ADDENDUM TO THE PAPER "DIRICHLET FORMS AND MARKOV PROCESSES: A GENERALIZED FRAMEWORK INCLUDING BOTH ELLIPTIC AND PARABOLIC CASES"

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Abstract. We extend the main result in [A/M/R], which is a complete characterization of all Dirichlet forms defined on some L^2–space $L^2(E,m)$ associated with m–tight special standard processes, to the framework of generalized Dirichlet forms.

0. Introduction

In our previous work [St] we introduced the class of *generalized Dirichlet forms* which contains symmetric and coercive Dirichlet forms as treated in [Fu] resp. [M/R] and time dependent Dirichlet forms as considered in [O], but also examples of an entirely new kind such as fractional diffusion operators and h–transformations (cf. [St] for a detailed discussion). The main result in [St] is the identification of an analytic property of *generalized Dirichlet forms* defined on $L^2(E,m)$, called *quasi–regularity* in analogy with [A/M/R] and [M/R], which allows the construction of an associated m–tight special standard process. This construction generalizes one half of the above mentioned result in [A/M/R] (namely the "sufficiency part"). In the present paper, we also carry over the "necessity part" of the result in [A/M/R] to our more general situation. More precisely: let $\mathcal{E}$ be a generalized (semi–) Dirichlet form on $L^2(E,m)$ and assume that there exists an m–tight special standard process $\mathbb{M}$ associated with $\mathcal{E}$. Then we shall prove in Theorem 1.6 that $\mathcal{E}$ is quasi–regular and, moreover, $\mathcal{E}$ is properly associated in the resolvent sense with $\mathbb{M}$ (cf. Definition 1.4 for the precise meaning).

The proof of the necessity of quasi–regularity of Dirichlet forms associated with m–tight special standard processes given in [A/M/R] and [M/R] cannot be carried over directly to the case of *generalized Dirichlet forms*. The main difference to the classical case lies in the fact that 1–excessive functions w.r.t. m–tight special standard processes associated with *generalized Dirichlet forms* are not quasi–continuous in general as seen for example in the case of the uniform motion on the real line.

1. Preliminaries and main results

Let us briefly recall the framework of generalized (semi–) Dirichlet forms: Let E be a topological Hausdorff space whose Borel–σ–algebra $\mathcal{B}(E)$ is generated by the set $C(E)$ of all continuous functions on E and m a σ–finite measure on $(E, \mathcal{B}(E))$ such that $\mathcal{H} = L^2(E, \mathcal{B}(E), m)$ is separable. As usual denote with $\mathcal{B}_b(E)$ (resp. $\mathcal{B}_b^+(E)$) the set of all bounded (resp. bounded and non–negative) $\mathcal{B}(E)$–measurable functions on E.

Let $(\mathcal{A}, \mathcal{V})$ be a real valued coercive closed form on $\mathcal{H}$ with sector constant $K \geq 1$. Note that $\mathcal{V}$ equipped with inner product $\tilde{\mathcal{A}}_1(\cdot,\cdot)$ (where $\tilde{\mathcal{A}}_\alpha(u,v) := 1/2(\mathcal{A}(u,v) + \mathcal{A}(v,u)) + \alpha(u,v)_{\mathcal{H}}$ for $u, v \in \mathcal{V}, \alpha > 0$) is again a separable real Hilbert space. Denote with $\|\cdot\|_{\mathcal{V}}$ the corresponding norm.

Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$, we obtain that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densely and continuously. Denote by $\langle\cdot,\cdot\rangle$ the dualization between $\mathcal{V}$ and $\mathcal{V}'$.

Let Λ with domain $D(\Lambda, \mathcal{V}')$ be a linear operator on $\mathcal{V}'$ satisfying the following conditions:

D1 (i) $(\Lambda, D(\Lambda, \mathcal{V}'))$ generates a strongly continuous semigroup $(U_t)_{t\geq 0}$ of bounded linear operators on $\mathcal{V}'$.

(ii) $(U_t)_{t\geq 0}$ can be restricted both to a strongly continuous *contraction* semigroup on $\mathcal{H}$ and to a strongly continuous semigroup of bounded linear operators on $\mathcal{V}$.

Denote with $(\Lambda, D(\Lambda, \mathcal{H}))$ and $(\Lambda, D(\Lambda, \mathcal{V}))$ the generators corresponding to the restricted semigroups and with $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{V}'))$ the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. Define the Hilbert spaces

$$\mathcal{F} := D(\Lambda, \mathcal{V}') \cap \mathcal{V} \text{ with norm } \|u\|_{\mathcal{F}}^2 := \|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}'}^2, \text{ and}$$
$$\widehat{\mathcal{F}} := D(\hat{\Lambda}, \mathcal{V}') \cap \mathcal{V} \text{ with norm } \|u\|_{\widehat{\mathcal{F}}}^2 := \|u\|_{\mathcal{V}}^2 + \|\hat{\Lambda} u\|_{\mathcal{V}'}^2 .$$

Since $U_t : \mathcal{H} \to \mathcal{H}$ is a contraction for $t \geq 0$ it follows $(\Lambda u, u)_{\mathcal{H}} \leq 0$ for $u \in D(\Lambda, \mathcal{H})$. By the density of $D(\Lambda, \mathcal{V})$ in $\mathcal{F}$ (cf. [St, II.1]) we can find for each $u \in \mathcal{F}$ a sequence $(u_n)_{n\geq 1} \subset D(\Lambda, \mathcal{V})$ converging to u in $\mathcal{F}$. Consequently $\langle \Lambda u, u\rangle = \lim_{n\to\infty} \langle \Lambda u_n, u_n\rangle = \lim_{n\to\infty} (\Lambda u_n, u_n)_{\mathcal{H}} \leq 0$. Similarly we obtain $\langle \hat{\Lambda} u, u\rangle \leq 0$ for $u \in \widehat{\mathcal{F}}$.

Then we can define the form

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{if } u \in \mathcal{F}, v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle & \text{if } u \in \mathcal{V}, v \in \widehat{\mathcal{F}}, \end{cases} \tag{1.1}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{\mathcal{H}}$ for $\alpha > 0$ (cf. [St, II.2])). $\mathcal{E}$ is well–defined and positive definite (i.e. $\mathcal{E}(u,u) \geq 0\ \forall u \in \mathcal{F} \cup \widehat{\mathcal{F}}$).

Denote with $(G_\alpha)_{\alpha>0}$ the associated strongly continuous contraction resolvent on $\mathcal{H}$ (i.e. $(G_\alpha)_{\alpha>0}$ is a strongly continuous contraction resolvent on $\mathcal{H}$ such that

$G_\alpha(\mathcal{H}) \subset \mathcal{F}\ \forall \alpha > 0$, and $\mathcal{E}_\alpha(G_\alpha f, u) = (f, u)_{\mathcal{H}}\ \forall f \in \mathcal{H}, u \in \mathcal{V}$) (cf. [St, II.4 and II.6]).

Consider the following assumptions D2, D3 and D3':

D2 $u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V}$ and $\mathcal{E}(u, u - u^+ \wedge 1) \geq 0$.

D3 For a dense linear subspace $\mathcal{Y} \subset \mathcal{F}, u \wedge \alpha \in \mathcal{Y}$ if $u \in \mathcal{Y}$ and $\alpha \geq 0$.

D3' For a dense linear subspace $\mathcal{Y} \subset \mathcal{F}$ of bounded functions $u \cdot v \in \mathcal{Y}$ if $u, v \in \mathcal{Y}$.

We say that the form $\mathcal{E}$ is a *generalized semi–Dirichlet form* (cf. [St, II.10]), if $\mathcal{E}$ satisfies in addition to D1 the assumptions D2 and D3 (resp. D2 and D3').

In order to state our main result precisely we recall some potential theoretic notions (cf. [St, Section III]) as well as some notations and notions from the general theory of Markov processes (cf. [St, Section IV]).

An element $h \in \mathcal{H}$ is called *1–excessive* if $\beta G_{\beta+1} h \leq h$ for all $\beta \geq 0$. For $h \in \mathcal{H}$ with $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$ denote with e_h the *1–reduced function of* h, where $\mathcal{L}_h = \{u \in \mathcal{H} \mid u \geq h\}$ (cf. [St, III.7 and III.8]). For an open subset $U \subset E$ and an element $h \in \mathcal{H}$ with $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$ let $h_U := e_{(h \cdot 1\!\!1_U)}$.

Definition 1.

(i) An increasing sequence of closed subsets $(F_k)_{k\geq 1}$ is called an *$\mathcal{E}$–nest*, if for every 1–excessive element $h \in \mathcal{F}$ it follows that $h_{F_k^c} \to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$.

(ii) A subset $N \subset E$ is called *$\mathcal{E}$–exceptional* if there is an $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ such that $N \subset \cap_{k\geq 1} E \setminus F_k$.

(iii) A function $f : A \to \mathbb{R}$, where $A \subset E$, is called *$\mathcal{E}$–quasi–continuous* ($\mathcal{E}$–q.c.) (resp. *$\mathcal{E}$–quasi–lower–semicontinuous* ($\mathcal{E}$–q.l.s.c.)) if there is some $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ such that $\bigcup_{k\geq 1} F_k \subset A$ and $f|_{F_k}$ is continuous (resp. lower semicontinuous) $\forall k$.

We can now define the quasi–regularity property of generalized semi–Dirichlet forms in analogy with [A/M/R] and [M/R] (cf. [St, IV.1]).

Definition 2. The generalized semi–Dirichlet form $\mathcal{E}$ is called *quasi–regular* if:

(i) There exists an $\mathcal{E}$–nest $(E_k)_{k\geq 1}$ consisting of compact sets.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$–q.c. m–versions.

(iii) There exists $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having $\mathcal{E}$–q.c. m–versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$–exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

As usual let $E_\Delta = E \cup \{\Delta\}$ where Δ is adjoined as an isolated point to E and extend every function f on E to E_Δ by setting $f(\Delta) := 0$. Let $\mathbb{M} = (\Omega, \mathcal{F}, (Y_t)_{t\geq 0},$

$(P_x)_{x\in E_\Delta})$ be a right process with state space E and life time ζ. Denote with $(\mathcal{F}_t)_{t\geq 0}$ the corresponding natural filtration. For an arbitrary finite positive measure μ on $(E_\Delta, \mathcal{B}(E_\Delta))$ let $P_\mu := \int P_x \, \mu(dx)$ and for a sub–σ–algebra $\mathcal{A} \subset \mathcal{F}$ denote with $\mathcal{A}^{P_\mu}$ its P_μ–completion in $\mathcal{F}$. For a subset $A \in \mathcal{B}(E)$ let $\sigma_A := \inf\{t > 0 \mid Y_t \in A\}$ be the first hitting time.

Definition 3.

(i) $\mathbb{M}$ is called *μ–tight*, if there exists an increasing sequence $(K_n)_{n\geq 1}$ of compact metrizable sets in E such that

$$P_\mu\left[\lim_{n\to\infty} \sigma_{E\setminus K_n} < \zeta\right] = 0.$$

(ii) $\mathbb{M}$ is called *μ–special standard*, if

(a) $Y_{t-} := \lim_{\substack{s\uparrow t\\ s<t}} Y_s$ exists in E for all $t \in (0,\zeta)$ P_μ–a.s.

(b) $\lim_{n\to\infty} Y_{\tau_n} = Y_\tau$ P_μ–a.s. on $\{\tau < \zeta\}$ and Y_τ is $\bigvee_{n\geq 1} \mathcal{F}_{\tau_n}^{P_\mu}$–measurable for every increasing sequence $(\tau_n)_{n\geq 1}$ of $(\mathcal{F}_t)_{t\geq 0}$–stopping times with limit τ.

(iii) $\mathbb{M}$ is called *special standard*, if $\mathbb{M}$ is μ–special standard for all probability measures μ on $(E_\Delta, \mathcal{B}(E_\Delta))$.

Definition 4. A right process $\mathbb{M}$ with state space E and resolvent $(R_\alpha)_{\alpha>0}$ is called *associated with $\mathcal{E}$* if $R_\alpha f$ is an m–version of $G_\alpha f$ for all $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$. $\mathbb{M}$ is called *properly associated in the resolvent sense with $\mathcal{E}$* if in addition $R_\alpha f$ is $\mathcal{E}$–q.c. for $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$.

Denote with $(T_t)_{t\geq 0}$ the strongly continuous contraction semigroup corresponding to $(G_\alpha)_{\alpha\geq 0}$ (cf. [M/R, I.1.12]). If $\mathbb{M}$ is associated with $\mathcal{E}$ and $(p_t)_{t\geq 0}$ denotes the transition semigroup of $\mathbb{M}$ then it is easy to see that also $p_t f$ is an m–version of $T_t f$ for all $t \geq 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$.

We recall that the main result in our previous paper [St] is contained in the following Theorem (cf. [St, IV.8]).

Theorem 5. *Let $\mathcal{E}$ be a quasi–regular generalized semi–Dirichlet form. Then there exists an m–tight special standard process $\mathbb{M}$ with state space E which is properly associated in the resolvent sense with $\mathcal{E}$.*

The main result of this paper now is the following converse to Theorem 5:

Theorem 6. *Let $\mathcal{E}$ be a generalized semi–Dirichlet form. Assume that there is an m–tight special standard process $\mathbb{M}$ with state space E which is associated with the form $\mathcal{E}$. Then $\mathcal{E}$ is quasi–regular, and, moreover, $\mathbb{M}$ is properly associated in the resolvent sense with $\mathcal{E}$.*

Proof. See Section 2 below.

Together with Theorem 5 we obtain the following

Corollary 7. *Let $\mathcal{E}$ be a generalized semi–Dirichlet form. Then $\mathcal{E}$ is quasi–regular if and only if there exists an m–tight special standard process $\mathbb{M}$ associated with $\mathcal{E}$.*

Remark. Let E be a metrizable Lusin space (i.e. a space that is topological isomorphic to a Borel subset of some complete separable metric space) then a special standard process on E is automatically m–tight (cf. [M/R, IV.1.15]).

Remark. Let $(G_\alpha)_{\alpha>0}$ be a sub–Markovian strongly continuous contraction resolvent on $\mathcal{H}$ with generator $(\Lambda, D(\Lambda))$ on $\mathcal{H}$. Denote with $(\widehat{\Lambda}, D(\widehat{\Lambda}))$ the adjoint operator. If there is a dense linear subspace $\mathcal{Y} \subset D(\Lambda)$ (w.r.t. the graph norm) satisfying D3 (resp. D3') then the corresponding form

$$\mathcal{E}(u,v) := \begin{cases} -(\Lambda u, v) & \text{if } u \in D(\Lambda), v \in \mathcal{H} \\ -\left(\widehat{\Lambda} v, u\right) & \text{if } u \in \mathcal{H}, v \in D(\widehat{\Lambda}) \end{cases}$$

is a generalized semi–Dirichlet form. We say that $(G_\alpha)_{\alpha>0}$ *is quasi–regular* if the form $\mathcal{E}$ is a quasi–regular generalized semi–Dirichlet form. A right process $\mathbb{M}$ with state space E is called *associated with* $(G_\alpha)_{\alpha>0}$ if it is associated with $\mathcal{E}$ and it is called *properly associated with* $(G_\alpha)_{\alpha>0}$ if it is properly associated in the resolvent sense with $\mathcal{E}$. Corollary 7 now implies that $(G_\alpha)_{\alpha>0}$ is quasi–regular if and only if there exists an m–tight special standard process $\mathbb{M}$ associated with $\mathcal{E}$.

Note that this is not the optimal result: for example consider the case $\mathcal{H} = L^2(\mathbb{R}^d, \varphi^2 dx)$ where $\varphi \in H^{1,2}(\mathbb{R}^d)$ such that

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, \varphi^2 \, dx \quad ; u, v \in C_0^\infty(\mathbb{R}^d)\,,$$

is closable in $\mathcal{H}$ with closure $(\mathcal{E}, \mathcal{V})$. Denote with $(G_\alpha)_{\alpha>0}$ the corresponding sub–Markovian strongly continuous contraction reslovent. It is now easy to see that the form $\mathcal{E}$ is a quasi–regular Dirichlet form in the sense of [A/M/R], hence in particular a quasi–regular *generalized* semi–Dirichlet form. But it is of course much more difficult to check that $(G_\alpha)_{\alpha>0}$ is quasi–regular.

2. Proof of Theorem 1.6

Denote with $(G_\alpha)_{\alpha>0}$ (resp. $(T_t)_{t\geq 0}$) the strongly continuous contraction resolvent (resp. semigroup) associated to $\mathcal{E}$. Let $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ with life time ζ be an m–tight special standard process associated with $\mathcal{E}$. Denote with $(R_\alpha)_{\alpha>0}$ (resp. $(p_t)_{t\geq 0}$) the corresponding resolvent (resp. transition semigroup).

(a) Proof of 1.2 (i).

Denote with $\tilde{C}$ the set of all $f \in \mathcal{B}_b(E)$ such that $t \mapsto f(X_t)$ is right continuous on $[0,\infty)$ and $t \mapsto f(X_{t-})$ is left continuous on $(0,\zeta) P_m$–a.s.

Proposition 1. *$R_\alpha f \in \tilde{C}$ for all $f \in \mathcal{B}_b(E)$, $\alpha > 0$.*

For the proof of Proposition 1 we can carry over the proof of [M/R, IV.5.14] which extends immediately to our more general situation.

A function $f \in \mathcal{B}_b(E)$ is called *1–supermedian for* $(R_\alpha)_{\alpha>0}$ if $\alpha R_{\alpha+1} f \leq f$, $\alpha \geq 0$. Note that 1–supermedian functions $f \in \mathcal{B}_b(E) \cap \mathcal{H}$ are m–versions of 1–excessive elements in $\mathcal{H}$. $f \in \mathcal{B}_b(E)$ is called *1–excessive for* $(R_\alpha)_{\alpha>0}$ if it is 1–supermedian for $(R_\alpha)_{\alpha>0}$ and $\lim\limits_{\alpha\to\infty} \alpha R_{\alpha+1} f = f$.

For $g \in \mathcal{B}_b(E)$ and $U \subset E$ open let $H^U g(x) := E_x[e^{-\sigma_U} g(X_{\sigma_U})]$.

Proposition 2. *Let $f \in \mathcal{B}_b^+(E) \cap \mathcal{H}$ and $U \subset E$ open. Then $H^U R_1 f$ is an m–version of $(G_1 f)_U$.*

Proof. 1. Step: $H^U R_1 f \geq (G_1 f)_U$ m–a.e.

The strong Markov property of $\mathbb{M}$ implies

$$
\begin{aligned}
H^U R_1 f(x) &= E_x\left[e^{-\sigma_U} E_{X_{\sigma_U}}\left[\int_0^\infty e^{-t} f(X_t)\,dt\right]\right] \\
&= E_x\left[\int_{\sigma_U}^\infty e^{-t} f(X_t)\,dt\right] .
\end{aligned}
\tag{2.1}
$$

Since

$$
\sigma_U \leq \sigma_U \circ \theta_t + t
\tag{2.2}
$$

we obtain

$$
\begin{aligned}
e^{-t} p_t H^U R_1 f(x) &= E_x\left[e^{-t} E_{X_t}\left[\int_{\sigma_U}^\infty e^{-s} f(X_s)\,ds\right]\right] \\
&= E_x\left[\int_{\sigma_U\circ\theta_t+t}^\infty e^{-s} f(X_s)\,ds\right] \\
&\leq E_x\left[\int_{\sigma_U}^\infty e^{-s} f(X_s)\,ds\right] = H^U R_1 f(x)\ \forall x .
\end{aligned}
\tag{2.3}
$$

It follows that $H^U R_1 f$ is 1–supermedian for $(R_\alpha)_{\alpha>0}$ because

$$
\alpha R_{\alpha+1} H^U R_1 f(x) = \int_0^\infty \alpha e^{-(\alpha+1)t} p_t H^U R_1 f(x)\,dt \leq H^U R_1 f(x) .
\tag{2.4}
$$

Since $H^U R_1 f \leq R_1 f$ (cf. (2.1)) we obtain that $H^U R_1 f$ is the m–version of some

1–excessive element in $\mathcal{H}$ and

$$\begin{aligned}
&\mathcal{A}_1\left(R_1 f-\alpha G_{\alpha+1} H^U R_1 f, R_1 f-\alpha G_{\alpha+1} H^U R_1 f\right) \\
&\leq \mathcal{E}_1\left(R_1 f-\alpha G_{\alpha+1} H^U R_1 f, R_1 f-\alpha G_{\alpha+1} H^U R_1 f\right) \\
&=\mathcal{E}_1\left(R_1 f, R_1 f-\alpha G_{\alpha+1} H^U R_1 f\right) \\
&\quad-\alpha\left(H^U R_1 f-\alpha G_{\alpha+1} H^U R_1 f, R_1 f-\alpha G_{\alpha+1} H^U R_1 f\right)_{\mathcal{H}} \\
&\leq \mathcal{E}_1\left(R_1 f, R_1 f-\alpha G_{\alpha+1} H^U R_1 f\right) \\
&\leq 2K\|R_1 f\|_{\mathcal{F}}\ \|R_1 f-\alpha G_{\alpha+1} H^U R_1 f\|_{\mathcal{V}}\ .
\end{aligned}$$

Hence

$$\begin{aligned}
\|\alpha G_{\alpha+1} H^U R_1 f\|_{\mathcal{V}} &\leq \|\alpha G_{\alpha+1} H^U R_1 f-R_1 f\|_{\mathcal{V}}+\|R_1 f\|_{\mathcal{V}} \\
&\leq 3K\|R_1 f\|_{\mathcal{F}}
\end{aligned}$$

which implies $\sup\limits_{\alpha>0}\|\alpha G_{\alpha+1} H^U R_1 f\|_{\mathcal{V}}<+\infty$ and consequently $H^U R_1 f \in \mathcal{V}$ (cf. [M/R, I.2.12]).

Clearly, $\sigma_U=0$ P_x–a.s. for $x \in U$ and therefore $H^U R_1 f(x)=R_1 f(x)$. This implies $H^U R_1 f \geq (G_1 f)_U$ m–a.e. (cf. [St, III.7 (i)]).

2. Step: $H^U R_1 f \leq (G_1 f)_U$ m–a.e.

Let $\{t_i\}_{i\geq 1}\subset[0,\infty)$ dense and define the stopping times

$$\sigma_n:=\min\{t_i \mid 1\leq i\leq n, X_{t_i}\in U\}, n\geq 1\ ,$$

(with the convention $\min\emptyset=+\infty$). Set $e^{-\infty}g(X_\infty):=0\ \forall g\in\mathcal{B}_b(E)$. Note that

$$E_x\left[e^{-\sigma_n} g(X_{\sigma_n})\right]=0 \quad m\text{–a.e.} \tag{2.5}$$

for $g\in\mathcal{B}_b(E)\cap\mathcal{H}$ with $g=0$ m–a.e. on U since

$$\begin{aligned}
\left|E_x\left[e^{-\sigma_n} g(X_{\sigma_n})\right]\right| &\leq \sum_{i=1}^n\left|E_x\left[e^{-t_i} g(X_{t_i})\mathbb{1}_{\{\sigma_n=t_i\}}\right]\right| \\
&\leq \sum_{i=1}^n E_x\left[e^{-t_i}|g\mathbb{1}_U|(X_{t_i})\right] \\
&=\sum_{i=1}^n e^{-t_i} p_{t_i}|g\mathbb{1}_U|(x)=0 \quad m\text{–a.e.}
\end{aligned}$$

because $p_t|g\mathbb{1}_U|$ is an m–version of $T_t|g\mathbb{1}_U|$.

Observe that $(e^{-t}R_1 g(X_t),\mathcal{F}_t,P_x)_{t\in[0,\infty]}$ is a nonnegative supermartingale for all x and $g\in\mathcal{B}_b^+(E)$ (where $(\mathcal{F}_t)_{t\geq 0}$ denotes the natural filtration corresponding to $\mathbb{M}$) since $e^{-t}p_t R_1 g\leq R_1 g$ and therefore

$$\begin{aligned}
E_x\left[e^{-(t+s)} R_1 g(X_{t+s})\middle|\mathcal{F}_s\right] &= E_{X_s}\left[e^{-(t+s)} R_1 g(X_t)\right] \\
&= e^{-(t+s)} p_t R_1 g(X_s)\leq e^{-s} R_1 g(X_s)\ .
\end{aligned}$$

Now, let $g \in \mathcal{B}_b^+(E)$ be an m–version of $(G_1 f)_U$. Since $(G_1 f)_U$ is 1–excessive, we obtain that $f_n := n(g - nR_{n+1}g) \geq 0 \quad m$–a.e., so that $\lim_{n\to\infty} R_1(f_n^+) = g \quad m$–a.e. Hence there is a set $A \in \mathcal{B}(E)$ with $m(E \setminus A) = 0$ and $\lim_{n\to\infty} R_1(f_n^+)(x) = g(x)\, \forall x \in A$. Therefore $\overline{g} := g\mathbb{1}_A$ is an m–version of $(G_1 f)_U$ too, and consequently $\overline{g} = R_1 f$ m–a.e. on U. By Proposition 1, (2.5) and the discrete version of the optional sampling theorem (cf. [Ba, Satz 17.4]) for m–a.e. $x \in A$

$$\begin{aligned}
H^U R_1 f(x) = \;& E_x\left[e^{-\sigma_U} R_1 f(X_{\sigma_U})\right] \\
=\;& \lim_{n\to\infty} E_x\left[e^{-\sigma_n} R_1 f(X_{\sigma_n})\right] \\
=\;& \lim_{n\to\infty} E_x\left[e^{-\sigma_n} \overline{g}(X_{\sigma_n})\right] \\
=\;& \lim_{n\to\infty} \lim_{m\to\infty} E_x\left[e^{-\sigma_n} R_1(f_m^+)\mathbb{1}_A(X_{\sigma_n})\right] \\
\leq\;& \lim_{n\to\infty} \limsup_{m\to\infty} E_x\left[e^{-\sigma_n} R_1(f_m^+)(X_{\sigma_n})\right] \\
\leq\;& \lim_{n\to\infty} \limsup_{m\to\infty} R_1(f_m^+)(x) = g(x)\ .
\end{aligned}$$

Hence $H^U R_1 f \leq (G_1 f)_U$ m–a.e. $\square$

Since $\mathbb{M}$ is m–tight there exists an increasing sequence $(K_n)_{n\geq 1}$ of compact metrizable sets in E with

$$P_m\left[\lim_{n\to\infty} \sigma_{K_n^c} < \zeta\right] = 0\ .$$

Let $\varphi \in \mathcal{B}_b^+(E) \cap \mathcal{H}$, $0 < \varphi(x) \leq 1\ \forall x$, and $h := R_1\varphi$. Then we can define the φ–capacity for all subsets of E as follows (cf. [St, III.12]): For $U \subset E$ open let $\mathrm{Cap}_\varphi(U) := (h_U, \varphi)_{\mathcal{H}}$ and for arbitrary $A \subset E$ let $\mathrm{Cap}_\varphi(A) := \inf\limits_{\substack{A\subset U\\ U \text{open}}} \mathrm{Cap}_\varphi(U)$.

Proposition 3. *$(K_n)_{n\geq 1}$ is an $\mathcal{E}$–nest.*

Proof. Due to [St, III.17] it is enough to show $\lim\limits_{n\to\infty} \mathrm{Cap}_\varphi(K_n^c) = 0$. But Proposition 2 implies

$$\begin{aligned}
\mathrm{Cap}_\varphi(K_n^c) &= \int_E E_x\left[e^{-\sigma_{K_n^c}} R_1\varphi\left(X_{\sigma_{K_n^c}}\right)\right] \varphi(x) m(dx) \\
&= \int_E E_x\left[\int_{\sigma_{K_n^c}}^{\zeta} e^{-s}\varphi(X_s)\, ds\right] \varphi(x) m(dx)\ .
\end{aligned}$$

Hence it suffices to show that

$$E_x\left[\int_{\sigma_{K_n^c}}^{\zeta} e^{-s}\varphi(X_s)\, ds\right] \to 0 \text{ in } \mathcal{H}\ .$$

Clearly, $P_x\left[\lim\limits_{n\to\infty} \sigma_{K_n^c} < \zeta\right] = 0 \quad m$–a.e., implies

$$\int_{\sigma_{K_n^c}}^{\zeta} e^{-s}\varphi(X_s)\, ds \to 0 \quad P_x\text{–a.s.} \quad m\text{–a.e.}$$

Therefore $E_x\left[\int_{\sigma_{K_n^c}}^{\zeta} e^{-s}\varphi(X_s)\,ds\right] \to 0 \quad m$–a.e. by Lebesgues theorem and consequently

$$E_x\left[\int_{\sigma_{K_n^c}}^{\zeta} e^{-s}\varphi(X_s)\,ds\right] \to 0 \text{ in } \mathcal{H}$$

by Lebesgues theorem again. □

Clearly, Proposition 3 implies 1.2 (i).

Remark. The proof of the proposition above shows that an increasing sequence of closed sets $(F_k)_{k\geq 1}$ is an $\mathcal{E}$–nest if and only if $P_m[\lim_{k\to\infty}\sigma_{F_k^c} < \zeta] = 0$.

(b) Proof of 1.2 (ii).

For arbitrary $A \subset E$ denote with $\tau_A := \inf\{0 \leq t < \zeta \mid \overline{X_0^t} \cap A \neq \emptyset\} \wedge \zeta$ the *first touching time of* A, where $\overline{X_0^t}(\omega)$ is the closure of $\{X_s(\omega) \mid 0 \leq s \leq t\}$ in E. A subset $N \subset E$ is called $\mathbb{M}$–*exceptional* if there exists $\tilde{N} \in \mathcal{B}(E)$ such that $N \subset \tilde{N}$ and $P_m[\tau_{\tilde{N}} < \zeta] = 0$. Note that this implies $m(\tilde{N}) = 0$, so that $\mathbb{M}$–exceptional sets $N \in \mathcal{B}(E)$ are zero sets w.r.t. m.

Proposition 4. *Let $f \in \tilde{C}$, then there exists an $\mathcal{E}$–quasi continuous function $\tilde{f}$ on E such that $f = \tilde{f}$ outside some $\mathbb{M}$–exceptional set.*

The proof of Proposition 4 is the same as the proof of Proposition IV.5.24 in [M/R] if one replaces $\text{Cap}_{h,g}$ in [M/R] by Cap_φ and one observes that the assumption $\int \varphi\, dm = 1$ (cf. [M/R, IV.5.4]) is not needed in the proof of [M/R, IV.5.24].

Since $\{G_\alpha f | f \in \mathcal{H}\} \subset \mathcal{F}$ dense (cf. [St, II.7]) Propositions 1 and 4 imply 1.2 (ii).

(c) Proof of 1.2 (iii).

Theorem 5. *(i) A subset N of E is $\mathbb{M}$–exceptional if and only if it is $\mathcal{E}$–exceptional.*

(ii) If $f \in \tilde{C}$ then f is $\mathcal{E}$–q.c. In particular, $R_\alpha f$ is $\mathcal{E}$–q.c. for all $f \in \mathcal{B}_b(E), \alpha > 0$, and $\mathbb{M}$ is properly associated in the resolvent sense with $\mathcal{E}$.

The proof of Theorem 5 is the same as the proof of Theorem IV.5.29 in [M/R] if one replaces $\text{Cap}_{h,g}$ in [M/R] by Cap_φ. Note that Cap_φ is a (finite) Choquet capacity (cf. [St, III.15]).

Lemma 6. *Let $f \in \mathcal{B}_b^+(E) \cap \mathcal{H}$ and $U \subset E$ open. Then $H^U R_1 f$ is an $\mathcal{E}$–q.l.s.c. m–version of $(G_1 f)_U$.*

Proof. We already showed that $H^U R_1 f$ is 1–supermedian for $(R_\alpha)_{\alpha>0}$ (cf. (2.4)).

Since $\lim_{t\downarrow 0} \sigma_U \circ \theta_t(\omega) + t = \sigma_U(\omega)\ \forall \omega$, we obtain by Lebesgues Theorem

$$\begin{aligned} \lim_{t\downarrow 0} e^{-t} p_t H^U R_1 f(x) &= \lim_{t\downarrow 0} E_x \left[\int_{\sigma_U \circ \theta_t + t}^{\infty} e^{-s} f(X_s)\, ds \right] \\ &= E_x \left[\int_{\sigma_U}^{\infty} e^{-s} f(X_s)\, ds \right] = H^U R_1 f(x)\ \forall x\ . \end{aligned}$$

It is easy to see that this implies

$$\lim_{\alpha\to\infty} \alpha R_{\alpha+1} H^U R_1 f(x) = H^U R_1 f(x)\ \forall x\ . \tag{2.6}$$

Therefore $\sup_{n\geq 1} n R_{n+1} H^U R_1 f = H^U R_1 f$. Since $\mathbb{M}$ is properly associated in the resolvent sense with $\mathcal{E}$ we obtain that $H^U R_1 f$ is $\mathcal{E}$–q.l.s.c. $\square$

Lemma 7. *$P_x \left[\lim_{n\to\infty} \sigma_{K_n^c} < \zeta \right] = 0$ $\mathcal{E}$–q.e., where $(K_n)_{n\geq 1}$ is as in Proposition 3.*

Proof. Let φ be as in Proposition 3. Set $h = R_1\varphi$ and $V_n := E \setminus K_n$. Clearly $H^{V_n} h \to 0$ in $\mathcal{H}$, as shown in the proof of Proposition 3. Since $H^{V_n} h$ is $\mathcal{E}$–q.l.s.c. there is a subsequence $\left(H^{V_{n_k}} h\right)_{k\geq 1}$ such that

$$\overline{\lim_{k\to\infty}}\, H^{V_{n_k}} h \leq 0 \quad \mathcal{E}\text{–q.e.}$$

(cf. [St, III.21]). But $H^{V_{n_k}} h \geq 0$ implies $\lim_{k\to\infty} H^{V_{n_k}} h = 0$ $\quad\mathcal{E}$–q.e. Since $H^{V_{n+1}} h \leq H^{V_n} h$ we conclude that $\lim_{n\to\infty} H^{V_n} h = 0$ $\quad\mathcal{E}$–q.e. It follows $P_x[\lim_{n\to\infty} \sigma_{V_n} < \zeta] = 0$ $\quad\mathcal{E}$–q.e. because

$$H^{V_n} h(x) = E_x \left[\int_{\sigma_{V_n}}^{\zeta} e^{-t} \varphi(X_t)\, dt \right]$$

and $\varphi > 0$ on E. $\square$

Let $N \subset E$ be $\mathcal{E}$–exceptional such that

$$P_x \left[\lim_{n\to\infty} \sigma_{K_n^c} < \zeta \right] = 0 \quad \forall x \in E \setminus N\ .$$

Since K_n is a compact metrizable set (w.r.t. the trace topology) there exist open sets U_n in E such that $(U_m \cap K_n)_{m\geq 1}$ separates the points of K_n $\forall n$ (i.e. if $x, y \in K_n$, $\ x \neq y$, there exist $U, V \in \mathcal{U} := \{U_n \mid n \geq 1\}$ such that $U \cap K_n \cap V = \emptyset$ and $x \in U$, $y \in V$).

Lemma 8. *$(h - H^U h)_{U\in\mathcal{U}}$ separates the points of $\bigcup_{n\geq 1} K_n \cap (E \setminus N)$.*

Proof. Let $x, y \in \bigcup_{n\geq 1} K_n \cap (E \setminus N)$, $\ x \neq y$. Then we may assume $x, y \in K_m$ for some m with $P_y \left[\sigma_{K_m^c} > 0\right] > 0$ and that there exist $U, V \in \mathcal{U}$ such that $x \in U$,

$y \in V$ and $U \cap K_m \cap V = \emptyset$. Let $\overline{\Omega} := \{\omega \mid \sigma_{K_m^c}(\omega) > 0, X_0(\omega) = y\}$. For $\omega \in \overline{\Omega}$ the map $X_\cdot(\omega) : [0, \sigma_{K_m^c}(\omega)) \to K_m$ is right–continuous w.r.t. the trace topology on K_m. Since $V \cap K_m$ is open in K_m and $U \cap K_m \cap V = \emptyset$ this implies $\sigma_U(\omega) > 0$. Hence $P_y[\sigma_U > 0] > 0$ and therefore

$$H^U h(y) = E_y\left[\int_{\sigma_U}^{\infty} e^{-t}\varphi(X_t)\,dt\right] < E_y\left[\int_0^{\infty} e^{-t}\varphi(X_t)\,dt\right] = h(y)\ .$$

On the other hand clearly $H^U h(x) = h(x)$, which implies the assertion. □

Lemma 9. $\{nR_{n+1}(h - H^U h) \mid U \in \mathcal{U}, n \geq 1\}$ *separates the points of* $\bigcup_{n\geq 1} K_n \cap (E \setminus N)$.

Proof. Clearly, $\lim_{n\to\infty} nR_{n+1}(h - H^U h)(x) = (h - H^U h)(x)\ \forall x$ (cf. (2.6)). Let $x, y \in \bigcup_{n\geq 1} K_n \cap (E \setminus N)$, $x \neq y$. By Lemma 8 there is some $U \in \mathcal{U}$ with $(h - H^U h)(x) \neq (h - H^U h)(y)$. Then $nR_{n+1}(h - H^U h)(x) \neq nR_{n+1}(h - H^U h)(y)$ for large n. Hence the assertion follows. □

Since $N \cup \left(\bigcap_{n\geq 1}(E \setminus K_n)\right)$ is an $\mathcal{E}$–exceptional set and $nR_{n+1}(h - H^U h)$ is $\mathcal{E}$–q.c. (due to Theorem 5) we see that 1.2 (iii) holds.

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