Generic modules over artin algebras

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Generic modules have been introduced by Crawley-Boevey in order to provide a better understanding of finite dimensional algebras of tame representation type. In fact he has shown that the generic modules correspond to the one-parameter families of indecomposable finite dimensional modules over a tame algebra [5]. The Second Brauer-Thrall Conjecture provides another reason to study generic modules because the existence of a generic module over an artin R-algebra Λ ($R/\operatorname{rad} R$ an infinite field) implies that Λ has strongly unbounded representation type, i.e. there are infinitely many pairwise non-isomorphic Λ-modules of length n for infinitely many $n \in \mathbb{N}$ [6].

The aim of this paper is to develop further the analysis of existence and properties of generic modules. Our approach depends to a large extent on the embedding of a module category into a bigger functor category. These general concepts are explained in the first two sections. We continue in Section 3 with a new characterization of the pure-injective modules which occur as the source of a minimal left almost split morphism. This is of interest in our context because generic modules are pure-injective. Next we consider indecomposable endofinite modules. Recall that a module is endofinite if it is of finite length when regarded in the natural way as a module over its endomorphism ring. Changing slightly the original definition, we say that a module is generic if it is indecomposable endofinite but not finitely presented. Section 4 is devoted to several characterizations of generic modules in order to justify the choice of the non-finitely presented modules as the generic objects. We prove them for dualizing rings, i.e. a class of rings which includes noetherian algebras and artinian PI-rings. Existence results for generic modules over dualizing rings follow in Section 5. Several results in this paper depend on the fact that a functor $f\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(\Lambda)$ which commutes with direct limits and products, preserves certain finiteness conditions. For example, if a Γ-module M is endofinite then $f(M)$ is endofinite. If in addition $\mathrm{End}_\Gamma(M)$ is a PI-ring, then $\mathrm{End}_\Lambda(N)$ is a PI-ring for every indecomposable direct summand N of $f(M)$. This material is collected in Section 6 and 7. In Section 8 we introduce an effective method to construct generic modules over artin algebras from so-called generalized tubes. The special case of a tube in the Auslander-Reiten quiver is discussed in the following section. We obtain as a consequence a parametrization of the stable tubes by indecomposable Σ-pure-injective modules. These results are combined with Crawley-Boevey's work on tame algebras. It is shown that for a tame algebra Λ and a fixed $n \in \mathbb{N}$ almost all n-dimensional indecomposable Λ-modules belong to a disjoint union $\cup_G \operatorname{ind}_G(\Lambda)$ of naturally defined sets such that each $\operatorname{ind}_G(\Lambda)$ corresponds to a generic module G. Moreover we present a new characterization of tameness in terms of homogeneous tubes in the Auslander-Reiten quiver. We continue in Section 10 with a discussion of representation embeddings. A pure representation embedding $\operatorname{Mod}(\Gamma) \to \operatorname{Mod}(\Lambda)$ induces a well-behaved map between the corresponding Ziegler spectra and we apply this to algebras of wild representation type. The final section is devoted to the Krull dimension of the category of finitely presented modules over an artin algebra. We obtain a characterization of those algebras having dimension 1 which allows one to deduce that this cannot occur for a finite dimensional algebra over an algebraically closed field.

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1. Filtrations and dimensions for abelian categories

Let $\mathcal{C}$ be an abelian category. Recall that a full subcategory $\mathcal{S}$ of $\mathcal{C}$ is a *Serre subcategory* provided that for every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$ the object X belongs to $\mathcal{S}$ iff X' and X'' belong to $\mathcal{S}$. For every Serre subcategory $\mathcal{S}$ of $\mathcal{C}$ one can form the *quotient category* $\mathcal{C}/\mathcal{S}$ as follows. The objects of $\mathcal{C}/\mathcal{S}$ are those of $\mathcal{C}$ and $\operatorname{Hom}_{\mathcal{C}/\mathcal{S}}(X,Y) = \varinjlim \operatorname{Hom}_{\mathcal{C}}(X', Y/Y')$ with $X' \subseteq X$, $Y' \subseteq Y$ and $X/X', Y' \in \mathcal{S}$. Again the category $\mathcal{C}/\mathcal{S}$ is abelian and there is canonically defined the *quotient functor* $q\colon \mathcal{C} \to \mathcal{C}/\mathcal{S}$ such that $q(X) = X$; it is exact with $\operatorname{Ker}(q) = \mathcal{S}$ [7]. Here the *kernel* $\operatorname{Ker}(f)$ of a functor $f\colon \mathcal{C} \to \mathcal{D}$ is, by definition, the full subcategory of all objects X such that $f(X) = 0$. If f is exact, then $\operatorname{Ker}(f)$ is automatically a Serre subcategory and there is induced an exact faithful functor $\mathcal{C}/\operatorname{Ker}(f) \to \mathcal{D}$. A Serre subcategory is called *localizing* provided that the corresponding quotient functor admits a right adjoint. For each object X in $\mathcal{C}$ we denote by $\mathsf{L}(X)$ the lattice of its subobjects in $\mathcal{C}$; it is a modular lattice with 0 and 1.

Now suppose that $\mathcal{L}$ is a class of modular lattices which is closed under intervals, quotients and isomorphisms. An object X is called an *$\mathcal{L}$-object* of $\mathcal{C}$ if there exists a finite filtration

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = X$$

of X with $\mathsf{L}(X_i/X_{i-1}) \in \mathcal{L}$ for all $i \in \{1, \ldots, n\}$. Note that the $\mathcal{L}$-objects form a Serre subcategory of $\mathcal{C}$. The *$\mathcal{L}$-filtration* $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ is defined recursively as follows.

– $\mathcal{C}_{-1} = 0$;

– if α is an ordinal of the form $\alpha = \beta + 1$, then $\mathcal{C}_\alpha$ is the the Serre subcategory of all objects in $\mathcal{C}$ which become $\mathcal{L}$-objects in $\mathcal{C}/\mathcal{C}_\beta$;

– if α is a limit ordinal, then $\mathcal{C}_\alpha$ is the smallest Serre subcategory containing all $\mathcal{C}_\gamma$ with $\gamma < \alpha$.

We denote by $\mathcal{C}_\infty$ the smallest Serre subcategory containing all $\mathcal{C}_\alpha$ and define the *$\mathcal{L}$-dimension* of $\mathcal{C}$ as follows. If $\mathcal{C}_\infty = \mathcal{C}$, then the least ordinal α with $\mathcal{C}_\alpha = \mathcal{C}$ is denoted by $\dim_{\mathcal{L}} \mathcal{C}$. Otherwise let $\dim_{\mathcal{L}} \mathcal{C} = \infty$.

Now suppose that L is a modular latice with 0 and 1. Given a pair of elements $x, y \in L$ we define $x \sim_{\mathcal{L}} y$, or simply $x \sim y$, if there exists a finite filtration

$$x \wedge y = z_0 \leq z_1 \leq \ldots \leq z_n = x \vee y$$

with $[z_{i-1}, z_i] \in \mathcal{L}$ for all $i \in \{1, \ldots, n\}$. Note that $\sim$ defines a congruence relation on L. Therefore the set of equivalence classes $L/{\sim}$ carries again the structure of a modular lattice and the canonical map $L \to L/{\sim}$ is a morphism. The *$\mathcal{L}$-cofiltration* $(L_\alpha)_\alpha$ of L is defined recursively as follows.

– $L_{-1} = L$;

– if α is an ordinal of the form $\alpha = \beta + 1$, then $L_\alpha = L_\beta/{\sim}$;

– if α is a limit ordinal, then $L_\alpha = \varinjlim_{\gamma < \alpha} L_\gamma$.

We denote by L_∞ the colimit of this cofiltration and define the *$\mathcal{L}$-dimension* of L as follows. If $L_\infty = 0$, then the least ordinal α with $L_\alpha = 0$ is denoted by $\dim_{\mathcal{L}} L$. Otherwise let $\dim_{\mathcal{L}} L = \infty$.

We collect some elementary facts. Let $(\mathcal{C}_\alpha)_\alpha$ be the $\mathcal{L}$-filtration of an abelian category $\mathcal{C}$ and denote for each α by $q_\alpha \colon \mathcal{C} \to \mathcal{C}/\mathcal{C}_\alpha$ the quotient functor. Also let $(L_\alpha)_\alpha$ be the $\mathcal{L}$-cofiltration of a modular lattice L and denote for each α by $\pi_\alpha \colon L \to L_\alpha$ the canonical map.

Lemma 1.1 *Let $X \in \mathcal{C}$.*

(1) $\mathsf{L}(q_\alpha(X)) \cong \mathsf{L}(X)_\alpha$ *for every ordinal* α.

(2) *If there is no proper Serre subcategory of* $\mathcal{C}$ *containing* X, *then* $\dim_{\mathcal{L}} \mathcal{C} = \dim_{\mathcal{L}} \mathsf{L}(X)$.

(3) *If* $\dim_{\mathcal{L}} L < \infty$, *then* $\dim_{\mathcal{L}} L = \beta + 1$ *for some ordinal* β.

Proof: (1) follows directly from the definitions.

(2) is a consequence of (1) since $\mathcal{C}_\alpha = \mathcal{C}$ iff $q_\alpha(X) = 0$.

(3) Let $I_\alpha = \{x \in L \mid \pi_\alpha(x) = 0\}$ for each ordinal α and note that $I_\alpha = \cup_{\gamma<\alpha} I_\gamma$ for a limit ordinal α. The assertion follows from this fact since $L_\alpha = 0$ iff $1 \in I_\alpha$.

Now let art (noeth) be the class of modular lattices having the descending (ascending) chain condition and let fin $=$ art $\cap$ noeth. We assume for the rest of this section that $\mathcal{L}$ is one of those three classes. Recall that a *dense chain* in a lattice L is a sublattice isomorphic to $\{m \cdot 2^{-n} \in [0,1] \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ with its usual ordering. The following lemma is well-known.

Lemma 1.2 $\dim_{\mathcal{L}} L = \infty$ *if and only if there is a dense chain in* L.

Proof: Let $\pi: L \to L_\infty$ be the canonical map and suppose first $L_\infty \neq 0$. We can construct inductively a dense chain in L_∞ since for any pair $x < y$ in L_∞ there exists z with $x < z < y$. Of course, a dense chain in L_∞ is the image under π of a dense chain in L. Now suppose that there is a dense chain in L, say between x and y. Using induction one shows that $\pi(x) \neq \pi(y)$ and therefore $\dim_{\mathcal{L}} L = \infty$.

Lemma 1.3 *Let* $f: \mathcal{D} \to \mathcal{C}$ *be a faithful exact functor between abelian categories. Then* $\dim_{\mathcal{L}} \mathcal{D} \leq \dim_{\mathcal{L}} \mathcal{C}$.

Proof: The $\mathcal{L}$-filtration $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ induces a filtration $(f^{-1}(\mathcal{C}_\alpha))_\alpha$ of $\mathcal{D}$ and it is easily checked that $f^{-1}(\mathcal{C}_\alpha) \subseteq \mathcal{D}_\alpha$ for every α, where $(\mathcal{D}_\alpha)_\alpha$ denotes the $\mathcal{L}$-filtration of $\mathcal{C}$.

2. The functor category

Let Λ be a ring. We denote by $\mathrm{Mod}(\Lambda)$ the category of (right) Λ-modules and by $\mathrm{mod}(\Lambda)$ the full subcategory of finitely presented Λ-modules. Let $D(\Lambda) = (\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ be the category of additive functors $\mathrm{mod}(\Lambda^{\mathrm{op}}) \to \mathrm{Ab}$ into the category of abelian groups. The fully faithful functor

$$\mathrm{Mod}(\Lambda) \longrightarrow D(\Lambda), \quad M \mapsto M \otimes_\Lambda -$$

will play an important role in our considerations. An exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{Mod}(\Lambda)$ is said to be *pure-exact* if its image

$$0 \longrightarrow L \otimes_\Lambda - \longrightarrow M \otimes_\Lambda - \longrightarrow N \otimes_\Lambda - \longrightarrow 0$$

under this functor is exact and $M \in \mathrm{Mod}(\Lambda)$ is *pure-injective* if $M \otimes_\Lambda -$ is injective. Note that any injective object in $D(\Lambda)$ is isomorphic to $M \otimes_\Lambda -$ for some module M. We shall also use the full subcategory $C(\Lambda) = \mathrm{fp}(\mathrm{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab})$ of finitely presented functors in $D(\Lambda)$. Observe that $C(\Lambda)$ is an abelian subcategory, equivalently that finitely generated subfunctors of finitely presented functors again are finitely presented. For details and proofs of the basic properties of the category $D(\Lambda)$ we refer to Crawley-Boevey's exposition [6].

Now fix a Λ-module M and let $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$. Identifying $\mathrm{End}(M \otimes_\Lambda -)^{\mathrm{op}}$ in $D(\Lambda)$ with Γ we obtain a functor

$$h_M : C(\Lambda) \longrightarrow \mathrm{Mod}(\Gamma), \quad X \mapsto \mathrm{Hom}(X, M \otimes_\Lambda -)$$

which is exact. We denote by $\mathcal{S}_M$ the kernel of h_M which is a Serre subcategory of $C(\Lambda)$. The *endocategory* $\mathcal{E}_M$ of M is, by definition, the image of the induced functor $C(\Lambda)/\mathcal{S}_M \to \mathrm{Mod}(\Gamma)$ and we shall tacitly assume that this subcategory of $\mathrm{Mod}(\Gamma)$ is closed under isomorphisms.

We consider now subobjects of M. Denote by $\mathsf{L}(M) = \mathsf{L}(M_\Lambda)$ the lattice of Λ-submodules of M. Recall that an additve subgroup of M is of *finite definition* if it arises as the kernel of a morphism $M \to M \otimes_\Lambda X$, $m \mapsto m \otimes x$ for some finitely presented Λ^{op}-module X, and some element $x \in X$. Note that a subgroup of finite definition is a Γ-submodule of M. Moreover, the set of all these subgroups is a sublattice of $\mathsf{L}(M_\Gamma)$. Following the notation in the preceding section we denote by $\dim_{\mathcal{L}} M$ the $\mathcal{L}$-dimension of the lattice of subgoups of finite definition. The following lemma collects the essential properties of $\mathcal{E}_M$.

Lemma 2.1 (1) *The endocategory $\mathcal{E}_M$ of M is abelian and the inclusion $\mathcal{E}_M \to \mathrm{Mod}(\Gamma)$ is exact.*

(2) *The functor* $\mathrm{Hom}(\ , M \otimes_\Lambda -)$ *induces an equivalence* $(C(\Lambda)/\mathcal{S}_M)^{\mathrm{op}} \to \mathcal{E}_M$.

(3) *The* Γ*-module* M *belongs to* $\mathcal{E}_M$ *and the subobjects of* M *in* $\mathcal{E}_M$ *are precisely the subgroups of finite definition.*

(4) *Every object in* $\mathcal{E}_M$ *is a subquotient of a finite coproduct of copies of* M*.*

Proof: (1) and (2) follow immediately from the definition of $\mathcal{E}_M$.

(3) We observe first that M belongs to $\mathcal{E}_M$ since $M \cong \mathrm{Hom}(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, \ ), M \otimes_\Lambda -)$ by Yoneda's lemma. Now suppose that U is a subobject of M in $\mathcal{E}_M$. Thus there is a an epimorphism $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, \ ) \to F$ in $C(\Lambda)$ such that $h_M(F) = U$. Choose a presentation $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(X, \ ) \to \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, \ ) \to F \to 0$ and apply Yoneda's lemma to obtain U as kernel of a morphism $M \to M \otimes_\Lambda X$. Conversely, suppose that $U = \mathrm{Ker}(M \otimes_\Lambda \varphi)$ for some $\varphi \colon \Lambda \to X$ in $\mathrm{mod}(\Lambda^{\mathrm{op}})$. It follows that $U = h_M(F)$ for $F = \mathrm{Coker}(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\varphi, \ ))$.

(4) The assertion follows from the fact that every object in $C(\Lambda)$ is a subquotient of a finite coproduct of copies of $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, \ )$.

Now let $\mathcal{S}$ be a Serre subcategory of $C(\Lambda)$. We call a full subcategory of $\mathrm{Mod}(\Lambda)$ *definable* if it consists of all modules M satisfying $\mathrm{Hom}(\mathcal{S}, M \otimes_\Lambda -) = 0$. A definable subcategory is automatically closed under direct limits, products and pure submodules, and it can be shown that this property characterizes a definable subcategory [17, Corollary 4.6]. Let $\mathcal{X}$ be a class of Λ-modules. We shall frequently identify $\mathcal{X} = M$ if $\mathcal{X}$ consists of a single object M. We denote by $\mathcal{S}_{\mathcal{X}}$ the Serre subcategory of all objects $X \in C(\Lambda)$ satisfying $\mathrm{Hom}(X, M \otimes_\Lambda -) = 0$ for all M in $\mathcal{X}$ and we call the definable subcategory corresponding to $\mathcal{S}_{\mathcal{X}}$ *generated* by $\mathcal{X}$.

Finally, let us recall from [24] that the *Ziegler spectrum* $\mathrm{Zsp}(\Lambda)$ of Λ is the set of isomorphism classes of indecomposable pure-injective Λ-modules. The following proposition gives a description of the topology on $\mathrm{Zsp}(\Lambda)$.

Proposition 2.2 *The assignment*

$$\mathcal{S} \mapsto \{M \in \mathrm{Zsp}(\Lambda) \mid \mathrm{Hom}(\mathcal{S}, M \otimes_\Lambda -) = 0\}$$

defines a bijection between the Serre subcategories of $C(\Lambda)$ *and the closed subsets of* $\mathrm{Zsp}(\Lambda)$*. Moreover, for any object* $X \in C(\Lambda)$ *the set* $\{M \in \mathrm{Zsp}(\Lambda) \mid \mathrm{Hom}(X, M \otimes_\Lambda -) = 0\}$ *is closed.*

Proof: See [13, Theorem 4.2].

Given a class $\mathcal{X}$ of modules we denote by $\overline{\mathcal{X}}$ the closed subset corresponding to $\mathcal{S}_{\mathcal{X}}$ and call $\overline{\mathcal{X}}$ the *Ziegler closure* of $\mathcal{X}$.

We end this section with some examples which illustrate the preceding definitions. Recall that a ring Λ is *right coherent* if every finitely generated submodule of a finitely presented Λ-module is finitely presented. Furthermore a Λ-module M is *fp-injective* if $\operatorname{Ext}^1_\Lambda(X, M) = 0$ for every finitely presented X.

Example 2.3 *Λ is right coherent iff the fp-injective Λ-modules form a definable subcategory. In this case the indecomposable injective Λ-modules form a closed subset of* $\operatorname{Zsp}(\Lambda)$. *The corresponding Serre subcategory $\mathcal{S}$ of $C(\Lambda)$ is the kernel of the functor $C(\Lambda) \to$* $\operatorname{mod}(\Lambda)$, *$F \mapsto F(\Lambda)$, which induces an equivalence between $C(\Lambda)/\mathcal{S}$ and* $\operatorname{mod}(\Lambda)$.

Proof: It is well-known that Λ is right coherent iff the full subcategory $\operatorname{fpinj}(\Lambda)$ of fp-injective Λ-modules is closed under direct limits. Moreover $\operatorname{fpinj}(\Lambda)$ is automataically closed under pure submodules and products. It follows that Λ is right coherent iff $\operatorname{fpinj}(\Lambda)$ is definable. An fp-injective Λ module is pure-injective iff it is injective and therefore $\operatorname{fpinj}(\Lambda) \cap \operatorname{Zsp}(\Lambda)$ is precisely the set of indecomposable injective Λ-modules. We conclude that this set is closed if Λ is right coherent. Given a functor $F \in C(\Lambda)$ there is an exact sequence

$$0 \longrightarrow F \longrightarrow X \otimes_\Lambda - \xrightarrow{\varphi \otimes_\Lambda -} Y \otimes_\Lambda -$$

with $X, Y \in \operatorname{mod}(\Lambda)$. Thus $F(\Lambda) = 0$ iff φ is a monomorphism and therefore $\mathcal{S}_{\operatorname{fpinj}(\Lambda)} = \{F \in C(\Lambda) \mid F(\Lambda) = 0\}$. We leave it to the reader to verify that the evaluation

$$D(\Lambda) = (\operatorname{mod}(\Lambda^{\mathrm{op}}), \mathrm{Ab}) \longrightarrow (\Lambda^{\mathrm{op}}, \mathrm{Ab}) = \operatorname{Mod}(\Lambda), \quad F \mapsto F(\Lambda)$$

induces an equivalence $C(\Lambda)/\mathcal{S} \to \operatorname{mod}(\Lambda)$.

Suppose now that Λ is right noetherian. Thus every finitely generated Λ-module is finitely presented and therefore Baer's criterion implies that every fp-injective Λ-module is injective. We proceed with a description of the closed subsets of the injective modules in $\operatorname{Zsp}(\Lambda)$. Let us assume for simplicity that Λ is commutative and denote by $\operatorname{Spec}(\Lambda)$ the *prime spectrum* of Λ. We shall identify every prime ideal $\mathfrak{p} \in \operatorname{Spec}(\Lambda)$ with the injective envelope $E(\Lambda/\mathfrak{p}) \in \operatorname{Zsp}(\Lambda)$. Recall that every localizing subcategory $\mathcal{T}$ of $\operatorname{Mod}(\Lambda)$ is completely described by the corresponding *Gabriel topology* $\mathcal{F} = \{\mathfrak{a} \subseteq \Lambda \mid \Lambda/\mathfrak{a} \in \mathcal{T}\}$.

Example 2.4 (1) *The assignment* $\mathcal{F} \mapsto \mathrm{Spec}(\Lambda) \setminus \mathcal{F}$ *defines a bijection between the Gabriel topologies on* Λ *and the closed subsets of* $\mathrm{Spec}(\Lambda) \subseteq \mathrm{Zsp}(\Lambda)$.

(2) *If* M *is an injective cogenerator for* $\mathrm{Mod}(\Lambda)$, *then* $\dim_{\mathrm{fin}} M$ *equals* $\dim_{\mathrm{fin}} \mathsf{L}(\Lambda)$ *which is the Krull dimension of* Λ.

Proof: (1) The assignment $\mathcal{T} \mapsto \mathcal{T} \cap \mathrm{mod}(\Lambda)$ defines a bijection between the localizing subcategories of $\mathrm{Mod}(\Lambda)$ and the Serre subcategories of $\mathrm{mod}(\Lambda)$ [7, Proposition III.10]. But the Serre subcategories of $\mathrm{mod}(\Lambda)$ are in bijective correspondence to the Serre subcategories of $C(\Lambda)$ containing $\mathcal{S}_{\mathrm{fpinj}(\Lambda)}$ by the preceding example. The assertion now follows from Proposition 2.2.

(2) We have $\dim_{\mathrm{fin}} M = \dim_{\mathrm{fin}} \mathrm{mod}(\Lambda)$ since $\mathcal{S}_M = \mathcal{S}_{\mathrm{fpinj}(\Lambda)}$. Moreover $\dim_{\mathrm{fin}} \mathrm{mod}(\Lambda) = \dim_{\mathrm{fin}} \mathsf{L}(\Lambda)$ by Lemma 1.1 and this equals the Krull dimension of Λ in the sense of Gabriel [7, Corollary V.4.2].

3. Almost split sequences for pure-injective modules.

We characterize the indecomposable pure-injective modules which occur as a source of an almost split sequence. This property will play a significant role in our discussion of indecomposable endofinite modules. We need to recall the following terminology. A morphism $\varphi\colon M \to N$ is *left almost split* if it is not a split monomorphism, and any morphism $M \to X$ which is not a split monomorphism factors through φ. Analogously, a *right almost split* morphism is defined. An exact sequence $0 \to M \xrightarrow{\varphi} N \xrightarrow{\psi} L \to 0$ is *almost split* if φ is left almost split and ψ is right almost split. A morphism $\varphi\colon M \to N$ is *left minimal* if any endomorphism ψ of N satisfying $\varphi = \psi \circ \varphi$ is an automorphism. Note that a morphism $\varphi\colon M \to N$ is minimal left almost split if and only if one of the following two cases occurs [2, II, Proposition 4.4]:

(1) M is injective with simple socle and φ is isomorphic to $M \to M/\operatorname{soc} M$.

(2) There is an almost split sequence $0 \to M \xrightarrow{\varphi} N \xrightarrow{\psi} L \to 0$.

Now let M be a finitely presented Λ-module with local endomorphism ring $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$. Then we denote by DM the Λ^{op}-module $\mathrm{Hom}_\Gamma(M, I)$ where I is an injective envelope of the unique simple Γ-module. Note that DM has automatically a local endomorpism ring. If M

is pure-injective and DM is finitely presented, then $DDM \cong M$ [25, Lemma 5]. We are now in position to state the main result of this section.

Theorem 3.1 *Let Λ be a semi-perfect ring. Then the following are equivalent for a Λ-module M.*

(1) *M is pure-injective and occurs as a source of a minimal left almost split morphism.*

(2) *$M \cong DN$ for some finitely presented Λ^{op}-module N having a local endomorphism ring.*

The proof is based on an existence result for almost split sequences which is due to Auslander. We need the following definition. If X is a finitely presented Λ-module without non-zero projective direct summand and $Q \xrightarrow{\varphi} P \to X \to 0$ is a minimal projective presentation, then the *transpose* $\operatorname{Tr} X$ is the finitely presented Λ^{op}-module which is the cokernel of $\operatorname{Hom}_\Lambda(\varphi, \Lambda)$. Note that

$$\operatorname{Hom}_\Lambda(P, \Lambda) \xrightarrow{\operatorname{Hom}_\Lambda(\varphi,\Lambda)} \operatorname{Hom}_\Lambda(Q, \Lambda) \longrightarrow \operatorname{Tr} X \longrightarrow 0$$

is again a minimal projective presentation. Therefore $\operatorname{End}_\Lambda(X)$ is local iff $\operatorname{End}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} X)$ is local.

Proposition 3.2 *Let X be a finitely presented non-projective Λ-module with local endomorphism ring. Then there exists an almost split sequence $0 \to D\operatorname{Tr} X \to E \to X \to 0$.*

Proof: See [2, II, Proposition 5.1].

The following lemma is well-known.

Lemma 3.3 *A sequence $0 \to M \to N \to L \to 0$ of Λ-modules is pure-exact iff the induced sequence $0 \to \operatorname{Hom}_\Lambda(X, M) \to \operatorname{Hom}_\Lambda(X, N) \to \operatorname{Hom}_\Lambda(X, L) \to 0$ is exact for every finitely presented Λ-module X.*

Proof: Use the isomorphism $\underline{\operatorname{Hom}}_\Lambda(X, \ ) \cong \operatorname{Tor}_1^\Lambda(\operatorname{Tr} X, \ )$ [2, I, Proposition 3.2] where $\underline{\operatorname{Hom}}$ denotes the goup of morphisms modulo the subgroup of those which factor through a projective.

Proof of Theorem 3.1: (1) $\Rightarrow$ (2) Suppose first that M is injective and let $S = \operatorname{soc} M$. Using the fact that Λ is semi-perfect we find an indecomposable projective Λ^{op}-module N such that $S \otimes_\Lambda N \neq 0$. The isomorphism

$$\operatorname{Hom}_\Lambda(\ ,\operatorname{Hom}_\Gamma(N, I)) \cong \operatorname{Hom}_\Gamma(- \otimes_\Lambda N, I)$$

shows that $DN = \operatorname{Hom}_\Gamma(N, I)$ is injective with $\operatorname{Hom}_\Lambda(S, DN) \neq 0$. Thus DN is an injective envelope of S and therefore isomorphic to M. Now suppose that $0 \to M \xrightarrow{\varphi} E \xrightarrow{\psi} L \to 0$ is an almost split sequence. The sequence is not pure-exact since M is pure-injective and therefore $\operatorname{Hom}_\Lambda(X, \psi)$ is not an epimorphism for some finitely presented X. The minimality of φ now implies that L is isomorphic to a direct summand of X and therefore finitely presented. Moreover, $\operatorname{End}_\Lambda(L)$ is local since ψ is right almost split. We can apply the above proposition and infer that $M \cong D \operatorname{Tr} L$.

(2) $\Rightarrow$ (1) Let $M \cong DN$. Suppose first that N is projective. The isomorphism used in the first part of the proof shows that M is injective. Moreover, if the Λ-module $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(N, \Lambda)$ is the projective cover of the simple module S, then the isomorphism

$$S \otimes_\Lambda N \cong \operatorname{Hom}_\Lambda(\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(N, \Lambda), S)$$

combined with the above isomorphism shows that there is a non-zero morphism $S \to DN$. The cokernel morphism then gives a minimal left almost split morphism with source DN. If N is non-projective, then $\operatorname{Tr} N$ is non-projective and the above proposition yields an almost split sequence with source DN since $\operatorname{Tr} \operatorname{Tr} N \cong N$. Moreover, the sequence is not pure-exact since the end term is finitely presented, and therefore DN is pure-injective.

4. Endofinite modules over dualizing rings

Given a Λ-module M we denote by $\ell_{\mathrm{end}}(M)$ the length of M when regarded in the natural way as a $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$-module. The module is called *endofinite* if its *endolength* is finite. Any endofinite module is automatically Σ-pure-injective since it satisfies the descending chain condition on subgroups of finite definition. Recall that M is Σ*-pure-injective* if any coproduct of copies of M is pure-injective. We call a module *generic* if it is endofinite and indecomposable but not finitely presented. Now suppose that M is endofinite and also finitely presented indecomposable. Then DM denotes the Λ^{op}-module $\mathrm{Hom}_\Gamma(M,I)$ where I is an injective envelope of the unique simple Γ-module. Note that DM is automatically indecomposable and endofinite. Following [16] the ring Λ is said to be *right dualizing* if it satisfies the following condition:

(D) If M is a finitely presented indecomposable endofinite Λ-module, then its dual DM is a finitely presented indecomposable endofinite Λ^{op}-module.

If Λ and Λ^{op} both satisfy (D), then we call Λ *dualizing*. Important examples of dualizing rings are noetherian algebras [5] and artinian PI-rings [23]. Recall that Λ is a *noetherian algebra* if the center $Z(\Lambda)$ is noetherian and Λ is a finitely generated $Z(\Lambda)$-module. Noetherian algebras over complete local rings are examples of Krull-Schmidt rings [2]. A ring Λ is *Krull-Schmidt* if every finitely presented Λ-module decomposes into a coproduct of indecomposable modules having local endomorphism rings. Observe that this notion is right-left symmetric since Λ needs to be semi-perfect. Of course every artinian ring is Krull-Schmidt.

Our aim in this section is a characterization of the endofinite modules which are not generic. We shall use the following description of the injective envelopes of simple functors in $D(\Lambda)$ which is due to Crawley-Boevey [6].

Proposition 4.1 *For an indecomposable pure-injective Λ-module M the following are equivalent.*

(1) *$M \otimes_\Lambda -$ is the injective envelope of a simple functor.*

(2) *M occurs as a source of a left almost split morphism.*

(3) *$M \cong \mathrm{Hom}_\Gamma(N,I)$ with N some finitely presented Λ^{op}-module, $\Gamma = \mathrm{End}_{\Lambda^{\mathrm{op}}}(N)^{\mathrm{op}}$ and I the injective envelope of a simple Γ-module.*

Proof: See [6, Theorem 2.3].

Having determined in the preceding section the indecomposable pure-injective modules which occur as a source of a minimal left almost split morphism, we observe now that there is no difference between the existence of a left almost split and a minimal left almost split morphism if Λ is a Krull-Schmidt ring.

Proposition 4.2 *Let Λ be a Krull-Schmidt ring. Then the following are equivalent for an indecomposable pure-injective Λ-module M.*

(1) *M occurs as a source of a left almost split morphism.*

(2) *M occurs as a source of a minimal left almost split morphism.*

Proof: Suppose that M is the source of a left almost split morphism. Then $M \cong \mathrm{Hom}_\Gamma(N, I)$ for some $N \in \mathrm{mod}(\Lambda^{\mathrm{op}})$ by the preceding lemma. In fact $M \otimes_\Lambda -$ is an injective envelope of a simple factor of $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(N,\ )$ which needs to be isomorphic to a factor of $\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(N',\ )$ for some indecomposable direct summand N' of N by our assumption on the ring. The assertion now follows from Theorem 3.1 since $M \cong DN'$.

We call an indecomposable pure-injective Λ module M *isolated* if $\{M\}$ is an open set in $\mathrm{Zsp}(\Lambda)$.

Lemma 4.3 *Let M be an indecomposable pure-injective Λ-module.*

(1) *If M is isolated, then $M \otimes_\Lambda -$ is the injective envelope of a simple functor.*

(2) *If M is finitely presented and the injective envelope of a simple functor, then M is isolated.*

Proof: (1) The module M is isolated if there is a non-zero $X \in C(\Lambda)$ such that $\mathrm{Hom}(X, N \otimes_\Lambda -) = 0$ for every $N \in \mathrm{Zsp}(\Lambda)$ different from M. Choosing a maximal subobject $U \subseteq X$ in $D(\Lambda)$ we infer that $M \otimes_\Lambda -$ is an injective envelope of X/U.

(2) A simple subfunctor S of $M \otimes_\Lambda -$ is obviously finitely generated and therefore also finitely presented since $M \otimes_\Lambda -$ is finitely presented and $C(\Lambda)$ is abelian. Thus $\{M\} = \{N \in \mathrm{Zsp}(\Lambda) \mid \mathrm{Hom}(S, N \otimes_\Lambda -) \neq 0\}$ since an injective envelope is unique up to isomorphism, and it follows from Proposition 2.2 that $\{M\}$ is open.

Lemma 4.4 *Let M be a finitely presented indecomposable pure-injective Λ-module and suppose that DM is finitely presented. Then $M \otimes_\Lambda -$ is the injective envelope of a finitely presented and simple functor in $D(\Lambda)$, and therefore M is isolated.*

Proof: The assumption implies that $M \cong DDM$ by [25, Lemma 5]. Thus $M \otimes_\Lambda -$ is the injective envelope of a simple functor, say S, by Proposition 4.1. The arguments used in the preceding lemma show that S is finitely presented and that M is isolated.

Lemma 4.5 *A non-zero module M is endofinite if and only if* $\dim_{\mathrm{fin}} M = 0$.

Proof: See [6, Proposition 4.1].

Lemma 4.6 *Let M be an indecomposable endofinite Λ-module. If M occurs as a source of an almost split morphism, then $M \cong DN$ for some finitely presented indecomposable endofinite Λ^{op}-module N.*

Proof: It follows from Proposition 4.1 that $M \otimes_\Lambda -$ is the injective envelope of a simple functor S and $M \cong \mathrm{Hom}_\Gamma(L, I)$ with L some finitely presented Λ^{op}-module, $\Gamma = \mathrm{End}_{\Lambda^{\mathrm{op}}}(L)^{\mathrm{op}}$ and I the injective envelope of a simple Γ-module. More precisely, there is an epimorphism $\pi\colon \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(L,\ ) \to S$ and the kernel of $\pi_L\colon \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(L, L) \to S(L)$ is the maximal ideal such that I is the injective envelope of $\Gamma/\operatorname{Ker}(\pi_L)$. We shall identify $\mathrm{End}_\Gamma(I)^{\mathrm{op}}$ with $\Sigma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ via $\varphi \mapsto \mathrm{Hom}_\Gamma(L, \varphi)$. Taking an injective envelope J of $I \coprod \Sigma/\operatorname{rad}\Sigma$ in $\mathrm{Mod}(\Sigma)$ we define $P = \mathrm{Hom}_\Sigma(M, J)$. The module J is an injective cogenerator for Σ and therefore $\mathrm{Hom}_\Sigma(\ , J)$ induces an anti-isomorphism between the lattice of subgroups of finite definition of M and P. This follows from the well-known isomorphism

$$X \otimes_\Lambda \mathrm{Hom}_\Sigma(M, J) \longrightarrow \mathrm{Hom}_\Sigma(\mathrm{Hom}_\Lambda(X, M), J)$$

for all $X \in \mathrm{mod}(\Lambda)$ which is functorial in X [4, VI, Proposition 5.2], if one uses the alternative description of a subgroup of finite definition of M as $\mathrm{Im}(\mathrm{Hom}_\Lambda(\varphi, M))$ for some morphism $\varphi\colon \Lambda \to X$ in $\mathrm{mod}(\Lambda)$. Thus P is endofinite by the preceding lemma and there is a decomposition $P = \coprod_{i\in K} P_i$ into indecomposable endofinite Λ^{op}-modules [6, Proposition 4.5]. We claim that $M \cong DP_i$ for some i. To this end consider the morphism $\alpha\colon L \to P$ defined by $\alpha(x)(\varphi) = \varphi(x)$. The following commutative diagram shows that $M \otimes_\Lambda \alpha$ is a monomorphism.

$$\begin{array}{ccccc}
\mathrm{Hom}_\Gamma(L, I) \otimes_\Lambda L & \longrightarrow & \mathrm{Hom}_\Gamma(\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(L, L), I) & \longrightarrow & I \\
\downarrow {\scriptstyle M\otimes_\Lambda \alpha} & & & & \downarrow \\
M \otimes_\Lambda \mathrm{Hom}_\Sigma(M, J) & \longrightarrow & \mathrm{Hom}_\Sigma(\mathrm{Hom}_\Lambda(M, M), J) & \longrightarrow & J
\end{array}$$

In fact, the horizontal morphisms are the obvious evaluation maps and the upper ones are isomorphisms since L is finitely presented. We consider now the composition

$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(P,\ ) \xrightarrow{\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\alpha,\ )} \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(L,\ ) \xrightarrow{\pi} S.$$

Applying $\operatorname{Hom}(\ ,M \otimes_\Lambda -)$ it follows with Yoneda's lemma that $\pi \circ \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\alpha,\ ) \neq 0$. Thus for some $i \in K$ the corresponding morphism $\alpha_i \colon L \to P_i$ induces an epimorphism

$$\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(P_i,\ ) \xrightarrow{\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\alpha_i,\ )} \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(L,\ ) \xrightarrow{\pi} S.$$

Using again Yoneda's lemma there is a morphism $\beta \colon P_i \to L$ such that

$$\pi = \pi \circ \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\alpha_i,\ ) \circ \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\beta,\ ).$$

The composition $\gamma = \alpha_i \circ \beta$ is an automorphism of P_i since

$$\pi \circ \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\alpha_i,\ ) = \pi \circ \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\alpha_i,\ ) \circ \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\gamma,\ )$$

and $\operatorname{End}_{\Lambda^{\mathrm{op}}}(P_i)$ is local. Thus P_i is isomorphic to a direct summand of L and therefore finitely presented. Moreover $M \cong DP_i$ and this finishes the proof.

We collect our findings and obtain the following characterization of endofinite modules which are not generic. This generalizes a result of Crawley-Boevey for noetherian algebras [6, Proposition 7.7] and includes also a result of Schmidmeier [23]. However, their methods are quite different.

Theorem 4.7 *Let Λ be a dualizing ring. Then the following are equivalent for an indecomposable endofinite Λ-module M.*

(1) *M is finitely presented.*

(2) *M is isolated.*

(3) *M occurs as a source of a left almost split morphism.*

(4) *$M \cong DN$ for some finitely presented indecomposable endofinite Λ^{op}-module N.*

Proof: (1) $\Rightarrow$ (2) follows from Lemma 4.4 since Λ is right dualizing.

(2) $\Rightarrow$ (3) This part needs no assumption on the ring. It follows from Lemma 4.3 and Proposition 4.1.

(3) $\Rightarrow$ (4) This part needs no assumption on the ring. It follows from Lemma 4.6.

(4) $\Rightarrow$ (1) follows from the assumption that Λ is left dualizing.

The assumption on the ring to be dualizing is necessary in the preceding theorem. In fact, one obtains the following characterization of dualizing rings.

Corollary 4.8 *For a ring Λ the following are equivalent.*

(1) *Λ is dualizing.*

(2) *Every indecomposable endofinite Λ- or Λ^{op}-module is finitely presented if and only if it is isolated.*

(3) *Every indecomposable endofinite Λ- or Λ^{op}-module is finitely presented if and only if it occurs as a source of a left almost split morphism.*

Proof: (1) $\Rightarrow$ (2), (3) follows from the preceding theorem.

(3) $\Rightarrow$ (1) Let M be a finitely presented indecomposable and endofinite Λ-module. We want to show that DM is finitely presented. It follows from Lemma 4.6 that $M \cong DN$ for some finitely presented indecomposable endofinite Λ^{op}-module N. Thus $DM \cong DDN \cong N$, and therefore DM is finitely presented. We conclude that Λ is right dualizing and the dual argument shows that Λ is also left dualizing.

(2) $\Rightarrow$ (1) Adapt the argument of (3) $\Rightarrow$ (1).

5. Existence of generic modules

This section is devoted to existence results for endofinite modules. Our first criterion needs no assumption on the ring.

Proposition 5.1 *Let M be a non-zero Λ-module and suppose that M has no dense chain of subgroups of finite definition. Then there exists an endofinite Λ-module in the Ziegler closure of M.*

Proof: Let $\mathcal{C} = C(\Lambda)/\mathcal{S}_M$. The assumption on M implies that $\dim_{\mathrm{fin}} \mathcal{C} = \dim_{\mathrm{fin}} M = \beta + 1$ for some ordinal β by Lemma 2.1 and Lemma 1.1. Let $(\mathcal{C}_\alpha)_\alpha$ be the fin-filtration of $\mathcal{C}$ and denote by q the composition of the quotient functors $C(\Lambda) \to C(\Lambda)/\mathcal{S}_M$ and $\mathcal{C} \to \mathcal{C}/\mathcal{C}_\beta$. The kernel $\mathcal{S} = \mathrm{Ker}(q)$ is a proper Serre subcategory of $C(\Lambda)$ and the bijective correspondence between Serre subcategories and closed subsets of $\mathrm{Zsp}(\Lambda)$ allows to choose $N \in \mathrm{Zsp}(\Lambda)$ such that $\mathrm{Hom}(\mathcal{S}, N \otimes_\Lambda -) = 0$. We have $\mathcal{S}_M \subseteq \mathcal{S} \subseteq \mathcal{S}_N$. Clearly, the inclusion implies that N belongs to the definable subcategory generated by M. The second inclusion implies that N is endofinite. In fact there is an induced quotient functor $\mathcal{C}/\mathcal{C}_\beta \to C(\Lambda)/\mathcal{S}_N$ and therefore

$$\dim_{\mathrm{fin}} N = \dim_{\mathrm{fin}} C(\Lambda)/\mathcal{S}_N \leq \dim_{\mathrm{fin}} \mathcal{C}/\mathcal{C}_\beta = 0.$$

We conclude from Lemma 4.5 that N is endofinite.

Our next aim is to show that under certain additional assumptions the endofinite module constructed above is a generic module. We need some lemmas.

Lemma 5.2 *Let M and N be Λ-modules. Suppose that N is finitely presented indecomposable pure-injective and that DN is finitely presented. Then the following are equivalent.*

(1) *N is isomorphic to a direct summand of M.*

(2) *N belongs to the Ziegler closure of M.*

Proof: One direction is trivial. Therefore suppose (2). It follows from Lemma 4.4 that $N\otimes_\Lambda -$ has a finitely presented and simple socle, say S, and (2) implies $\mathrm{Hom}(S, M\otimes_\Lambda -) \neq 0$. A non-zero morphism $S \to M \otimes_\Lambda -$ extends to a morphism $\varphi: N \otimes_\Lambda - \to M \otimes_\Lambda -$ since $\mathrm{Hom}(\ , M \otimes_\Lambda -)$ is exact on $C(\Lambda)$ and $N \otimes_\Lambda -$ is finitely presented. Moreover, φ is a monomorphism since $N \otimes_\Lambda -$ is the injective envelope of S. We conclude that φ induces a split monomorphism $N \to M$ since N is pure-injective, and therefore (1) holds.

We need to study the lattice of subgroups of finite definition of a Λ-module M. To this end consider again the exact functor

$$h_M: C(\Lambda) \longrightarrow \mathrm{Mod}(\Gamma), \quad X \mapsto \mathrm{Hom}(X, M \otimes_\Lambda -)$$

where $\Gamma = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$ and let $F = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda,\ )$. A chain

$$0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = F$$

of subobjects in $C(\Lambda)$ is denoted by $(F_i)_{0\leq i\leq n}$.

Lemma 5.3 *M is endofinite of endolength at most n iff for every chain $(F_i)_{0\leq i\leq n+1}$ there is $i \in \{0, \ldots, n\}$ such that $h_M(F_{i+1}/F_i) = 0$.*

Proof: We use the fact that all Γ-submodules are of finite definition if M is endofinite [6, Proposition 4.1]. The assertion now follows with Lemma 4.5 since h_M induces an epimorphism from the lattice of subobjects of F onto the lattice of subgroups of finite definition of M by Lemma 2.1.

The following is due to Herzog [9].

Proposition 5.4 *For every $n \in \mathbb{N}$ the modules of endolength at most n form a closed subset of the Ziegler spectrum.*

Proof: For every chain $\varphi = (F_i)_{0 \leq i \leq n+1}$ we define $\mathcal{U}_{\varphi,i} = \{M \in \operatorname{Zsp}(\Lambda) \mid h_M(F_{i+1}/F_i) = 0\}$ for all i and $\mathcal{U}_\varphi = \cup_{i=0}^n \mathcal{U}_{\varphi,i}$. These are closed subsets of $\operatorname{Zsp}(\Lambda)$ and therefore also the intersection $\mathcal{U} = \cap_\varphi \mathcal{U}_\varphi$ is closed where φ runs through all chains $\varphi = (F_i)_{0 \leq i \leq n+1}$. But $\mathcal{U}$ is precisely the set of all modules of endolength at most n by the preceding lemma, and this finishes the proof.

We shall also use the compactness of $\operatorname{Zsp}(\Lambda)$ which is due to Ziegler [24].

Proposition 5.5 $\operatorname{Zsp}(\Lambda)$ *is a quasi-compact topological space.*

Proof: The proof is based on our description of the Ziegler topology in Proposition 2.2. Let $(\mathcal{S}_i)_{i \in I}$ be a family of Serre subcategories. Then $\cup_{i \in I} \mathcal{S}_i = C(\Lambda)$ iff $F = \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\Lambda, \ ) \in \cup_{i \in I} \mathcal{S}_i$ since every finitely presented functor is a subquotient of F^n for some natural n. But the latter condition is equivalent to $F \in \cup_{i \in J} \mathcal{S}_i$ for some finite $J \subseteq I$ since $F \in \cup_{i \in I} \mathcal{S}_i$ iff there exists a finite chain $(F_j)_{0 \leq j \leq n}$ with elements $i_j \in I$ such that $F_{j+1}/F_j \in \mathcal{S}_{i_j}$ for all j.

We are now in position to prove the following main result. Note that for artin algebras the second criterion is due to Crawley-Boevey [6].

Theorem 5.6 *Let Λ be a right dualizing ring. Then there exists a generic Λ-module provided that one of the following conditions holds.*

(1) *There is a non-zero Λ-module having no finitely presented indecomposable endofinite direct summand and no dense chain of subgroups of finite definition.*

(2) *For some natural number n there is an infinite number of finitely presented indecomposable Λ-modules of endolength at most n.*

Proof: (1) It is shown in Proposition 5.1 that there exists an indecomposable endofinite module N in the Ziegler closure of M. To show that N is not finitely presented we assume the converse. Then N is isomorphic to a direct summand of M by Lemma 5.2 since Λ is right dualizing. This contradicts condition (1) and therefore N is a generic module.

(2) We follow an idea of Herzog. Suppose that $\mathcal{U} = \{M \in \operatorname{Zsp}(\Lambda) \mid \ell_{\mathrm{end}}(M) \leq n\}$ is infinite. It is a quasi-compact space according to the preceding propositions and therefore

$\{M\}$ is not open for some $M \in \mathcal{U}$. We conclude from Lemma 4.4 that $\mathcal{U}$ contains a generic module.

Our next aim is to prove a refinement of the preceding theorem. It depends on an analysis of injective envelopes in certain quotient categories of $D(\Lambda)$.

Lemma 5.7 *Let $\mathcal{S}$ be a Serre subcategory of $C(\Lambda)$ and denote by $\vec{\mathcal{S}}$ the full subcategory of $D(\Lambda)$ formed by all direct limits of objects in $\mathcal{S}$.*

(1) *$\vec{\mathcal{S}}$ is a localizing subcategory of $D(\Lambda)$.*

(2) *The quotient functor $q\colon D(\Lambda) \to D(\Lambda)/\vec{\mathcal{S}}$ induces a fully faithful and exact functor $C(\Lambda)/\mathcal{S} \to D(\Lambda)/\vec{\mathcal{S}}$.*

(3) *Let $X \in C(\Lambda)$ and suppose that $q(X)$ is simple in $C(\Lambda)/\mathcal{S}$. Then $q(X)$ is also simple in $D(\Lambda)/\vec{\mathcal{S}}$.*

(4) *The quotient functor q sends an injective object $M \in D(\Lambda)$ satisfying* $\operatorname{Hom}(\mathcal{S}, M) = 0$ *to an injective object and induces an isomorphism* $\operatorname{Hom}(X, M) \to \operatorname{Hom}(q(X), q(M))$ *for every $X \in D(\Lambda)$.*

Proof: (1) See [13, Theorem 2.8].

(2) We need to show that $\vec{\mathcal{S}} \cap C(\Lambda) = \mathcal{S}$. Then the kernel of the canonical functor

$$C(\Lambda) \longrightarrow D(\Lambda) \xrightarrow{q} D(\Lambda)/\vec{\mathcal{S}}$$

is $\mathcal{S}$ and there is induced a faithful exact functor $C(\Lambda)/\mathcal{S} \to D(\Lambda)/\vec{\mathcal{S}}$. Now let $X = \varinjlim X_i \in \vec{\mathcal{S}} \cap C(\Lambda)$ with $X_i \in \mathcal{S}$ for all i. Denote by Y_i the image of $X_i \to X$ and observe that $Y_i \in \mathcal{S}$ for all i. Clearly, $X = \sum Y_i$ and therefore $X = Y_j$ for some j since X is finitely generated. Thus $X \in \mathcal{S}$. For the fact that the induced functor is full we refer to [13, Theorem 2.6].

(3) We shall view $C(\Lambda)/\mathcal{S}$ as an abelian subcategory of $D(\Lambda)/\vec{\mathcal{S}}$ which is possible by (2). Now suppose that X is simple in $C(\Lambda)/\mathcal{S}$ and let U be a proper subobject of $q(X)$ in $D(\Lambda)/\vec{\mathcal{S}}$. We claim that $U = 0$. We find a subobject $Y \subseteq X$ such that $q(Y) = U$ and write $Y = \sum Y_i$ as sum of finitely presented subobjects. It follows that $U = \sum q(Y_i) = 0$ since every $q(Y_i)$ is a proper subobject of $q(X)$ in $C(\Lambda)/\mathcal{S}$.

(4) These are well-known facts which can be found in [7].

Lemma 5.8 *Let $\mathcal{S}$ be a Serre subcategory of $C(\Lambda)$ and let $X \in C(\Lambda)$ be simple in $C(\Lambda)/\mathcal{S}$. If M_1 and M_2 is a pair of indecomposable injective objects in $D(\Lambda)$ satisfying $\operatorname{Hom}(\mathcal{S}, M_i) = 0$ and $\operatorname{Hom}(X, M_i) \neq 0$ for $i = 1, 2$, then $M_1 \cong M_2$.*

Proof: It follows from (3) and (4) in the preceding lemma that the quotient functor $q\colon D(\Lambda) \to D(\Lambda)/\vec{\mathcal{S}}$ sends M_1 and M_2 to injective envelopes of $q(X)$. Therefore $M_1 \cong M_2$.

Lemma 5.9 *Let $\mathcal{S}$ be a Serre subcategory of $C(\Lambda)$ and denote by $\mathcal{X}$ the class of Λ-modules M satisfying $\operatorname{Hom}(\mathcal{S}, M \otimes_\Lambda -) = 0$. Then the following are equivalent.*

(1) $\dim_{\text{fin}} C(\Lambda)/\mathcal{S} = 0$

(2) *$\mathcal{X}$ is a class of endofinite modules and $\mathcal{X} \neq 0$.*

(3) *The Ziegler closure $\overline{\mathcal{X}}$ is a finite non-empty set of endofinite modules.*

Proof: (1) $\Rightarrow$ (2), (3) If $M \in \mathcal{X}$, then $\mathcal{S} \subseteq \mathcal{S}_M$ and therefore

$$\dim_{\text{fin}} M = \dim_{\text{fin}} C(\Lambda)/\mathcal{S}_M \leq \dim_{\text{fin}} C(\Lambda)/\mathcal{S} = 0.$$

It follows from Lemma 4.5 that M is endofinite. The Ziegler closure of $\mathcal{X}$ is non-empty by Proposition 2.2 since $\mathcal{S}$ is a proper Serre subcategory of $C(\Lambda)$. To show that $\overline{\mathcal{X}}$ is finite observe first that there are only finitely many simple objects in $C(\Lambda)/\mathcal{S}$ since every simple object must occur as a composition factor of $\operatorname{Hom}_{\Lambda^{\text{op}}}(\Lambda, \ )$ in $C(\Lambda)/\mathcal{S}$. It is clear that for every $M \in \overline{\mathcal{X}}$ there is $X \in C(\Lambda)$ which is simple in $C(\Lambda)/\mathcal{S}$, such that $\operatorname{Hom}(X, M \otimes_\Lambda -) \neq 0$. Using the preceding lemma we conclude that $\overline{\mathcal{X}}$ is finite.

(3) $\Rightarrow$ (1) Taking $M = \prod_{N \in \overline{\mathcal{X}}} N$ we have $\mathcal{S} = \mathcal{S}_M$ by Proposition 2.2. The module M is endofinite and therefore $\dim_{\text{fin}} C(\Lambda)/\mathcal{S} = \dim_{\text{fin}} M = 0$ by Lemma 4.5.

(2) $\Rightarrow$ (1) Adapt the argument of (3) $\Rightarrow$ (1).

An indecomposable pure-injective Λ-module M is called *simply reflexive* if the endocategory of M contains a simple object. It is not hard to see that M is simply reflexive iff there is $\varphi\colon X \to Y$ in $\operatorname{mod}(\Lambda)$ such that the cokernel of $\operatorname{Hom}_\Lambda(\varphi, M)$ is a simple $\operatorname{End}_\Lambda(M)^{\text{op}}$-module [17]. Clearly, $\dim_{\text{fin}} M < \infty$ implies that M is simply reflexive; in particular a Σ-pure-injective module is simply reflexive. The following property will be needed later.

Lemma 5.10 *Let M be indecomposable pure-injective with $\Gamma = \operatorname{End}_\Lambda(M)^{\text{op}}$. If $X \in \mathcal{E}_M$ is simple, then $\Gamma/\operatorname{rad}\Gamma \cong \operatorname{End}(X)$.*

Proof: Let $\mathcal{S} = \mathcal{S}_M$. We apply Lemma 5.7 as follows. The quotient functor $D(\Lambda) \to D(\Lambda)/\vec{\mathcal{S}}$ sends $M \otimes_\Lambda -$ to an injective envelope N of the simple object $f(X)$ where f is the fully faithful functor

$$\mathcal{E}_M^{\mathrm{op}} \xrightarrow{\sim} C(\Lambda)/\mathcal{S} \longrightarrow D(\Lambda)/\vec{\mathcal{S}}.$$

We have therefore

$$\Gamma/\operatorname{rad}\Gamma \cong (\operatorname{End}(N)/\operatorname{rad}\operatorname{End}(N))^{\mathrm{op}} \cong \operatorname{End}(f(X))^{\mathrm{op}} \cong \operatorname{End}(X).$$

Theorem 5.11 *Let M be an indecomposable pure-injective Λ-module. Then* $\dim_{\mathrm{fin}} M = 1$ *if and only if M is simply reflexive and there is a finite number $n \geq 1$ of endofinite modules N_i such that $\{M, N_1, \ldots, N_n\}$ is the Ziegler closure of M. If Λ is right dualizing, then every N_i is a generic module.*

Proof: Let $\mathcal{C} = C(\Lambda)/\mathcal{S}_M$ and denote by $\mathcal{S}$ the kernel of the canonical functor $C(\Lambda) \to \mathcal{C}/\mathcal{C}_0$. We have $\dim_{\mathrm{fin}} M = \dim_{\mathrm{fin}} \mathcal{C} = 1$ iff $\dim_{\mathrm{fin}} C(\Lambda)/\mathcal{S} = \dim_{\mathrm{fin}} \mathcal{C}/\mathcal{C}_0 = 0$. By definition

$$\overline{M} = \{N \in \operatorname{Zsp}(\Lambda) \mid \operatorname{Hom}(\mathcal{S}_M, N \otimes_\Lambda -) = 0\}$$

and the first part of the assumption follows immediately from Lemma 5.9 once we have shown that $\overline{M} \setminus \{M\} = \mathcal{U}$ for

$$\mathcal{U} = \{N \in \operatorname{Zsp}(\Lambda) \mid \operatorname{Hom}(\mathcal{S}, N \otimes_\Lambda -) = 0\}.$$

Moreover, in this case every module in $\mathcal{U}$ is generic by Lemma 5.2 if Λ is right dualizing. To show that $\overline{M} \setminus \{M\} = \mathcal{U}$ observe first that $\overline{M} \setminus \mathcal{U}$ is non-empty since $\mathcal{C}_0 \neq 0$ by our assumption on M. Now suppose that $N \in \overline{M} \setminus \mathcal{U}$. It follows that $\operatorname{Hom}(X, N \otimes_\Lambda -) \neq 0$ for some object $X \in C(\Lambda)$ which becomes a non-zero object of finite length in $\mathcal{C}$, and we may even suppose that X is simple in $\mathcal{C}$. On the other hand $\operatorname{Hom}(X, M \otimes_\Lambda -) \neq 0$ and therefore $M = N$ by Lemma 5.8. This finishes the proof.

6. Functors between module categories

The purpose of this section is to collect all facts about functors between module categories which are needed throughout the rest of this paper.

Proposition 6.1 *Let* $f\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(\Lambda)$ *be a functor which commutes with direct limits and products, and denote by* $F\colon \mathrm{Mod}(\Gamma) \to \mathrm{Ab}$ *its composition with the forgetful functor.*

(1) f *preserves pure-exactness and pure-injectivity.*

(2) *If* $\mathcal{X}$ *is a definable subcategory of* $\mathrm{Mod}(\Lambda)$*, then the full subcategory* $f^{-1}(\mathcal{X})$ *of all* Γ*-modules* M *with* $f(M) \in \mathcal{X}$ *is a definable subcategory of* $\mathrm{Mod}(\Gamma)$*.*

(3) *If* $\mathcal{U} \subseteq \mathrm{Zsp}(\Lambda)$ *is closed, then the indecomposable direct summands of modules in* $f(\mathcal{U})$ *form a closed subset in* $\mathrm{Zsp}(\Lambda)$*.*

(4) *For every* Γ*-module* M *the Ziegler closure* $\overline{f(M)}$ *equals the set of indecomposable direct summands of modules in* $f(\overline{M})$*.*

(5) *If* $f(M)$ *is indecomposable for every* $M \in \mathrm{Zsp}(\Gamma)$*, then* f *induces a continuous and closed map* $\mathrm{Zsp}(\Gamma) \to \mathrm{Zsp}(\Lambda)$*.*

(6) *The canonical morphism* $\mathrm{End}_\Gamma(M) \to \mathrm{End}_\Lambda(f(M))$ *induces a faithful exact functor* $\mathcal{E}_{f(M)} \to \mathcal{E}_M$ *for all* $M \in \mathrm{Mod}(\Gamma)$*.*

(7) $\dim_{\mathrm{fin}} f(M) \le \dim_{\mathrm{fin}} M$ *for all* $M \in \mathrm{Mod}(\Gamma)$*.*

(8) *If* f *induces an injective map* $\mathrm{Zsp}(\Gamma) \to \mathrm{Zsp}(\Lambda)$*, then* $\dim_{\mathrm{fin}} f(M) = \dim_{\mathrm{fin}} M$ *for all* $M \in \mathrm{Mod}(\Gamma)$ *satisfying* $\dim_{\mathrm{fin}} M < \infty$*.*

(9) *There is a presentation* $\mathrm{Hom}_\Gamma(Y, \ ) \to \mathrm{Hom}_\Gamma(X, \ ) \to F \to 0$ *with* $X, Y \in \mathrm{mod}(\Gamma)$*.*

(10) *There is* $c \in \mathbb{N}$ *such that* $\ell_{\mathrm{end}}(f(M)) \le c \cdot \ell_{\mathrm{end}}(M)$ *for all* $M \in \mathrm{Mod}(\Gamma)$*.*

Proof: (1) See [11, Proposition 8.61] or [17, Theorem 6.6].

(2) This follows from the fact that a full subcategory of modules is definable iff it is closed under direct limits, products and pure submodules [17, Corollary 4.6].

(3) See [17, Theorem 6.6].

(4) Adapt the proof of [17, Theorem 6.6].

(5) f induces a map $\mathrm{Zsp}(\Gamma) \to \mathrm{Zsp}(\Lambda)$ by (1). It is continuous by (2) and closed by (3).

(6) See [17, Theorem 7.4].

(7) follows immediately from (6) with Lemma 1.3 and Lemma 2.1.

(8) If $\dim_{\mathrm{fin}} M < \infty$, then $\dim_{\mathrm{fin}} M$ equals the so-called Cantor-Bendixson rank of the space $\overline{M}$ by [17, Corollary 11.12]. The assumption on f implies that f induces a homeomorphism $\overline{M} \to \overline{f(M)}$ by (4) and (5). We conclude that $\dim_{\mathrm{fin}} f(M) = \dim_{\mathrm{fin}} M$.

(9) See [15, Corollary 12.2].

(10) is a direct consequence of (9). In fact, choosing an epimorphism $\Gamma^n \to X$ in $\mathrm{mod}(\Gamma)$ one takes $c = n$. Then $c \cdot \ell_{\mathrm{end}}(M)$ bounds the length of the $\mathrm{End}_\Gamma(M)^{\mathrm{op}}$-module $F(M)$ and therefore also the endolength of $f(M)$ since the $\mathrm{End}_\Gamma(M)^{\mathrm{op}}$-module structure on $F(M)$ is induced by that of $\mathrm{End}_\Lambda(f(M))^{\mathrm{op}}$ via the canonical homomorphism $\mathrm{End}_\Gamma(M) \to \mathrm{End}_\Lambda(f(M))$.

7. The PI-property

Recall that a ring Γ is a *polynomial identity ring*, or simply a *PI-ring*, if there exists a monic polynomial $p \in \mathbb{Z}\langle X_1, \ldots, X_n\rangle$ such that $p(x_1, \ldots, x_n) = 0$ for all $x_i \in \Gamma$. In our context this property leads to an interesting finiteness condition if one considers the endomorphism ring Γ of an endofinite module. The radical of Γ is nilpotent and therefore Γ is a PI-ring iff $\Gamma/\operatorname{rad}\Gamma$ is a PI-ring. This fact explains the following definition. Let $\mathcal{C}$ be a skeletally small abelian category with $\dim_{\mathrm{fin}} \mathcal{C} < \infty$. We say that $\mathcal{C}$ has the *PI-property* if for every Serre subcategory $\mathcal{S}$ of $\mathcal{C}$ the endomorphism ring $\mathrm{End}(X)$ is a PI-ring for every simple object $X \in \mathcal{C}/\mathcal{S}$. A Λ-module M with $\dim_{\mathrm{fin}} M < \infty$ is said to have the PI-property if the endocategory $\mathcal{E}_M$ has the PI-property. The aim of this section is to prove the following result.

Theorem 7.1 *Let M be a Λ-module with $\dim_{\mathrm{fin}} M < \infty$.*

(1) *M has the PI-property if and only if $\mathrm{End}_\Lambda(N)/\operatorname{rad}\mathrm{End}_\Lambda(N)$ is a PI-ring for every module N in the Ziegler closure of M.*

(2) *Let $f\colon \mathrm{Mod}(\Lambda) \to \mathrm{Mod}(\Gamma)$ be a functor commuting with direct limits and products. If M has the PI-property, then $f(M)$ has the PI-property.*

Let $\mathcal{C}$ be a skeletally small abelian category and denote by $\mathcal{A} = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the category of left exact functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$. The category $\mathcal{A}$ is a locally coherent Grothendieck category. Moreover if $\mathcal{C}$ is a *length category*, i.e. every object in $\mathcal{C}$ is of finite length, then the Yoneda functor $\mathcal{C} \to \mathcal{A}$, $X \to \mathrm{Hom}(\ ,X)$ induces an equivalence between $\mathcal{C}$ and the full subcategory of finite length objects in $\mathcal{A}$.

Lemma 7.2 *Let X be an object of finite length in any abelian category. If $\mathrm{End}(S)$ is a PI-ring for every composition factor S of X, then $\mathrm{End}(X)$ is a PI-ring.*

Proof: Let $S_1, \ldots, S_n$ be the simples which occur as composition factors of X and denote by $\mathcal{C}$ the abelian category of all finite length objects with composition factors in $\{S_1, \ldots, S_n\}$.

We identify $\mathcal{C}$ with the full subcategory of finite length objects in $\mathcal{A} = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$. Let I be the injective envelope of $\coprod_{i=1}^n S_i$ and let $\Lambda = \mathrm{End}(I)^{\mathrm{op}}$. The functor

$$h: \mathcal{C} \longrightarrow \mathrm{Mod}(\Lambda), \quad Y \mapsto \mathrm{Hom}(Y, I)$$

is fully faithful and $h(Y)$ is a finite length module for every $Y \in \mathcal{C}$. Now let $\mathfrak{a} = \mathrm{ann}\, h(X)$ and $\Gamma = \Lambda/\mathfrak{a}$. The assumption implies that $\Lambda/\operatorname{rad}\Lambda \cong \prod_{i=1}^n \mathrm{End}(S_i)^{\mathrm{op}}$ is a PI-ring. Therefore also $\Gamma/\operatorname{rad}\Gamma$ is a PI-ring and we conclude that Γ is a PI-ring since $\operatorname{rad}\Gamma$ is nilpotent by Nakayama's lemma. We conclude that $\mathrm{End}(X) \cong \mathrm{End}_\Gamma(h(Y))^{\mathrm{op}}$ is a PI-ring since $h(Y)$ is a finitely generated Γ-module [18, Corollary 13.4.9].

Lemma 7.3 *Let $\mathcal{C}$ be a skeletally small abelian category and suppose there is a filtration $(\mathcal{C}_\alpha)_{\alpha \geq -1}$ of Serre subcategories such that*

(i) $\mathcal{C}_{-1} = 0$ *and* $\cup_\alpha \mathcal{C}_\alpha = \mathcal{C}$*;*

(ii) *if* $\alpha = \beta + 1$*, then* $\mathcal{C}_\alpha/\mathcal{C}_\beta$ *is a length category;*

(iii) *if* α *is a limit ordinal, then* $\mathcal{C}_\alpha = \cup_{\gamma<\alpha}\mathcal{C}_\gamma$*.*

Then for every indecomposable injective object M in $\mathcal{A} = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ there is α and a simple object $X \in \mathcal{C}_{\alpha+1}/\mathcal{C}_\alpha \subseteq \mathcal{C}/\mathcal{C}_\alpha$ such that $\mathrm{End}(M)/\operatorname{rad}\mathrm{End}(M) \cong \mathrm{End}(X)$.

Proof: We shall identify $\mathcal{C}$ with a full subcategory of $\mathcal{A}$ via the Yoneda functor. Denote for every α by $\mathcal{A}_\alpha = \vec{\mathcal{C}}_\alpha$ the full subcategory of direct limits $\varinjlim X_i$ with $X_i \in \mathcal{C}_\alpha$ for all i. These are localizing subcategories of $\mathcal{A}$ and we obtain quotient functors $q_\alpha: \mathcal{A} \to \mathcal{A}/\mathcal{A}_\alpha$ [13, Theorem 2.8]. Now let M be an indecomposable injective object in $\mathcal{A}$. It follows from (i) that there is a minimal β such that $\mathrm{Hom}(Y, M) \neq 0$ for some $Y \in \mathcal{C}_\beta$ and (iii) implies that $\beta = \alpha + 1$. We apply Lemma 5.7 as follows. The quotient functor q_α sends M to an indecomposable injective object and $\mathrm{Hom}(q_\alpha(Y), q_\alpha(M)) \neq 0$. The object $X = q_\alpha(Y)$ is of finite length by (ii) and we may suppose that X is simple. Thus $q_\alpha(M)$ is an injective envelope of X and therefore

$$\mathrm{End}(M)/\operatorname{rad}\mathrm{End}(M) \cong \mathrm{End}(q_\alpha(M))/\operatorname{rad}\mathrm{End}(q_\alpha(M)) \cong \mathrm{End}(X).$$

Lemma 7.4 *Let $\mathcal{C}$ be a skeletally small abelian category with $\dim_{\mathrm{fin}}\mathcal{C} < \infty$. Then $\mathcal{C}$ has the PI-property iff* $\mathrm{End}(M)/\operatorname{rad}\mathrm{End}(M)$ *is a PI-ring for every indecomposable injective object M in* $\mathcal{A} = \mathrm{Lex}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

Proof: One direction follows immediately from the preceding lemma if one uses the fin-filtration of $\mathcal{C}$. To show the converse let $X \in \mathcal{C}/\mathcal{S}$ be a simple object. Using the same argument as above we identify X with a simple object in $\mathcal{A}/\vec{\mathcal{S}}$. Let I be an injective envelope of X and denote by M the image under the section functor $\mathcal{A}/\vec{\mathcal{S}} \to \mathcal{A}$. Then $\operatorname{End}(X) \cong \operatorname{End}(M)/\operatorname{rad}\operatorname{End}(M)$ and therefore $\mathcal{C}$ has the PI-property.

Lemma 7.5 *Let $f\colon \mathcal{D} \to \mathcal{C}$ be a faithful exact functor between skeletally small abelian categories and suppose that* $\dim_{\mathrm{fin}} \mathcal{C} < \infty$*. If $\mathcal{C}$ has the PI-property, then $\mathcal{D}$ has the PI-property.*

Proof: We need to show that $\operatorname{End}(M)/\operatorname{rad}\operatorname{End}(M)$ is a PI-ring for every indecomposable injective object $M \in \operatorname{Lex}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ by the preceding lemma. To this end consider the fin-filtration $(\mathcal{C}_\alpha)_\alpha$ of $\mathcal{C}$ and let $\mathcal{D}_\alpha = f^{-1}(\mathcal{C}_\alpha)$ for all α. We obtain a filtration $(\mathcal{D}_\alpha)_\alpha$ of $\mathcal{D}$ which satisfies the assumption (i) - (iii) in Lemma. The functor f induces for every α a faithful exact functor $f_\alpha\colon \mathcal{D}_{\alpha+1}/\mathcal{D}_\alpha \to \mathcal{C}_{\alpha+1}/\mathcal{C}_\alpha$ which sends a simple object X to a finite length object Y. Thus $\operatorname{End}(X)$ is identified with a subring of $\operatorname{End}(Y)$ and is therefore a PI-ring by our assumption on $\mathcal{C}$ and Lemma 7.2. We can now apply Lemma 7.4 and conclude that $\mathcal{D}$ has the PI-property.

Proof of Theorem 7.1: (1) Suppose that M has the PI-property and let $N \in \overline{M}$. The endocategory $\mathcal{E}_N$ is a quotient of $\mathcal{E}_M$ and has therefore the PI-property. It follows from Lemma 5.10 that $\operatorname{End}_\Lambda(N)/\operatorname{rad}\operatorname{End}_\Lambda(N)$ is a PI-ring. The converse follows similarly from Lemma 7.4 using the fact that $D(\Lambda)/\vec{\mathcal{S}}_M \cong \operatorname{Lex}(\mathcal{E}_M, \mathrm{Ab})$.

(2) follows from Proposition 6.1 and Lemma 7.5.

8. Generalized tubes

A collection $\mathcal{T} = (M_i, \varphi_i, \psi_i)_{i \in \mathbb{N}_0}$ of Λ-modules M_i and morphisms $\varphi_i\colon M_{i+1} \to M_i$ and $\psi_i\colon M_i \to M_{i+1}$ is called a *generalized tube* if $M_0 = 0$ and for every $i \in \mathbb{N}$ the sequence

$$0 \longrightarrow M_i \xrightarrow{\left[\begin{smallmatrix}\varphi_{i-1}\\ \psi_i\end{smallmatrix}\right]} M_{i-1} \coprod M_{i+1} \xrightarrow{[\psi_{i-1}\ -\varphi_i]} M_i \longrightarrow 0$$

is exact. We say that a Λ-module N *belongs* to $\mathcal{T}$ if $N \cong M_i$ for some i and we shall often simplify the notation by writing $\mathcal{T} = (M_i)_{i \in \mathbb{N}}$. A generalized tube induces the following directed system of exact sequences which is indexed by $i \in \mathbb{N}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & M_i & \xrightarrow{\varphi_{i-1}} & M_{i-1} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \psi_{i-1}} & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_{i+1} & \xrightarrow{\varphi_i} & M_i & \longrightarrow & 0 \end{array}$$

Taking direct limits gives an exact sequence $0 \to M_1 \to M \xrightarrow{\varphi} M \to 0$. We have also the following inverse system of exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_i & \xrightarrow{\psi_i} & M_{i+1} & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{i-1}} & & \downarrow{\scriptstyle \varphi_i} & & \| & & \\ 0 & \longrightarrow & M_{i-1} & \xrightarrow{\psi_{i-1}} & M_i & \longrightarrow & M_1 & \longrightarrow & 0 \end{array}$$

Taking inverse limits gives an exact sequence $0 \to \widehat{M} \xrightarrow{\psi} \widehat{M} \to M_1 \to 0$. This gives rise to the following directed system of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{M} & \xrightarrow{\psi^i} & \widehat{M} & \longrightarrow & M_i & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_i} & & \\ 0 & \longrightarrow & \widehat{M} & \xrightarrow{\psi^{i+1}} & \widehat{M} & \longrightarrow & M_{i+1} & \longrightarrow & 0 \end{array}$$

Taking direct limits gives an exact sequence $0 \to \widehat{M} \to E \to M \to 0$ which we call the *universal exact sequence* corresponding to $\mathcal{T}$.

Lemma 8.1 *There is, up to isomorphism, a bijective correspondence between generalized tubes and pairs (M, φ) consisting of a Λ-module M and an epimorphism $\varphi: M \to M$ such that $M = \varinjlim_{i \in \mathbb{N}} \operatorname{Ker}(\varphi^i)$.*

Proof: We have already constructed a pair (M, φ) from a generalized tube. Given a pair (M, φ) one constructs a generalized tube $\mathcal{T} = (M_i, \varphi_i, \psi_i)_{i \in \mathbb{N}_0}$ as follows. Put $M_i = \operatorname{Ker}(\varphi^i)$ and denote by $\psi_i: M_i \to M_{i+1}$ the inclusion. The morphism $\varphi_i: M_{i+1} \to M_i$ is obtained by restricting φ to M_{i+1}. It is easily checked that both constructions are mutually inverse.

Our aim is to show that the module M corresponding to a generalized tube is Σ-pure-injective under suitable assumptions. Let us mention before that M is never endofinite if it is indecomposable.

Lemma 8.2 *Let M be an indecomposable endofinite Λ-module. Then every epimorphism (monomorphism) $\varphi \in \operatorname{End}_\Lambda(M)$ is an isomorphism.*

Proof: The radical of $\operatorname{End}_\Lambda(M)$ is nilpotent by Nakayama's lemma. It follows that φ is invertible since $\varphi^n \neq 0$ for every $n \in \mathbb{N}$.

The following result is based on Ringel's use of the shift endomorphism in [21].

Proposition 8.3 *Let Λ be an R-algebra over some commutative artinian ring R and let M be a Λ-module. Let $\varphi \colon M \to M$ be an epimorphism and let $M_i = \operatorname{Ker}(\varphi^i)$. If $M = \varinjlim \operatorname{Ker}(\varphi^i)$ and M_1 is of finite length over R, then the following holds.*

(1) *M is artinian over $\Gamma = \operatorname{End}_\Lambda(M)^{\mathrm{op}}$ and $\dim_{\mathrm{fin}} \mathsf{L}(M_\Gamma) \leq 1$.*

(2) *M is Σ-pure-injective and $\dim_{\mathrm{fin}} M \leq 1$.*

(3) *If infinitely many of the M_i are indecomposable, then M has no non-zero direct summand which is finitely generated.*

The proof of the proposition requires some preparations.

Lemma 8.4 *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of Λ-modules. Then $\dim_{\mathrm{fin}} \mathsf{L}(M) = \max(\dim_{\mathrm{fin}} \mathsf{L}(M'), \dim_{\mathrm{fin}} \mathsf{L}(M''))$.*

Proof: Clear.

We discuss now a special case of the preceding proposition. Let $\Lambda = R[T]$ be the polynomial ring in one indeterminate and let $M = \coprod_{i \in \mathbb{N}} x_i R$ be a free R-module with basis $(x_i)_{i \in \mathbb{N}}$. Defining $x_i T = x_{i-1}$ for $i > 1$ and $x_1 T = 0$ induces a Λ-module structure on M.

Lemma 8.5 *The Λ-module M is artinian and $\dim_{\mathrm{fin}} \mathsf{L}(M) = 1$.*

Proof: We prove the lemma by induction on the length n of the R-module R. If $n = 1$, then R is a field and the assertion follows from the fact that every proper submodule of M is of the form $\coprod_{i=1}^{r} x_i R$ for some $r \in \mathbb{N}$. Now suppose $n > 1$ and choose a simple ideal $\mathfrak{a}$ of R. We consider the submodule $M' = \coprod_{i \in \mathbb{N}} x_i \mathfrak{a}$ of M. The assertion holds by induction for the Λ-modules M' and M/M' since $M' \cong \coprod_{i \in \mathbb{N}} x_i (R/\operatorname{ann} \mathfrak{a})$ and $M/M' \cong \coprod_{i \in \mathbb{N}} x_i (R/\mathfrak{a})$. Applying the preceding lemma we conclude that M is artinian and $\dim_{\mathrm{fin}} \mathsf{L}(M) = 1$.

Proof of Proposition 8.3: (1) The ring homomorphism $R[T] \to \Gamma$, $T \mapsto \varphi$ induces a $R[T]$-module structure on M. Therefore it suffices to show that M is artinian over $R[T]$ and that $\dim_{\mathrm{fin}} \mathsf{L}(M) \leq 1$ over $R[T]$. To this end choose elements $m_{11}, \ldots, m_{1n}$ which generate M_1 over R. For $i > 1$ choose inductively elements $m_{i1}, \ldots, m_{in}$ in M_i such that $\varphi(m_{ij}) = m_{(i-1)j}$ for every j. Now let $N_j = \coprod_{i \in \mathbb{N}} x_{ij} R$ be a copy of the $R[T]$-module $\coprod_{i \in \mathbb{N}} x_i R$ for every $j \in \{1, \ldots, n\}$. The assignment $x_{ij} \mapsto m_{ij}$ induces an epimorphism $\coprod_{j=1}^{n} N_j \to M$ of $R[T]$-modules by our construction. The assertion is now a consequence of the preceding lemmas.

(2) It is well-known that M is Σ-pure-injective iff M has the descending chain condition on subgroups of finite definition. But every subgroup of finite definition is a Γ-submodule, and therefore (2) is a direct consequence of (1).

(3) Suppose that N is a finitely generated direct summand of M. Then $N \subseteq M_n$ for some n and therefore N is a direct summand of M_i for every $i \geq n$. If infinitely many of the M_i are indecomposable, then $N = 0$ and this finishes the proof of the proposition.

We combine now Proposition 8.3 with a result from the preceding section and obtain the following consequence.

Corollary 8.6 *Let Λ be an artin algebra and suppose that $\mathcal{T} = (M_i)_{i \in \mathbb{N}}$ is a generalized tube of finitely presented Λ-modules. Suppose also that there is an infinite family of indecomposable modules belonging to $\mathcal{T}$ or, more generally, that $M = \varinjlim M_i \neq 0$ has no finitely presented indecomposable direct summand. Then there is a generic module in the Ziegler closure of M and therefore also in the Ziegler closure of $\mathcal{T}$.*

Proof: The module M has no finitely presented indecomposable direct summand and no dense chain of subgroups of finite definition by Proposition 8.3. It follows from Theorem 5.6 that there is a generic module N in the Ziegler closure of M.

Given a generalized tube $\mathcal{T}$ and the direct limit $M = \varinjlim M_i$ of modules M_i belonging to $\mathcal{T}$, it is natural to ask for an explicit method to construct the generic modules corresponding to M. The first approach simply uses the fact that the generic modules occur in the Ziegler closure of the Σ-pure-injective module M.

Proposition 8.7 *A Λ-module M is Σ-pure-injective if and only if every product of copies of M decomposes into a coproduct of copies of modules which belong to the Ziegler closure $\overline{M}$ of*

M. In this case an indecomposable Λ-module N belongs to $\overline{M}$ if and only if N is isomorphic to a direct summand of a product of copies of M.

Proof: It is well-known that M is Σ-pure-injective iff every product of copies of M decomposes into a coproduct of indecomposable modules. These modules belong to the definable subcategory generated by M since this category is closed under products and direct summands. But $\overline{M} = \{N \in \operatorname{Zsp}(\Lambda) \mid \operatorname{Hom}(\mathcal{S}_M, N \otimes_\Lambda -) = 0\}$ and therefore every indecomposable direct summand of $\prod_I M$ belongs to $\overline{M}$. The last assertion follows from the fact that the quotient functor $q\colon D(\Lambda) \to D(\Lambda)/\vec{\mathcal{S}}_M$ sends $M \otimes_\Lambda -$ to an injective cogenerator of the quotient category [17, Lemma 8.9]. This implies that for every $N \in \overline{M}$ there is a pure monomorphism $N \to \prod_I M$ for some I by Lemma 5.7. We conclude that N is a direct summand.

We present now another approach to the above question. For the rest of this section let Λ be a k-algebra over a field k. Let $\mathcal{T} = (M_i)_{i\in\mathbb{N}}$ be a generalized tube of finite dimensional Λ-modules and let $0 \to \widehat{M} \to E \to M \to 0$ be the corresponding universal exact sequence. Denote by d the k-dimension of M_1. Ringel has recently shown that for every product $P = \prod_I M$ the module $Q = \varinjlim(\prod_I M_i)$ is a pure submodule such that P/Q is of endolength at most d [22]. The module P/Q is non-zero if I is infinite, and P/Q decomposes into generic modules if M has no non-zero finitely presented direct summand. However, we use now a different construction which is based on the module $\widehat{M}$. Let $k[\![T]\!]$ be the ring of formal power series and denote by $k(\!(T)\!)$ its quotient field. Note that $k[\![T]\!] \cong \varprojlim k[\![T]\!]/(T^i)$ and $k(\!(T)\!)$ is obtained by forming the direct limit over the directed system

$$k[\![T]\!] \xrightarrow{\cdot T} k[\![T]\!] \xrightarrow{\cdot T} \cdots.$$

In other words, $0 \to k[\![T]\!] \to k(\!(T)\!) \to k(\!(T)\!)/k[\![T]\!] \to 0$ is the universal exact sequence corresponding to the tube $(k[\![T]\!]/(T^i))_{i\in\mathbb{N}}$. The following lemma is a consequence of these observations.

Lemma 8.8 (1) *$\widehat{M}$ is a $k[\![T]\!]$-Λ-bimodule which is free over $k[\![T]\!]$ of rank d.*

(2) *$\mathcal{T} \cong (k[\![T]\!]/(T^i) \otimes_{k[\![T]\!]} \widehat{M})_{i\in\mathbb{N}}$ and the universal exact sequence is isomorphic to*

$$0 \longrightarrow k[\![T]\!] \otimes_{k[\![T]\!]} \widehat{M} \longrightarrow k(\!(T)\!) \otimes_{k[\![T]\!]} \widehat{M} \longrightarrow k(\!(T)\!)/k[\![T]\!] \otimes_{k[\![T]\!]} \widehat{M} \longrightarrow 0.$$

Proof: Adapt the proof of Proposition 8.3.

We need the following facts about complete discrete valuation rings.

Lemma 8.9 *Let R be a commutative complete discrete valuation ring with maximal ideal $\mathfrak{m}$ and quotient field K.*

(1) $\mathrm{Zsp}(R) = \{R/\mathfrak{m}^i \mid i \in \mathbb{N}\} \cup \{R, K/R, K\}$.

(2) $\dim_{\mathrm{fin}} M \leq 1$ *for every* $M \in \mathrm{Zsp}(R)$.

(3) *Every R-module has the PI-property.*

Proof: (1) See [24, Lemma 5.1].

(2) $\dim_{\mathrm{fin}} M = 0$ if M is endofinite by Lemma 4.5. Otherwise $M = R$ or $M = K/R$. We have $\dim_{\mathrm{fin}} R = \dim_{\mathrm{fin}} \mathsf{L}(R) = 1$ and $\dim_{\mathrm{fin}} K/R = 1$ by Proposition 8.3.

(3) For every $M \in \mathrm{Zsp}(R)$ the ring $\mathrm{End}_R(M)/\operatorname{rad}\mathrm{End}_R(M)$ is either k or $k(\!(T)\!)$. The assertion therefore follows from Theorem 7.1.

Consider now the functor $-\otimes_{k[\![T]\!]} \widehat{M}\colon \mathrm{Mod}(k[\![T]\!]) \to \mathrm{Mod}(\Lambda)$. We apply our general results about functors commuting with direct limits and products, using the fact that $k[\![T]\!]$ is a complete discrete valuation ring.

Theorem 8.10 *Let $\mathcal{T} = (M_i)_{i\in\mathbb{N}}$ be a generalized tube of finite dimensional Λ-modules and let $0 \to \widehat{M} \to E \to M \to 0$ be the corresponding universal exact sequence. Let N be a Λ-module in the Ziegler closure of $\mathcal{T}$.*

(1) *E is endofinite of endolength at most the k-dimension of M_1.*

(2) *N is either finite dimensional or a direct summand of M, $\widehat{M}$ or E.*

(3) $\dim_{\mathrm{fin}} N \leq 1$ *and* $\mathrm{End}_\Lambda(N)/\operatorname{rad}\mathrm{End}_\Lambda(N)$ *is a PI-ring.*

(4) *If N is endofinite, then* $\mathrm{End}(N)$ *is a PI-ring.*

(5) *A module in the Ziegler closure of M is a direct summand of M or E.*

(6) *A module in the Ziegler closure of $\widehat{M}$ is a direct summand of $\widehat{M}$ or E.*

Proof: Most parts follow immediately from Proposition 6.1, Theorem 7.1 and the preceding lemmas. For (1) observe that $E \cong k(\!(T)\!) \otimes_{k[\![T]\!]} \widehat{M}$ is a $k(\!(T)\!)$-Λ-bimodule. Therefore the endolength of E is bounded by the length over $k(\!(T)\!)$ which equals the k-dimension of M_1.

(4) follows from (3) since $\operatorname{End}_\Lambda(N)$ is nilpotent if N is endofinite. Finally, (5) and (6) follow from the fact that

$$\overline{k[\![T]\!]} = \{k[\![T]\!], k(\!(T)\!)\} \quad \text{and} \quad \overline{k(\!(T)\!)/k[\![T]\!]} = \{k(\!(T)\!)/k[\![T]\!], k(\!(T)\!)\}$$

which can be deduced from Theorem 5.11.

9. Tubes in the Auslander-Reiten quiver

Let Λ be an artin algebra. Following [20] a component $\mathcal{C}$ of the Auslander-Reiten quiver of Λ is called *quasi-serial* if

(i) $\mathcal{C}$ contains neither projective nor injective modules;

(ii) if $0 \to X \to \coprod_{i=1}^k Y_i \to Z \to 0$ is an almost split sequence of modules in $\mathcal{C}$, then $k \leq 2$ and $k = 2$ with $\ell(Y_1) \leq \ell(Y_2)$ implies that $\ell(Y_1) < \ell(X) < \ell(Y_2)$.

The component $\mathcal{C}$ is called a *stable tube* if there exists a natural n such that $(D\operatorname{Tr})^n M \cong M$ for some M in $\mathcal{C}$. The *rank* of $\mathcal{C}$ is the minimal n having this property and a stable tube of rank 1 is called *homogeneous*. An indecomposable module M belonging to a quasi-serial component $\mathcal{C}$ determines, up to isomorphism, uniquely a sequence of irreducible morphisms $M_i \to M_{i+1}$, $i \in \mathbb{N}$, such that

(i) M_i belongs to $\mathcal{C}$ for all i and $M = M_j$ for some j;

(ii) M_1 is *quasi-simple*, i.e. there is no irreducible monomorphism ending in M_1.

Theorem 9.1 *Let $\mathcal{T}$ be a stable tube and suppose that $M_1 \to M_2 \to \ldots$ is a sequence of irreducible monomorphisms starting in a quasi-simple module M_1 belonging to $\mathcal{T}$. Denote by $M = \varinjlim M_i$ the corresponding direct limit.*

(1) *M is an indecomposable module which depends, up to isomorphism, only on M_1*

(2) *Let N_1 be a second quasi-simple module and denote by $N = \varinjlim N_i$ the corresponding direct limit. Then $M \cong N$ if and only if $M_1 \cong N_1$.*

(3) *M is Σ-pure-injective but not endofinite. There is a finite number $n \geq 1$ of generic modules G_i such that $\{M, G_1, \ldots, G_n\}$ is the Ziegler closure of M.*

Proof: (1) and (2) is [12, Theorem 8]. To prove (3) denote by r the rank of $\mathcal{T}$. Composing the irreducible morphisms between the modules belonging to $\mathcal{T}$ in the appropriate way we obtain a generalized tube $\mathcal{T}' = (M_{ri+1})_{i\in\mathbb{N}_0}$. Therefore the module M is Σ-pure-injective by Proposition 8.3 but not endofinite by Lemma 8.2. Finally, the description of the Ziegler closure of M follows from Theorem 5.11.

The author is indebted to S. Liu for pointing out the following application of Theorem 9.1.

Corollary 9.2 *Let Λ be a connected artin algebra of infinite representation type. Suppose there is a finitely generated indecomposable Λ-module M such that $D\operatorname{Tr} M \cong M$. Then there exists a generic Λ-module.*

Proof: M belongs to a homogeneous tube by a result of Hoshino [10, Theorem 1].

For the rest of this section let Λ be a finite dimensional algebra over a field k. Our next result is a refinement of Theorem 9.1.

Corollary 9.3 *Let $\mathcal{T}$ be a stable tube of rank r and let d be the maximum of the k-dimensions of the quasi-simple modules in $\mathcal{T}$. Then there are modules $M_1, \dots, M_r$ and $N_1, \dots, N_r$ with $\dim_{\mathrm{fin}} M_j = \dim_{\mathrm{fin}} N_j = 1$ for all j, and there is a finite number $n \geq 1$ of generic modules G_i of endolength at most d such that $\mathcal{T} \cup \{M_j, N_j \mid 1 \leq j \leq r\} \cup \{G_1, \dots, G_n\}$ is the Ziegler closure of $\mathcal{T}$.*

Proof: Let $\mathcal{T} = (M_{ij})_{i\in\mathbb{N}, 1\leq j\leq r}$ where $M_{11}, \dots, M_{1r}$ are the quasi-simple modules and $M_{1j} \to M_{2j} \to \dots$ is a sequence of irreducible monomorphisms with $M_j = \varinjlim M_{ij}$ for every j. Composing the irreducible morphisms between the modules belonging to $\mathcal{T}$ in the appropriate way we obtain for every pair (k, j) in $\{1, \dots, r\}$ a generalized tube $\mathcal{T}_{kj} = (M_{ri+k,j})_{i\in\mathbb{N}_0}$. Observe that $\overline{\mathcal{T}} = \cup_{k,j=1}^{r} \overline{\mathcal{T}_{kj}}$ and it is clear that the universal exact sequence corresponding to $\mathcal{T}_{kj}$ depends only on j; we denote it by $0 \to \widehat{M_j} \to E_j \to M_j \to 0$. We apply now Theorem 8.10 as follows. For every $j \in \{1, \dots, r\}$ the Ziegler closure of M_j consists of M_j and a finite number of generic modules which are direct summands of E_j and therefore of endolength at most d. Now let $(N_{ij})_{i\in\mathbb{N}, 1\leq j\leq r}$ be the homogeneous tube of Λ^{op}-modules obtained from $\mathcal{T}$ by applying the duality $\operatorname{Hom}_k(\ , k)$. We have

$$\widehat{M_j} = \varprojlim M_{ij} \cong \varprojlim \operatorname{Hom}_k(N_{ij}, k) \cong \operatorname{Hom}_k(\varinjlim N_{ij}, k)$$

and obtain therefore an endolength preserving bijection $X \mapsto DX$ between the Ziegler closure of $\varinjlim N_{ij}$ and $\widehat{M_j}$ from [14]. It follows that every module in the Ziegler closure of $\mathcal{T}$ is either endofinite, M_j or $N_j = D \varinjlim N_{ij}$ for some $j \in \{1, \dots, r\}$.

We call a generalized tube $\mathcal{T} = (M_i)_{i \in \mathbb{N}}$ *generic* if there exists a generic module G such that $\{M_i \mid i \in \mathbb{N}\} \cup \{\varinjlim M_i, \varprojlim M_i, G\}$ is the Ziegler closure of $\mathcal{T}$. Note that $\varinjlim M_i$ and $\varprojlim M_i$ cannot be endofinite by Lemma 8.2. The following is a consequence of Crawley-Boevey's work on generic modules over tame algebras [5].

Theorem 9.4 *Let Λ be a finite dimensional algebra over an algebraically closed field k and suppose that Λ is of tame representation type. Then for every $d \in \mathbb{N}$ almost all d-dimensional indecomposable Λ-modules belong to a homogeneous tube which is generic.*

To prove this theorem we need some information about the Ziegler spectrum of a Dedekind domain.

Lemma 9.5 *Let R be a Dedekind domain. Then for every prime ideal $\mathfrak{p} \neq 0$ the corresponding tube $(R/\mathfrak{p}^i)_{i \in \mathbb{N}}$ is generic. The unique generic R-module is the quotient field of R.*

Proof: See [24, Example 9.5].

Proof of Theorem 9.3: Crawley-Boevey has shown that for each generic Λ-module G there is an R_G-Λ-bimodule M_G having the following properties [5, p. 242].

(i) R_G is a finitely generated localization of $k[T]$ and M_G is free over R_G. The rank of M_G equals the endolength of G and if K_G is the quotient field of R_G, then $G \cong K_G \otimes_{R_G} M_G$.

(ii) The functor $- \otimes_{R_G} M_G \colon \mathrm{Mod}(R_G) \to \mathrm{Mod}(\Lambda)$ preserves isomorphism classes, indecomposabilty and Auslander-Reiten sequences.

(iii) For each $d \in \mathbb{N}$ almost all indecomposable Λ-modules of dimension d arise as $R_G/\mathfrak{p}^n \otimes_{R_G} M_G$ for some prime ideal $\mathfrak{p}$ and some $n \in \mathbb{N}$.

The assertion of the theorem is a direct consequence of this result, Proposition 6.1 and the above lemma. In fact, each functor $- \otimes_{R_G} M_G$ commutes with direct limits and products since M_G is a finitely presented R_G-module. Moreover one uses the fact that any localization of $k[T]$ is a Dedekind domain. Given a generic Λ-module G and a prime ideal $0 \neq \mathfrak{p} \subseteq R_G$, the image of the tube $(R_G/\mathfrak{p}^i)_{i\in\mathbb{N}}$ under the functor $-\otimes_{R_G} M_G$ is a homogeneous tube and $G \cong K_G \otimes_{R_G} M_G$ is the unique generic module in its Ziegler closure.

Let Λ be a finite dimensional algebra over an algebraically closed field and suppose that Λ is of tame representation type. Given a generic Λ-module G we denote by $\mathrm{ind}_G(\Lambda)$ the set of isomorphism classes of indecomposable Λ-modules belonging to a homogeneous tube such that G is the unique generic module in its Ziegler closure. With this definition we obtain the following consequence of the above theorem.

Corollary 9.6 *For every $d \in \mathbb{N}$ almost all d-dimensional indecomposable Λ-modules belong to the disjoint union $\cup_G \mathrm{ind}_G(\Lambda)$.*

Corollary 9.7 *Let $d \in \mathbb{N}$. Then a Λ-module M lies in the Ziegler closure of all d-dimensional indecomposable Λ-modules if and only if M is indecomposable of dimension d or generic of endolength $d \cdot n^{-1}$ for some $n \in \mathbb{N}$.*

Proof: Almost all d-dimensional indecomposable Λ-modules are of the form $R_G/\mathfrak{p}^n \otimes_{R_G} M_G$ for some generic Λ-module G of endolength $d \cdot n^{-1}$ and some prime ideal $\mathfrak{p}$. For fixed $d \in \mathbb{N}$ there are only finitely many generic modules involved [5, Theorem 4.4] and therefore the assertion follows from Theorem 6.1 and the fact that $\{R_G/\mathfrak{p}^n \mid \mathfrak{p} \neq 0 \text{ prime }\} \cup \{K_G\}$ is the Ziegler closure of $\{R_G/\mathfrak{p}^n \mid \mathfrak{p} \neq 0 \text{ prime }\}$, for instance by Proposition 5.4 and Theorem 5.6.

Having shown for an algebra of tame representation type that all generic Λ-modules lie in the Ziegler closure of a homogeneous tube, it is an obvious question to ask whether this property characterizes tameness. We have the following answer.

Corollary 9.8 *The following are equivalent for a finite dimensional algebra Λ over an algebraically closed field.*

(1) *Λ has tame representation type.*

(2) *Every generic Λ-module occurs as the unique generic module in the Ziegler closure of a homogeneous tube.*

(3) *Every indecomposable endofinite Λ-module occurs in the Ziegler closure of a generalized tube of finite dimensional modules.*

Proof: We have already shown that (1) implies (2) and it is clear that (2) implies (3). Therefore suppose (3). It follows from Theorem 8.10 that $\operatorname{End}_\Lambda(G)$ is a PI-ring for every generic Λ-module G. On the other hand, Crawley-Boevey has shown that a representation embedding $\operatorname{Mod}(k\langle X,Y\rangle) \to \operatorname{Mod}(\Lambda)$ sends the universal skew field of fractions of $k\langle X,Y\rangle$ to a generic module G such that $\operatorname{End}_\Lambda(G)$ is not a PI-ring [5, Theorem 4.4]. It follows that Λ is not wild and therefore tame by Drozd's Tame and Wild Theorem.

10. Pure representation embeddings

Let Λ and Γ be a pair of rings. In this section we compare the corresponding module categories using certain functors between $\operatorname{Mod}(\Gamma)$ and $\operatorname{Mod}(\Lambda)$. Recall that a functor $f\colon \operatorname{Mod}(\Gamma) \to \operatorname{Mod}(\Lambda)$ is said to be a *representation embedding* if

(i) f is exact and commutes with coproducts and products;

(ii) f sends indecomposable modules to indecomposable modules;

(iii) f sends non-isomorphic modules to non-isomorphic modules.

The following weaker conditions seem to be appropriate for a discussion of pure-injective modules. We call a functor $f\colon \operatorname{Mod}(\Gamma) \to \operatorname{Mod}(\Lambda)$ a *pure representation embedding* if

(i') f commutes with direct limits and products;

(ii') f sends indecomposable modules in $\operatorname{Zsp}(\Gamma)$ to indecomposable modules;

(iii') f sends non-isomorphic modules in $\operatorname{Zsp}(\Gamma)$ to non-isomorphic modules.

Note that a pure representation embedding preserves pure-exactness and pure-injectivity. It has been shown by Prest that any representation embedding induces a homeomorphism onto a closed subset between the Ziegler spectra [19]. Using different methods we can replace representation embeddings by pure representation embeddings and obtain the following characterization of wild representation type.

Theorem 10.1 *If Λ is a finite dimensional algebra over an algebraically closed field k, then the following are equivalent.*

(1) *Λ has wild representation type, i.e. there is a k-linear representation embedding* $\operatorname{Mod}(k\langle X,Y\rangle) \to \operatorname{Mod}(\Lambda)$.

(2) $\mathcal{U}_{\Lambda,n} = \{M \in \operatorname{Zsp}(\Lambda) \mid M \text{ is not isolated and } \ell_{\text{end}}(M) \leq n\}$ *is infinite for some* $n \in \mathbb{N}$.

(3) *There is a pure representation embedding* $\operatorname{Mod}(k\langle X,Y\rangle) \to \operatorname{Mod}(\Lambda)$.

(4) *For every finitely generated k-algebra Γ there is a homeomorphism $f\colon \mathrm{Zsp}(\Gamma) \to \mathrm{Zsp}(\Lambda)$ onto a closed subset of $\mathrm{Zsp}(\Lambda)$ and $c \in \mathbb{N}$ such that $\ell_{\mathrm{end}}(f(M)) \leq c \cdot \ell_{\mathrm{end}}(M)$ for all $M \in \mathrm{Zsp}(\Gamma)$.*

Proof: (1) $\Leftrightarrow$ (2) It has been shown by Crawley-Boevey that Λ has wild representation type iff there are infinitely many generic Λ-modules of endolength n for some $n \in \mathbb{N}$ [5, Theorem 4.4]. The assertion follows from this and the fact that a module $M \in \mathrm{Zsp}(\Lambda)$ is generic iff it is not isolated by Theorem 4.7.

(1) $\Rightarrow$ (3) Every representation embedding is pure.

(3) $\Rightarrow$ (4) If $\Gamma = k\langle X_1, \ldots, X_n\rangle/I$, then the composition

$$\mathrm{Mod}(\Gamma) \xrightarrow{g} \mathrm{Mod}(k\langle X_1, \ldots, X_n\rangle) \xrightarrow{h} \mathrm{Mod}(k\langle X, Y\rangle)$$

of the restriction functor g with the functor h used by Brenner [3] is a fully faithful functor which commutes with direct limits and products. In particular it preserves indecomposability and isomorphism classes. Therefore the composition of $h \circ g$ with the pure representation embedding $\mathrm{Mod}(k\langle X, Y\rangle) \to \mathrm{Mod}(\Lambda)$ is again a pure representation embedding. The assertion now follows from Proposition 6.1.

(4) $\Rightarrow$ (2) Choose a finite dimensional algebra Γ of wild representation type and a map $f\colon \mathrm{Zsp}(\Gamma) \to \mathrm{Zsp}(\Lambda)$ with $c \in \mathbb{N}$. Using (2) we know that $\mathcal{U}_{\Gamma,n}$ is infinite for some $n \in \mathbb{N}$. The properties of f and c imply that $f(\mathcal{U}_{\Gamma,n})$ is an infinite subset of $\mathcal{U}_{\Lambda,c\cdot n}$ and this shows (2).

We proceed now with an interesting numerical property of algebras having wild representation type.

Proposition 10.2 *Let Λ be any ring and suppose there is a pure representation embedding $\mathrm{Mod}(k\langle X, Y\rangle) \to \mathrm{Mod}(\Lambda)$. Then for every $n \in \mathbb{N}$ there is $M \in \mathrm{Zsp}(\Lambda)$ such that $\dim_{\mathrm{fin}} M = n$.*

Proof: Let $\Gamma = k[X_1, \ldots, X_n]_{\mathfrak{p}}$ be the localization at $\mathfrak{p} = (X_1, \ldots, X_n)$ and denote by I the injective envelope of $\Gamma/\operatorname{rad}\Gamma$. Then $\dim_{\mathrm{fin}} I = n$ by Example 2.4 since Γ has Krull dimension n. The argument used in the preceding proof shows that there is a pure representation embedding $f\colon \mathrm{Mod}(\Gamma) \to \mathrm{Mod}(\Lambda)$ and therefore $\dim_{\mathrm{fin}} f(I) = n$ by Proposition 6.1.

We end this section with a characterization of tame hereditary algebras.

Theorem 10.3 *For a finite dimensional hereditary algebra Λ of infinite representation type the following are equivalent.*

(1) Λ *is tame.*

(2) $\operatorname{End}_\Lambda(M)/\operatorname{rad}\operatorname{End}_\Lambda(M)$ *is a PI-ring for every* $M \in \operatorname{Zsp}(\Lambda)$.

(3) $\dim_{\text{fin}} M \leq 1$ *for every* $M \in \operatorname{Zsp}(\Lambda)$.

Proof: Suppose that Λ is tame. It follows, for example from Geigle's work [8], that every infinite dimensional indecomposable pure-injective Λ-module M lies in the Ziegler closure of a stable tube. Thus $\operatorname{End}_\Lambda(M)/\operatorname{rad}\operatorname{End}_\Lambda(M)$ is a PI-ring and $\dim_{\text{fin}} M \leq 1$ by Theorem 8.10. If Λ is not tame, then there exists a representation embedding $\operatorname{Mod}(k\langle X, Y\rangle) \to \operatorname{Mod}(\Lambda)$ by [6, Theorem 8.4]. The argument used in the proof of Corollary 9.8 shows that (2) fails and the preceding proposition shows that (3) fails.

11. Krull dimension and finite representation type

In this section we discuss the Krull dimension of the category $D(\Lambda)$ in the sense of Gabriel. We refer to [7] for the precise definition of the Krull dimension $\operatorname{Kdim} D(\Lambda)$. Denote by $\operatorname{ind}(\Lambda)$ the set of finitely presented modules in $\operatorname{Zsp}(\Lambda)$.

Proposition 11.1 *For an artin algebra Λ the following are equivalent.*

(1) $\operatorname{Kdim} D(\Lambda) = 1$.

(2) $\dim_{\text{fin}} C(\Lambda) = 1$.

(2) *Every module having no finitely presented indecomposable direct summand is endofinite and there exists a non-zero module with this property.*

(3) $\operatorname{Zsp}(\Lambda) \setminus \operatorname{ind}(\Lambda)$ *is a finite non-empty set of endofinite modules.*

We collect some well-known facts which are needed in the proof of this proposition.

Lemma 11.2 *Let Λ be an artin algebra. Then the simple functors in $D(\Lambda)$ are finitely presented, and a Λ-module M has a finitely presented indecomposable direct summand iff* $\operatorname{Hom}(S, M \otimes_\Lambda -) \neq 0$ *for some simple functor S.*

Proof: Observe first that if $M \in \operatorname{ind}(\Lambda)$, then the socle of $M \otimes_\Lambda -$ is finitely presented and simple by Lemma 4.4. Now suppose that $S \in D(\Lambda)$ is simple and let $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(X,\ ) \to S$ be an epimorphism with indecomposable X. The well-known duality between $C(\Lambda)$ and $C(\Lambda^{\mathrm{op}})$ sends the finitely presented socle of $- \otimes_\Lambda X$ to a finitely presented and simple quotient of $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(X,\ )$ which is isomorphic to S since $\operatorname{End}_{\Lambda^{\mathrm{op}}}(X)$ is local. The second part of the assertion now follows with the argument used in the proof of Lemma 5.2.

Proof of Proposition 11.1: (1) $\Leftrightarrow$ (2) Denote by $D(\Lambda)_0$ the smallest localizing subcategory of $D(\Lambda)$ containing all objects of finite length, and let $C(\Lambda)_0$ be the Serre subcategory of all objects of finite length in $C(\Lambda)$. Every object of finite length in $D(\Lambda)$ is finitely presented according to the preceding lemma, and therefore $D(\Lambda)_0 = \overrightarrow{C(\Lambda)}_0$ by Lemma 5.7. Moreover, the full subcategory of finitely presented objects in $D(\Lambda)/D(\Lambda)_0$ is equivalent to $C(\Lambda)/\mathcal{C}(\Lambda)_0$ by [13, Theorem 2.6]. Now, by definition, $\operatorname{Kdim} D(\Lambda) = 1$ if $D(\Lambda)/D(\Lambda)_0$ is locally finite but non-zero. This condition is equivalent to $\dim_{\mathrm{fin}} C(\Lambda)/\mathcal{C}(\Lambda)_0 = 0$. We conclude that (1) and (2) are equivalent.

(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) Let $\mathcal{S} = C(\Lambda)_0$. The preceding lemma implies that a Λ-module M has no finitely presented indecomposable direct summand iff $\operatorname{Hom}(\mathcal{S}, M \otimes_\Lambda -) = 0$. The assertion is therefore an immediate consequence of Lemma 5.9 since $\dim_{\mathrm{fin}} C(\Lambda) = 1$ iff $\dim_{\mathrm{fin}} C(\Lambda)/\mathcal{S} = 0$.

Corollary 11.3 *Let Λ be an artin algebra and suppose that $\mathcal{T} = (M_i)_{i \in \mathbb{N}}$ is a generalized tube of finitely presented Λ-modules. If $M = \varinjlim M_i$ is indecomposable, then* $\operatorname{Kdim} D(\Lambda) > 1$.

Proof: The module M is not endofinite by Lemma 8.2.

The following consequence gives an answer to a problem posed in the book of Jensen and Lenzing [11].

Corollary 11.4 *Let Λ be a finite dimensional algebra over an algebraically closed field k. Then* $\operatorname{Kdim} D(\Lambda) \neq 1$.

Proof: If Λ is of finite representation type, then $\operatorname{Kdim} D(\Lambda) = 0$ by a result of Auslander [1, Theorem 3.1]. Now suppose that Λ is of infinite type. We use Drozd's Tame and Wild Theorem. If Λ is wild, then a k-linear representation embedding $\operatorname{Mod}(k[T]) \to \operatorname{Mod}(\Lambda)$

produces generalized tubes of the above form. Otherwise Λ is tame, and the validity of the Second Brauer-Thrall Conjecture guarantees the existence of homogeneous tubes in the Auslander-Reiten quiver of Λ by the result of Crawley-Boevey discussed in Theorem 9.4.

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