Block P-Matrices

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ABSTRACT.
Several possibilities to generalize P-matrices to block form are discussed. Unfortunately certain equivalences holding for P-matrices do not carry over to the block case. We opt for one such generalization calling it block P-matrices . It has the most analogies to the usual P-matrices .

This class and its connections to other generalizations is studied in some detail.

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1 Introduction

From the point of view of applicability, real P-matrices are very interesting. We refer to their role in the linear complementarity problem (see [1]), in global invertibility of vector fields [2] and in nonsingularity characterization of interval matrices [6]. We recall that a square matrix is called a P-*matrix* if all its principal minors are positive.

In [4] Johnson and Tsatsomeros gave a new characterization of this class: an n-by-n real matrix A is a P-matrix if and only if the set $r(A)$ consisting of all matrices of the form $TA + (I - T)$, where $T = \operatorname{diag}(t_1, \ldots, t_n)$ with $t_i \in [0, 1]$, is nonsingular, i.e., if each matrix in $r(A)$ is nonsingular. Besides this they also give a characterization of all matrices A such that $tA+(1-t)I$ is nonsingular for all $t \in [0, 1]$. This suggests to study this problem for matrices T "between" diagonal matrices and scalar matrices (multiples of the identity matrix), namely those which are direct products of scalar matrices for a given partition of $\{1, \ldots, n\}$. Consider the following example.

Example. Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}$$

and let $T = \operatorname{diag}(t_1, t_2, t_2)$ with $t_1, t_2 \in [0, 1]$. Then a direct calculation yields

$$\det(TA + (I - T)) = 1 - t_2 + 2t_2^2 + t_1 t_2$$

which is always positive in the given range.

As $a_{22} = 0$, A is not a P-matrix but $TA + (I - T)$ is nonsingular for each T from the "restricted" set of T's.

This example motivates us to introduce a new matrix class below and investigate how it relates to the class of P-matrices .

2 Notation and preliminaries

Let $N(k)$ be a partition of the set $N = \{1, \ldots, n\}$ into k, $1 \leq k \leq n$, pair-wise disjoint subsets N_j of the cardinality n_j and let

$$\mathcal{M}_k = \{\bigcup_{j \in \alpha} N_j : \ \alpha \subseteq \{1, \ldots, k\}\}.$$

Observe that if $k = n$, i.e., if $N_j = \{j\}$ the set $\mathcal{M}_k$ becomes

$$\mathcal{M}_n = \{\alpha : \ \alpha \subseteq N\}.$$

For an n-by-n square matrix X and $\beta \subseteq N$ we will denote by $X[\beta]$ the principal submatrix of X with row and column indices in β and by $\sigma(X)$ the spectrum of X.

For given partition $N(k)$, we will denote by $\mathcal{T}^{(k)}$ ($\mathcal{D}^{(k)}$, respectively) the set of all diagonal n-by-n matrices T (D, respectively) such that for $i = 1, \ldots, k$ $T[N_i] = t_i I_{n_i}$ where $t_i \in [0, 1]$ ($D[N_i] = d_i I_{n_i}$, where $d_i \geq 0$, respectively). Here I_{n_i} is the n_i-by-n_i identity matrix. We quote a theorem characterizing real P-matrices.

THEOREM **1** *([3], [4]). Let $A \in \mathrm{I\!R}_{n\times n}$. The following are equivalent.*

(i) $\forall_{M\in\mathcal{M}_n}$ $\det(A[M]) > 0$.

(ii) For each nonzero real $x = (x_1, \ldots, x_n)^T$, some entry of the Hadamard product $x \circ (Ax)$ is positive, that is, there is some $k \in \{1, \ldots, n\}$ such that $x_k(Ax)_k > 0$.

(iii) For each nonzero real $x = (x_1, \ldots, x_n)^T$, there is some positive diagonal n-by-n matrix $D = D(x)$ such that $x^T DAx > 0$.

(iv) For each nonzero real $x = (x_1, \ldots, x_n)^T$, there is some nonnegative diagonal n-by-n matrix $\tilde{D} = \tilde{D}(x)$ such that $x^T \tilde{D}Ax > 0$.

(v) $\forall_{M\in\mathcal{M}_n}$ $\sigma(A[M]) \cap (-\infty, 0] = \emptyset$.

(vi) $\forall_{T\in\mathcal{T}^{(n)}}$ $TA + (I - T)$ *is nonsingular.*

It should be noted that a real P-matrix can be also characterized in terms of unique solvability of the linear complementarity problem [1].

We also mention that the class of real P-matrices is closed under inversion and positive diagonal multiplication.

Now, for given $N(k)$, we define the sets:

$\mathcal{P}_1^{(k)} = \{A \in \mathrm{I\!R}_{n\times n} : \ \forall_{M\in\mathcal{M}_k} \ \det(A[M]) > 0\}$,

$\mathcal{P}_2^{(k)} = \{A \in \mathrm{I\!R}_{n\times n} : \ \forall_{M\in\mathcal{M}_k} \ \sigma(A[M]) \cap (-\infty, 0] = \emptyset\}$,

$\mathcal{P}_3^{(k)} = \{A \in \mathrm{I\!R}_{n\times n} : \ \forall_{M\in\mathcal{M}_k}\forall_{D\in\mathcal{D}^{(k)}} \ \det((A + D)[M]) \neq 0\}$,

$\mathcal{P}_4^{(k)} = \{A \in \mathrm{IR}_{n\times n} : \ \forall_{T\in\mathcal{T}^{(k)}} \ \det(TA + (I-T)) \neq 0\}$.

We observe that one can as well replace "$\neq 0$" by "> 0" in the last two sets.

REMARK **1** .

If $k = n$ *then all our sets coincide and become the set of* P-*matrices . If* $k = 1$*, i.e. if* $N_1 = N$*, then*

$$\mathcal{P}_4^{(1)} = \mathcal{P}_3^{(1)} = \mathcal{P}_2^{(1)} \neq \mathcal{P}_1^{(1)}.$$

These relations follow from the fact that the determinant of a nonsingular real matrix with an even number of negative eigenvalues is positive and from Observation 3.1 in [4]. We note that the relation

$$\mathcal{P}_3^{(k)} \subset \mathcal{P}_2^{(k)} \subset \mathcal{P}_1^{(k)} \tag{1}$$

holds for any partition $N(k)$.

3 Results

We start by establishing that $\mathcal{P}_3^{(k)}$ and $\mathcal{P}_4^{(k)}$ coincide.

LEMMA. *For any partition* $N(k)$ *we have*

$$\mathcal{P}_3^{(k)} = \mathcal{P}_4^{(k)}. \tag{2}$$

PROOF. The proof will be split into two parts.

First let us show $\mathcal{P}_3^{(k)} \subseteq \mathcal{P}_4^{(k)}$. For this let $A \in \mathcal{P}_3^{(k)}$ and $T \in \mathcal{T}^{(k)}$ be given. We want to show that $TA + (I-T)$ is nonsingular.

If T is nonsingular then $D = T^{-1}(I-T) \in \mathcal{D}^{(k)}$, hence $A + D$ is nonsingular and so is $TA + (I-T) = T(A+D)$.

If $T = \mathrm{diag}(t_1, \ldots, t_n)$ with $t_i \neq 0$ on some $M \in \mathcal{M}_k$ and $t_i = 0$ on $N \setminus M$ then $TA + (I-T)$ is nonsingular if and only if $T[M]A[M]+ (I-T)[M]$ is nonsingular. But this follows as above, because also $A[M] + D[M]$ is nonsingular for all $D \in \mathcal{D}^{(k)}$. So, we have $A \in \mathcal{P}_4^{(k)}$.

For the other inclusion, let $A \in \mathcal{P}_4^{(k)}$, $M \in \mathcal{M}_k$ and let $D \in \mathcal{D}^{(k)}$. We want to show that $(A+D)[M]$ is nonsingular. Define $T = \mathrm{diag}(t_1, \ldots, t_n)$ by

$$t_i = \left\{ \begin{array}{ll} 0, & \text{if } i \in N \setminus M \\ (1+d_i)^{-1}, & \text{if } i \in M. \end{array} \right\}$$

Obviously $T \in \mathcal{T}^{(k)}$ and so $TA + (I - T)$ is nonsingular. But then $T[M]A[M] + (I - T)[M]$ is nonsingular too, and as $T[M]$ is nonsingular, also $A[M] + (T[M])^{-1}(I - T)[M] = A[M] + D[M]$ is nonsingular. This shows that $A \in \mathcal{P}_3^{(k)}$ and finishes the proof. ■

Before we establish relations between all our sets we show some properties of $\mathcal{P}_4^{(k)}$.

Theorem **2** . *Let $A \in \mathcal{P}_4^{(k)}$. Then*

(i) $A^{-1} \in \mathcal{P}_4^{(k)}$,

(ii) $A^T \in \mathcal{P}_4^{(k)}$,

(iii) $DA \in \mathcal{P}_4^{(k)}, AD \in \mathcal{P}_4^{(k)}$ for all nonsingular $D \in \mathcal{D}^{(k)}$,

(iv) $A + D \in \mathcal{P}_4^{(k)}$ for all $D \in \mathcal{D}^{(k)}$,

(v) $TA + (I - T) \in \mathcal{P}_4^{(k)}$ for all $T \in \mathcal{T}^{(k)}$,

(vi) $A \in \mathcal{P}_2^{(k)} \subset \mathcal{P}_1^{(k)}$,

Proof.

(i): Let $T \in \mathcal{T}^{(k)}$. Then, as also $I - T \in \mathcal{T}^{(k)}$, we have that $(I - T)A + T$ is nonsingular and so is $I - T + TA^{-1}$. This shows $A^{-1} \in \mathcal{P}_4^{(k)}$.

(ii): This is a trivial consequence of the Lemma.

(iii): Let $T \in \mathcal{T}^{(k)}$. As $D \in \mathcal{D}^{(k)}$, we see at once that $\hat{T} = I - T + TD$ is positive and $\hat{T}^{-1}TD \in \mathcal{D}^{(k)}$. Now from the equation $TDA + (I - T) = \hat{T}(\hat{T}^{-1}TDA + (I - \hat{T}^{-1}TD)$, where both factors on the right hand side are nonsingular, we get that $TDA + (I - T)$ is nonsingular, hence $DA \in \mathcal{P}_4^{(k)}$.

The second statement follows from $AD = (DA^T)^T$ and $A^T \in \mathcal{P}_4^{(k)}$.

(iv): Observe that if $D \in \mathcal{D}^{(k)}$ and $T \in \mathcal{T}^{(k)}$ then $DT + I$ is nonsingular and $\tilde{T} = (DT + I)^{-1}T \in \mathcal{T}^{(k)}$. So, $\tilde{T}A + (I - \tilde{T})$ is nonsingular and a direct calculation yields $\tilde{T}A + (I - \tilde{T}) = (DT + I)^{-1}(T(A + D) + (I - T))$. Hence, $A + D \in \mathcal{P}_4^{(k)}$.

(v): Observe that if $T, S \in \mathcal{T}^{(k)}$ then $TS \in \mathcal{T}^{(k)}$. So, if $A \in \mathcal{P}_4^{(k)}$ then $STA + (I - ST)$ is nonsingular. A direct calculation yields

$$STA + (I - ST) = S(TA + (I - T)) + I - S$$

and hence $TA + (I - T) \in \mathcal{P}_4^{(k)}$.

(vi): trivial by (1) and the Lemma. ■

Based on the above results we obtain the following theorem.

THEOREM **3** . *The following inclusions hold*

$$\mathcal{P}_1^{(k)} \supseteq \mathcal{P}_2^{(k)} \supseteq \mathcal{P}_3^{(k)} = \mathcal{P}_4^{(k)}.$$

In addition $\mathcal{P}_1^{(k)} \neq \mathcal{P}_2^{(k)}$ *for* $k < n$ *and* $\mathcal{P}_2^{(k)} \neq \mathcal{P}_3^{(k)} (= \mathcal{P}_4^{(k)})$ *for* $1 < k < n$.

PROOF. The inclusions are just restatements of (1), (2) and (vi) of Theorem 2.

We show now that $\mathcal{P}_1^{(k)} \neq \mathcal{P}_2^{(k)}$ for $k < n$. In this case there is $n_i > 1$ for some i. Take A as a blockdiagonal matrix where $A[N_j] = I_{n_j}$ for $j \neq i$, while $A[N_i]$ is an $n_i \times n_i$ matrix with exactly two negative eigenvalues. This matrix A is in $\mathcal{P}_2^{(k)}$ but not in $\mathcal{P}_1^{(k)}$.

For the last statement we consider first the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2.01 \\ 2 & 2.01 & 1 \end{bmatrix} \tag{3}$$

and let $k = 2$ with $N_1 = \{1\}$ and $N_2 = \{2, 3\}$. Then it is easy to verify that $A \in \mathcal{P}_2^{(2)}$. However, setting $D = \operatorname{diag}(0, x, x)$, we get

$$\det(A + D) = x^2 - 3x + 0.0001.$$

which has a positive zero. Hence $A \notin \mathcal{P}_3^{(2)}$. For general k with $1 < k < n$ we have at least two sets N_1 and N_2 where $n_2 > 1$. Embedding the first row of the matrix above in N_1, rows 2 and 3 in N_2 and filling up with zero elements outside the diagonal and ones on the diagonal, we get a matrix in $\mathcal{P}_2^{(k)}$, but not in $\mathcal{P}_3^{(k)}$. ■

Our next result provides some characterizations of $\mathcal{P}_4^{(k)}$ (and hence $\mathcal{P}_3^{(k)}$). Before formulating it we remark that the partition $N_1 \cup \ldots \cup N_k = N$ generates in a canonical manner a partition $x = x^{(1)} \oplus \ldots \oplus x^{(k)}$ of each vector $x \in \mathbb{R}^n$

THEOREM 4 . *Let $A \in \mathbb{R}_{n\times n}$. The following are equivalent.*

(i) $A \in \mathcal{P}_4^{(k)}$, i.e., $TA + (I - T)$ is nonsingular for all $T \in \mathcal{T}^{(k)}$.

(ii) For each nonzero real $x = (x_1, \ldots, x_n)^T = x^{(1)} \oplus \ldots \oplus x^{(k)}$, either there is some i, $1 \le i \le k$, such that $\mathrm{Rank}((Ax)^{(i)}, x^{(i)}) = 2$ *or there is some j, $1 \le j \le n$, such that $(Ax)_j x_j > 0$.*

(iii) For each nonzero real $x = (x_1, \ldots, x_n)^T = x^{(1)} \oplus \ldots \oplus x^{(k)}$, either there is some i, $1 \le i \le k$, such that Rank$((Ax)^{(i)}, x^{(i)}) = 2$ or there is some positive $D = D(x) \in \mathcal{T}^{(k)}$ such that $x^T DAx > 0$.

PROOF.

$(i) \Leftrightarrow (ii)$: We show that $\neg(i) \Leftrightarrow \neg(ii)$. $\neg(i)$ is equivalent to the existence of some $T = \mathrm{diag}(t_1, \ldots, t_n) \in \mathcal{T}^{(k)}$ and some nonzero $x = x^{(1)} \oplus \ldots \oplus x^{(k)} \in \mathbb{R}_n$ such that

$$TAx + (I - T)x = 0. \tag{4}$$

Observe that (4) holds if and only if for all i, $1 \le i \le k$,

$$t_i (Ax)^{(i)} + (1 - t_i) x^{(i)} = 0. \tag{5}$$

Finally, (5) holds if and only if for all i, $1 \le i \le k$, $\mathrm{Rank}((Ax)^{(i)}, x^{(i)}) \le 1$ and for each $j \in 1, \ldots, n$, $(Ax)_j x_j \le 0$. So, $\neg(i) \Leftrightarrow \neg(ii)$.

$(iii) \Rightarrow (ii)$: trivial.

$(ii) \Rightarrow (iii)$: Let $x = x^{(1)} \oplus \ldots \oplus x^{(k)} \in \mathbb{R}_n$ be given and let (ii) hold. If for some i, $1 \le i \le k$, $\mathrm{Rank}((Ax)^{(i)}, x^{(i)}) = 2$ then (iii) holds. Assume that, for all $i \in 1, \ldots, k$, $\mathrm{Rank}((Ax)^{(i)}, x^{(i)}) \le 1$ and let $j \in N_l$, $1 \le l \le k$, be such that

$$(Ax)_j x_j > 0. \tag{6}$$

Then there exist nonzero s and r such that

$$s(Ax)^{(l)} + rx^{(l)} = 0. \tag{7}$$

Since $j \in N_l$ and $sr \ne 0$, by (6) and (7), we get

$$sr < 0. \tag{8}$$

However (7) holds for all indices in N_l which, combined with (8), yields $(Ax)_p x_p \geq 0$ for each $p \in N_l$. So, for some $\varepsilon > 0$, we have

$$\sum_{p \in N_l} (Ax)_p x_p + \varepsilon \sum_{p \in N \setminus N_l} (Ax)_p x_p > 0. \tag{9}$$

Choosing $D = D(x) = \operatorname{diag}(t_1, \ldots, t_n) \in \mathcal{T}^{(k)}$ with

$$t_p = \left\{ \begin{array}{ll} 1 & \text{if } p \in N_l \\ \varepsilon & \text{otherwise,} \end{array} \right\}$$

from (9) it follows that $x^T D A x > 0$. ■

REMARK **2** . *If all* $N_i = \{i\}$ *then* $x^{(i)}$ *is a number, hence Rank*$((Ax)^{(i)}, x^{(i)}) \leq 1$, *and (ii) ((iii), respectively) becomes (ii) ((iii), respectively) of Theorem 1.*

Comparing Theorems 2 and 4 with the properties of P-matrix listed in Section 2 it seems to be natural to call the matrices from $\mathcal{P}_4^{(k)}$ (and hence also from $\mathcal{P}_3^{(k)}$) *block* P*-matrices* .

DEFINITION. Let $N(k)$ be a partition of N. A real n-by-n matrix A is called a *block* P*-matrix* , with respect to the partition $N(k)$, if $A \in \mathcal{P}_4^{(k)}$.

For later use we need

THEOREM **5** . *Let* $A \in \mathcal{P}_4^{(k)}$. *Then, for each* $M \in \mathcal{M}_k$, $A[M]$ *is a block* P*-matrix with respect to the induced partition of* M.

PROOF. The proof is an immediate consequence of the lemma. ■

THEOREM **6** . *Let* $A \in \mathcal{P}_4^{(k)}$ *and let, for some* $M \in \mathcal{M}_k$,

$$A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right]$$

with $A_{11} = A[M]$. *Then*

$$S := A_{22} - A_{21} A_{11}^{-1} A_{12},$$

i.e. the Schur complement of A_{11} *in* A, *is a block* P*-matrix with respect to the induced partition of* $N \setminus M$.

PROOF. By (ii) in Theorem 2 it follows that $A^{-1} \in \mathcal{P}_4^{(k)}$ and by Theorem 5 that A_{11} is nonsingular. From [5] we have

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}S^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}S^{-1} \\ -S^{-1}A_{21}A_{11}^{-1} & S^{-1} \end{bmatrix}.$$

The assertion follows by Theorem 5 applied to S^{-1} and by (i) in Theorem 2. ■

The above results suggests to investigate whether the set $\mathcal{P}_2^{(k)}$ is closed under inversion. As a byproduct of our considerations we have the following negative answer:

THEOREM **7** . *For $1 < k < n$ the set $\mathcal{P}_2^{(k)}$ is not closed under inversion.*

PROOF. Consider the matrix (3) with the same partitioning as in the proof of Theorem 3. Using MATLAB we get:

$$A^{-1} = \begin{bmatrix} 125.6883 & 75.3117 & -100 \\ -125.1870 & -74.8130 & 100 \\ 0.2494 & -0.2494 & 0 \end{bmatrix}.$$

The eigenvalues of $A^{-1}[2,3]$ are -74.4781 and -0.3348. So, for this partition, $A \in \mathcal{P}_2^{(2)}$ but $A^{-1} \notin \mathcal{P}_2^{(2)}$. Proceeding as in the proof of Theorem 3, we see that this example can be lifted to an example for any $N(k), 1 < k < n$. ■

Part (v) of Theorem 2 leads to the following characterization of the set $\mathcal{P}_4^{(k)}$.

THEOREM **8** . *Let $A \in \mathcal{P}_4^{(k)}$ be partitioned as*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{10}$$

where, for some nonempty $M \in \mathcal{M}_k$, $A_{11} = A[M]$. Then

$$\begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \in \mathcal{P}_4^{(k)}. \tag{11}$$

Conversely, if for some nonempty $M \in \mathcal{M}_k$, $A \in \mathbb{R}_{n\times n}$ has a block partition as in (10) *such that* (11) *holds, then $A \in \mathcal{P}_4^{(k)}$.*

Proof. Let $A \in \mathcal{P}_4^{(k)}$. Then, by (v) in Theorem 2,

$$TA + (I - T) \in \mathcal{P}_4^{(k)} \tag{12}$$

for all $T \in \mathcal{T}^{(k)}$. So, let $T_1 = \mathrm{diag}(t_1^{(1)}, \ldots, t_n^{(1)}) \in \mathcal{T}^{(k)}$ be such that

$$t_i^{(1)} = \begin{cases} 1 & \text{if } i \notin M \\ 0 & \text{otherwise,} \end{cases}$$

Define

$$C_1 = T_1 A + (I - T_1) \text{ and } C_2 = (I - T_1)A + T_1.$$

Observe that, by (v) in Theorem 2, C_1 and C_2 belong to $\mathcal{P}_4^{(k)}$. We show that

$$TC_1 + (I - T)C_2$$

is nonsingular for all $T \in \mathcal{T}^{(k)}$. Indeed, from the definition of C_1 and C_2, we get

$$TC_1 + (I - T)C_2 = (I - (T + T_1 - 2TT_1))A + T + T_1 - 2TT_1$$

and, by the definition of T_1, we have

$$I \geq T + T_1 - 2TT_1 \geq T^2 + T_1^2 - 2TT_1 = (T - T_1)^2 \geq 0.$$

So, $T + T_1 - 2TT_1 \in \mathcal{T}^{(k)}$ and, by (12), $TC_1 + (I - T)C_2$ is nonsingular for all $T \in \mathcal{T}^{(k)}$. To complete this part of the proof it suffices to observe that

$$C_1 = \begin{bmatrix} I & 0 \\ A_{21} & A_{22} \end{bmatrix} \text{ and } C_2^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

and so $C_1 C_2^{-1}$ is equal to the matrix in (11). Hence (11) holds.

To show the second statement of the theorem observe that if one applies the procedure leading from the matrix (10) to the matrix (11) again to the matrix in (11), one ends up at the matrix (10). So, A is in $\mathcal{P}_4^{(k)}$ if the matrix in (11) is in $\mathcal{P}_4^{(k)}$. Therefore, the reverse implication holds. ■

Remark **3** .

Notice that the block $(2,2)$ *in* (10) *is the Schur complement of* A_{11} *in* A. *This observation, combined with Theorem* 5, *implies Theorem* 4, *too. It should be also noted that, for* $k = n$, *the first statement of Theorem* 5 *can be found also in (*[1], p. 292*).*

We shall close the paper with a sufficient condition for $A \in \mathbb{R}_{n\times n}$ to be in $\mathcal{P}_4^{(k)}$.

THEOREM **9** . *Let $N(k)$ be a partition of N. If for each k-tuple $(m_1, \ldots, m_k)$ of nonnegative integers m_i, where $0 \le m_i \le n_i$, we have*

$$\sum_{\substack{M\in\mathcal{M}_k,\\ \mathrm{card}(M\cap N_i)=m_i,\, i=1,\ldots,k}} det(A[M]) > 0, \tag{13}$$

then $A \in \mathcal{P}_4^{(k)}$ (we set $det(A[\emptyset]) = 1$ and $det(A[N]) = detA$).

PROOF. Firstly we observe that if $\tilde{M}, \hat{M} \in \mathcal{M}_k$ are such that

$$\mathrm{card}(\tilde{M} \cap N_i) = \mathrm{card}(\hat{M} \cap N_i), \;\; i = 1, \ldots, k, \tag{14}$$

and if $T = \mathrm{diag}(t_1, \ldots, t_n) \in \mathcal{T}^{(k)}$ then

$$\prod_{j\notin\tilde{M}} t_j \prod_{j\in\tilde{M}} (1-t_j) = \prod_{j\notin\hat{M}} t_j \prod_{j\in\hat{M}} (1-t_j). \tag{15}$$

Indeed, by (14) and by the definition of $\mathcal{T}^{(k)}$ we get

$$\prod_{j\notin\tilde{M}} t_j \prod_{j\in\tilde{M}} (1-t_j) = t_1^{\mathrm{card}(N_1\setminus(\tilde{M}\cap N_1))} \cdot \ldots \cdot t_n^{\mathrm{card}(N_k\setminus(\tilde{M}\cap N_k))} \times$$

$$(1-t_1)^{\mathrm{card}(\tilde{M}\cap N_1)} \cdot \ldots (1-t_n)^{\mathrm{card}(\tilde{M}\cap N_k)} = t_1^{\mathrm{card}(N_1\setminus(\hat{M}\cap N_1))} \cdot \ldots \cdot t_n^{\mathrm{card}(N_k\setminus(\hat{M}\cap N_k))} \times$$

$$(1-t_1)^{\mathrm{card}(\hat{M}\cap N_1)} \cdot \ldots (1-t_n)^{\mathrm{card}(\hat{M}\cap N_k)} = \prod_{j\notin\hat{M}} t_j \prod_{j\in\hat{M}} (1-t_j).$$

Now, from the determinantal expansion given in [4], we have

$$\det(T + (I-T)A) = \sum_{M\in\mathcal{M}_k} (\det(A[M]) \prod_{j\notin M} t_j \prod_{j\in M} (1-t_j),$$

which, by (13) and (15), becomes positive. So, $T + (I - T)A$ is nonsingular for all $T \in \mathcal{T}^{(k)}$ and, as also $I - T \in \mathcal{T}^{(k)}$, so is $TA + (I - T)$. Hence, $A \in \mathcal{P}_4^{(k)}$ and the proof is completed. ■

REMARK **4** . *In the case $n = k$ the condition* (13) *becomes the well known principal minors characterization for* P*-matrices ((ii) in Theorem 1). In particular, for a* 3*-by-*3 *real matrix $A = (a_{ij})$ and for the partition $N_1 = \{1\}$ and $N_2 = \{2, 3\}$ the inequalities* (13) *become:* $a_{11} > 0$, $a_{22}+a_{33} > 0$, $det(A[2, 3]) > 0$, $det(A[1, 2]) + det(A[1, 3]) > 0$ *and* $detA > 0$.

REMARK **5** . *The condition offered by Theorem* 9 *is sufficient for $A \in \mathbb{R}_{n\times n}$ to be in $\mathcal{P}_4^{(k)}$ but it is not necessary. Consider the matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix} \tag{16}$$

and let $N_1 = \{1\}$ and $N_2 = \{2, 3\}$. Then it is easy to see that

$$det(TA + (I - T)) = (1 - t_2)^2 + 2t_2^2.$$

So, $TA + (I - T)$ is nonsingular for all $T = diag(t_1, t_2, t_3)$ with $t_i \in [0, 1]$ but the matrix (16) *does not satisfy the condition* (13).

4 Conclusion

We have proposed here a generalization of P-matrices to block form. While many of the properties of P-matrices carry over to this case, we were unable to find equivalent conditions consisting of *finite* many equations. Such conditions are the necessary conditions $\mathcal{P}_2^{(k)}$ and $\mathcal{P}_1^{(k)}$ and the sufficient conditions of Theorem 9, they are however not equivalent except in the trivial cases $k = 1$ and $k = n$. Another such set of conditions is: A and $A^{-1} \in \mathcal{P}_2^{(k)}$. It can be proved that this is equivalent to A being a block P-matrix in the case that all $n_i = 1$ except one, which is allowed to be greater than 1.

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