_{\{F_1,F_2,F_3\}}$ for certain $F_i \in S\mathcal{K}$, and let $\Phi_1 = \{A, B\}$. Lemma 17 shows that one can find $k \in \{1,2,3\}$ such that $\Phi_1 - F_k$ and $\Phi_1 > F_t$ for $t \neq k$. But then $S\mathcal{K}$ contains the subset $G = \{P, F_k, F_t < A > F_s < B > F_t\}$, where $t \neq k, s \neq k, s \neq t$, which is impossible because, by the assumption, $\mathcal{K}$ satisfies all the conditions of Theorem 1. We therefore get $X_1 \in OL\partial_P\mathcal{K}$. From Lemma 27 we now see that $X_2, \ldots, X_n \in OL\partial_P\mathcal{K}$ and hence $\mathcal{K}$ does not satisfy the condition (g) of Theorem 1, a contradiction.

H. We will prove that $\partial_P\mathcal{K}$ satisfies the condition (h) of Theorem 1. Suppose, contrary to our claim, that $S\partial_P\mathcal{K}$ contains one of the subsets $W_1, \ldots, W_6, G$. Let Ω denote it. From Lemma 13 we conclude that $\Omega > Z$ or $\Omega < Z$ for any $Z \in L\partial_P\mathcal{K}$. Assume that there exists $X \in OL\partial_P\mathcal{K}$ such that $\Omega < X$. Lemma 25 shows that $S_Y \in \Omega$ for a certain $Y \in L\mathcal{K}$. Lemma 13 then implies $\Omega > Y$. Since $S_Y < X$, Lemma 16 gives $\mathcal{K}(\{X,Y\}) \cong \mathcal{I}$ and hence, according to Lemma 11, Ω is the union of two semichains, which is impossible because of the form of the sets $W_1, \ldots, W_6$. From Lemma 20 it follows that $\Omega \neq G$.

We therefore may assume that $\Omega > X$ for any $X \in OL\partial_P\mathcal{K}$. Let $Z = \max OL\partial_P\mathcal{K}$. Lemma 25 then yields $S_Z \in \Omega$. Suppose $S\partial_P\mathcal{K}^{Z,V}$ contains the subset $(1,1)$ for a certain $V \in \mathbf{P}^1(Z)$. Since $\partial_P\mathcal{K}$ has already been proved to satisfy the condition (f) of Theorem 1, we get $\operatorname{ord}\Omega\partial_P\mathcal{K}^Z = 2$. On the other hand, $\partial_P\mathcal{K}$ has been proved to satisfy the condition (e) of Theorem 1. Since $Z^P \in \Omega\mathcal{K}^Z$, Lemma 19 therefore shows that S_Z does not satisfy the condition $(\star)$ in Ω, which is impossible. Hence we may assume that $S\partial_P\mathcal{K}^{Z,V}$ does not contain the subset $(1,1)$ for any $V \in \mathbf{P}^1(Z)$ and $\operatorname{ord}\Omega\partial_P\mathcal{K}^Z \geq 3$. From the definition of $\partial_P\mathcal{K}$ it now follows that $S_Z \in \min\Omega$. Moreover, we have $S_{\{A_1,A_2,A_3\}} \notin \Omega$ for any $A_i \in S\mathcal{K}$. Indeed, if $S_{\{A_1,A_2,A_3\}} \in \Omega$ for certain $A_i \in S\mathcal{K}$, then Lemma 13 gives $\Omega > L_{\{A_1,A_2,A_3\}}$. But then we get $S_Z \notin \Omega$, which contradicts Lemma 25. Let Φ denote the set of the elements Ω which are incomparable with S_Z. Since $S\partial_P\mathcal{K}^{Z,V}$ does not contain the subset $(1,1)$ for any $V \in \mathbf{P}^1(Z)$, we obtain $\Phi \subseteq OS\partial_P\mathcal{K} \cap S\partial_P\mathcal{K}^Z$.

H.1. Let $\Omega = \{A_1 < \cdots < A_l, B_1 < \cdots < B_m, S_Z < C_1 < \cdots < C_n\}$. Since Ω satisfies the condition $(\star)$, we may assume that $l \geq 2, m \geq 1$.

(a) We will show that $\operatorname{ord}\Omega\mathcal{K}^Z = 3$. Conversely, suppose that there exists an element $G \in S\mathcal{K}^{Z,U}$, where $U \neq Z^{A_1}, U \neq Z^{B_1}, U \neq Z^P$. Then $S\mathcal{K}$ contains the wild subset $\{A_1 < A_2, B_1, G, P\}$, which contradicts the assumption.

(b) We will show that $G \in S\mathcal{K}^Z$ if $G \in \psi(\Omega \setminus \{S_Z\})$. Indeed, if it is not the case, then $\dim_k\mathcal{K}(G,Z) = 2$ and hence $G - P, G - A_i, G - B_j, 1 \leq i \leq l, 1 \leq j \leq m$. Then $S\mathcal{K}$ contains the wild subset $\{G, P, A_1 < A_2, B_1)$, which contradicts the assumption.

(c) If $C_i \in O\partial_P\mathcal{K}, 1 \leq i \leq n$, then $S\mathcal{K}$ contains the wild subset $\{A_1 < \cdots < A_l, B_1 < \cdots < B_m, C_1 < \cdots < C_n < P\}$, which contradicts the assumption.

Let $C_i = S_{\{F_i,G_i\}}$ for a certain i and some $F_i, G_i \in S\mathcal{K}$. If $j < i$, then $C_j < S_{\{F_i,G_i\}}$ and we therefore get either $F_i \geq C_j$ or $G_i \geq C_j$. We then have $C_j \notin O\partial_P\mathcal{K}$ because otherwise we would have $G_i \notin S\mathcal{K}^Z$ or $F_i \notin S\mathcal{K}^Z$, which is impossible (see (b)).

Therefore $C_j = S_{\{F_j,G_j\}}$ for certain $F_j, G_j \in S\mathcal{K}^Z$. If $j > i$ and $C_j \in O\partial_P\mathcal{K}$, then Lemma 15 gives $C_j \geq F_i, C_j \geq G_i$ and hence $\dim_k\mathcal{K}(C_j, Z) = 2$, which contradicts what was stated the above (see (b)).

Therefore $C_j = S_{\{F_j,G_j\}}$ for certain $F_j, G_j \in S\mathcal{K}^Z$. We will show that $S\mathcal{K}$ contains the subset $(1, l+1, m+n)$ or the subset $(1, l+n, m+1)$. By virtue of what was stated above, we may assume that $C_i = S_{\{F_i, G_i\}}, F_i \in S\mathcal{K}^{Z, Z^{A_1}}, G_i \in S\mathcal{K}^{Z, Z^{B_1}}$ for any $1 \leq i \leq n$. If $\psi(\{C_1 < \cdots < C_n\})$ contains the subset $\{F_{i_1} < F_{i_2}, G_{j_1} < G_{j_2}\}$, then $S\mathcal{K}$ contains the wild subset $\{P, F_{i_1} < F_{i_2} < A_1 < A_2, G_{j_1} < G_{j_2} < B_1\}$, which contradicts the assumption.

We therefore get either $\psi(\{C_1 < \cdots < C_n\}) = \{F_1 < \cdots < F_n, G_1\}$ or $\psi(\{C_1 < \cdots < C_n\}) = \{G_1 < \cdots < G_n, F_1\}$, and hence $S\mathcal{K}$ contains the subset $(1, l+1, m+n) = \{P, F_1 < A_1 < \cdots < A_l, G_1 < \cdots < G_n < B_1 < \cdots < B_m\}$ or the subset $(1, l+n, m+1) = \{P, F_1 < \cdots < F_n < A_1 < \cdots < A_l, G_1 < B_1 < \cdots < B_m\}$. Taking into account the form of the sets $W_1, \ldots, W_6, G$ we see that if Ω is of the form $\{A_1 < \cdots < A_l, B_1 < \cdots < B_m, S_Z < C_1 < \cdots < C_n\}$, then the following cases are possible.

Case 1. $\Omega = (2,2,3), l = 2, m = 2, n = 2$. Here case (c) shows that $S\mathcal{K}$ contains the subset $(1,3,4)$, which is impossible by the assumption.

Case 2. $\Omega = (2,2,3), l = 3, m = 2, n = 1$. Here case (c) shows that $S\mathcal{K}$ contains the subset $(1,3,4)$, which is impossible by the assumption.

Case 3. $\Omega = (1,3,4), l = 4, m = 1, n = 2$. Here case (c) shows that $S\mathcal{K}$ contains either the subset $(1,2,6)$ or the subset $(1,3,5)$, which is impossible by the assumption.

Case 4. $\Omega = (1,3,4), l = 3, m = 1, n = 3$. Here case (c) shows that $S\mathcal{K}$ contains either the subset $(1,2,6)$ or the subset $(1,4,4)$, which is impossible by the assumption.

Case 5. $\Omega = (1,3,4), l = 4, m = 3, n = 0$. Here case (c) shows that $S\mathcal{K}$ contains either the subset $(1,4,4)$ or the subset $(1,3,5)$, which is impossible by the assumption.

Case 6. $\Omega = (1,2,6), l = 6, m = 1, n = 1$. Here case (c) shows that $S\mathcal{K}$ contains the subset $(1,2,7)$, which is impossible by the assumption.

Case 7. $\Omega = (1,2,6), l = 6, m = 2, n = 0$. Here case (c) shows that $S\mathcal{K}$ contains either the subset $(1,3,6)$ or the subset $(1,2,7)$, which is impossible by the assumption.

Case 8. $\Omega = (1,2,6), l = 2, m = 1, n = 5$. Here case (c) shows that $S\mathcal{K}$ contains either the subset $(1,2,7)$ or the subset $(1,3,6)$, which is impossible by the assumption.

H.2. Let $\Omega = \{F_1 < F_2 > G_1 < G_2, H_1 < \ldots < H_5\}$. Then we have either $F_1 = S_Z$ or $G_1 = S_Z$ for a certain $Z \in L\mathcal{K}$.

Case 1. Let $G_1 = S_Z$. If $G_2 \in N\partial_P\mathcal{K}$, then from case **H.1** it follows that $S\mathcal{K}$ contains either $(1,3,4)$ or $(1,2,6)$, which contradicts the assumption. Therefore, we may assume that $G_2 \in O\partial_P\mathcal{K}$. If $G_2 - P$, then $S\mathcal{K}$ contains the wild subset $(1,1,1,2)$, which is impossible. Therefore $C_2 < P$. If $F_2 \in N\partial_P\mathcal{K}$, then either $F_2 = S_{\{F,G,H\}}$ or $F_2 = S_{\{F,G\}}$. But then we have $F < H_1$ or $G < H_1$ or $H < H_1$. Hence $S\mathcal{K}$ contains the wild subset $(1,2,6)$, which is impossible by the assumption. Finally, if $F_2 \in O\partial_P\mathcal{K}$, then $S\mathcal{K}$ contains the wild subset $\{F_1 < F_2, G_2 < P, H_1 < H_2 < H_3\}$, a contradiction.

Case 2. Let $F_1 = S_Z$. If $F_2 \in O\partial_P\mathcal{K}$, then $F_2 - P$ and $S\mathcal{K}$ contains the wild subset $(1,1,1,2)$, which is impossible. If $F_2 \in N\partial_P\mathcal{K}$, then as in the Case 1, $S\mathcal{K}$ contains the wild subset $(1,2,6)$, which contradicts the assumption.

H.3. Let $\Omega = \{F_1 < F_2 > G_1 < G_2 > F_1, H_1, H_2\}$. Then we have either $F_1 = S_Z$ or $G_1 = S_Z$.

Let $F_1 = S_Z$ (the case $G_1 = S_Z$ can be treated similarly). If $F_2 \in O\partial_P\mathcal{K}$, then $F_2 - P$. Hence $S\mathcal{K}$ contains the wild subset $\{P, F_2 > G_1, H_1, H_2\}$, which is impossible. Let $F_2 \in N\partial_P\mathcal{K}$. If $F_2 = S_{\{F,G\}}$, then $F < H_1$ or $F < H_2$ or $G < H_1$ or $G < H_2$. Hence $S\mathcal{K}$ contains the wild subset $(1,1,1,2)$, which is impossible.

H.4. Let $\Omega = \{ S_Z, B, C, D, E \}$. Then $S\mathcal{K}$ contains the wild subset $\{P, B, C, D, E\}$, which contradicts the assumption.

3.5

Suppose $\mathcal{K}$ satisfies all the conditions of Theorem 1. Let P be an admissible object and let $Z = \max \operatorname{ind} \mathcal{K}$, $Z \in L\mathcal{K}$, $\operatorname{ord} \Omega\mathcal{K}_Z \leq 3$. We will show that $\partial_P\mathcal{K}$ also satisfies all the conditions of Theorem 1.

From the definition of $\partial_P\mathcal{K}$ it follows that $\psi_Z \subseteq SN\partial_P\mathcal{K}$ and $\psi_Z > X$ for any $X \in L\partial_P\mathcal{K}$. Hence from (3.4) we conclude that the conditions (a)-(d) of Theorem 1 are fulfilled for the category $\partial_P\mathcal{K}$.

From the definition of $\partial_P\mathcal{K}$ it follows that if $S_{Z,0} \in S\partial_P\mathcal{K}^X$ for $X \in L\partial_P\mathcal{K}$, then $X \in OL\partial_P\mathcal{K}$ and $\mathcal{K}(\{X,Z\}) \cong \mathcal{I}$. Suppose $\mathcal{K}(\{X,Z\}) \ncong \mathcal{I}$ for any $X \in L\mathcal{K}$. Then $\psi_Z \cap S\partial_P\mathcal{K}^Y = \emptyset$, $\psi_Z \cap S\partial_P\mathcal{K}_Y = \emptyset$ for any $Y \in L\partial_P\mathcal{K}$ and hence from (3.4) it follows that the conditions (e)-(g) of Theorem 1 are fulfilled for the category $\partial_P\mathcal{K}$.

Suppose $\mathcal{K}(\{X,Z\}) \cong \mathcal{I}$ for a certain $X \in L\mathcal{K}$ and $Y \in \psi_Z \cap S\partial_P\mathcal{K}^{X,V}$ for a certain $V \in \mathbf{P}^1(X)$. Then Lemma 11 implies that $S\partial_P\mathcal{K}^{X,V}$ does not contain the subsets $(1,2)$ and $(1,1,1)$. Moreover, if $(1,1) \subseteq S\partial_P\mathcal{K}^{X,V}$, then $(1,1) \subseteq S\mathcal{K}^{X,V}$ for any $X \in L\mathcal{K}$. From the above it follows that the category $\partial_P\mathcal{K}$ satisfies the conditions (e)-(g) of Theorem 1.

We will prove that the category $\partial_P\mathcal{K}$ satisfies the condition (h) of Theorem 1. Suppose, contrary to our claim, that $S\partial_P\mathcal{K}$ contains one of the subsets $W_1, \ldots, W_6, G$. Let Ω denote it. If $\Omega \cap \psi_Z = \emptyset$, then from (3.4) it follows that the category $\partial_P\mathcal{K}$ satisfies the condition (h).

Therefore we may assume that $\Omega \cap \psi_Z \neq \emptyset$. Fix $A \in \Omega$ such that $A \in \psi_Z$. From the definition of $\partial_P\mathcal{K}$ it follows that $A > Y$ for any $Y \in L\partial_P\mathcal{K}$. Then Lemma 13 implies that $\Omega > Y$ for any $Y \in L\partial_P\mathcal{K}$.

Suppose $S_X \in \Omega$ for a certain $X \in L\mathcal{K}$. If $\mathcal{K}(\{X,Z\}) \cong \mathcal{I}$, then from Lemma 11 it follows that S_X does not satisfy the condition $(*)$, which is impossible. Therefore we may assume that $\mathcal{K}(\{X,Z\}) \ncong \mathcal{I}$. Suppose that there exist $C, B \in S\mathcal{K}_Z$ such that $S_{\{C,B\}} \in \Omega$, $C, B \in S\mathcal{K}^{X,V}$ for a certain $V \in \mathbf{P}^1(X)$. We denote by $N(S_X)$ the set of the elements of $S\partial_P\mathcal{K}$, which are incomparable with S_X. From the conditions (h), (f) of Theorem 1 for $\mathcal{K}$ it follows that $\Omega\mathcal{K}^X = \{X^P, V\}$. Hence $N(S_X)$ does not contain the subsets $(1,2)$ and $(1,1,1)$, which is impossible, because any element of Ω satisfies the condition $(*)$.

Therefore we may assume that if $S_{\{C,B\}} \in \Omega$, $C, B \in S\mathcal{K}^X$, then $X^C \neq X^B$. This implies that any element of Ω is comparable with S_X or A, which is impossible. Therefore we may assume that $S_X \notin \Omega$ and $\psi(\Omega) > X$ for any $X \in L\mathcal{K}, X \neq Z$.

Denote by $(s, \mathcal{N}_{n,m})$ the cardinal sum of the sets (s) and $S\mathcal{N}_{n,m}$. Since the sets $S\mathcal{N}_{3,3}$, $(2, \mathcal{N}_{1,3})$, $S\mathcal{N}_{2,5}$, $(1, \mathcal{N}_{1,5})$, $(4, \mathcal{N}_{1,2})$, $(1, \mathcal{N}_{2,2})$ are of polynomial growth, there exists $B \in \Omega$ such that $\dim_k \partial_P\mathcal{K}(Z, B_1) = 2$ for some $B_1 \in \psi(B)$. We fix element B. It is clear that $B < A$, $Z \notin \psi(B)$ and $\Omega \neq (1,1,1,1,1)$. We denote by U the set of all the elements of Ω, which are incomparable with A and B. Since $Z \in \psi(A)$ and $U - \{B < A\}$, we get $Y \in S\mathcal{K}_{Z,Z_P} \setminus \{P\}$ or $Y = S_{\{F,G\}}, F, G \in S\mathcal{K}_Z$ for any $Y \in U$.

3.5.1

Suppose $B = S_{\{B_1,B_2,B_3\}}, \dim_k \mathcal{K}(Z, B_1) = 2$ for some $B_i \in S\mathcal{K}$. We shall show that if $H \in S\mathcal{K}_Z \setminus S\mathcal{K}_{Z,Z_P}, H \notin \{B_1, B_2, B_3\}$, then H is comparable with no less than two elements of $\psi(B)$. Indeed, if $H - \{B_1, B_2, B_3\}$, then $S\mathcal{K}$ contains the wild subset $\{P, B_1, B_2, B_3, H\}$, which is impossible.

If $H - B_i, H - B_j, H < B_k$ (resp., $H > B_k$), $i \neq j, i, j, k \in \{1,2,3\}$, then $S\mathcal{K}$ contains the wild subset $\{P, B_i, B_j, H < B_k\}$ (resp., $\{P, B_i, B_j, B_k < H\}$), which is impossible.

If $H_1, H_2 \in S\mathcal{K}_Z \setminus S\mathcal{K}_{Z,Z_P}$, $H_1, H_2 \notin \{B_1, B_2, B_3\}$, $H_1 - H_2$, $\{H_1, H_2\} - B_i$ for some i, then $S\mathcal{K}$ contains the subset of non-polynomial growth $\{P, B_i, H_1 < B_j > H_2 < B_k > H_1\}$ in the case $\{H_1, H_2\} < \{B_j, B_k\}$ and the subset of non-polynomial growth $\{P, B_i, H_1 > B_j < H_2 > B_k < H_1\}$ in the case $\{H_1, H_2\} > \{B_j, B_k\}$, where $j \neq k, i, j, k \in \{1, 2, 3\}$. A contradiction.

From the above information and Lemma 15 it follows that if $Y = S_{\{F,G\}}$, $F, G \in S\mathcal{K}_Z$, $B = S_{\{B_1,B_2,B_3\}}$, then $B < Y$.

3.5.2

Suppose $\psi(U \cap N\partial_P\mathcal{K})$ contains the subset $U_1 = \{C_1 < C_2, D_1 < D_2 < D_3\}$. We assume also that for any $X \in U_1$ there exists $Y \in U_1$ such that $S_{\{X,Y\}} \in U$. Then one of the following two situations occur:

α). $B \in O\partial_P\mathcal{K}$. Since $B - U$ and $\dim_k\mathcal{K}(Z, B) = 2$, we get $B - U_1$. If $B - P$, then $S\mathcal{K}$ contains the wild subset $\{P, B, C_1, D_1 < D_2\}$, a contradiction. If $B < P$, then $S\mathcal{K}$ contains the wild subset $\{B < P, C_1 < C_2, D_1 < D_2 < D_3\}$, a contradiction.

β). $B \in N\partial_P\mathcal{K}$. Then it follows from (3.5.1) that $B = S_{\{B_1,B_2\}}$, $\dim_k\mathcal{K}(Z, B_1) = 2$ for some $B_1, B_2 \in S\mathcal{K}$. It is clear that $B_1 \ngtr T$ for any $T \in \psi(U)$. We distinguish several cases.

$\beta\alpha$). $B_1 < C_1$. From Lemma 15 it follows that $B_2 - \{C_1 < C_2\}$, $B_2 \nleq D_3$, because $B - U$. If $B_2 - D_3$, then $S\mathcal{K}$ contains the wild subset $\{P, B_2, D_3, C_1 < C_2\}$, a contradiction. If $B_2 > D_3$, then $B_1 - \{D_1 < D_2 < D_3\}$ and hence $S\mathcal{K}$ contains the wild subset $\{P, B_1 < C_1 < C_2, D_1 < D_2 < D_3 < B_2\}$, a contradiction.

$\beta\beta$). $B_1 < D_1$. From Lemma 15 it follows that $B_2 - \{D_1 < D_2 < D_3\}$, $B_2 \nleq C_2$, because $B - U$. If $B_2 - C_2$, then $S\mathcal{K}$ contains the wild subset $\{P, B_2, C_2, D_1 < D_2\}$, a contradiction. If $B_2 > C_2$, then $B_1 - \{C_1 < C_2\}$ and hence $S\mathcal{K}$ contains the wild subset $\{P, B_1 < D_1 < D_2 < D_3, C_1 < C_2 < B_2\}$, a contradiction.

$\beta\gamma$). $B_1 - D_1, B_1 - C_1$. If $B_1 - C_2$ (resp., $B_1 - D_2$), then $S\mathcal{K}$ contains the wild subset $\{P, B_1, D_1, C_1 < C_2\}$ (resp., $\{P, B_1, C_1, D_1 < D_2\}$), a contradiction. Hence we get $C_2 > B_1, D_2 > B_1$. From Lemma 15 it follows that $B_2 - \{C_2, D_2 < D_3\}$, because $B - U$. Therefore $S\mathcal{K}$ contains the wild subset $\{P, B_2, C_2, D_2 < D_3\}$, a contradiction.

3.5.3

Suppose $\psi(U \cap N\partial_P\mathcal{K})$ contains the subset $U_1 = \{C, D_1 < D_2\}$ and there exists $E \in O\partial_P\mathcal{K} \cap U$. We assume also that $S_{\{C,D_i\}} \in U$ for $1 \leq i \leq 2$. Then one of the following two situations occur:

α). $B \in O\partial_P\mathcal{K}$. Since $B - U$ and $\dim_k\mathcal{K}(Z, B) = 2$, we get $B - \{C, D_1 < D_2, E\}$ and hence $S\mathcal{K}$ contains the wild subset $\{B, C, E, D_1 < D_2\}$, a contradiction.

β). $B \in N\partial_P\mathcal{K}$. Then it follows from (3.5.1) that $B = S_{\{B_1,B_2\}}$, $\dim_k\mathcal{K}(Z, B_1) = 2$ for some $B_1, B_2 \in S\mathcal{K}$. It is clear that $B_1 \ngtr T$ for any $T \in \psi(U)$. We distinguish several cases.

$\beta\alpha$). $B_1 < C$. From Lemma 15 it follows that $B_2 - C$, $B_2 \nleq D_2$, because $B - U$. If $B_2 - D_2$, then $S\mathcal{K}$ contains the wild subset $\{B_2, D_2, C, E < P\}$, a contradiction.

If $B_2 > D_2$, then $B_1 - \{D_1 < D_2\}$ and hence $S\mathcal{K}$ contains the wild subset $\{E < P, B_1 < C, D_1 < D_2 < B_2\}$, a contradiction.

$\beta\beta$). $B_1 < D_1$. From Lemma 15 it follows that $B_2 - \{D_1 < D_2\}$, $B_2 \nleq C$, because $B - U$. If $B_2 - C$, then $S\mathcal{K}$ contains the wild subset $\{B_2, C, D_1, E < P\}$, a contradiction.

If $B_2 > C$, then $B_1 - C$ and hence $S\mathcal{K}$ contains the wild subset $\{E < P, B_1 < D_1 < D_2, C < B_2\}$, a contradiction.

$\beta\gamma$). $B_1 - D_1, B_1 - C$. Then $S\mathcal{K}$ contains the wild subset $\{E < P, B_1, C, D_1\}$, a contradiction.

3.5.4

Suppose $\psi(U \cap N\partial_P\mathcal{K})$ contains the subset $U_1 = \{C, D\}$ and $O\partial_P\mathcal{K} \cap U$ contains the subset $\{E_1 < E_2\}$. We assume also that $S_{\{C,D\}} \in U$. Then one of the following two situations occur:

α). $B \in O\partial_P\mathcal{K}$. Since $B - U$ and $\dim_k\mathcal{K}(Z, B) = 2$, we get $B - \{C, D, E_1 < E_2\}$ and hence $S\mathcal{K}$ contains the wild subset $\{B, C, D, E_1 < E_2\}$, a contradiction.

β). $B \in N\partial_P\mathcal{K}$. Then it follows from (3.5.1) that $B = S_{\{B_1,B_2\}}$, $\dim_k\mathcal{K}(Z, B_1) = 2$ for some $B_1, B_2 \in S\mathcal{K}$. It is clear that $B_1 \not\geq T$ for any $T \in \psi(U)$. If $B_1 - C, B_1 - D$, then $S\mathcal{K}$ contains the wild subset $\{B_1, C, D, E_1 < E_2\}$, a contradiction.

Therefore we may assume that $B_1 < C$ (the case $B_1 < D$ can be treated similarly). From Lemma 15 it follows that $B_2 - C$, $B_2 \not\leq D$, because $B - U$. If $B_2 - D$, then $S\mathcal{K}$ contains the wild subset $\{B_2, D, C, E_1 < E_2\}$, a contradiction.

If $B_2 > D$, then $B_1 - D$ and hence $S\mathcal{K}$ contains the wild subset $\{E_1 < E_2 < P, B_1 < C, D < B_2\}$, a contradiction.

3.5.5

Suppose $\Omega = (2, 2, 3)$ or $\Omega = (1, 3, 4)$ or $\Omega = (1, 2, 6)$. Then U contains the subset $\{F < G, H\}$. Since $\mathcal{K}$ satisfies the condition (e) of Theorem 1, we get $H \in N\partial_P\mathcal{K}$ or $\{F < G\} \subseteq N\partial_P\mathcal{K}$. If $\{F < G, H\} \subseteq N\partial_P\mathcal{K}$, then $\psi(U \cap N\partial_P\mathcal{K})$ contains the subset $\{C_1 < C_2, D_1 < D_2 < D_3\}$ for some $C_i, D_j \in S\mathcal{K}$, which is impossible because of (3.5.2).

If $\{F < G\} \subseteq N\partial_P\mathcal{K}, H \in O\partial_P\mathcal{K}$, then $\psi(U \cap N\partial_P\mathcal{K})$ contains the subset $\{C, D_1 < D_2\}$ for some $C, D_i \in S\mathcal{K}$, which is impossible because of (3.5.3).

If $\{F < G\} \subseteq O\partial_P\mathcal{K}, H \in N\partial_P\mathcal{K}$, then $\psi(U \cap N\partial_P\mathcal{K})$ contains the subset $\{C, D\}$ for some $C, D \in S\mathcal{K}$, which is impossible because of (3.5.4).

3.5.6

Suppose $\Omega = \{F, G, H, B < A\}$. Since $\mathcal{K}$ satisfies the condition (e) of Theorem 1, we get $F \in N\partial_P\mathcal{K}$ or $G \in N\partial_P\mathcal{K}$ or $H \in N\partial_P\mathcal{K}$. Therefore we may assume that $F, G \in N\partial_P\mathcal{K}$, because $\mathcal{K}$ satisfies the condition (f) of Theorem 1.

If $\{F, G, H\} \subseteq N\partial_P\mathcal{K}$, then $\psi(U \cap N\partial_P\mathcal{K})$ contains the subset $\{C_1 < C_2 < C_3, D_1 < D_2 < D_3\}$ for some $C_i, D_j \in S\mathcal{K}$, which is impossible because of (3.5.2).

If $\{F, G\} \subseteq N\partial_P\mathcal{K}, H \in O\partial_P\mathcal{K}$, then $\psi(U \cap N\partial_P\mathcal{K})$ contains the subset $\{C_1 < C_2, D_1 < D_2\}$ for some $C_i, D_j \in S\mathcal{K}$, which is impossible because of (3.5.3).

3.5.7

Suppose $\Omega = \{F < G > H < T, L_1 < \ldots < L_5\}$. If $A \neq G$ or $B \neq H$, then U contains the subset $(1, 2)$, which is impossible because of (3.5.5). Therefore we may assume that $A = G$, $B = H$. Since $F - B$, then $Z \notin \psi(F)$. If there exists $F_1 \in \psi(F)$ such that $\dim_k\mathcal{K}(Z, F_1) = 2$, then we can put $B = F$, which is impossible in accordance with the above. Therefore we may assume that $\psi(F) \subseteq S\mathcal{K}_Z$.

Since $T - A$, we have $T \neq S_{Z,0}$. If $T > S_{Z,0}$, then $S_{Z,0} - F$ and hence $S\partial_P\mathcal{K}$ contains the wild subset $\{F, H < S_{Z,0} < T,\ L_1 < \ldots < L_4\}$, which is impossible in accordance with (3.5.5). Hence $Z \notin \psi(T)$.

If $S_{Z,0} - F$ and $S_{Z,0} - T$, then $S\partial_P\mathcal{K}$ contains the wild subset $\{F, T, S_{Z,0}, L_1 < L_2\}$, which is impossible by (3.5.6). If $S_{Z,0} - F$ and $T < S_{Z,0}$, then $S\partial_P\mathcal{K}$ contains the wild subset $\{F, H < T < S_{Z,0}, L_1 < \ldots < L_4\}$, which is impossible by (3.5.5). Therefore we may assume that $F < S_{Z,0}$ and hence $F \in S\mathcal{K}_Z$, $F \notin S\mathcal{K}_{Z,Z_P}$. Since $T - A$, we have $\psi(T) \subseteq S\mathcal{K}_Z$.

One of the following situations occur:

(A). $\{L_1 < \ldots < L_5\} \subset S\mathcal{K}_{Z,Z_P}$. If $T = S_{\{T_1,T_2\}}$, then we may assume that $T_1 < F$, $T_2 - F$ because $T - F$. Hence $S\mathcal{K}$ contains the wild subset $\{T_1 < F,\ T_2,\ L_1 < \ldots < L_5 < P\}$, a contradiction. Thus $T \in S\mathcal{K}_Z$. Since $T - A$, we obtain $T \in S\mathcal{K}_{Z,Z_P}$. Hence $S\mathcal{K}_{Z,Z_P}$ contains the subset $\{T, L_1 < L_2\}$, a contradiction.

(B). $\{L_1 < \ldots < L_5\} \not\subset S\mathcal{K}_{Z,Z_P}$. We distinguish two cases.

(BA). $\{M_1 < M_2, N_1 < N_2\} \subseteq \psi(\{L_1 < \ldots < L_5\})$ for some $M_i, N_j \in S\mathcal{K}_Z$. Since $F - \{L_1 < \ldots < L_5\}$, we may assume that $F > N_2$, $F - \{M_1 < M_2\}$. Then one of the following situations occur:

α). $B \in O\partial_P\mathcal{K}$. Since $B - U$ and $\dim_k\mathcal{K}(Z,B) = 2$, we get $B - U_1$. If $B - P$, then $S\mathcal{K}$ contains the wild subset $\{P, B, M_1, N_1 < N_2\}$, a contradiction. If $B < P$, then $S\mathcal{K}$ contains the wild subset $\{B < P, M_1 < M_2, N_1 < N_2 < F\}$, a contradiction.

β). $B \in N\partial_P\mathcal{K}$. Then it follows from (3.5.1) that $B = S_{\{B_1,B_2\}}$, $\dim_k\mathcal{K}(Z,B_1) = 2$ for some $B_1, B_2 \in S\mathcal{K}$. Hence $B_1 \ngeq X$ for any $X \in S\mathcal{K}_Z$. We distinguish several cases.

$\beta\alpha$). $B_1 < M_1$. From Lemma 15 it follows that $B_2 - \{M_1 < M_2\}$, $B_2 \ngeq F$, $B_2 \nleq N_2$, because $B - U$, $B - F$. If $B_2 - F$ (resp., $B_2 - N_2$), then $S\mathcal{K}$ contains the wild subset $\{P, B_2, F, M_1 < M_2\}$ (resp., $\{P, N_2, B_2, M_1 < M_2\}$), a contradiction. If $N_2 < B_2 < F$, then $B_1 - \{N_1 < N_2 < F\}$ and hence $S\mathcal{K}$ contains the wild subset $\{P, B_1 < M_1 < M_2, N_1 < N_2 < B_2 < F\}$, a contradiction.

$\beta\beta$). $B_1 < N_1$. From Lemma 15 it follows that $B_2 - \{N_1 < N_2 < F\}$, $B_2 \nleq M_2$, because $B - U$, $B - F$. If $B_2 - M_2$, then $S\mathcal{K}$ contains the wild subset $\{P, B_2, M_2, N_1 < N_2\}$, a contradiction. If $B_2 > M_2$, then $B_1 - \{M_1 < M_2\}$ and hence $S\mathcal{K}$ contains the wild subset $\{P, B_1 < N_1 < N_2 < F, M_1 < M_2 < B_2\}$, a contradiction.

$\beta\gamma$). $B_1 - N_1, B_1 - M_1$. If $B_1 - M_2$ (resp., $B_1 - N_2$), then $S\mathcal{K}$ contains the wild subset $\{P, B_1, N_1, M_1 < M_2\}$ (resp., $\{P, B_1, M_1, N_1 < N_2\}$), a contradiction. Hence we get $M_2 > B_1, N_2 > B_1$. From Lemma 15 it follows that $B_2 - \{M_2, N_2 < F\}$, because $B - U$, $B - F$. Therefore $S\mathcal{K}$ contains the wild subset $\{P, B_2, M_2, N_2 < F\}$, a contradiction.

(BB). $\{C, D_1 < \ldots < D_5\} \subseteq \psi(\{L_1 < \ldots < L_5\})$ for some $C, D_i \in S\mathcal{K}_Z$. Since $F - \{L_1 < \ldots < L_5\}$, then one of the following situations occur:

(BBA) $F > C$, $F - \{D1 < \ldots < D_5\}$. We distinguish several cases.

α). $B \in O\partial_P\mathcal{K}$. Since $B - U$ and $\dim_k\mathcal{K}(Z,B) = 2$, we get $B - U_1$. If $B - P$, then $S\mathcal{K}$ contains the wild subset $\{P, B, C, D_1 < D_2\}$, a contradiction. If $B < P$, then $S\mathcal{K}$ contains the wild subset $\{B < P, C < F, D_1 < D_2 < D_3\}$, a contradiction.

β). $B \in N\partial_P\mathcal{K}$. Then it follows from (3.5.1) that $B = S_{\{B_1,B_2\}}$, $\dim_k\mathcal{K}(Z,B_1) = 2$ for some $B_1, B_2 \in S\mathcal{K}$. Hence $B_1 \ngeq X$ for any $X \in S\mathcal{K}_Z$. We distinguish several cases.

$\beta\alpha$). $B_1 < C$. From Lemma 15 it follows that $B_2 - \{C < F\}$, $B_2 \nleq D_5$, because $B - U$, $B - F$. If $B_2 - D_5$, then $S\mathcal{K}$ contains the wild subset $\{P, B_2, D_5, C < F\}$, a contradiction. If $B_2 > D_5$, then $B_1 - \{D_1 < \ldots < D_5\}$ and hence $S\mathcal{K}$ contains the wild subset $\{P, B_1 < C < F, D_1 < D_2 < D_3 < B_2\}$, a contradiction.

$\beta\beta$). $B_1 < D_1$. From Lemma 15 it follows that $B_2 - \{D_1 < \ldots < D_5\}$, $B_2 \nleq C$, $B_2 \ngeq F$, because $B - U$, $B - F$. If $B_2 - F$ (resp., $B_2 - C$), then $S\mathcal{K}$ contains the wild subset $\{P, B_2, F, D_1 < D_2\}$ (resp., $\{P, B_2, C, D_1 < D_2\}$), a contradiction. If $C < B_2 < F$, then $B_1 - \{C < F\}$ and hence $S\mathcal{K}$ contains the wild subset $\{P, B_1 < D_1 < D_2 < D_3, C < B_2 < F\}$, a contradiction.

$\beta\gamma$). $B_1 - D_1, B_1 - C$. If $B_1 - F$ (resp., $B_1 - D_2$), then $S\mathcal{K}$ contains the wild subset $\{P, B_1, D_1, C < F\}$ (resp., $\{P, B_1, C, D_1 < D_2\}$), a contradiction. Hence we get $F > B_1, D_2 > B_1$. From Lemma 15 it follows that $B_2 - \{F, D_2 < D_3\}$, because $B - U$, $B - F$. Therefore $S\mathcal{K}$ contains the wild subset $\{P, B_2, F, D_2 < D_3\}$, a contradiction.

(BBB) $F > D_5$, $F - C$. Suppose $B \in O\partial_P\mathcal{K}$. Then $\dim_k \mathcal{K}(Z, B) = 2$ and hence $B - \{C,\, D_1 < \ldots < D_5 < F\}$.

If $B - P$, then $S\mathcal{K}$ contains the wild subset $\{B, P, C, D_1 < D_2\}$.

If $B < P$, then $S\mathcal{K}$ contains the wild subset $\{C, B < P, D_1 < \ldots < D_5 < F\}$.

Suppose $B \in NS\partial_P\mathcal{K}$. Then it follows from (3.5.1) that $B = S_{\{B_1, B_2\}}$, $\dim_k \mathcal{K}(Z, B_1) = 2$ for some $B_1, B_2 \in S\mathcal{K}$. It is clear that $B_1 \ngeq X$ for any $X \in S\mathcal{K}_Z$. We distinguish several cases:

(BBBA) $B_1 < C$. From Lemma 15 it follows that $B_2 - C$, $B_2 \nleq D_5$, $B_2 \ngeq F$, because $B - U$, $B - F$. If $B_2 - D_4$, then $B_2 - D_5$ and hence $S\mathcal{K}$ contains the wild subset $\{P, C, B_2, D_4 < D_5\}$, a contradiction. Therefore we may assume that $B_2 > D_5$. Then $B_1 - \{D_1 < \ldots < D_5\}$ and hence $S\mathcal{K}$ contains the wild subset $\{P, B_1 < C,\, D_1 < \ldots < D_5 < B_2\}$, a contradiction.

(BBBB) $B_1 < D_1$. From Lemma 15 it follows that $B_2 - \{D_1 < \ldots < D_5 < F\}$, $B_2 \nleq C$, because $B - U$, $B - F$. If $B_2 - C$, then $S\mathcal{K}$ contains the wild subset $\{P, C, B_2, D_1 < D_2\}$, a contradiction. If $B_2 > C$, then $B_1 - C$ and hence $S\mathcal{K}$ contains the wild subset $\{P, C < B_2,\, B_1 < D_1 < \ldots < D_5\}$, a contradiction.

(BBBC) $B_1 - C, B_1 - D_1$. If $B_1 - D_2$, then $S\mathcal{K}$ contains the wild subset $\{P, C, B_1, D_1 < D_2\}$, a contradiction. Therefore we may assume that $B_1 < D_2$. Then $B_2 \nleq C$, $B_2 - \{D_2 < \ldots < D_5 < F\}$ since $B - U$, $B - F$. If $B_2 - C$, then $S\mathcal{K}$ contains the wild subset $\{P, C, B_2, B_1 < D_2\}$, a contradiction. If $B_2 > C$, then $S\mathcal{K}$ contains the wild subset $\{P, C < B_2,\, B_1 < D_2 < \ldots < D_5 < F\}$, a contradiction.

3.5.8

Suppose $\Omega = \{B < A > F < G > B, H_1, H_2\}$. Then $U = \{H_1, H_2\}$. If $\operatorname{ord} \Omega\mathcal{K}_Z < 3$, then A does not satisfy the condition $(*)$, a contradiction.

Therefore we may assume that $H_1 \in N\partial_P\mathcal{K}$ or $H_2 \in N\partial_P\mathcal{K}$, because $\mathcal{K}$ satisfies the condition (f) of Theorem 1. Since $F - B$, we get $Z \notin \psi(F)$. If $\psi(F) \subseteq S\mathcal{K}_Z$ and $F \in N\partial_P\mathcal{K}$, then $S\partial_P\mathcal{K}$ contains the wild subset $\{S_{Z,0} > B, F, H_1, H_2\}$, which is impossible in accordance with (3.5.6).

Suppose $F \in O\partial_P\mathcal{K}$ and $F \in S\mathcal{K}_Z$. Then one of the following situations occur:

(a). $H_1, H_2 \in N\partial_P\mathcal{K}$. Then we may assume that $H_1 = S_{\{L,M\}}$, $H_2 = S_{\{S,T\}}$, $L < S$, $T < M$, $\{L < S\} - \{T < M\}$ for some $L, S, T, M \in S\mathcal{K}_Z$. Since $F - \{H_1, H_2\}$, we may assume that $F > M$, $F - \{L < S\}$ (the case $F > S$, $F - \{T < M\}$ can be treated similarly). We distinguish several cases.

α). $B \in O\partial_P\mathcal{K}$. Since $B - U$ and $\dim_k \mathcal{K}(Z, B) = 2$, we get $B - U_1$. If $B - P$, then $S\mathcal{K}$ contains the wild subset $\{P, B, L, T < M\}$, a contradiction. If $B < P$, then $S\mathcal{K}$ contains the wild subset $\{B < P, L < S, T < M < F\}$, a contradiction.

β). $B \in N\partial_P\mathcal{K}$. Then it follows from (3.5.1) that $B = S_{\{B_1,B_2\}}$, $\dim_k\mathcal{K}(Z, B_1) = 2$ for some $B_1, B_2 \in S\mathcal{K}$. Hence $B_1 \not\geq X$ for any $X \in S\mathcal{K}_Z$. We distinguish several cases.

$\beta\alpha$). $B_1 < L$. From Lemma 15 it follows that $B_2 - \{L < S\}$, $B_2 \not\leq M$, $B_2 \not\geq F$, because $B - U$, $B - F$. If $B_2 - F$ (resp., $B_2 - M$), then $S\mathcal{K}$ contains the wild subset $\{P, B_2, F, L < S\}$ (resp., $\{P, B_2, M, L < S\}$), a contradiction. If $M < B_2 < F$, then $B_1 - \{T < M < F\}$ and hence $S\mathcal{K}$ contains the wild subset $\{P, B_1 < L < S, T < M < B_2 < F\}$, a contradiction.

$\beta\beta$). $B_1 < T$. From Lemma 15 it follows that $B_2 - \{T < M < F\}$, $B_2 \not\leq S$, because $B - U$, $B - F$. If $B_2 - S$, then $S\mathcal{K}$ contains the wild subset $\{P, B_2, S, T < M\}$, a contradiction. If $B_2 > S$, then $B_1 - \{L < S\}$ and hence $S\mathcal{K}$ contains the wild subset $\{P, B_1 < T < M < F, L < S < B_2\}$, a contradiction.

$\beta\gamma$). $B_1 - T, B_1 - L$. If $B_1 - S$ (resp., $B_1 - M$), then $S\mathcal{K}$ contains the wild subset $\{P, B_1, T, L < S\}$ (resp., $\{P, B_1, L, T < M\}$), a contradiction. Hence we get $S > B_1, M > B_1$. From Lemma 15 it follows that $B_2 - \{S, M < F\}$, because $B - U$, $B - F$. Therefore $S\mathcal{K}$ contains the wild subset $\{P, B_2, S, M < F\}$, a contradiction.

(b). $H_1 = S_{\{C,D\}}$, $H_2 \in O\partial_P\mathcal{K}$ for some $C, D \in S\mathcal{K}_Z$ (the case $H_1 \in O\partial_P\mathcal{K}$, $H_2 \in N\partial_P\mathcal{K}$ is similar). Since $F - H_1$, we may assume that $F > D$, $F - C$. Then one of the following two situations occur:

α). $B \in O\partial_P\mathcal{K}$. Since $B - U$, $B - F$ and $\dim_k\mathcal{K}(Z, B) = 2$, we get $B - \{C, D < F, H_2\}$ and hence $S\mathcal{K}$ contains the wild subset $\{B, C, H_2, D < F\}$, a contradiction.

β). $B \in N\partial_P\mathcal{K}$. Then it follows from (3.5.1) that $B = S_{\{B_1,B_2\}}$, $\dim_k\mathcal{K}(Z, B_1) = 2$ for some $B_1, B_2 \in S\mathcal{K}$. Hence $B_1 \not\geq X$ for any $X \in S\mathcal{K}_Z$. We distinguish several cases.

$\beta\alpha$). $B_1 < C$. From Lemma 15 it follows that $B_2 - C$, $B_2 \not\leq D$, $B_2 \not\geq F$, because $B - U$, $B - F$. If $B_2 - F$ (resp., $B_2 - D$), then $S\mathcal{K}$ contains the wild subset $\{B_2, F, C, H_2 < P\}$ (resp., $\{B_2, D, C, H_2 < P\}$), a contradiction.

If $D < B_2 < F$, then $B_1 - \{D < F\}$ and hence $S\mathcal{K}$ contains the wild subset $\{H_2 < P, B_1 < C, D < B_2 < F\}$, a contradiction.

$\beta\beta$). $B_1 < D$. From Lemma 15 it follows that $B_2 - \{D < F\}$, $B_2 \not\leq C$, because $B - U$, $B - F$. If $B_2 - C$, then $S\mathcal{K}$ contains the wild subset $\{B_2, C, D, H_2 < P\}$, a contradiction.

If $B_2 > C$, then $B_1 - C$ and hence $S\mathcal{K}$ contains the wild subset $\{H_2 < P, B_1 < D < F, C < B_2\}$, a contradiction.

$\beta\gamma$). $B_1 - D, B_1 - C$. Then $S\mathcal{K}$ contains the wild subset $\{H_2 < P, B_1, C, D\}$, a contradiction.

Therefore we may assume that there exists $F_1 \in \psi(F)$ such that $\dim_k\mathcal{K}(Z, F_1) = 2$. Then one of the following situations occur:

A. $H_1, H_2 \in N\partial_P\mathcal{K}$. Then we may assume that $H_1 = S_{\{C_1,C_2\}}$, $H_2 = S_{\{D_1,D_2\}}$, $C_1 < D_1$, $D_2 < C_2$, $\{C_1 < D_1\} - \{D_2 < C_2\}$. One of the following situations occur:

α). $B, F \in O\partial_P\mathcal{K}$. Since $\{B, F\} - U$, $\dim_k\mathcal{K}(Z, B) = 2$, $\dim_k\mathcal{K}(Z, F) = 2$, we get $\{B, F\} - \psi(U)$ and hence $S\mathcal{K}$ contains the wild subset $\{B, F, C_1, D_2 < C_2\}$, a contradiction.

β). $B \in O\partial_P\mathcal{K}, F \in N\partial_P\mathcal{K}$ (the case $B \in N\partial_P\mathcal{K}, F \in O\partial_P\mathcal{K}$ can be treated in the analogous manner). As in case (α) we get $B - \psi(U)$. If $B - P$, then $S\mathcal{K}$ contains the wild subset $\{B, P, C_1, D_2 < C_2\}$, a contradiction. Therefore we may assume that $B < P$. From (3.5.1) it follows that $F = S_{\{F_1,F_2\}}$ for some $F_1, F_2 \in S\mathcal{K}$ and $\dim_k\mathcal{K}(Z, F_1) = 2$. Since $B - F$ and $B < P$, we obtain $B - \{F_1, F_2\}$. It is clear that $F_1 \not\geq T$ for any $T \in \psi(U)$. If $F_1 - C_1, F_1 - D_2$, then $S\mathcal{K}$ contains the wild subset $\{F_1, C_1, D_2, B < P\}$.

Assume that $\{F_1 < C_1\}$ (the case $F_1 < D_2$ can be treated in the analogous manner). From Lemma 15 it follows that $F_2 - \{C_1 < D_1\}, F_2 \not\leq C_2$, because $F - U$. If $F_2 - C_2$, then $S\mathcal{K}$ contains the wild subset $\{F_2, C_1, C_2, B < P\}$, a contradiction.

If $F_2 > C_2$, then $F_1 - \{D_2 < C_2\}$ and hence $S\mathcal{K}$ contains the wild subset $\{F_1 < C_1 < D_1, D_2 < C_2, B < P\}$, a contradiction.

γ). $B, F \in N\partial_P\mathcal{K}$. From (3.5.1) it follows that $B = S_{\{B_1,B_2\}}$, $F = S_{\{F_1,F_2\}}$ for certain $B_1, B_2, F_1, F_2 \in S\mathcal{K}$ and $\dim_k\mathcal{K}(Z, B_1) = 2$, $\dim_k\mathcal{K}(Z, F_1) = 2$. It is clear that $F_1 \not\geq T$, $B_1 \not\geq T$ for any $T \in \psi(U)$. We distinguish several cases.

$\gamma\alpha$). $F_1 < C_1, B_1 < C_1$ (the case $F_1 < D_2$, $B_1 < D_2$ can be treated in the analogous manner). From Lemma 15 it follows that $F_2 - \{C_1 < D_1\}$, $B_2 - \{C_1 < D_1\}$, $F_2 \not\leq C_2$, $B_2 \not\leq C_2$. If $F_2 - C_2$ (resp., $B_2 - C_2$), then $S\mathcal{K}$ contains the wild subset $\{F_2, C_2, P, F_1 < C_1\}$ (resp., $\{B_2, C_2, P, B_1 < C_1\}$), a contradiction. Therefore we may assume that $F_2 > C_2$ and $B_2 > C_2$. Hence $F_1 - \{D_2 < C_2\}$, $B_1 - \{D_2 < C_2\}$. If $B_1 - F_1$, then $S\mathcal{K}$ contains the wild subset $\{B_1, F_1, P, D_2 < C_2\}$, a contradiction.

If $B_1 < F_1$ (the case $F_1 < B_1$ is similar), then $F_1 - B_2$ and hence $S\mathcal{K}$ contains the wild subset $\{F_1 < B_1 < C_1 < D_1, D_2 < C_2 < B_2, P\}$, a contradiction.

$\gamma\beta$). $F_1 < C_1, B_1 < D_2$ (the case $F_1 < D_2$, $B_1 < C_1$ is similar). From Lemma 15 it follows that $F_2 - \{C_1 < D_1\}$, $B_2 - \{D_2 < C_2\}$, $F_2 \not\leq C_2$, $B_2 \not\leq D_1$. If $F_2 - C_2$ (resp., $B_2 - D_1$), then $S\mathcal{K}$ contains the wild subset $\{F_2, C_2, P, C_1 < D_1\}$ (resp., $\{B_2, D_1, P, D_2 < C_2\}$), a contradiction. Therefore we may assume that $F_2 > C_2$ and $B_2 > D_1$. Then $F_1 - \{D_2 < C_2\}$, $B_1 - \{C_1 < D_1\}$ and hence $S\mathcal{K}$ contains the wild subset $\{B_1 < D_2 < C_2 < F_2, F_1 < C_1 < D_1, P\}$, a contradiction.

$\gamma\gamma$). $F_1 - C_1, F_1 - D_2$ (the case $B_1 - C_1, B_1 - D_1$ is similar). If $F_1 - D_1$ (resp., $F_1 - C_2$), then $S\mathcal{K}$ contains the wild subset $\{C_1 < D_1, F_1, D_2, P\}$ (resp., $\{D_2 < C_2, C_1, F_1, P\}$), a contradiction.

Therefore we obtain $D_1 > F_1, C_2 > F_1$. Since $F - \{H_1, H_2\}$, we get $F_2 - \{C_1 < D_1, D_2 < C_2\}$. But then $S\mathcal{K}$ contains the wild subset $\{F_1, F_2, C_1, D_2, P\}$, a contradiction.

B. $H_1 = S_{\{C,D\}}, H_2 \in O\partial_P\mathcal{K}$ (the case $H_1 \in O\partial_P\mathcal{K}$, $H_2 \in N\partial_P\mathcal{K}$ is similar). One of the following situations occur:

(α). $B, F \in O\partial_P\mathcal{K}$. Since $\{B, F\} - U$, $\dim_k\mathcal{K}(Z, B) = 2$, $\dim_k\mathcal{K}(Z, F) = 2$, we get $\{B, F\} - \psi(U)$ and hence $S\mathcal{K}$ contains the wild subset $\{B, F, C, D, H_2\}$, a contradiction.

(β). $B \in O\partial_P\mathcal{K}, F \in N\partial_P\mathcal{K}$ (the case $B \in N\partial_P\mathcal{K}$, $F \in O\partial_P\mathcal{K}$ can be treated in the analogous manner). As in case (α) we get $B - \psi(U)$. If $B - P$, then $S\mathcal{K}$ contains the wild subset $\{B, C, D, H_2 < P\}$, a contradiction.

Therefore we may assume that $B < P$. From (3.5.1) it follows that $F = S_{\{F_1,F_2\}}$ for certain $F_1, F_2 \in S\mathcal{K}$ and $\dim_k\mathcal{K}(Z, F_1) = 2$. Since $B < P$ and $B - F$, we obtain $B - \{F_1, F_2\}$. It is clear that $F_1 \not\geq T$ for any $T \in \psi(U)$. If $F_1 - C$, $F_1 - D$, then $S\mathcal{K}$ contains the wild subset $\{F_1, C, D, B, H_2\}$, a contradiction.

Assume that $F_1 < C$ (the case $F_1 < D$ can be treated in analogous manner). From Lemma 15 it follows that $F_2 - C$, $F_2 \not\leq D$, because $F - U$. If $F_2 - D$, then $S\mathcal{K}$ contains the wild subset $\{F_2, C, D, B, H_2\}$, a contradiction.

If $F_2 > D$, then $F_1 - D$ and hence $S\mathcal{K}$ contains the wild subset $\{F_1 < C, D, B, H_2\}$, a contradiction.

γ). $B, F \in N\partial_P\mathcal{K}$. From (3.5.1) it follows that $B = S_{\{B_1,B_2\}}$, $F = S_{\{F_1,F_2\}}$ for certain $B_1, B_2, F_1, F_2 \in S\mathcal{K}$ and $\dim_k\mathcal{K}(Z, B_1) = 2$, $\dim_k\mathcal{K}(Z, F_1) = 2$. It is clear that $F_1 \not\geq T$, $B_1 \not\geq T$ for any $T \in \psi(U)$. We distinguish several cases.

$\gamma\alpha$). $F_1 < C, B_1 < C$ (the case $F_1 < D$, $B_1 < D$ can be treated in the analogous manner). From Lemma 15 it follows that $F_2 - C$, $B_2 - C$, $F_2 \not\leq D$, $B_2 \not\leq D$.

If $F_2 - D$ (resp., $B_2 - D$), then $S\mathcal{K}$ contains the wild subset $\{F_2, D, P, F_1 < C\}$ (resp.,

$\{B_2, D, P, F_1 < C\}$), a contradiction. Therefore we may assume that $F_2 > D$ and $B_2 > D$. Hence $F_1 - D$, $B_1 - D$. If $B_1 - F_1$, then $S\mathcal{K}$ contains the wild subset $\{B_1, F_1, D, H_2 < P\}$, a contradiction. If $B_1 < F_1$ (the case $F_1 < B_1$ is similar), then $F_1 - B_2$ and hence $S\mathcal{K}$ contains the wild subset $\{B_1 < F_1 < C, D < B_2, H_2 < P\}$, a contradiction.

$\gamma\beta$). $F_1 < C, B_1 < D$ (the case $F_1 < D$, $B_1 < C$ is similar). From Lemma 15 it follows that $F_2 - C$, $B_2 - D$, $F_2 \not\leq D$, $B_2 \not\leq C$. If $F_2 - D$ (resp., $B_2 - C$), then $S\mathcal{K}$ contains the wild subset $\{F_2, C, D, H_2 < P\}$ (resp., $\{B_2, C, D, H_2 < P\}$), a contradiction. Therefore we may assume that $F_2 > D$ and $B_2 > C$. Then $F_1 - D$, $B_1 - C$ and hence $S\mathcal{K}$ contains the wild subset $\{B_1 < D < F_2, F_1 < C, H_2 < P\}$, a contradiction.

$\gamma\gamma$). $F_1 - C, F_1 - D$ (the case $B_1 - C$, $B_1 - D$ is similar). Then $S\mathcal{K}$ contains the wild subset $\{F_1, C, D, H_2 < P\}$, a contradiction.

3.6

Suppose $\mathcal{K}$ satisfies all the conditions of Theorem 1. Let P be an admissible object and let $Z = \max \operatorname{ind} \mathcal{K}$, $Z \in L\mathcal{K}$, $\operatorname{ord} \Omega\mathcal{K}_Z = 4$, $S\mathcal{K}_Z = \{B, C, D, P\}$. From the definition of the category $\partial_P\mathcal{K}$ and (3.4) it immediately follows that the conditions (a)-(g) of Theorem 1 are fulfilled for the category $\partial_P\mathcal{K}$.

We will prove that the category $\partial_P\mathcal{K}$ satisfies the condition (h) of Theorem 1. Suppose, contrary to our claim, that $S\partial_P\mathcal{K}$ contains one of the subsets $W_1, \ldots, W_6, G$. Let Ω denote it. If $\Omega \cap \psi_Z = \emptyset$, then from (3.4) it follows that the category $\partial_P\mathcal{K}$ satisfies the condition (h). Therefore we may assume that $\Omega \cap \psi_Z \neq \emptyset$. Then from Lemma 13 it follows that $\Omega > L_{\{B,C,D\}}$ and hence $\Omega \subseteq O$, where $\partial_P\mathcal{K}(O) \cong \mathcal{O}_\lambda$ (see (2.4)), a contradiction.

3.7

In this subsection we prove Theorem 4. From (3.1) it follows that if $\mathcal{K}$ is a schurian vector space category of polynomial growth, then $\mathcal{K}$ satisfies all the conditions of Theorem 1. The proof is carried out in the class of all schurian vector space categories satisfying all the conditions of Theorem 1. We proceed by induction on the dimension d for which a sincere series of indecomposable factor spaces exists.

For $d = 1$, the statement of the theorem is obviously true, because there is no a vector space category with a series of factor spaces in dimension d. For $d = 2$, there is one series $\mathcal{A}_{A,\lambda,1}$. So in this case the statement is also true.

Suppose the theorem is valid for all $d \leq d^0$. Let $\mathcal{K}$ be a nonlinear schurian vector space category which satisfies all the conditions of Theorem 1 and has a sincere series $\mathcal{T}$ of indecomposable factor spaces in the dimension $d > d^0$. According to the results in Section 2, we can consider the derived category $\partial_P\mathcal{K}[P]$ (see for details (3.3)). It is clear that $\mathcal{K}$ is infinitely sincere iff $\mathcal{K}^{op}$ is infinitely sincere. Therefore we may assume that either $\mathcal{K}[P] = \mathcal{K}$ or $\mathcal{K}[P](\operatorname{ind}\mathcal{K}[P] \setminus \{P\}) \cong \mathcal{K}$. According to Proposition 4 and Remark in (3.3), the series $\mathcal{T}$ turns into the series $\widetilde{\mathcal{T}}$ in the dimension $d \leq d^0$. Set $\widetilde{\mathcal{K}} = \partial_P\mathcal{K}[P](\operatorname{supp}\widetilde{\mathcal{T}})$. By the inductive assumption, $\widetilde{\mathcal{K}}$ is isomorphic to one of the categories $\mathcal{A}, \mathcal{E}, \mathcal{E}^{op}, \mathcal{F}, \mathcal{F}^{op}, \mathcal{G}, \mathcal{G}^{op}, \mathcal{H}_\lambda, \mathcal{H}_\lambda^{op}$ or $\widetilde{\mathcal{K}}$ is linear and $S(\widetilde{\mathcal{K}})$ is the critical or defining poset. We consider all possible cases. Since $\mathcal{T}$ is sincere, we have $\mathcal{K} \cong \psi(\widetilde{\mathcal{K}})$.

A. $P \in \max\operatorname{ind}\mathcal{K}[P]$. According to (3.3), we may assume that $\mathcal{K}[P] = \mathcal{K}$.

A.1. $\widetilde{\mathcal{K}} \cong \mathcal{A}, L\widetilde{\mathcal{K}} = Z$. If $Z \in NL\partial_P\mathcal{K}$, then $\mathcal{K} = \mathcal{K}(K_1)$ since $\psi(\widetilde{\mathcal{K}}) = \mathcal{K}$, i. e. $\mathcal{K}$ is linear, which contradicts the assumption.

Suppose that $Z \in OL\partial_P\mathcal{K}$. We then have $\operatorname{ind}\mathcal{K} = \{Z, P\}$. If $P \in S\mathcal{K}^Z$, then Proposition 1 shows that $\mathcal{K}$ does not have sincere series, a contradiction. If $\dim_k \mathcal{K}(P, Z) = 2$, then from the definition of the mapping $\mathbf{i}_P$ we see that $\mathcal{K}$ does not have sincere factor space, a contradiction.

A.2. Suppose that $\widetilde{\mathcal{K}}$ is isomorphic to one of the following categories: $\mathcal{E}^{op}$, $\mathcal{F}^{op}$, $\mathcal{G}^{op}$, $\mathcal{H}_\lambda^{op}$, $\lambda \in k \setminus \{0, 1\}$. Let $L\widetilde{\mathcal{K}} = \{Z\}$. The following cases are possible.

A.2.1. $Z \in NL\partial_P\mathcal{K}$. Lemma 26 shows that if $A \in S\widetilde{\mathcal{K}}$, then $A \neq S_D$ for any $D \in L\mathcal{K}$. Therefore $\mathcal{K} = \psi(\widetilde{\mathcal{K}})$ is linear, which contradicts the assumption.

A.2.2. $Z \in OL\partial_P\mathcal{K}$, $\widetilde{\mathcal{K}} \ncong \mathcal{H}_\lambda^{op}$. Since $\operatorname{ord}\Omega\partial_P\mathcal{K}^Z = 3$, we have $\operatorname{ord}\Omega\mathcal{K}^Z = 3$. The category $\mathcal{K}$ satisfies the condition (f) of Theorem 1, and therefore $S\mathcal{K}^{Z,V}$ does not contain the subset $(1, 1)$ for any $V \in \mathbf{P}^1(Z)$. Hence $A = S_Z$ and $NS\widetilde{\mathcal{K}} = \{A\}$ if $A \in NS\widetilde{\mathcal{K}}$. From this it follows that $\mathcal{K} = \psi(\widetilde{\mathcal{K}}) \cong \widetilde{\mathcal{K}}$, where $P \to C_n, C_m \to C_{m-1}$ for $m > 1, C_1 \to S_Z$ if $S\mathcal{K}^{Z,Z^P} = \{C_1 < \cdots < C_n < P\}$, and $T \to T$ for $T \in \operatorname{ind}\mathcal{K} \setminus S\mathcal{K}^{Z,Z^P}$. We therefore may assume that $NS\widetilde{\mathcal{K}} = \emptyset$, i. e. $\operatorname{ind}\widetilde{\mathcal{K}} = \operatorname{ind}\mathcal{K} \setminus \{P\}$.

Since $S\mathcal{K}$ is tame, we conclude that $\dim_k \mathcal{K}(P, Z) = 2$. From the fact that $\mathcal{T}$ is sincere it follows that there is an element $A \in \max\operatorname{ind}\mathcal{K} \cap S\mathcal{K}$ such that $A - P$. Therefore, we can consider $\partial_A\mathcal{K}$. According to Proposition 4, the series $\mathcal{T}$ turns into the series $\check{\mathcal{T}}$ in the dimension $d \leq d^0$. Set $\check{\mathcal{K}} = \partial_A\mathcal{K}(\operatorname{supp}\check{\mathcal{T}})$. By the induction assumption, $\check{\mathcal{K}}$ is isomorphic to one of the categories $\mathcal{A}, \mathcal{E}, \mathcal{E}^{op}, \mathcal{F}, \mathcal{F}^{op}, \mathcal{G}, \mathcal{G}^{op}, \mathcal{H}_\lambda, \mathcal{H}_\lambda^{op}$ or $\check{\mathcal{K}}$ is linear and $S\check{\mathcal{K}}$ is the critical or defining poset. If $NL\check{\mathcal{K}} \neq \emptyset$, then $Z \notin \operatorname{ind}\check{\mathcal{K}}$ and hence, $S_Z \in \operatorname{ind}\check{\mathcal{K}}$. From the proof of Lemma 23 it follows that if $X \in NL\check{\mathcal{K}}$, then $\dim_k \check{\mathcal{K}}(X, S_Z) = 2$ (since $P \in \psi(X)$), which contradicts the assumption.

If $Z \in \operatorname{ind}\check{\mathcal{K}}$, then $P \notin \psi(Y)$ for any $Y \in \operatorname{ind}\check{\mathcal{K}}$ (otherwise $\dim_k \check{\mathcal{K}}(Y, Z) = 2$), which contradicts the fact that $\mathcal{T}$ is sincere. From the above it follows that $\check{\mathcal{K}}$ is linear and hence, $S(\check{\mathcal{K}})$ is either critical or defining poset. Since $\mathcal{T}$ is sincere, we have $P, S_Z \in S(\check{\mathcal{K}})$. Consider the following posets:

1. $S_1^{\mathcal{E}} = \{A_1 < A_2, B_1 < B_2, C_1 < C_2, A_1 < P, B_1 < P, C_1 < P\}$;
2. $S_2^{\mathcal{E}} = \{A_1 < A_2 < P > B_1 < B_2, C_1 < C_2\}$;
3. $S_1^{\mathcal{F}} = \{A_1, B_1 < B_2 < B_3, C_1 < C_2 < C_3, B_2 < P > C_2\}$;
4. $S_2^{\mathcal{F}} = \{A_1, B_1 < B_2 < B_3, C_1 < C_2 < C_3 < P > B_1\}$;
5. $S_3^{\mathcal{F}} = \{A_1 < P, B_1 < B_2 < B_3, B_2 < P, C_1 < C_2 < C_3\}$;
6. $S_4^{\mathcal{F}} = \{A_1 < P, B_1 < B_2 < B_3, C_1 < C_2 < C_3, B_1 < P, C_1 < P\}$;
7. $S_1^{\mathcal{G}} = \{A_1, B_1 < B_2, C_1 < \ldots < C_5, B_1 < P > C_4\}$;
8. $S_2^{\mathcal{G}} = \{A_1, B_1 < B_2 < P, C_1 < \ldots < C_5, P > C_2\}$;
9. $S_3^{\mathcal{G}} = \{A_1 < P, B_1 < B_2 < P, C_1 < \ldots < C_5\}$;
10. $S_4^{\mathcal{G}} = \{A_1 < P, B_1 < B_2, P > B_1, C_1 < \ldots < C_5, C_1 < P\}$;
11. $S_5^{\mathcal{G}} = \{A_1 < P, B_1 < B_2, C_1 < \ldots < C_5, P > C_3\}$.

From the tameness of $S\mathcal{K}$ it follows that $S\mathcal{K}$ is either one of the sets $S_i^{\mathcal{E}}$, $S_j^{\mathcal{F}}$, $S_l^{\mathcal{G}}$ or can be obtained from them by adding a few relations $T < P$, where $T \in S\mathcal{K} \setminus \{P\}$. Clearly, $\mathcal{K}$ is uniquely determined by $S\mathcal{K}$. By immediate verification (it suffices to consider the cases where $S\mathcal{K} = S_i^L$ with $L \in \{\mathcal{E}, \mathcal{F}, \mathcal{G}\}$) it is seen that if Φ is a critical or defining subset in $S\check{\mathcal{K}}$, then either $P \notin \Phi$ or $S_Z \notin \Phi$, a contradiction.

A.2.3. $Z \in OL\partial_P\mathcal{K}$, $\widetilde{\mathcal{K}} \cong \mathcal{H}_\lambda^{op}, \lambda \in k \setminus \{0, 1\}$. Since $\operatorname{ord}\Omega\partial_P\mathcal{K}^Z = 4$, we have $\operatorname{ord}\Omega\mathcal{K}^Z = 4$. $S\mathcal{K}$ is the tame poset and we therefore see that $S\mathcal{K}^{Z,V}$ does not contain the subset $(1, 1)$ for any $V \in \mathbf{P}^1(Z)$. Suppose $NS\widetilde{\mathcal{K}} \neq \emptyset$ and $A \in NS\widetilde{\mathcal{K}}$. Then $A = S_Z$ and $NS\widetilde{\mathcal{K}} = \{A\}$. From this it follows that $\mathcal{K} = \psi(\widetilde{\mathcal{K}}) \cong \widetilde{\mathcal{K}}$, where $P \to S_Z, T \to T$ for $T \in \operatorname{ind}\mathcal{K} \setminus \{P\}$, i. e. $\mathcal{K} \cong \mathcal{H}_\lambda^{op}$ as required. We therefore may assume that $NS\widetilde{\mathcal{K}} = \emptyset$,

i. e. $\operatorname{ind}\widetilde{\mathcal{K}} = \operatorname{ind}\mathcal{K} \setminus \{P\}$. Let $S\widetilde{\mathcal{K}} = \{A, B, C, D\}$. Since $S\mathcal{K}$ is tame, we see that P is comparable with at least two points from $S\widetilde{\mathcal{K}}$. If P is comparable with three or four points from $S\widetilde{\mathcal{K}}$, then the construction of differentiation shows that $\mathcal{K}$ is not faithful. Therefore, we can assume that $P - A, P - B, P > C, P > D$. Set $\mathcal{K}_1 = \partial_A\mathcal{K}$. It is easily seen that $\mathcal{K}_1$, $S_Z \in S\mathcal{K}_1$ and $L_{\{B,C,D\}} \in L\mathcal{K}_1$ satisfy the conditions of Lemma 28. Since $P \in L^{-}_{\{B,C,D\}}$ and $P \notin S\mathcal{K}_1^{L_{\{B,C,D\}}, L^{S_Z}_{\{B,C,D\}}}$, from Lemma 28 it follows that if $S_Z \subseteq Y$ or $Z \subseteq Y$, then $P \nsubseteq Y$ for any $(Y, U, \psi) \in \operatorname{ind}\mathcal{V}(\mathcal{K}_1)$. This implies that if $Z \subseteq X$, then $P \nsubseteq X$ for any $(X, V, \varphi) \in \operatorname{ind}\mathcal{V}(\mathcal{K})$. Hence $\mathcal{K}$ is not sincere, a contradiction.

A.3. Suppose that $\widetilde{\mathcal{K}}$ is isomorphic to one of the following categories $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda, \lambda \in k \setminus \{0, 1\}$. Let $L\widetilde{\mathcal{K}} = \{Z\}$. The following cases are possible.

A.3.1. $Z \in O\partial_P\mathcal{K}$. From Lemma 26 it follows that $S_Y \notin \operatorname{ind}\widetilde{\mathcal{K}}$ for any $Y \in L\mathcal{K}$. Therefore $L\mathcal{K} = \{Z\}$ because $\mathcal{T}$ is sincere. Since $\operatorname{ord}\Omega\partial_P\mathcal{K}_Z \geq 3$, we have $\operatorname{ord}\Omega\mathcal{K}_Z \geq 3$. Therefore $S\mathcal{K}_{Z,V}$ does not contain the subset $(1, 1)$ for any $V \in \mathbf{P}^1(Z)$. Consequently, $S\widetilde{\mathcal{K}} \subseteq O\partial_P\mathcal{K}$. From the above it follows that $\mathcal{K}^{op}$ satisfies the conditions of Lemma 28 and hence $\mathcal{K}$ is not sincere, a contradiction.

A.3.2. $Z \in N\partial_P\mathcal{K}$ and $S_X \notin \operatorname{ind}\widetilde{\mathcal{K}}$ for any $X \in L\mathcal{K}$. Then $\mathcal{K} = \psi(\widetilde{\mathcal{K}})$ is linear, which contradicts the assumption.

A.3.3. $Z = L_{\{A,B,C\}}$ for some $A, B, C \in S\mathcal{K}$ and $S_X \in \operatorname{ind}\widetilde{\mathcal{K}}$ for a certain $X \in L\mathcal{K}$. Since $S_X \in S\partial_P\mathcal{K}_Z$, from the proof of Lemma 23 it follows that one of the following two situations occur:

(a). $A, B, C \in S\mathcal{K}^X$, $X^A \neq X^B$, $X^A \neq X^C$, $X^B \neq X^C$. Since $S\mathcal{K}$ is tame, we obtain $S\mathcal{K}^X = \{A, B, C, P\}$ and $S\partial_P\mathcal{K}_Z = \{A, B, C, S_X\}$. Hence $\mathcal{K} \cong \mathcal{H}_\lambda^{op}$, $\lambda \in k$, $\lambda \notin \{0, 1\}$.

(b). $A, B \in S\mathcal{K}^{X,U}$ for some $U \in \mathbf{P}^1(X)$ and $\dim_k\mathcal{K}(C, X) = 2$. From the proof of Lemma 23 it follows that in this case $S\partial_P\mathcal{K}_Z$ does not contain the subsets $(1, 2, 2)$ and $(1, 1, 1, 1)$, a contradiction.

A.4. Let $\widetilde{\mathcal{K}}$ be a linear vector space category and $S\widetilde{\mathcal{K}}$ be a critical or defining poset. If $S_Z \notin \operatorname{ind}\widetilde{\mathcal{K}}$ for a certain $Z \in L\mathcal{K}$, then $\mathcal{K} = \psi(\widetilde{\mathcal{K}})$ is linear, which contradicts the assumption. Therefore, we may assume that $S_Z \in \operatorname{ind}\widetilde{\mathcal{K}}$ for a certain $Z \in L\mathcal{K}$. From Lemma 13 and the fact that $\mathcal{T}$ is sincere we conclude that $L\mathcal{K} = \{Z\}, Z \in \min\operatorname{ind}\mathcal{K}$.

Suppose that $\mathcal{K}$ contains an object $A \in S\mathcal{K}$ such that $\dim_k\mathcal{K}\,(A, Z) = 2$. Let A be the maximal object with this property. Let us consider $\partial_A\mathcal{K}$. By Proposition 4, the series $\mathcal{T}$ turns into the series $\check{\mathcal{T}}$ in the dimension $d \leq d^0$. Set $\check{\mathcal{K}} = \partial_A\mathcal{K}(\operatorname{supp}\check{\mathcal{T}})$. By the induction assumption, $\check{\mathcal{K}}$ is isomorphic to one of the categories $\mathcal{A}, \mathcal{E}, \mathcal{E}^{op}, \mathcal{F}, \mathcal{F}^{op}, \mathcal{G}, \mathcal{G}^{op}, \mathcal{H}_\lambda, \mathcal{H}_\lambda^{op}$ or $\check{\mathcal{K}}$ is linear and $S\check{\mathcal{K}}$ is the critical or defining poset. If $\check{\mathcal{K}}$ is nonlinear, then from cases **A.1-A.3** it follows that Theorem 4 is valid for $\mathcal{K}$.

If $\check{\mathcal{K}}$ is linear, then $\psi(\check{\mathcal{K}})$ is linear since $\dim_k\mathcal{K}\,(A, Z) = 2$. But this contradicts the assumption. Therefore, we may assume that $\dim_k\mathcal{K}\,(A, Z) = 1$ for any $A \in S\mathcal{K}$. From the fact that $S\mathcal{K}$ is tame and [Ri2] we see that $\mathcal{K}$ is isomorphic to one of the categories $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda$, $\lambda \in k \setminus \{0, 1\}$.

B. $Z = \max\operatorname{ind}\mathcal{K}[P]$, $Z \in L\mathcal{K}[P]$, $\operatorname{ord}\Omega\mathcal{K}[P]_Z \leq 3$. From the fact that $\mathcal{T}$ is sincere it follows that there exists $A \in \operatorname{ind}\widetilde{\mathcal{K}}$ such that $Z \in \psi(A)$. Therefore we conclude that $\widetilde{\mathcal{K}}$ cannot be isomorphic to one of the categories $\mathcal{A}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda$.

B.1. Let $\widetilde{\mathcal{K}}$ be isomorphic to one of the categories $\mathcal{E}^{op}$, $\mathcal{F}^{op}$,$\mathcal{G}^{op}$,$\mathcal{H}_\lambda^{op}$, $\lambda \in k \setminus \{0, 1\}$ and let $X \in L\widetilde{\mathcal{K}}$.

B.1.1. $X \in NL\partial_P\mathcal{K}[P]$. From Lemma 26 it follows that $S_Y \notin \operatorname{ind}\widetilde{\mathcal{K}}$ for any $Y \in L\mathcal{K}[P]$. Therefore, from the fact that $\mathcal{T}$ is sincere it follows that $L\mathcal{K} = L\mathcal{K}[P] = \{Z\}$. If the category $\mathcal{K}^{op}$ satisfies the conditions of the case **A**, then Theorem 4 is valid for it and hence, for the category $\mathcal{K}$ too.

If the category $\mathcal{K}^{op}$ does not satisfy the conditions of the case **A**, then there exists $V \in \mathbf{P}^1(Z)$ such that $Z^+ = S\mathcal{K}_{Z,V}$ and $\operatorname{ord}\min S\mathcal{K}_{Z,V} = 2$. Fix a P-basis $\mathcal{B}$ of the category $\mathcal{K}[P]$. Consider the set with involution $(S,\sigma) = (\mathcal{B}_0, \sigma)$, where $\sigma(b_Z^1) = b_Z^2$, $\sigma(b_L^1) = b_L^1$ for $L \in \operatorname{ind}\mathcal{K}[P]$, $L \neq Z$. It is easily to seen that $\mathcal{K}[P] \cong \mathcal{K}(S,\sigma)$. Therefore, from [NBR] it follows that $\mathcal{K} \cong \mathcal{A}$.

B.1.2. $X \in OL\partial_P\mathcal{K}[P]$. From the construction of differentiation it follows that if $\mathcal{K}[P](\{Z,X\}) \ncong \mathcal{I}$, then $\dim_k \partial_P\mathcal{K}[P](A,X) = 2$, which is impossible. Therefore we may assume that $\mathcal{K}[P](\{Z,X\}) \cong \mathcal{I}$. Lemma 11 now implies that A does not satisfy the condition $(*)$, which is impossible.

B.2. Let $\widetilde{\mathcal{K}}$ be a linear vector space category. Analogously to the case **B.1.1** we may assume that $S_X \in S\widetilde{\mathcal{K}}$ for a certain $X \in L\mathcal{K}[P]$. If $\mathcal{K}[P](\{X,Z\}) \cong \mathcal{I}$, then from Lemma 11 it follows, that S_X does not satisfy the condition $(*)$, which is impossible. Therefore we may assume that $\mathcal{K}[P](\{X,Z\}) \ncong \mathcal{I}$.

Suppose that there exist $C, B \in S\mathcal{K}[P]_Z$ such that $S_{\{C,B\}} \in S\widetilde{\mathcal{K}}$, $C, B \in S\mathcal{K}[P]^{X,V}$ for a certain $V \in \mathbf{P}^1(X)$. We denote by $N(S_X)$ the set of the elements of $S\partial_P\mathcal{K}[P]$, which are incomparable with S_X. From the conditions (h),(f) of Theorem 1 for $\mathcal{K}[P]$ it follows that $\Omega\mathcal{K}[P]^X = \{X^P, V\}$. Hence $N(S_X)$ does not contain the subsets $(1,2)$ and $(1,1,1)$, which is impossible, because $S\widetilde{\mathcal{K}}$ is either critical or defining set and any element of $S\widetilde{\mathcal{K}}$ satisfies the condition $(*)$ (see (1.1)). Therefore we may assume that if $S_{\{C,B\}} \in S\widetilde{\mathcal{K}}, C, B \in S\mathcal{K}[P]^X$, then $X^C \neq X^B$. This implies that S_X is comparable with all the elements of $S\partial_P\mathcal{K}[P] \setminus \Phi$, where $\Phi = \cup_{Z_P \neq V \in \mathbf{P}^1(Z)} S\mathcal{K}[P]_{Z,V}$. But we have $A > \Phi$ and hence any element of $S\widetilde{\mathcal{K}}$ is comparable with S_X or A, which is impossible, because $S\widetilde{\mathcal{K}}$ is either critical or defining set.

C. $Z = \max\operatorname{ind}\mathcal{K}[P]$, $Z \in L\mathcal{K}[P]$, $\operatorname{ord}\Omega\mathcal{K}[P]_Z = 4$, $S\mathcal{K}[P]_Z = \{B, C, D, P\}$. Since $\mathcal{T}$ is sincere, we see that there is $A \in \operatorname{ind}\widetilde{\mathcal{K}}$ such that $Z \in \psi(A)$. Since $Z \in \psi(A)$, we obtain $A > L_{\{B,C,D\}}$ and hence $S\widetilde{\mathcal{K}} > L_{\{B,C,D\}}$ using Lemma 13.

Suppose $L\widetilde{\mathcal{K}} \neq \emptyset, X \in L\widetilde{\mathcal{K}}$ and $X < L_{\{B,C,D\}}$. Then $\widetilde{\mathcal{K}}$ is isomorphic to one of the categories $\mathcal{E}^{op}$, $\mathcal{F}^{op}$,$\mathcal{G}^{op}$,$\mathcal{H}_\lambda^{op}$. Since $\dim_k\widetilde{\mathcal{K}}(T,X) = 1$ for any $T \in S\widetilde{\mathcal{K}}$ and $S\widetilde{\mathcal{K}} > L_{\{B,C,D\}}$, from Lemma 10 it follows that there exists $V \in \mathbf{P}^1(X)$ such that $S\widetilde{\mathcal{K}} \subseteq S\partial_P\mathcal{K}[P]^{X,V}$, which is impossible. Therefore we may assume that $X \geq L_{\{B,C,D\}}$ for any $X \in \operatorname{ind}\widetilde{\mathcal{K}}$. Hence $\psi(\widetilde{\mathcal{K}}) = \{Z, P, B, C, D\}$. If $\mathcal{K}[P] \neq \mathcal{K}$, then $\mathcal{K} \cong \mathcal{D}$, which is impossible by Proposition 2. Therefore $\mathcal{K}[P] = \mathcal{K}$ and hence $\mathcal{K} \cong \mathcal{H}_\lambda$ for some $\lambda \in k \setminus \{0,1\}$.

Thus, the Theorem 4 is proved. ∎

3.8

Now Theorem 1 follows from (3.1), (3.7) and the fact that the categories $\mathcal{A}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda$, $\lambda \in k \setminus \{0,1\}$ are of polynomial growth (see [Ri2]).

Theorem 2 follows from Theorem 4 and the fact that the statement of Theorem 2 is valid for the categories $\mathcal{A}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda$, $\lambda \in k \setminus \{0,1\}$ (see [Ri2]) and for the linear schurian vector space categories of polynomial growth (see [ZN1],[ZN2]).

Theorem 3 follows from Theorem 4, Theorem 2 and the fact that the linear schurian vector space categories of polynomial growth and categories $\mathcal{A}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda$, $\lambda \in k \setminus \{0,1\}$ are of linear growth (see [Si;Theorem 15.89],[Ri2]).

Theorem 6 follows from [Ro1], Theorem 4 and the fact that the statement of Theorem 6 is valid for the categories $\mathcal{A}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda$, $\lambda \in k \setminus \{0,1\}$ (see [Ri2]) and for the linear schurian vector space categories of polynomial growth (see [BZN]).

Theorems 5 and 7 follow from Theorem 4 and the fact that the categories $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}_\lambda$, $\lambda \in k \setminus \{0,1\}$ are not domestic (see [Ri2]).

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