Absolutely continuous flows generated by Sobolev class vector fields in finite and infinite dimensions

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Abstract

We prove the existence of the global flow $\{U_t\}$ generated by a vector field A from a Sobolev class $W^{1,1}(\mu)$ on a finite or infinite dimensional space X with a measure μ, provided μ is sufficiently smooth, and ∇A and $|\delta_\mu A|$ (where $\delta_\mu A$ is the divergence with respect to μ) are exponentially integrable. In addition, the measure μ is shown to be quasi-invariant under $\{U_t\}$. In the case $X = \mathbb{R}^n$ and $\mu = pdx$, where $p \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ is a locally uniformly positive probability density, a sufficient condition is: $\exp(c\|\nabla A\|) + \exp(c|(A, \frac{\nabla p}{p})|) \in L^1(\mu)$ for all c. In the infinite dimensional case we get analogous results for measures differentiable along sufficiently many directions. Examples of measures which fit our framework, important for applications, are symmetric invariant measures of infinite dimensional diffusions and Gibbs measures. Typically, in both cases such measures are essentially non-Gaussian. Our result in infinite dimensions significantly extends previously studied cases where μ was a Gaussian measure. Finally, we study flows generated by vector fields whose values are not necessarily in the Cameron–Martin space.

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Contents

1 Introduction

One of the fundamental problems of the theory of dynamical systems concerns the existence and uniqueness of a global flow $\{U_t\}$ generated by a vector field A. It is often desirable to have a measure μ which is invariant or quasi-invariant under $\{U_t\}$. A standard result says that if A is uniformly Lipschitzian on $\mathbb{R}^n$ or on a Banach space, then there is a unique flow $\{U_t\}$ generated by A. In addition, in the finite dimensional case the Lebesgue measure is quasi-invariant under $\{U_t\}$ (hence this is true for every measure with strictly positive density). However, in many applications, e.g., in fluid mechanics, statistical physics, stochastic analysis (especially in infinite dimensions) one has to deal with vector fields which are not even locally Lipschitzian, but still have certain regularity properties such as belonging to some Sobolev class. This kind of regularity not always implies continuity, so the very existence of solutions is not granted in advance. Recent infinite dimensional analysis research, in particular in infinite dimensional manifolds, gives rise to questions of the same sort (see, e.g., [21], [22], [26], [28], [31]).

Even if we deal with uniformly bounded continuous fields on $\mathbb{R}^n$ it may happen that such a weak regularity is not enough, e.g., for uniqueness. An obvious example: $n = 1$, $A(x) = 0$ on $(-\infty, 0]$, $A(x) = \min(x^\alpha, 1)$ on $[0, \infty)$, with $\alpha \in (0, 1)$. Then A is in $W^{p,1}_{\mathrm{loc}}(\mathbb{R}^1)$ for $p < \frac{1}{1-\alpha}$, but the corresponding equation has many solutions. It is shown in [25] that for each $p > 1$ there is a field $A \in C_b(\mathbb{R}^2) \cap W^{p,1}_{\mathrm{loc}}(\mathbb{R}^2)$ generating infinitely many flows (that is, solutions with the group property) under which the Lebesgue measure is quasi-invariant. It also worth mentioning that in infinite dimensions the continuity of A does not imply even local solvability (see, e.g., [8, Section 2]).

Before proceeding further let us state with some precision what we mean by a solution to our dynamical systems:

Definition 1.1 *If X is a topological vector space equipped with a positive Radon measure μ on its Borel sigma field and a measurable vector field $A : X \longrightarrow X$, a mapping $(t,x) \in \mathbb{R} \times X \longrightarrow U_t^A(x) \in X$ is said to be a solution of the equation*

$$U_t(x) = x + \int_0^t A(U_s(x))\, ds \tag{1.1}$$

if

a) for μ–almost every x (1.1) holds with $U = U^A$ for all $t \in \mathbb{R}$ (its right hand side being implicitly well defined), and

b) for every $t \in \mathbb{R}$ $\mu \circ U_t^{-1}$ is absolutely continuous with respect to μ. The Radon–Nikodym derivative $\frac{d(\mu \circ U_t^{-1})}{d\mu}$ will always be denoted by r_t.

Moreover, $U = U^A$ is a flow solution *if, in addition, μ–a.e.*

$$U_{t+s} = U_t \circ U_s \quad \forall s,t \in \mathbb{R}. \tag{1.2}$$

The quasiinvariance (requirement b) above) is essential when dealing with Lebesgue classes of vector fields, where we want solutions to be independent of the particular representative in the equivalence class:

Proposition 1.2 *If $A(x) = B(x)$ μ–a.e. then U^B also solves (1.1).*

Proof: By the quasiinvariance $\mu\{x :\ A(U_s^B(x)) \neq B(U_s^B(x))\} = 0$ for all $s \in \mathbb{R}$. By Fubini's theorem, thus, the set $\Omega_x = \{s :\ A(U_s^B(x)) \neq B(U_s^B(x))\} \subset \mathbb{R}$ has zero Lebesgue measure for μ–almost every x so that

$$U_t^B(x) = x + \int_0^t B(U_s^B(x))\, ds = x + \int_0^t A(U_s^B(x))\, ds.$$

for all $t \in \mathbb{R}$, μ–a.e. $\square$

A detailed investigation of the described problem was undertaken in [16]–[20], [25], [34], [14]. The results obtained by A.-B. Cruzeiro in [16]–[18] for the Wiener measure have found interesting applications in stochastic analysis (see [31]) and considerably influenced subsequent research in this direction, in particular our own. It was shown in [16] that if μ is a measure on $\mathbb{R}^n$ with a positive density $\varrho \in C^1$ and a vector field $A \in C^3$ is such that $\|A\|$ and $|\delta_\mu A|$ (where $\delta_\mu A = \mathrm{div} A + (A, \frac{\nabla \varrho}{\varrho})$ is the divergence of A with respect to μ) are exponentially integrable, then μ-a.e. there exists a flow $\{U_t\}$ under which μ is quasi-invariant, and, in addition, an explicit expression for the Radon–Nikodym derivative $d(\mu \circ U_t^{-1})/d\mu$ was derived. The same is true if $A \in C^4 \cap L^4(\mu)$, $\|\nabla A\| \in L^3(\mu)$ and $\delta_\mu A \in L^\infty(\mu)$ or, for Gaussian μ, if $A \in W_2^\infty(\mu, \mathbb{R}^n)$ and $\exp(\|\nabla A\|)$ and $\exp(|\delta_\mu A|)$ are in all $L^p(\mu)$. This result was extended in [17] to the infinite dimensional case where μ is the Wiener measure on the classical Wiener space (it could just as well have been any abstract Wiener space) and A is a field taking values in the Cameron–Martin space H of μ, belonging to the Sobolev class $W_2^\infty(\mu, H)$ over μ and satisfying the exponential integrability condition $\exp\{|\delta_\mu A| + \|A\| + \|\nabla A\|_{HS}\} \in L^p(\mu)$ for all p, where $\|\nabla A\|_{HS}$ stands for the Hilbert–Schmidt norm (in fact, all the proofs in [17] go through equally well with the operator norm $\|\nabla A\|_{L(H)}$). (These conditions have been relaxed in [34], where the Hilbert–Schmidt norm of $\|\nabla A\|$ has been replaced by the usual operator norm). An example constructed in [16] in the case where μ is the standard Gaussian measure on $\mathbb{R}^2$ shows that even if $A \in C^\infty$ and $\delta_\mu A = 0$ it may happen that there is no global flow (see also Example 7.1 below).

In infinite dimensions there exists an exact analogue of the notion of a measure with a smooth density: this is a measure differentiable in Fomin's sense (see [12], [24] for references). As opposed to the finite dimensional situation, however, the study of such measures cannot be reduced to that of Gaussian measures in the sense that, as shown in [6] (see also [12]), there exist smooth measures on infinite dimensional spaces mutually singular with respect to all Gaussian measures. Moreover, and from the technical point of view, a fact which significantly simplifies calculations in the Gaussian case — in contrast with the general case considered here — is that the gradient commutes with finite dimensional conditioning. For both these reasons, Cruzeiro's theorem does not extend in a simple manner beyond the Gaussian case.

One of the motivations for the study of this class of measures is that, as shown in [10] (Theorem 4.1), it covers important applications such as all the symmetric invariant probability measures of infinite dimensional diffusions generated by stochastic differential equations $d\xi_t = dW_t + B(\xi_t)dt$ (see also [1] for a related discussion of Dirichlet operators) as well as Gibbs distributions arising in typical models (see [2]). It should be noted that in non-trivial examples, measures of both these types are essentially non-Gaussian.

Unlike [17]–[20] and [34], the paper [25] deals with the finite dimensional case and with the Lebesgue measure λ (instead of μ). The local regularity condition there is weaker than in the papers cited above: $A \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, but it is assumed that $A \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, and the exponential integrability of $\|\nabla A\|$ and $\delta_\mu A$ is replaced by the condition $\mathrm{div} A \in L^\infty(\mathbb{R}^n)$. Under these conditions the existence of a flow $\{U_t\}$ with $\lambda \circ U_t^{-1} \ll \lambda$ is established. Moreover, the corresponding Radon–Nikodym densities are uniformly bounded and locally uniformly positive.

These investigations show the natural role played by the exponential-type conditions on $\delta_\mu A$. The aim of our work is to show that similar "exponential integrabilities" are indeed the conditions (rather close to optimal) under which it is possible to guarantee the existence of absolutely continuous flows both in finite and infinite dimensions for broad classes of measures important for applications. In addition, we derive some estimates of the solutions which are of independent interest on their own. As a byproduct result we obtain explicit expressions for the corresponding Radon–Nikodym derivatives in the spirit of [5].

To simplify our exposition, in infinite dimensions we impose such conditions on measure μ and field A which in finite dimensions imply the existence of a strictly positive continuous density and the existence of solutions for all starting points. Weaker conditions under which μ–a.e. defined flows arise in finite dimensions will be investigated in a forthcoming paper (see, however, Theorem 2.16).

Our results in finite dimensions, to be dealt with in Section 2, can be summarized as follows:

Theorem A *Let μ be a measure on $\mathbb{R}^n$ possessing a density $\varrho \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ which is locally uniformly positive, and let $A \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be a vector field such that either $e^{\|\nabla A\|} \in \bigcap_{p>1} L^p(\mu)$ or $e^{\|\nabla A\|} \in \bigcap_{p>1} L^p_{\mathrm{loc}}(\mu)$ and $\|A\| \in L^{1+\varepsilon}(\mathbb{R}^n)$, and such that $e^{|\delta_\mu A|} \in \bigcap_{p>1} L^p(\mu)$, where $\delta_\mu A = \mathrm{div} A + (A, \nabla\varrho/\varrho)$. Then equation (1.1) possesses a flow solution.*

In Section 3 the necessary Sobolev structure in infinite dimensions is established. We consider a measure μ on a locally convex space X such that there is a separable Hilbert space H continuously embedded into X. This gives rise to the embedding $j : X^* \to H$. The measure μ is assumed to be Fomin differentiable along all vectors in $j(X^*)$ such that the corresponding logarithmic derivatives have all exponential moments. Then one can define the Sobolev–type classes $G_{p,1}(H, \mu, H)$ of H-valued vector fields on X differentiable along H, as well as the divergence $\delta_\mu A$ with respect to μ of a vector field A. Here our main result, considered in Sections 4 and 5, is

Theorem B *Let $A \in G_{\alpha,1}(\mu, H)$ for some $\alpha > 1$. Assume that $e^{\|\nabla A\|} \in \bigcap_{p>1} L^p(\mu)$ and that $e^{|\delta_\mu A|} \in \bigcap_{p>1} L^p(\mu)$ in a somewhat strengthened form* **A3)** *(or* **A3')***) specified below. Then the assertion of Theorem A holds true in X.*

(In the Gaussian case this "strengthened form" reduces to usual integrability).

Motivated by constructions developed in [26], [21], [22], we introduce a new concept of *weak* solutions for equations with values not necessarily in the Cameron–Martin space (or its analogues) and consider a class of vector fields which in the case of a Gaussian measure γ can be written (formally) as $A(x) = L(x)x$, where $L(\,\cdot\,)$ is a map with values in the space of antisymmetric operators on the Cameron–Martin space H (thus, the very existence of such a field needs studying). A concrete example in the case of the classical Wiener space $C([0,1], \mathbb{R}^d)$ is $A(w)(t) = \int_0^t C(w(s))\,dw(s)$, where $C(\,\cdot\,)$ takes values in antisymmetric operators on $\mathbb{R}^d$. If $L(x) \equiv L \in L(H)$, then we get the linear equation $x'(t) = Lx(t)$, which, however, has to be made sense of. In fact, we study an even more general linear case, where L is the generator of an arbitrary strongly continuous semigroup $\{T_t\}$ on H. For our concept of a solution, an existence result is obtained by applying a recent result from [11]. Our approach also gives simple necessary and sufficient conditions for the quasiinvariance of γ along the corresponding flow. We extend a result from [34] and prove the solvability of the equation $x'(t) = A(x(t)) + Lx(t)$, $x(0) = x$, where L is the generator of a strongly continuous semigroup $\{Q_t\}$ of orthogonal operators on H (extended to a measurable linear map according to [11]). Finally, we discuss a more general case where $C(\,\cdot\,)$ need not be constant. The corresponding flows are analogous to the "rotations" introduced in [41], [42].

Thus, in comparison with the results in [16]–[20], [34], and [14] both in finite and infinite dimensions, our theorems cover much broader classes of measures (in particular, no moment conditions on measures are imposed which allows us to deal with general locally convex spaces and manifolds), only one-fold weak differentiability of A is required, and the integrability conditions are weakened. Another novelty is the new concept mentioned above of a weak solution introduced for dealing with non-Cameron–Martin space valued vector fields. It should be mentioned that our results are formulated intrinsically, in terms of the space H which plays the role of a "tangent space". It is also worth noting that assuming the weakest possible assumptions in finite dimensions is not only interesting in itself, but is also useful for the study of those infinite dimensional measures which fail to be differentiable but possess differentiable conditional measures on finite dimensional subspaces.

The problems described above are closely connected to the classical Malliavin calculus, nonlinear transformations of random processes ([15], [27], [36], [40], [44]) and conditional expectations preserving smoothness ([30], [32], [33], [39]).

Finally, Section 7 contains some examples, remarks and extensions related to the results described above.

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2 Finite dimensional case

2.1 Notation, terminology and auxiliary results

Throughout this section $|\Omega|$ will denote the Lebsegue measure of a measurable set $\Omega \subset \mathbb{R}^n$. The corresponding classes of integrable functions are denoted by $L^p(\Omega)$ without indicating the measure explicitly. The space of functions k times differentiable in Ω (with compact support) is denoted by $C^k(\Omega)$ (respectively, $C_0^k(\Omega)$); as usual, when $k = 0$ it is omitted in the notation.

The space of operators on $\mathbb{R}^n$ is equipped with the operator norm. If F is a measurable vector- or operator-valued map on $(\mathbb{R}^n, \mu)$, we use the same notation $\|F\|_{L^p(\mu)}$ as for scalar functions to denote the L^p-norm of F's norm in the corresponding target space.

Let $W^{p,r}(\Omega)$ be the Sobolev class of all functions on a domain $\Omega \subset \mathbb{R}^n$ which, together with its generalized partial derivatives up to order r, belong to $L^p(\Omega)$. This class is equipped with its natural Banach norm which is the sum of $\|\nabla^k f\|_{L^p}$, $k = 1, \ldots, r$. In a natural way one defines analogous Sobolev classes $W^{p,r}(\Omega, \mathbb{R}^m)$ of $\mathbb{R}^m$-valued mappings. The notation $W^{p,r}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ is used for the mappings whose restrictions belong to $W^{p,r}(D, \mathbb{R}^m)$, for every open ball D in $\mathbb{R}^n$.

For a measure μ with density $p \in W^{1,1}_{\rm loc}(\mathbb{R}^n)$ on $\mathbb{R}^n$ we define its logarithmic gradient (vector logarithmic derivative) β by

$$\beta(x) = \frac{\nabla p(x)}{p(x)},$$

where the ratio above is by definition zero on the set $\{x : p(x) = 0\}$. Clearly, $\|\beta\| \in L^1_{\rm loc}(\mu)$. For every h one has $\partial_h p(x)/p(x) = (\beta(x), h)$.

Let μ be a measure on $\mathbb{R}^n$ and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a μ-measurable map such that $\|A\| \in L^1_{\rm loc}(\mu)$. We say that the vector field A has a divergence $\delta_\mu A$ with respect to μ if $\delta_\mu A$ is a function in $L^1_{\rm loc}(\mu)$ such that for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ one has

$$\int\limits_{\mathbb{R}^n} (\nabla\varphi(x), A(x))\, \mu(dx) = -\int\limits_{\mathbb{R}^n} \varphi(x)\delta_\mu A(x)\, \mu(dx). \tag{2.1}$$

Note that if μ has a density $\varrho \in W^{1,1}_{\rm loc}(\mathbb{R}^n)$ and $A \in W^{1,1}_{\rm loc}(\mathbb{R}^n, \mathbb{R}^n)$ is such that $\varrho\|\nabla A\| + \|A\|\|\nabla\varrho\| \in L^1_{\rm loc}(\mathbb{R}^n)$, then $\delta_\mu A$ exists and

$$\delta_\mu A(x) = {\rm div} A(x) + \Big(A(x), \frac{\nabla\varrho}{\varrho}(x)\Big), \tag{2.2}$$

as one can check by the integration by parts formula. (Actually, the standard divergence operator div which appears in (2.2) is the divergence with respect to Lebesgue measure).

The concept of the divergence of a vector field with respect to a given measure will be considered in Section 3.3 in infinite dimensional spaces.

Lemma 2.1 *Let f be an integrable function on $[0, t]$, where $t \geq 0$. Then*

$$e^{\int_0^t f(s)\, ds} \leq 1 + \int_0^t e^{(t\vee 1)f(s)}\, ds. \tag{2.3}$$

Proof: By Jensen's inequality $\exp\Big(t^{-1}\int_0^t tf(s)\, ds\Big) \leq t^{-1}\int_0^t e^{tf(s)}\, ds$. If $t \geq 1$ this yields (2.3) immediately. If $t < 1$ we apply the estimate above on the interval $[0, 1]$ to the function g defined to be f on $[0, t]$ and zero on $(t, 1]$. Since $e^{g(s)} = 1$ on $(t, 1]$ we again obtain (2.3). $\square$

Lemma 2.2 *Let ν be a finite nonnegative measure on a space Ω and let $\{U_t\}_{|t|\leq T}$ be a family of measurable transformations of Ω such that $\nu \circ U_t^{-1} = r_t\nu$, where $r_t(x) = \exp\Big\{\int\limits_0^t f(U_{-s}(x))\, ds\Big\}$ with $\exp(|f|) \in L^p(\nu)$ for all $p \in (0, \infty)$. Then if (denote throughout $\|\cdot\|_\alpha = \|\cdot\|_{L^\alpha(\nu)}$)*

$$\int\limits_{-T}^{T} \|r_t\|_{1+\varepsilon}\, dt < \infty \tag{2.4}$$

holds for some $\varepsilon > 0$ it also holds for every $\varepsilon > 0$. Furthermore, for every $p > 1$ and $t \in [-T, T]$

$$\|r_t\|_p \leq (2 + 2\nu(\Omega))e^{C(p,T)|t|}, \tag{2.5}$$

where $C(p, T) = \Big(\int\limits_\Omega e^{qp(T\vee 1)|f(x)|}\, \nu(dx)\Big)^{1/q}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: By Lemma 2.1, for any $t \in [0,T]$ we get $r_t(x)^p \le 1 + \int\limits_0^t \exp\Big\{p(t\vee 1) f\left(U_{-s}(x)\right)\Big\}\,ds$, whence by Hölder's inequality with k defined by $\frac{1}{k} + \frac{1}{1+\varepsilon} = 1$

$$\int\limits_\Omega r_t(x)^p\,\nu(dx) \le \nu(\Omega) + \int\limits_\Omega \int\limits_0^t \exp\Big\{p(t\vee 1) f\left(U_{-s}(x)\right)\Big\}\,ds\,\nu(dx)$$

$$= \nu(\Omega) + \int\limits_0^t \int\limits_\Omega \exp\Big\{p(t\vee 1) f(x)\Big\} r_{-s}(x)\,\nu(dx)\,ds \le \nu(\Omega) + \|\exp\{p(t\vee 1)f\}\|_k \int\limits_0^t \|r_{-s}\|_{1+\varepsilon}\,ds.$$

Similarly, for negative t,

$$\int_\Omega r_t(x)^p\,\nu(dx) \le \nu(\Omega) + \|\exp\{p(|t|\vee 1)f\}\|_k \int\limits_0^{|t|} \|r_{-s}\|_{1+\varepsilon}\,ds.$$

Thus, $\|r_t\|_p$ is uniformly bounded on $[-T,T]$ and (2.4) indeed holds for every $\epsilon > 0$.

Let I_t denote the interval $[0,|t|]$. Then, the estimate above holds with $1+\varepsilon = p$ (and $k = q$) so that we get for all $|t| \le T$

$$\int\limits_\Omega r_t(x)^p\,\nu(dx) \le \nu(\Omega) + \|\exp\{p(T\vee 1)f\}\|_q \int_{I_t} \|r_{-s}\|_p\,ds.$$

Since $a \le 1 + a^p$ for $a \ge 0$, it follows that $\|r_t\|_p \le 1 + \nu(\Omega) + C(p,T)\int_{I_t}\|r_{-s}\|_p\,ds$. Letting $\psi(t) = \|r_t\|_p + \|r_{-t}\|_p$ we get $\psi(t) \le 2 + 2\nu(\Omega) + C(p,T)\int_0^t \psi(s)\,ds$, which implies the desired estimate by Gronwall's inequality. □

The following lemma is an obvious modification of a standard result from the theory of linear differential equations (see [7] for references). For the reader's convenience we include a proof.

Lemma 2.3 *Let X be a separable Banach space, L, L_n, $n \in \mathbb{N}$, mappings from $[0,T]$ to the space $L(X)$ of bounded operators on X such that the maps $t \mapsto L_n(t)x$ are measurable for all $x \in X_0$, where X_0 is a dense subset of X. Assume that for almost all t one has $L_n(t)x \to L(t)x$ $\forall x \in X_0$ and that the sequence of functions $\psi_n : t \mapsto \|L_n(t)\|_{L(X)}$ is uniformly integrable on $[0,T]$ (e.g., $\sup_n \|\psi_n\|_{L^2} < \infty$). Then for every x the equation $x(t) = x + \int_0^t L(s)x(s)\,ds$ is uniquely solvable on $[0,T]$ and $\sup_{t\in[0,T]} \|x(t) - x_n(t)\| \to 0$, where $x_n(t)$ are the (unique) solutions of the equations $x_n(t) = x + \int_0^t L_n(s)x_n(s)\,ds$.*

Proof: By virtue of the uniform integrability, $\sup_n \int_0^T \psi_n(t)\,dt \le C < \infty$. Clearly, $\|L(t)\|_{L(X)}$ is majorized by $\liminf \|L_n(t)\|_{L(X)}$ for almost all t. Note that the function $\psi : t \mapsto \|L(t)\|_{L(X)}$ is automatically measurable, since so are the functions $t \mapsto \|L(t)x\|$, $x \in X_0$, and X is separable. By Fatou's lemma, $\int_0^T \psi(t)\,dt \le C$. As is well-known (and follows easily from the classical fixed point theorem), for each $x \in X$ there exist unique solutions $x(t)$ and $x_n(t)$ of the linear equations mentioned in the formulation above. Then

$$\|x(t) - x_n(t)\| \le \int_0^t \|L_n(s)\|_{L(X)}\|x(s) - x_n(s)\|\,ds + \int_0^t \|(L(s) - L_n(s))x(s)\|\,ds.$$

By Gronwall's inequality, $\|x(t)\| \le \|x\| \exp\Big(\int_0^t \|L(s)\|_{L(X)}\,ds\Big) \le \|x\|e^C$ and

$$\|x(t) - x_n(t)\| \le e^C \int_0^T \|(L(s) - L_n(s))x(s)\|\,ds \to 0,$$

since $\|(L(s) - L_n(s))x(s)\| \to 0$ for almost all s and

$$\|(L(s) - L_n(s))x(s)\| \le e^C \|x\| (\|L(s)\|_{L(X)} + \|L_n(s)\|_{L(X)}),$$

which is uniformly integrable on $[0, T]$. $\square$

Note that the above result may fail if only $\sup_n \|\psi\|_{L^1} < \infty$ is assumed.

2.2 Global flows

This section contains our main results on global flows in finite dimensional spaces. As it has been noted in Introduction, for a Sobolev class vector field A on $\mathbb{R}^n$ with a smooth (say, Gaussian) measure μ it can happen that

(i) $A \in C^\infty$, $\|A\|$ and $\|\nabla A\|$ are in $L^{1-\varepsilon}(\mu)$, $\delta_\mu A = 0$, but the solution is explosive for every starting point (see Example 7.1);

(ii) $A \in C_b(\mathbb{R}^n, \mathbb{R}^n) \cap W^{p,1}(\mathbb{R}^n, \mathbb{R}^n)$ with p arbitrary large, but there are many flow solutions which preserve measure equivalence;

(iii) $A \in C_b(\mathbb{R}^n, \mathbb{R}^n) \cap W^{p,1}(\mathbb{R}^n, \mathbb{R}^n)$ but there is a global solution which is neither a flow nor preserves measure equivalence (this can be constructed by means of a nonuniqueness example).

Therefore, one needs extra conditions to ensure the existence of global flows preserving measure equivalence. It can be shown that for smooth fields the corresponding global solutions are automatically flows preserving measure equivalence. Standard results on the existence of global solutions involve typically certain estimates of Lipschitz or Lyapunov type and turn out to be too restrictive in many applications (especially for non-Lipschitzian fields). An alternative approach which we pursue here is to control the growth of solutions by means of some strong integrability conditions on the fields.

Lemma 2.4 *Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of the class $C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ and let $V : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$, $(t, x) \mapsto V_t(x)$, be the corresponding flow. Then the maps V_t are diffeomorphisms, and the image of the Lebesgue measure under V_t has the Radon–Nikodym density*

$$\varrho_t(x) = \exp\Bigl\{ - \int_0^t \mathrm{div} B(V_{-s}(x))\, ds \Bigr\}.$$

In addition, for every ball D that contains $supp(B)$ and every $p > 1$ one has

$$\|\varrho_t\|_p := \|\varrho_t\|_{L^p(D)} \le M_D e^{C(p,t)|t|}, \tag{2.6}$$

with $M_D = 2(1 + |D|)$ and $C(p, t) := \| \exp\{p(|t| \vee 1)\mathrm{div} B\}\|_{L^q(D)}$, where $1/q + 1/p = 1$. Finally,

$$\|\nabla V_t\|_{L^p(D)} \le 2M_D \| \exp\{(|t| \vee 1)\|\nabla B\|\}\|_{L^{2p}(D)} e^{(C(2,t)+1)|t|/p}$$

,

$$\Bigl\| \frac{\partial}{\partial t} V_t \Bigr\|_{L^p(D)} \le M_D \|B\|^p_{L^{2p}(D;\mathbb{R}^n)} e^{C(2,t)|t|}.$$

Proof: It is well-known that the global flow $\{V_t\}$ exists and that the mappings V_t are diffeomorphisms of $\mathbb{R}^n$. Clearly, $V_t(x) = x$ for all t and all $x \notin D$, where D is any ball containing $supp(B)$. The image of the Lebesque measure under V_t admits a density ϱ_t which is continuous in both variables (since $V_t(x)$ is continuously differentiable in both arguments). For any $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have $\frac{\partial}{\partial t}\varphi \circ V_t = \frac{\partial}{\partial \tau}\varrho \circ V_t \circ V_\tau|_{\tau=0} = (\nabla(\varphi \circ V_t), B)$. Hence

$$\int \varphi(x)\varrho_t(x)\, dx = \int \varphi(x)\, dx + \int \int_0^t (\nabla(\varphi \circ V_s)(x), B(x))\, ds\, dx$$

$$= \int \varphi(x)\,dx + \int_0^t \int (\nabla(\varphi\circ V_s)(x), B(x))\,dx\,ds = \int \varphi(x)\,dx - \int_0^t \int \mathrm{div}B(x)\varphi(V_s(x))\,dx\,ds$$

$$= \int \varphi(x)\,dx - \int_0^t \int \mathrm{div}B(V_{-s}(y))\varphi(y)\varrho_s(y)\,dy.$$

Since φ was arbitrary, we conclude that for all t and x $\varrho_t(x) = 1 - \int_0^t \mathrm{div}B(V_{-s}(x))\varrho_s(x)\,ds$,and we arrive at the desired expression. Estimate (2.6) follows from Lemma 2.2. The map $\eta_t(x) = \nabla V_t(x)$ satisfies the equation

$$\eta_t(x) = I + \int_0^t \nabla B(V_s(x))\eta_s(x)\,ds.$$

Therefore, $\|\eta_t(x)\| \le 1 + \int_0^t \|\nabla B(V_s(x))\|\,\|\eta_s(x)\|\,ds$, for $t \ge 0$, and by Gronwall's inequality $\|\eta_t(x)\| \le \exp\Bigl\{\int_0^t \|\nabla B(V_s(x))\|\,ds\Bigr\}$. Hence, by Lemma 2.1

$$\int_D \|\eta_t(x)\|^p dx \le \int_D \Bigl\{1 + \int_0^t e^{p(t\vee 1)\|\nabla B(V_s(x))\|}\,ds\Bigr\}\,dx = |D| + \int_0^t \int_D e^{p(t\vee 1)\|\nabla B(x)\|}\varrho_s(x)\,dx\,ds$$

$$\le |D| + \Bigl(\int_D e^{2p(t\vee 1)\|\nabla B(x)\|}\,dx\Bigr)^{\frac12} \int_0^t \|\varrho_s\|_2\,ds \le |D| + M_D\left(\int_D e^{2p(t\vee 1)\|\nabla B(x)\|}\,dx\right)^{\frac12} e^{(C(2,t)+1)t}.$$

The case where $t<0$ is analogous. Similarly,

$$\begin{aligned}\int_D \|\frac{\partial}{\partial t}V_t(x)\|^p\,dx &= \int_D \|B(V_t(x))\|^p\,dx = \int_D \|B(x)\|^p\varrho_t(x)\,dx\\ &\le \|B\|_{2p}^p\|\varrho_t\|_2 \le M_D\|B\|_{2p}^p\exp(C(2,t)t).\end{aligned}$$

□

Corollary 2.5 *Let B be as in Lemma 2.4 and let μ be a probability measure with density $\varrho \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ on $\mathbb{R}^n$ such that $\exp(c(B,\beta)) \in L^1_{\mathrm{loc}}(\mu)$ for all c. Then for every t the measure $\mu\circ V_t^{-1}$ is absolutely continuous with respect to μ and its Radon–Nikodym density is given by*

$$r_t(x) = \exp\Bigl\{-\int_0^t \delta_\mu B(V_{-s}(x))\,ds\Bigr\}.$$

In addition, if $\Lambda(p,t) = \|\exp\{(|t|\vee 1)p|\delta_\mu B|\}\|_{L^q(\mu)}$, $\frac1p + \frac1q = 1$, the following estimates hold:

$$\|r_t\|_{L^p(\mu)} \le 4e^{\Lambda(p,t)|t|}, \qquad (2.7)$$

$$\|\nabla V_t\|_{L^p(\mu)} \le 4\|e^{(|t|\vee 1)\|\nabla B\|}\|_{L^{2p}(\mu)} e^{(\Lambda(2,t)+1)|t|/p}, \qquad \Bigl\|\frac{\partial}{\partial t}V_t\Bigr\|_{L^p(\mu)} \le 4\|B\|^p_{L^{2p}(\mu;\mathbb{R}^n)} e^{\Lambda(2,t)|t|}. \qquad (2.8)$$

Proof: Note that since $B=0$ outside of some ball, one has $\delta_\mu B(V_{-s}(x)) = 1$ for all $s\in[0,t]$ and all x with sufficiently big norm. The expression for $r_t(x)$ can be obtained exactly in the same way as the formula for $\varrho_t(x)$ above. Then the same arguments based on Lemma 2.1, Lemma 2.2, and Gronwall's inequality lead to the estimates (2.7) and (2.8). □

We say that a measurable function f on $\mathbb{R}^n$ is locally uniformly positive if $\mathrm{essinf}_D f > 0$ for every bounded set D.

Theorem 2.6 *Let μ be a probability measure on $\mathbb{R}^n$ with a locally uniformly positive density $\varrho \in W^{1,1}_{\text{loc}}(\mathbb{R}^n)$ and let $A \in W^{1,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ be such that for each positive λ*

$$\int \exp(\lambda |\delta_\mu A(x)|)\, \mu(dx) < \infty. \tag{2.9}$$

$$\int \exp(\lambda \|\nabla A(x)\|)\, \mu(dx) < \infty. \tag{2.10}$$

Then (1.1) has a flow solution $U_t(x)$, jointly continuous in both its arguments. Moreover,

(i) For each $t \in \mathbb{R}$ and $p \geq 1$, the flow's Radon–Nikodym derivative statisfies

$$\|r_t\|_{L^p(\mu)} \leq 4 e^{\Lambda(p,t)|t|}, \tag{2.11}$$

where $\Lambda(p,t) = \left(\int e^{(|t|\vee 1)pq|\delta_\mu A(x)|}\, \mu(dx)\right)^{1/q}$, $\frac{1}{p} + \frac{1}{q} = 1$;

(ii) $U_t \in W^{p,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ for each $t \in \mathbb{R}$ and $p \geq 1$, and

$$\|\nabla U_t\|_{L^p(\mu)} \leq 4 \Big(\int_{\mathbb{R}^n} e^{2(|t|\vee 1)p\|\nabla A(x)\|}\, \mu(dx) \Big)^{\frac{1}{2p}} e^{(\Lambda(2,t)+1)|t|/p}; \tag{2.12}$$

(iii)

$$r_t(x) = \exp\Big\{ - \int_0^t \delta_\mu A(U_{-s}x)\, ds \Big\} \qquad \forall t \in \mathbb{R}. \tag{2.13}$$

Remark 2.7 *In view of (2.10)and Proposition 1.2 we may and shall assume that A is continuous. In this case (1.1) and (1.2) will hold* for every *t and x.*

Proof: Step 1. Assume first that A is uniformly bounded. Then equation (1.1) is solvable for every x on the whole line. In order to get a solution with the claimed properties let us choose a sequence $A_j \in C_b^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $A_j \to A$ in all $W^{p,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$, $p > 1$, and for every ball D

$$\|A_j(x)\| \leq \|A(x)\| \leq L, \quad \|\nabla A_j\|_{L^p(D)} \leq \|\nabla A\|_{L^p(D)}.$$

A possible choice of such a sequence is $A_j = A * \varphi_j$, where $\varphi_j(x) = j^n \varphi(jx)$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ is a probability density (see [45], Theorem 1.6.1). In this case, for every ball D the $L^p(D)$-norm of $\exp(\lambda\|\nabla A_j\|)$ is majorized by that of $\exp(\lambda\|\nabla A\|)$. In addition, $\{A_j\}$ converges to A locally uniformly by virtue of the convergence in $W^{p,1}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ with $p > n$ (see [45], Theorem 2.4.1). Clearly, the smooth fields A_j generate global flows $\{U_t^j\}$. We shall prove that the maps U^j converge uniformly on bounded subsets of $\mathbb{R}^1 \times \mathbb{R}^n$ to a continuous flow solution U which satifies the properties (i)–(iii).

Now let us fix a centered ball $D \in \mathbb{R}^n$ of radius R and a time interval $J = [-T, T]$. For every $x \in D$ one has $\|U_t^j(x)\| \leq \|x\| + L|t|$, where $L = \sup_y \|A(y)\|$. Thus, for all $x \in D$ and $t \in J$ the trajectories $U_t^j(x)$ remain within the ball D_0 of radius $R + 2TL$. The same is true for all possible solutions of the equation with A. Therefore, if we replace all the fields A_j and A by $B_j = kA_j$ and $B = kA$ with a function $k \in C_0^\infty(\mathbb{R}^n)$ such that $k|_{D_0} = 1$, this does not influence the behavior of solutions for $(t, x) \in [-2T, 2T] \times D$. Since $\|\nabla B_j(x)\| \leq \text{const}(\|\nabla A_j(x)\| + \|A_j(x)\|)$, by virtue of our choice of A_j we have the uniform estimates for every positive λ

$$\int_{D_0} e^{\lambda\|\nabla B_j(x)\|}\, dx \leq K(\lambda) < \infty.$$

Let V^j be the flows generated by B_j. According to Lemma 2.4, the sequence $\{V^j\}$ is bounded in $W^{p,1}([-2T,2T]\times D_0,\mathbb{R}^n)$ for every $p>1$. Hence, its subsequence converges uniformly on the set $J\times D$, on which we have $V_t^j(x)=U_t^j(x)$. Clearly, by a diagonal procedure we can choose a subsequence of U^j which converges locally uniformly on $\mathbb{R}^1\times\mathbb{R}^n$. In fact, this is true for the whole sequence U^j as it follows from the uniqueness assertion proven below. Therefore, we get a continuous flow solution U for which (2.11) holds.

Step 2. Let us verify properties (i), (ii) and (iii), still assuming that A is bounded. Assume first that A has bounded support in the centered ball D of radius r. Fix $p>1$ and $T>0$. Let $A_j=A*\varphi_j$ be the smooth approximations indicated above. For sufficiently big j the supports of A_j belong to the centered ball of radius $r+1$ and $\|A_j(x)-A(x)\|\le 1$ for all x. Hence, for all $x\in D$ and $t\in[-T,T]$ all vectors $U_t^j(x)$ are contained in the centered ball D_0 of radius $r+1+T(C+1)$, where $C=\sup_x\|A(x)\|$. As mentioned above, by our choice of A_j

$$\int_{D_0} e^{\lambda\|\nabla A_j(x)\|}\,dx\le\int_{D_0} e^{\lambda\|\nabla A(x)\|}\,dx.$$

Hence, according to Lemma 2.4, the Lebesgue measure is quasi-invariant under the flows $\{U_t^j\}$ and for the corresponding Radon–Nikodym derivatives ϱ_t^j we get the uniform estimates

$$\sup_j\sup_{-T\le t\le T}\|\varrho_t^j\|_{L^p(D_0)}\le c(p,T).$$

Clearly, the Radon-Nikodym densities r_t^j of $\mu\circ(U_t^j)^{-1}$ can be written as $r_t^j=\varrho_t^j/\varrho$. Note that for all $x\notin D_0$ one has $r_t^j(x)=1$ if $t\in[-T,T]$, since $U_t^j(x)=x$ for such x. The density ϱ is separated from zero on D_0, whence

$$\sup_j\sup_{-T\le t\le T}\|r_t^j\|_{L^p(\mu)}\le c_0(p,T).$$

For any $g\in C_0(\mathbb{R}^n)$ one has

$$\int g(U_t(x))\,\mu(dx)=\lim_{j\to\infty}\int g(U_t^j(x))\,\mu(dx)=\lim_{j\to\infty}\int g(x)r_t^j(x)\,\mu(dx),$$

which is majorized by $c_0(q,T)\|g\|_{L^p(\mu)}$. Therefore, there is a function $r_t\in L^q(\mu)$ such that the left hand side coincides with $\int g(x)r_t(x)\,\mu(dx)$, which means that r_t is the Radon–Nikodym density of $\mu\circ U_t^{-1}$ with respect to μ. In addition, we get a uniform estimate of $\|r_t\|_{L^p(\mu)}$ on $[-T,T]$. As has already been mentioned, Lemma 2.4 implies that $U_t\in W^{p,1}_{\text{loc}}(\mathbb{R}^n,\mathbb{R}^n)$. Therefore, the same arguments as in Lemma 2.4 lead to the explicit expression (2.13) for r_t. Applying Lemma 2.2, we obtain the estimates (2.11).

The solutions we have constructed so far possess the following important local property: if two fields A and B of the type above with bounded supports coincide on a ball (or an open set) D, then $U_t^A(x_0)=U_t^B(x_0)$ provided $U_s^A(x_0)$, $U_s^B(x_0)\in D$ for all $s\in[0,t]$. Indeed, let D_0 be a smaller ball in D. Then for all sufficiently big j we have that $A_j:=A*\varphi_j=B_j:=B*\varphi_j$ on D_0 and $U_s^{A_j}(x_0)$, $U_s^{B_j}(x_0)\in D_0$ (which follows from the locally uniform convergence of U^{A_j} to U^A and U^{B_j} to U^B). Then $U_s^{A_j}(x_0)=U_s^{B_j}(x_0)$ on $[0,t]$ (since both are governed by the same equation with smooth field), whence the desired assertion.

If A is not assumed to have bounded support (but still is bounded), we choose a sequence of functions $k_i(x)=g_i(\|x\|)$ such that $g_i(t)=1$ on $[0,i]$, $g_i(t)=0$ on $[i+1,\infty)$, and $g_i(t)=i+1-t$ on $[i,i+1]$. The fields $C_i=k_iA$ satisfy the conditions of the theorem, and as we have already shown they generate the flows W^i (constructed as above). Fix a ball $D\subset\mathbb{R}^n$ and $T>0$. Since $W_t^i(x)$ are uniformly bounded on bounded sets, it follows from the local property mentioned above that for each $t \in [-T,T]$, $x \in D$ and all sufficiently big i, the vectors $W_t^i(x)$ coincide with one and the same element which is denoted by $U_t(x)$.

We have now obtained a flow solution with the desired expression for r_t. Moreover, the estimates (2.11) follow as well. Indeed, since

$$\delta_\mu C_i = k_i \delta_\mu A + (\nabla k_i, A), \qquad 0 \le k_i \le 1, \quad |(\nabla k_i(x), A(x))| \le \|A(x)\|,$$

we get some uniform bound on $\|r_t^{W_i}\|_{L^p(\mu)}$ for the Radon–Nikodym derivatives of the images of μ under W_t^i. This leads to the same estimates for $\|r_t\|_{L^p(\mu)}$ and enables us to apply Lemma 2.2 to obtain (2.11).

In order to get the estimates (2.12) note that it follows from what has already been proven that $\eta_t(x) = \nabla U_t(x)$ exists in the Sobolev sense, belongs to $W^{p,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ and satisfies the equation

$$\eta_t(x) = I + \int_0^t \nabla A(U_s(x))\eta_s(x)\, ds. \tag{2.14}$$

Indeed, for smooth fields $A_{j,i} = (k_i) * \varphi_j$ (in the notation above) this is obvious. It is easy to see that for a.e. x one has

$$\|\nabla A(U_s(x)) - \nabla A_{j(i),i}(U_s^i(x))\|_{L^2[0,t]} \to 0,$$

where U^i is the flow generated by $A_{j(i),i}$ and $j(i)$ is chosen so that $\|\nabla A - \nabla A_{j(i),i}\|_{L^2(\mu,\mathbb{R}^n)} \to 0$ (see Lemma 5.11 for the proof of this in the infinite dimensional case). Now Lemma 2.1 applies. Therefore, (2.12) follows in the same way as in the smooth case considered in Lemma 2.4.

It should be noted that the solutions obtained at this stage still have the aforementioned local property.

Step 3. Finally, in the general case where A is not assumed to be bounded we can consider the fields

$$D_j(x) = g_j(\|A(x)\|)\frac{A(x)}{\|A(x)\|},$$

where $g_j(t) = t$ if $t \le t_j$, $g_j(t) = t_j$ if $t \ge t_j$, and $\{t_j\}$ is a sequence increasing to infinity such that the sets $\{x : \|A(x)\| = t_j\}$ have measure zero. It is easy to see that these new fields satisfy the conditions of the theorem, and we can construct the corresponding flows Y^j. Note that $D_j(x) = \alpha_j(x)A(x)$ with $0 \le \alpha_j(x) \le 1$,

$$\delta_\mu D_j = trace(\nabla D_j) + (D_j, \beta) = \alpha_j \delta_\mu A + trace(\nabla D_j) - \alpha_j trace(\nabla A),$$

and that

$$\|\nabla D_j(x)\| \le \|\nabla A(x)\|.$$

The latter follows from the fact that $D_j(x) = G_j(A(x))$, where $G_j(x) = g_j(\|x\|)x/\|x\|$, and the chain rule for Sobolev maps (see [45]), taking into account that G_j is Lipschitzian with $\|\nabla G_j(x)\| \le 1$ if $\|x\| \ne t_j$. Hence

$$|\delta_\mu D_j(x)| \le |\delta_\mu A(x)| + 2n\|\nabla A(x)\|,$$

and since the right hand side has all positive moments, we get uniform (in j) estimates of the exponential moments of $|\delta_\mu D_j|$.

Hence, we arrive at the uniform estimates of the type (2.12) for Y^j (the terms $\Lambda(2,t)$, which appear in (2.12) are indeed exponential moments of $|\delta_\mu D_j|$ of bounded order), which implies that for every fixed time interval J and every ball D the mappings $(t,x) \mapsto Y_t^j(x)$ have uniformly bounded norms in $W^{p,1}(D, \mathbb{R}^n)$ for each $p \ge 1$. In particular, these mappings are uniformly bounded on $J \times D$. Therefore, by construction and the local property established above, if $(t,x) \in J \times D$ then for all sufficiently big j the vectors $Y_t^j(x)$ coincide with one and the same element denoted by $U_t(x)$. Clearly, we get a solution with all the desired properties and estimates. $\square$

Remark 2.8 *An important by-product follows from the arguments above applying Lemma 2.3 to (2.14): for every fixed open ball D and bounded interval J, there exist smooth approximations A_j of A in the Sobolev class $W^{p,1}(D_0, \mathbb{R}^n)$ with a bigger ball D_0, such that $U_t^j \to U_t$ in $W^{p,1}(D, \mathbb{R}^n)$ for all $t \in J$.*

Proposition 2.9 *Let $\{T_t(x)\}$ is another solution such that $\mu \circ T_t^{-1} = \psi_t \mu$ and $\|\psi_t\|_{L^p(\mu)}$ is locally integrable for some $p > 1$.*

(i) If $\{T_t(x)\}$ is continuous in both variables, then $T_t(x) = U_t(x)$ for all (t, x);

(ii) If $\|A\| \in L^\alpha(\mu)$ for some $\alpha > 1$, then for every t one has $T_t(x) = U_t(x)$ μ-a.e.

Proof: Note that for all t and almost all x one has

$$A(U_t(x)) = \nabla U_t(x) A(x). \tag{2.15}$$

Indeed, let A_j be the smooth fields mentioned in Remark 2.8. For them (2.15) holds since

$$A_j(U_t^j(x)) = \frac{\partial}{\partial t} U_t^j(x) = \frac{\partial}{\partial \tau} U_t^j(U_\tau^j(x))|_{\tau=0} = \nabla U_t^j(x) \frac{\partial}{\partial \tau} U_\tau^j(x)|_{\tau=0} = \nabla U_t^j(x) A_j(x).$$

Using the convergence of U^j to U in the Sobolev classes $W^{p,1}(D, \mathbb{R}^n)$ for open balls D, (2.15) is obtained for all t and almost all x by passing to the limit (possibly for a subsequence of U^j) as $j \to \infty$.

In the case where $T_t(x)$ is jointly continuous, for every ball D and time interval J there is a bigger ball D_0 such that $U_t(x)$ and $T_t(x)$ remain within D_0 for all $x \in D$ and $t \in J$. Therefore, multiplying A by a smooth function $g : \mathbb{R}^n \to [0, 1]$ with bounded support such that $g = 1$ on D_0, we do not change both solutions on J with starting points in D. Clearly, our new field gA satisfies the conditions of the same type as A, since $|(gA, \beta)| \le |(A, \beta)|$ and $\|\nabla(gA)\| \le \|\nabla A\| + const$ as $|g| \le 1$ and g has compact support on which A is bounded. Thus, we can assume that A has bounded support.

Now let $T_t(x)$ be another solution of our equation such that $\mu \circ T_t^{-1} = \psi_t \mu$. Let us prove that for all t, $F(t, x) := U_t(T_{-t}(x)) = x$ for a.e. x. which implies that $T_{-t}(x) = U_{-t}(x)$. It suffices to check that $F(t, x)$ is absolutely continuous in t (for a.e. x) and that its partial derivative in t vanishes.

Appealing once again to the approximations of Remark 2.8, $U_t(T_{-t}(x)) = \lim_{j \to \infty} U_t^j(T_{-t}(x))$ for all t and x. Differentiating by the chain rule we get

$$\frac{\partial}{\partial t} U_t^j(T_{-t}(x)) = A_j(U_t^j \circ T_{-t}(x)) - \nabla U_t^j(T_{-t}(x)) A(T_{-t}(x)).$$

It remains to check that $\int_J \|\partial U_t^j(T_{-t}(x))/\partial t\|^2 \, dt \to 0$ for almost all x and every time interval J. Since $A_j(U_t^j \circ T_{-t}(x)) \to A(U_t \circ T_{-t}(x))$ locally uniformly in t and we have (2.15), it suffices to show that

$$\int_J \|\nabla U_t^j(T_{-t}(x)) A(T_{-t}(x)) - \nabla U_t(T_{-t}(x)) A(T_{-t}(x))\|^2 \, dt \to 0.$$

Since A is assumed to have bounded support, and hence is bounded, this follows from

$$\begin{aligned} \int_J \int_{\mathbb{R}^n} \|\nabla U_t^j(T_{-t}(x)) - \nabla U_t(V_{-t}(x))\|^2 \, \mu(dx) \, dt \quad &\le \quad \int_J \int_{\mathbb{R}^n} \|\nabla U_t^j(y) - \nabla U_t(y)\|^2 \psi_{-t}(y) \, \mu(dy) \, dt \\ \le \int_J \|\psi_t\|_p \, dt \int_J \|\nabla U_t^j - \nabla U_t\|_{4q}^4 \, dt \quad &\to \quad 0, \end{aligned}$$

which holds with our choice of A_j. Indeed, the functions $\|\nabla U_t^j - \nabla U_t\|_{4q}^4$ are uniformly integrable on J and converge to zero pointwise by the convergence $U_t^j \to U_t$ in the local Sobolev norms combined with the uniform integrability of the sequence $\{\|\nabla U_t^j\|\}$ for every fixed t.

For the case (ii) the reasoning is similar. Instead of a reduction to the bounded support case (which requires more justification), one can choose the sequence $A_j \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that $\{A_j\}$ converges to A in $L^\alpha(\mu)$ and in $W^{p,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ for every $p \geq 1$. To this end, we take first the cutt-off $D_j(x)$ as in Step 3 of the proof of Theorem 2.6, then consider $k(x)D_j(x)$ as in Step 2, and, finally, construct a smooth approximation for kD_j. The justification of the relationship $\int_J \|\partial U_t^j(T_{-t}(x))/\partial t\|^p\, dt \to 0$ for $p = (1+\alpha)/2 > 1$ is then similar to the case considered above with $p = 2$ with the only difference that instead of uniform boundedness of A one has to use Hölder's inequality (this case is considered in detail in the infinite dimensional setting below). □

In fact, the following has been shown in the proof of Theorem 2.6:

Corollary 2.10 *Let A and B be two continuous vector fields that satisfy conditions of Theorem 2.6. Assume that for some x_0 the trajectories $U_s^A(x_0)$ and $U_s^B(x_0)$, $s \in [0,t]$, belong to an open set on which $A = B$. Then $U_t^A(x_0) = U_t^B(x_0)$.*

Moreover, a closer look at the proof yields the following:

Corollary 2.11 *Assume that estimate (2.9) holds for some fixed $\lambda = \lambda_0 > 0$. Let $p_0 > 1$ and $T > 0$ satisfy $(T \vee 1)\frac{p_0^2}{p_0 - 1} < \lambda_0$. If estimate (2.10) holds with $\lambda = 2(T \vee 1)p_0$, then the existence of a flow solution to (1.1) still follows, while the other assertions of Theorem 2.6 remain valid for $t \in [-T, T]$ and $p = p_0$.*

Proof: Indeed, in this case all the considerations above apply to a sufficiently small time interval $[-\tau, \tau]$ (including the group property for those $t, s \in [-\tau, \tau]$ with $t + s \in [-\tau, \tau]$), which together with uniqueness implies the existence of a flow solution on the whole line. □

Theorem 2.12 *The existence statement as well as claims (i) and (iii) of Theorem 2.6 remain valid if $\int_D \exp(\lambda\|\nabla A(x)\|)\, \mu(dx) < \infty$ is required to hold only for any ball D if, in addition, $\|A\| \in L^{1+\varepsilon}(\mu)$ for some $\varepsilon > 0$. In addition, the uniqueness statement of Proposition 2.9 holds true.*

Proof: Assume first that A is bounded. The same arguments as above work with the following modification. The reduction to fields with bounded supports is achieved by multiplying A by a sequence of functions $g_k \in C_0^\infty(\mathbb{R}^n)$ with $\sup_k \|\nabla g_k\| \leq C < \infty$, $0 \leq g_k \leq 1$ and $g_k = 1$ on the centered ball of radius k. Then the fields $A_k = g_k A$ have divergences $\delta_\mu A_k$ with $\sup_k \int e^{\lambda|\delta_\mu A_k(x)|}\, \mu(dx) < \infty$ $\forall \lambda > 0$, since

$$\delta_\mu A_k = trace\, \nabla(g_k A) + g_k(A, \beta) = g_k \delta_\mu A + (\nabla g_k, A),$$

and $|(\nabla g_k, A)| \leq C\|A\|$. In addition, ∇A_k has bounded support and $\|\nabla A_k\|$ is estimated by $\|\nabla A\| + C\|A\|$, which is exponentially integrable on the support of A_k. All the conditions of Theorem 2.6 are thus fulfilled, which implies the existence of the flow and enables us to estimate the L^p-norms of the corresponding Radon–Nikodym derivatives (expressed by (2.13)) by means of the exponential moments of $|\delta_\mu A_k|$ according to (2.11). These L^p-norms are therefore uniformly bounded in k. Let D be the ball of radius d. Since $\|A_k(x)\| \leq \|A(x)\| \leq C$, for every fixed time interval $[-T, T]$ all the solutions $U_t^k(x)$, $x \in D$, $t \in [-T, T]$, remain within the ball of radius $R = d + CT$. Hence, for $k > R$ all these solutions coincide (see Corollary 2.10). This gives rise to the flow solution $U_t(x)$ generated by A. Clearly, μ is quasi-invariant under $\{U_t\}$ with the Radon-Nikodym densities given by (2.13), since on D the Radon–Nikodym derivatives corresponding to

U_t^k coincide for all $k > R$ as $\delta_\mu A_k(U_{-t}(x)) = \delta_\mu A(U_{-t}(x))$ if $(t,x) \in [-T,T] \times D$. In addition, we get the estimates (2.11). Note also, that by construction $U_t \in W^{p,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ for all t and $p \geq 1$.

In the general case (i.e. A not necessarily bounded), for every $k \geq 1$ let $\varphi_k : [0,\infty) \to [0,1]$ be defined by $\varphi_k(t) = 1$ if $t \leq k$, $\varphi_k(t) = (t-k+1)^{-1}$ if $t \geq k$. Clearly, φ_k is Lipschitzian. Put $f_k(x) = \varphi_k(\|A(x)\|)$. Then $f_k \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ by [45, Theorem 2.1.11]. In addition, for every $h \in \mathbb{R}^n$ one has

$$(\nabla f_k(x), h) = \varphi_k'(\|A(x)\|)\Big(\frac{A(x)}{\|A(x)\|}, \nabla A(x)(h)\Big).$$

Note that on the set $\{x : \|A(x)\| \leq k\}$ one has $f_k(x) = 1$ and hence $\nabla f_k(x) = 0$ a.e. on this set.

Let us define approximating bounded fields by $A_k(x) = f_k(x)A(x)$. Clearly, $A_k \in W^{p,1}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ for all $p \geq 1$. However, the uniform exponential estimate of the divergences may be lost. For this reason, following [16], we shall consider equivalent measures $\mu_k := \frac{1}{f_k}\mu$. This is possible, since $1/f_k \in L^1(\mu)$. Indeed, $f_k(x)^{-1} \leq 1 + \|A(x)\|$.

Denote by ϱ the density of μ. Recall that by the Gagliardo–Nirenberg theorem, $\varrho \in L^{\frac{n}{n-1}}_{\mathrm{loc}}(\mathbb{R}^n)$ (see [45, Theorem 2.4.1 and Theorem 1.6.1]). Since the functions f_k are continuous and positive, and $\|\nabla f_k\|$ together with $\|\nabla A\|$ is in all $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, we can conclude that the densities ϱ/f_k of the measures μ_k are in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. In addition, these densities are locally uniformly positive, since so is ϱ and $1/f_k \geq 1$. Finally, we have:

$$\delta_{\mu_k} A_k = f_k \mathrm{div} A + (\nabla f_k, A) + \Big(f_k A, \beta + \frac{\nabla(1/f_k)}{1/f_k}\Big) = f_k \mathrm{div} A + f_k(A,\beta) = f_k \delta_\mu A,$$

since $(f_k A, \nabla(\frac{1}{f_k})/\frac{1}{f_k}) = -(A, \nabla f_k)$. For every $\lambda > 0$ one thus obtains

$$\begin{aligned}\int \exp(\lambda|\delta_{\mu_k} A_k(x)|)\, \mu_k(dx) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int f_k(x)^n \lambda^n |\delta_\mu A(x)|^n \, \mu_k(dx) \\ &\leq \mu_k(\mathbb{R}^n) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \lambda^n |\delta_\mu A(x)|^n \, \mu(dx) \leq 1 + \|A\|_{L^1(\mu)} + \int \exp(\lambda|\delta_\mu A(x)|)\, \mu(dx).\end{aligned}$$

Therefore, according to the bounded case considered above, there exist μ_k-quasi-invariant flows $\{U_t^k\}$ generated by A_k. In addition, the Radon–Nikodym densities $\varrho_t^k = \frac{d(\mu_k \circ (U_t^k)^{-1})}{d\mu_k}$ satisfy for each $q > 1$ and $T > 0$

$$\sup_{-T \leq t \leq T} \|\varrho_t^k\|_{L^q(\mu_k)} \leq C(q,T),$$

where the constants $C(q,T)$ do not depend on k. By the equivalence of measures, μ is quasi-invariant under these flows.

Now set $p = 1 + \varepsilon/2$, $c = \frac{1+\varepsilon}{p}$, $q = 2\frac{1+\varepsilon}{\varepsilon}$. Then $c^{-1} + q^{-1} = 1$. Letting $V_t^k(x) = U_t^k(x) - x$, and by Hölder's inequality, one has for each t in a fixed interval $[-T,T]$

$$\begin{aligned}\int \|V_t^k(x)\|^p \, \mu(dx) &\leq \int \|V_t^k(x)\|^p \, \mu_k(dx) \leq t^{\varepsilon/2} \int_0^t \int \|A_k(U_s^k(x))\|^p \, \mu_k(dx) \\ &\leq t^{\varepsilon/2} \Big(\int \|A_k(x)\|^{1+\varepsilon} \, \mu_k(dx)\Big)^{\frac{1}{c}} \int_0^t \|\varrho_s^k\|_{L^q(\mu_k)}\, ds \\ &\leq t^p \Big(\int f_k(x)^\varepsilon \|A(x)\|^{1+\varepsilon} \, \mu(dx)\Big)^{\frac{1}{c}} \sup_{|s| \leq t} \|\varrho_s^k\|_{L^q(\mu_k)} \\ &\leq t^p \Big(\int \|A(x)\|^{1+\varepsilon} \, \mu(dx)\Big)^{\frac{1}{c}} C(q,T) = t^p C(q,T) \|A\|_{1+\varepsilon}^p.\end{aligned}$$

Thus, the sequence of mappings $V_t^k(\cdot)$ is uniformly integrable. Similarly

$$\int\int_{-T}^{T}\left\|\frac{\partial U_t^k}{\partial t}(x)\right\|^p dt\,\mu(dx) = \int_{-T}^{T}\int \|A_k(U_t^k(x))\|^p\,\mu(dx)\,dt \quad\leq\quad \int_{-T}^{T}\int \|A_k(U_t^k(x))\|^p\,\mu_k(dx)\,dt$$
$$= \int_{-T}^{T}\int \|A_k(y)\|^p \varrho_t^k(y)\,\mu_k(dx)\,dt \quad\leq\quad \|A_k\|_{L^{1+\varepsilon}(\mu_k)}^{1/c}\int_{-T}^{T}\|\varrho_t^k\|_{L^q(\mu_k)}\,dt,$$

which is bounded uniformly in k. It follows by Fatou's lemma that,

$$\liminf \int_{-T}^{T}\left\|\frac{\partial U_t^k}{\partial t}(x)\right\|^p dt < \infty \quad \mu\text{-a.e.}$$

There is thus a.s. a subsequence k_i (which may depend on x) such that the sequence $\{U_t^{k_i}(x)\}$ converges uniformly on $[-T,T]$ to some limit $U_t(x)$. In particular, there is a centered ball D containing all the points $U_t^{k_i}(x)$, $t\in[-T,T]$. This implies that there is N such that $U_t^{k_i}(x) = U_t(x)$ for all $k_i \geq N$, $t\in[-T,T]$. Indeed, let N_1 be such that $\sup_D \|A(y)\| < N_1$. Then $A_k \equiv A$ on D if $k\geq N_1$. Let us pick m with $N := k_m > N_1$. According to Corollary 2.10 $U_t^{k_i}(x) = U_t^N(x)$ $\forall t\in[-T,T]$, $\forall k_i \geq N$. If $\{k_i'\}$ is another subsequence for which $U_t^{k_i'}(x)$ converges uniformly on $[-T,T]$ to some limit $V_t(x)$, then again there exist a centered ball D' and an integer N' such that $U_t^{k_i'}(x) = U_t^{N'}(x) \in D'$ for all $k_i' \geq N'$, $t\in[-T,T]$. We may assume that $D'\subset D$. Putting $N_0 = \max(N,N')$, we get that $A_{k_i} = A_{k_i'}$ on D and $U_t^{k_i}(x)\in D$, $U_t^{k_i'}(x)\in D$ for all $k_i,\ k_i'\geq N_0$, $t\in[-T,T]$. By the local property, $U_t^{k_i}(x) = U_t^{k_i'}(x)$ for such indices, whence $V_t(x) = U_t(x)$.

It follows from our reasoning that every subsequence of the initial sequence admits a selection of such a uniformly convergent subsequence. By the uniqueness of all possible limits that we have established we get the uniform convergence and stabilization of the whole sequence $\{U_t^k(x)\}$.

Thus, for a.e. x the sequence $\{U_t^k(x)\}$ converges uniformly on $[-T,T]$ to $U_t(x)$. From the uniform integrability established above we get the L^p-convergence of $\{U_t^k(x)\}$ to $U_t(x)$ for all $p<1+\varepsilon$. Hence, $U_t(x)$ is a solution with the group property.

Set now $r_t^k = \frac{d(\mu\circ(U_t^k)^{-1})}{d\mu}$. Then

$$r_t^k(x) = \varrho_t^k(x)\frac{f_k(U_{-t}^k(x))}{f_k(x)} \leq \frac{\varrho_t^k(x)}{f_k(x)}.$$

Applying Hölder's inequality with $p = 1+\varepsilon/2$ and the uniform estimate of the norms $\|\varrho_t^k\|_{L^q(\mu)}$ (which are bounded above by $\|\varrho_t^k\|_{L^q(\mu_k)}$), one obtains

$$\sup_{-T\leq t\leq T}\|r_t^k\|_{L^p(\mu)} \leq c(T).$$

For any bounded continuous function f we have

$$\int f(U_t(x))\,\mu(dx) = \lim_{k\to\infty}\int f(U_t^k(x))\,\mu(dx),$$

which is majorized by $c(T)\|f\|_{L^{p'}(\mu)}$, $\frac{1}{p}+\frac{1}{p'} = 1$. Hence, there is a function $r_t\in L^p(\mu)$ such that the right hand side of the equality above equals $\int f(x)r_t(x)\,\mu(dx)$. This means that $\mu\circ U_t^{-1} = r_t\mu$.

It remains to verify the formula for r_t. Since the sequence $\{r_t^k\}$ is uniformly integrable and by the above converges to r_t weakly in $L^p(\mu)$, it suffices to prove that it converges in measure to the right hand side of (2.13). The sequence of continuous functions f_k converges locally uniformly to

1. As $U^k_{-t}(x) \to U_{-t}(x)$ a.e., we get the a.e. convergence of $f_k(U^k_{-t}(x))/f_k(x)$ to 1. Thus, it suffices to verify that

$$\int_0^t \delta_{\mu_k} A_k(U^k_{-s}(x))\, ds \to \int_0^t \delta_\mu A(U_{-s}(x))\, ds \quad \text{in} \quad L^1(\mu).$$

It follows from Hölder's inequality that the sequence

$$\delta_{\mu_k} A_k(U^k_{-s}(x)) = f_k(U^k_{-s}(x))\delta_\mu A(U^k_{-s}(x))$$

is uniformly $ds \otimes \mu$-integrable on $[-T,T] \times \mathbb{R}^n$. So it suffices to show that for every fixed s it converges to $\delta_\mu A(U_{-s}(x))$ in $L^1(\mu)$. As has already been mentioned, $f_k(U^k_{-s}(x)) \to 1$ a.e. Since $|f_k| \le 1$, it remains to show the L^1-convergence of $g_k(x) = |\delta_\mu A(U^k_{-s}(x)) - \delta_\mu A(U_{-s}(x))|$ to zero. Let $\alpha > 0$. As before, let $p = 1 + \varepsilon/2$, $1/p + 1/q = 1$. By the uniform integrability, there is $\eta > 0$ and k_1 such that for every set M with $\mu(M) < \eta$ one has $\int_M g_k(x)\,\mu(dx) < \alpha$ for all $k > k_1$. We know that $c = \|r^k_{-s}\|_{L^p(\mu)} + \|r_{-s}\|_{L^p(\mu)}$ is finite. There exists a compact set $S_1 \in \mathbb{R}^n$ on which $\delta_\mu A$ is uniformly continuous such that $\mu(\mathbb{R}^n \backslash S_1) < (\eta/c)^q$. Let $\gamma > 0$ be such that

$$|\delta_\mu A(x) - \delta_\mu A(y)| < \alpha \qquad \text{if} \quad x,\ y \in S_1,\ \|x - y\| \le \gamma.$$

There is also a compact set $S_2 \in \mathbb{R}^n$ with $\mu(\mathbb{R}^n \backslash S_2) < \eta$ on which the sequence $U^k_{-s}(x)$ converges uniformly to $U_{-s}(x)$. Let us pick $k_2 > k_1$ such that $\|U^k_{-s}(x) - U_{-s}(x)\| < \gamma$ for all $x \in S_2$ and all $k > k_2$. By Hölder's inequality,

$$\mu\Big((U^k_s)^{-1}(S_1)\Big) \le \mu(S_1)^{1/q} \|r^k_{-s}\|_{L^p(\mu)},$$

and the same estimate holds for $U_s^{-1}(S_1)$. Therefore, the integral of g_k equals the sum of the integrals over the set $S_2 \cap (U^k_{-s})^{-1}(S_1) \cap (U_{-s})^{-1}(S_2)$ and its complement, and thus is estimated by 4α.

The proof of uniqueness is analogous to that of Proposition 2.9. Considering functions $F(t,x) = \varphi(U_{-t} \circ T_t(x))$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, it is sufficient to have only the local integrability of $\exp(\|\nabla A\|)$ in that proof. $\square$

Example 2.13 *All the assumptions above are satisfied if μ has continuous positive density in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, A is locally Lipschitzian, $\|A\| \in L^{1+\varepsilon}(\mu)$ and $\exp(|\delta_\mu A|) \in \bigcap_{p>1} L^p(\mu)$.*

One might ask how far are the conditions imposed above essential for the validity of the results. In particular, since the estimates of r_t involve only the exponential moments of $\delta_\mu A$, it might be tempting to relaxe the integrability conditions on A and ∇A. Recall that our Theorems 2.6 and 2.12 require that either $\|A\| \in L^{1+\varepsilon}(\mu)$ or $\|\nabla A\|$ must be exponentially integrable. Example 7.1 below shows, however, that the inclusions $\|A\|$, $\|\nabla A\| \in L^p(\mu)$ for every $p < 1$ are not sufficient for the existence of the flow even if $\delta_\mu A = 0$ and A is a C^∞ map on $\mathbb{R}^2$ with the standard Gaussian measure μ.

Note also that according to Corollary 2.11, for the existence of a flow solution it suffices that $\exp\|\nabla A(x)\|$ be locally integrable and for some $c > 0$ the functions $\exp(c|\mathrm{div}A|)$, $\exp(c\|A\|^2)$ and $\exp(c\|\beta\|^2)$ be in $L^1(\mu)$. Also, for some measures (which satisfy a Poincaré–type inequality, such as Gaussian measures), $\|\nabla A\| \in L^{1+\varepsilon}(\mu)$ implies $\|A\| \in L^{1+\varepsilon}(\mu)$. It is worth mentioning that even for smooth vector fields the existence of an absolutely continuous flow defined almost everywhere does not imply the global existence for every starting point. Indeed,

Example 2.14 *Let A be the field on $\mathbb{R}^2$ with identically zero second component and the first component of the form $x_1^2 f(x_1,x_2)$, where $f \in C_b^\infty(\mathbb{R}^2)$ is identically 1 when $x_2 = 0$ and has support in the domain restricted by the curves $x_1x_2 = 1$ and $x_1x_2 = -1$.*

Also, uniqueness for Sobolev class vector fields need not hold everywhere.

Example 2.15 *Let A on $\mathbb{R}^2$ be defined by $(x_1, x_2) \mapsto (f(x_1, x_2,), 0)$, where $f \in C_0(\mathbb{R}^2)$ in a small neighborhood of the origin has the form $f(x_1, x_2) = \sqrt{|x_1| + 4(\log|x_2|)^{-2}}$, $f(x_1, 0) = \sqrt{|x_1|}$, and is locally Lipschitzian outside of the first coordinate line. Let μ be the standard Gaussian measure on $\mathbb{R}^2$. Then $A \in W^{1,1}(\mu, \mathbb{R}^2)$, $\exp(|\delta_\mu A|) \in \bigcap_{p>1} L^p(\mu)$, but for the initial data $x = (0, 0)$ there is no uniqueness.*

The next result deals with more general differentiable measures (not necessarily possessing strictly positive or continuous densities).

Theorem 2.16 *Let μ be a probability measure on $\mathbb{R}^n$ with density $\varrho \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and let A be a vector field on $\mathbb{R}^n$ such that $A \in L^p(\mu)$ for some $p > 1$ and there is a sequence $A_i \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$ with the following properties: $\|A_i - A\|_{L^p(\mu)} \to 0$ and for every positive λ*

$$\sup_i \int_{\mathbb{R}^n} \exp\{\lambda|\delta_\mu A_i(x)|\}\, \mu(dx) < \infty.$$

Then (1.1) possesses a flow solution, and $\sup_{|t|\le T} \|r_t\|_{L^p(\mu)} \le C(T, p)$ for all $T > 0$ and $p \ge 1$.

Proof: For the approximating fields A_i one constructs the global flows U_t^i in the same way as above (where, in fact, the existence of strictly positive density was only used for deriving some local regularity of A and U_t). In particular, all the uniform estimates like in Theorem 2.6 hold true. Then the existence of the flow for A follows in the same way as in the infinite dimensional case below (see Lemma 5.11). □

The next technical result is a preparation for the infinite dimensional case.

Lemma 2.17 *Let A and B two vector fields satisfying the conditions of Theorem 2.6. Put $D = B - A$ and assume that $\|D\| \in L^\alpha(\mu)$ for some $\alpha > 1$. Denote by U^A and U^B the corresponding flows. Then for every $p < \alpha$ one has*

$$\|U_t^B - U_t^A\|_p := \|U_t^B - U_t^A\|_{L^p(\mu, \mathbb{R}^n)} \le C(T)\|D\|_\alpha, \tag{2.16}$$

where $C(T)$ is a number depending only on the exponential moments of $|\delta_\mu A|$, $|\delta_\mu B|$, $\|\nabla A\|$, and $\|\nabla B\|$ (up to a certain finite order depending on T, α, and p).

Proof: Let $U_t^\lambda(x)$ be the solution corresponding to the field $A + \lambda D$, $\lambda \in [0, 1]$. This solution exists, since $A + \lambda D = (1 - \lambda A)A + \lambda B$ satisfies the conditions of Theorem 2.6. Moreover, since for every positive c

$$\int \exp\{c|\delta_\mu(A + \lambda D)(x)|\}\, \mu(dx) \le \int \exp c|\delta_\mu A(x)| + c|\delta_\mu B(x)|\}\, \mu(dx) < \infty,$$

we get from (2.11), for every T and $q > 1$, the uniform estimates

$$\sup_{-T\le t\le T} \|r_t^\lambda\|_q \le L(q, T) < \infty, \tag{2.17}$$

where $L(q, t) = \|\exp(q(|t| + 1)|\delta_\mu A|)\|_{2q'} \|\exp(q(|t| + 1)|\delta_\mu A|)\|_{2q'}$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Approximating initially A and B by smooth flows as in the proof of Theorem 2.6 one can prove that U_t^λ is differentiable with respect to λ and the operator valued mapping $\eta_t^\lambda(x) := \partial U_t^\lambda(x)/\partial\lambda$ satisfies the equation

$$\eta_t^\lambda(x) = \int_0^t \nabla A(U_s^\lambda(x))\eta_s^\lambda(x) ds + \int_0^t D(U_s^\lambda(x))\, ds + \int_0^t \lambda\nabla D(U_s^\lambda(x))\eta_s^\lambda(x)\, ds. \tag{2.18}$$

For smooth fields, (2.18) follows by differentiating in λ the equation

$$U_t^\lambda(x) = x + \int_0^t A(U_s^\lambda(x))\,ds + \int_0^t \lambda D(U_s^\lambda(x))\,ds.$$

The general case follows by taking suitable smooth approximations (it is also possible to prove the whole assertion for smooth fields and then approximate at the end).

¿From the linear equation (2.18) we get by Gronwall's inequality

$$\|\eta_t^\lambda(x)\| \le \int_0^t \|D(U_s^\lambda(x))\|\,ds \exp\Big\{\int_0^t \{\|\nabla A(U_s^\lambda(x))\| + \lambda\|\nabla D(U_s^\lambda(x))\|\}\,ds\Big\},$$

so that by Lemma 2.1

$$\|\eta_t^\lambda(x)\|^p \le \left(\int_0^t \|D(U_s^\lambda(x))\|\,ds\right)^p \left(1 + \int_0^t \exp\left\{p(t\vee 1)\left(\|\nabla A(U_s^\lambda(x))\| + \lambda\|\nabla D(U_s^\lambda(x))\|\right)\right\}\,ds\right).$$

If $k = \frac{p+\alpha}{2p}$, $1/k + 1/k' = 1$, then the L^1-norm of the right hand side is bounded by the product

$$\left\|\left(\int_0^t \|D(U_s^\lambda(\,\cdot\,))\|\,ds\right)^p\right\|_k \left\|1 + \int_0^t \exp\left\{p(t\vee 1)\left(\|\nabla A(U_s^\lambda(x))\| + \lambda\|\nabla D(U_s^\lambda(x))\|\right)\right\}\,ds\right\|_{k'}.$$

Letting $\gamma = \frac{2\alpha}{p+\alpha}$, $\gamma' = \frac{2\alpha}{\alpha-p}$, (thus $kp = \frac{p+\alpha}{2}$, $\gamma kp = \alpha$), the first factor is bounded by

$$\begin{aligned} t^{p-\frac{1}{k}}\left(\int_0^t\int \|D(U_s^\lambda(x))\|^{kp}\,\mu(dx)\,ds\right)^{1/k} &= t^{p-\frac{1}{k}}\left(\int_0^t\int \|D(x)\|^{kp} r_s^\lambda(x)\,\mu(dx)\,ds\right)^{1/k} \\ &\le t^{p-\frac{1}{k}}\left(\int \|D(x)\|^\alpha\,\mu(dx)\right)^{\frac{1}{\gamma k}}\left(\int_0^t \|r_s^\lambda\|_{\gamma'}\,ds\right)^{\frac{1}{k}} \le C_1(T)\|D\|_\alpha^p, \end{aligned}$$

where $C_1(T) = T^p L(\gamma', T)^{1/k}$ depends only on T, α, p and the exponential moments of $|\delta_\mu A|$, $|\delta_\mu B|$, $\|\nabla A\|$, and $\|\nabla B\|$ (see (2.17)).

Similarly, the second factor above is majorized by

$$\begin{aligned} &\|\exp\{2p(t\vee 1)(\|\nabla A\| + \|\nabla B\|)\}\|_{2k'}\ \|r_s^\lambda\|_2^{1/k'} \\ &\quad \le \|\exp\{2p(t\vee 1)\|\nabla A\|\}\|_{4k'}\ \|\exp\{2p(t\vee 1)\|\nabla B\|\}\|_{4k'}\ L(2,T)^{1/k'}, \end{aligned}$$

which is a constant that depends only on the exponential moments of $|\delta_\mu A|$, $|\delta_\mu B|$, $\|\nabla A\|$, and $\|\nabla B\|$ (up to a finite order depending on T, α, and p).

Thus, $U_t^B(x) - U_t^A(x) = \int_0^1 \eta_t^\lambda(x)\,d\lambda$, so that

$$\|U_t^B - U_t^A\|_p \le \int_0^1 \|\eta_t^\lambda\|_p\,d\lambda \le C(T)\|D\|_\alpha,$$

where $C(T)$ depends only on the exponential moments of $|\delta_\mu A|$, $|\delta_\mu B|$, $\|\nabla A\|$, and $\|\nabla B\|$ (up to a certain finite order depending on T, α, and p). □

We conclude this section by giving a convenient sufficient condition for the existence of a strictly positive density expressed in terms of the logarithmic derivative (which improves a well-known result of Skorohod [37]). Its formulation is entirely intrinsic (in particularly, it does not refer to the Lebesgue measure). In infinite dimensions the corresponding result (see Corollary 5.2) will serve as a reasonable substitute for the existence of strictly positive densities.

Proposition 2.18 *Let μ be a positive measure on $X = \mathbb{R}^n$ which has density $p \in W^{1,1}(\mathbb{R}^n)$ and assume that for every $h \in \mathbb{R}^n$ there is $c = c(h) > 0$ such that*

$$\int_{\mathbb{R}^n} e^{c|\beta_h(x)|}\mu(dx) < \infty, \tag{2.19}$$

where $\beta_h = \frac{\partial_h p}{p}$. Then p admits a continuous strictly positive modification.

Remark 2.19 *It is easy to check that (2.19) will hold for every h once it does for* n *independent directions and moreover we can take $c(h) = \frac{c_0}{|h|}$ $\forall h \neq 0$, for some $c_0 > 0$.*

Proof: Without loss of generality we assume that $\mu(\mathbb{R}^n) = 1$. Set $\gamma = \min_{1\le i\le n} c(e_i)$ and $M = \max_{1\le i\le n} \int_{\mathbb{R}^n} e^{\gamma|\beta_i|(x)}d\mu < \infty$, where $\beta_i := \beta_{e_i}$. Then using the inequality $x^k \le \left(\frac{k}{e}\right)^k e^x$, $k \in \mathbb{N}$, $x \ge 0$, we get

$$\|\beta_i\|^k_{L^k(\mu)} \le \left(\frac{k}{\gamma e}\right)^k \int_{\mathbb{R}^n} \exp^{\gamma|\beta_i(x)|}\, d\mu \le \left(\frac{k}{\gamma e}\right)^k M. \tag{2.20}$$

For $k \in \mathbb{N}$ set $f_k = p^{1/k}$. Then $\partial_i f_k = k^{-1}p^{1/k-1}\partial_i p$ for each i so that

$$\|\partial_i f_k\|^k_{L^k(\mathbb{R}^n)} = k^{-k}\int_{\mathbb{R}^n} \frac{(\partial_i p(x))^k}{p(x)^{k-1}}\,dx = k^{-k}\|\beta_i\|^k_{L^k(\mu)} \le \frac{M}{(\gamma e)^k}. \tag{2.21}$$

Fix now an arbitrary bounded domain $D \subset \mathbb{R}^n$ with smooth boundary ∂D and Lebesgue measure $|D|$. For every $m \le k$ Hölder's inequality yields $\|f_k\|_{L^m(D)} \le |D|^{\frac{1}{m}} \vee 1$, while applying it to (2.21) we obtain

$$\|\partial_i f_k\|_{L^m(D)} \le |D|^{\frac{k-m}{km}}\|\partial_i f_k\|_{L^k(D)} \le \frac{(M\vee 1)(|D|^{\frac{1}{m}}\vee 1)}{\gamma e}.$$

The sequence $\{f_k\}_{k=m}^{\infty}$ is thus bounded in $W^{m,1}(D)$, so that it follows from Sobolev's compact embedding theorem ([45, Theorem 2.5.1]) that if m is large enough (any $m > n$ suffices) this sequence is compact in $C(\bar{D})$. In particular each f_k, and thus p itself, has a continuous version and there is a subsequence $\{f_{k_i}\}$ which converges uniformly to a function f^D which is continuous on D. But by the definition of f_k, $f^D = \mathbf{1}_{\{p>0\}}$.

Now, $p|_{D_0} \not\equiv 0$ for some large enough bounded D_0, so that for any $D \supset D_0$ we conclude by f^D's continuity that $f^D \equiv 1$. From this it follows that $p(x) > 0$ $\forall x \in \mathbb{R}^n$. □

3 Non-Gaussian stochastic analysis

3.1 Infinite dimensional notation and terminology

Let X be a (real) Hausdorff locally convex space with the dual X^* and the Borel σ-field $\mathcal{B}(X)$. The duality pairing between X^* and X will be denoted by ${}_{X^*}\langle\cdot,\cdot\rangle_X$.

Let $H \subset X$ be a linear subspace endowed with an inner product making it to a separable Hilbert space with continuous embedding into X (such H will be called a continuously embedded Hilbert space). The embedding $H \to X$ defines the embedding $j_H : X^* \to H$ (denoted by j if there is no risk of ambiguity) as any element $k \in X^*$ gives a continuous functional on H and thus is represented by an element $j_H(k) \in H$ via the Riesz representation in H. If H is dense in X, then j_H is injective and we get what is usually called a rigged triple $X^* (= j_H(X^*)) \subset H \subset X$. In the case where X is also Hilbert, there is a symmetric nonnegative operator T on X such that $H = T(X)$. Then $j_H(X^*) = T^2(X)$.

Denote the inner product in H by $(\cdot,\cdot)_H$ and the norm in H by $\|\cdot\|_H$. If $E\subset H$ is a closed linear subspace, P_E denotes the orthogonal projection from H onto E.

The space of all bounded operators on a normed space Y with values in a normed space Z is denoted by $L(Y,Z)$, and the corresponding operator norm by $\|A\|_{L(Y,Z)}$ (or just by $\|A\|$ if there is no risk of confusion). If $Y=Z$ we write $L(Y)$. The Hilbert–Schmidt norm of an operator A between Hilbert spaces is always specified as $\|A\|_{HS}$. By $HS(H)$ we denote the space of all Hilbert–Schmidt operators on a Hilbert space H.

Let $\mathcal{M}(X)$ be the space of all Radon measures on $\mathcal{B}(X)$. In the sequel, μ-measurable means measurable with respect to the μ-completion $\mathcal{B}_\mu(X)$ of $\mathcal{B}(X)$. All the measures we consider on locally convex spaces are assumed to be finite and Radon.

If F is a measurable mapping from (X,μ) to a normed space E $\|F\|_{L^p(\mu,E)}$ (or $\|F\|_p$ if there is no risk of ambiguity) will stand for $\left(\int\limits_X \|F(x)\|_E^p\,\mu(dx)\right)^{\frac{1}{p}}$. In the case where $E=L(Y,Z)$ by measurability of F we always mean the measurability with respect to the strong operator topology, that is, the measurability of all mappings $x\mapsto F(x)(y)$, $y\in Y$.

Denote by $\mathcal{S}(X)$ the class of all functions on a locally convex space X such that $f(x)=\varphi(l_1(x),\dots,l_n(x))$, where $\varphi\in C_b^\infty(\mathbb{R}^n)$, $l_1,\dots,l_n\in X^*$. Such functions are called smooth cylindrical.

A measure $\mu\in\mathcal{M}(X)$ is said to be differentiable (in Fomin's sense) along a vector $h\in X$ if there exists $\beta_h=\beta_h^\mu\in L^1(\mu)$, μ's logarithmic derivative along h, such that $\forall f\in\mathcal{S}(X)$

$$\int_X \partial_h f(x)\,\mu(dx) = -\int_X f(x)\beta_h(x)\,\mu(dx) \tag{3.1}$$

where we use the standard notation $\partial_h f(x)=\frac{d}{dt}f(x+th)|_{t=0}$. It is worth mentioning that (3.1) remains valid for all μ–measurable functions f such that for a.e. x the function $t\mapsto f(x+th)$ is locally absolutely continuous and $f\beta_h,\ \partial_h f\in L^1(\mu)$ (see [12, Theorem 2.6]).

In the above setup, if H is dense in X and μ is differentiable along any $j_H(k)$, $k\in X^*$ and if there exists a measurable map $\beta: X\to X$ such that for any $k\in X^*$

$$\beta_{j(k)}(x)=\langle k,\beta(x)\rangle \qquad \mu\text{–a.e.},$$

then this map will be called the vector logarithmic derivative (logarithmic gradient) of μ associated with H (note that inclusion $j(X^*)\subset X$ which is implicit in β's definition is dependent on H).

Examples

(i) Let $X=\mathbb{R}^n$. Then μ is differentiable along all directions if and only if μ is absolutely continuous and admits a density p in the Sobolev class $W^{1,1}(\mathbb{R}^n)$. In this case its logarithmic gradient β is given by $\beta=\nabla p/p$.

(ii) Let μ be a centered Radon Gaussian measure on a locally convex space X and let H be the Cameron–Martin space (the reproducing kernel Hilbert space) of μ ([13], [7]). Then β exists and $\beta(x)=-x$.

(iii) Let $X=\mathbb{R}^\infty$, $H=l^2$ and let μ be the countable product of the probability measures μ_n on the line which have absolutely continuous densities p_n with integrable p_n'. Then β exists and $\beta(x)=(-p_n'(x_n)/p_n(x_n))$.

In more general spaces, sufficient conditions for the existence of β can be found in [7], [10]. For example, β exists if X is a Hilbert space and $H=:H(\mu):=\{h:\beta_h\in L^2(\mu)\}$. However, in infinite dimensions the vector logarithmic derivative does not always exist. For this reason, we do not assume its existence in our main results.

In a similar way it is possible to define differentiability of vector-valued measures.

For later use let us recall a characterization of the directional differentiability by means of conditional measures. Let $h \neq 0$ and let Y be a closed hyperplane in X complementary to $\mathbb{R}^1 h$. Then there is the natural continuous linear projection $\pi : X \to Y$. To be more precise, let $Y = \ker l$, where $l \in X^*$ and $l(h) = 1$. Then $\pi(x) = x - l(x)h$. Denote by μ_Y the image of μ under π. It is well-known that in this case there exist measures $\mu(\,\cdot\,|y) \in \mathcal{M}(X)$, $y \in Y$, supported on the lines $y + \mathbb{R}^1 h$ (called conditional measures) such that for every Borel set B in X one has

$$\mu(B) = \int_Y \int_{y+\mathbb{R}^1 h} \mu(B|y)\,\mu_Y(dy).$$

Then μ is differentiable along h if and only if for μ_Y-a.e. y the measure $\mu(\,\cdot\,|y)$ is differentiable along h and

$$\int_Y \int_{y+\mathbb{R}^1 h} \left|\beta_h^{\mu(\,\cdot\,|y)}(x)\right| \mu(\,\cdot\,|y)(dx)\,\mu_Y(dy) < \infty.$$

In this case, the restriction of β_h^μ to $y + \mathbb{R}^1 h$ can be taken for $\beta_h^{\mu(\,\cdot\,|y)}$ (for μ_Y-a.e. y). The same is true for the differentiability along a finite dimensional subspace (see [12, Theorem 2.5]). Clearly, in such a case it is possible to choose differentiable conditional measures for every $y \in Y$.

Except for Gaussian measures the logarithmic derivative represents in a way a nonlinearity which the measure induces on the space. Its explicit use in stochastic analysis can thus be seen as a non-Gaussian feature. In contrast, the Sobolev differentiable structure which we now introduce is essentially the same as the one traditionally used in Wiener spaces.

3.2 Sobolev classes

In infinite dimensions the class of Hilbert–Schmidt operators is smaller than the class of all bounded operators. This leads to two different directions along which the theory of Sobolev classes may be developed depending on whether the derivatives of vector-valued mappings are supposed to be Hilbert–Schmidt or just bounded operators. Below we discuss in more detail the first approach (see also [7] for additional information), but first we give a construction connected with the second direction because it is involved in the formulations of our main results and leads to wider classes of vector fields.

First definition

Let G be another Hilbert (or Banach) space. Just as in the case $G = \mathbb{R}$ considered in Subsection 3.1 a function f defined on X and taking values in G will be said to be smooth cylindrical if can be represented

$$f(x) = \sum_{j=1}^{m} F_j(\langle y_1, x\rangle, ..., \langle y_n, x\rangle) g_j, \tag{3.2}$$

where $y_1, \dots, y_n \in X^*, F_1, \dots, F_m \in C_b^\infty(\mathbb{R}^n)$ and $g_1, \dots, g_m \in G$. Denote the linear space of all such smooth functions by $\mathcal{S}(X, G)$.

Definitions 3.1 *(i) Let μ be a measure differentiable along a vector h and let F be a μ-integrable map with values in a normed (or locally convex) space G. We say that F has the stochastic partial derivative along h if $F\beta_h$ is μ-integrable and there is a μ-integrable G-valued map $D_h F$, also denoted $\partial_h F$, such that for all $\varphi \in \mathcal{S}(X)$*

$$\begin{aligned}\int_X \varphi(x) D_h F(x)\,\mu(dx) \\ = -\int_X \partial_h \varphi(x) F(x)\,\mu(dx) - \int_X \varphi(x) F(x) \beta_h(x)\,\mu(dx).\end{aligned} \tag{3.3}$$

(ii) Let E be a Hilbert (or Banach) space continuously embebbed into X such that μ is differentiable along all vectors from some linear subspace E_0 of E which is dense in E. We say that F has the stochastic derivative (gradient) $\nabla_E F$ if the partial stochastic derivatives $\partial_h F$ exist for all $h \in E_0$ and $\nabla_E F : X \to L(H, G)$ is a map measurable in the sense above such that for every $h \in E_0$

$$\nabla_E F(x)h = \partial_h F(x) \quad \mu\text{-a.e.}$$

If $E = H$, then we write ∇F instead of $\nabla_H F$.

Denote $G_{p,1}(E, \mu, G) := \Big\{ F \in L^p(\mu, G) : \ \|\nabla_E F\|_{L(E,G)} \in L^p(\mu) \Big\}.$

Let us consider the following examples which can be readily deduced from the integration by parts formula (see [12] or [7], Theorem 4.1.2).

Examples. (i) Let $F \in L^p(\mu, H)$ have the Gâteaux derivative $\nabla_H F$ along H at every point. Assume that $\|\nabla_H F(x)\|_{L(H)}$ is in $L^p(\mu)$. Then $F \in G_{p,1}(H, \mu, H)$.

(ii) More generally, let $F \in L^p(\mu, H)$ be such that for every $h \in j_H(X^*)$ there is a version of F which is locally absolutely continuous on the lines parallel to h and that the corresponding partial derivatives along h coincide a.s. with $T(x)(h)$, where $T \in L^p(\mu, L(H))$. Then $F \in G_{p,1}(H, \mu, H)$ and $\nabla F = T$ (we shall see below that this is in fact equivalent to being an element of $G_{p,1}(H, \mu, H)$).

Clearly, one can iterate this definition to produce classes $G_{p,k}(\mu)$.

Below we shall deal with vector fields of the class $G_{p,1}(H, \mu, H)$, and thus we do not need this iterative procedure. However, it is worth mentioning that an obvious incovenience of dealing with higher classes of this type is that $L(H, G)$ is not a Hilbert space and, hence, one cannot remain within the class of Hilbert spaces when carrying out this iterations. For some applications it might be preferable to have higher order gradients which are elements of Hilbert spaces.

<u>Second definition</u>

A traditional approach furnishing such a possibility is to consider mappings whose derivatives are Hilbert–Schmidt mappings. This approach is based on a fundamental property that the space of all Hilbert–Schmidt operators between two Hilbert spaces is again a Hilbert space provided it is equipped with the Hilbert–Schmidt norm. We now recall this construction, delibaretly using the same notation ∂_h and ∇_E as it will turn out that they are restrictions of their respective counterparts defined above.

Let $E \subset H$ be a linear subspace. The gradient ∇_E along E is defined for $f \in \mathcal{S}(X, G)$ by

$$\nabla_E f = \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\langle y_1, x\rangle, ..., \langle y_n, x\rangle) \ P_E j_H(y_i) \otimes g_j \in \mathcal{S}(X, G_1). \tag{3.4}$$

Here $G_1 = \mathit{HS}(E, G)$ is the space of Hilbert–Schmidt operators from E to G equipped with the norm $\|T\|_{HS}^2 = \sum_i \|Te_i\|_G^2$ for any complete orthonormal basis $\{e_i\}$ in E. For $h \in E$ the G–valued mapping $(\nabla_E f)h$ will also be denoted $\partial_h f$. (If $G = \mathbb{R}$, and since $\mathit{HS}(E, \mathbb{R}) \approx E$, $(\nabla_E f)h$ can also be written $(\nabla_E f, h)$).

Higher order gradients can now be defined on $\mathcal{S}(X, G)$ iteratively by setting $G_0 = G$, $\nabla_E^0 = I$ and for $k \in \mathbb{N}$

$$G_k = \mathit{HS}(E, G_{k-1}), \qquad \nabla_E^k = \nabla_E \circ \nabla_E^{k-1} : \mathcal{S}(X, G) \longrightarrow \mathcal{S}(X, G_k).$$

(Note that G_k can be identified with the space of symmetric k–linear Hilbert–Schmidt operators on E with range in G).

Assume that μ is a measure on X such that if $f, \ g \in \mathcal{S}(X)$, $f = g$ μ-a.e., then $\nabla_H f = \nabla_H g$ μ-a.e. (e.g., this is the case if μ has full support). Since for any $k \in \mathbb{N}$ and $p \in [1, \infty)$ ∇_E^k is closable under the norm

$$\|f\|_{E,k,p} = \sum_{i=0}^{k} \|\nabla_E^i f\|_{L^p(\mu, G_i)}$$

the completions $W^{p,k}(E,\mu,G)$ of $\mathcal{S}(X,G)$ under these norms form, for each p, a decreasing sequence of linear subspaces of $L^p(\mu,G)$, and the gradients ∇_E^k can be extended naturally (retaining the same notation with no risk of confusion), i.e. $\nabla_E^k : W^{p,k}(E,\mu,G) \longrightarrow L^p(\mu,G_k)$ $\forall k \in \mathbb{N}, p \geq 1$. Such extensions are called stochastic gradients.

One can check that $W^{p,1}(E,\mu,G) \subset G_{p,1}(E,\mu,G)$, where $p > 1$, provided $\beta_h \in L^q(\mu)$ $(\frac{1}{p}+\frac{1}{q} = 1)$ for all $h \in j_H(X^*)$, and that in this case both definitions of the gradients coincide on $W^{p,1}(E,\mu,G)$.

Notation: The explicit mention of G will be ommitted in all of the notation above whenever $G = \mathbb{R}$. For example, we have already encountered $\mathcal{S}(X) = \mathcal{S}(X,\mathbb{R})$ earlier in this section.
For any family of random variables $\mathcal{L} \subset L^0(\mu)$, denote by $\mathcal{F}_{\mathcal{L}}$ the sigma algebra generated by $\{\varphi, \varphi \in \mathcal{L}\}$.
Let L stand for any function space on X (e.g. $W^{p,k}(H,\mu,G)$). Given a subsigma algebra $\mathcal{G} \subset \mathcal{B}(X)$ denote by $L_{\mathcal{G}}$ (e.g. $W^{p,k}_{\mathcal{G}}(H,\mu,G)$) the space of those elements of L which are $\mathcal{G}$–measurable.

3.3 Divergence of vector fields

Let us consider a vector field $A \in L^1(\mu,H)$, and generalizing the constant case $A^h(x) \equiv h \in H$, denote for any $f \in \mathcal{S}(X)$

$$\partial_A f(x) = (\nabla_H f(x), A(x)).$$

If there exists an element $\delta_\mu A$ of $L^1(\mu)$ such that the integration by parts formula

$$\int_X \partial_A f(x) d\mu = -\int_X f(x)\delta_\mu A(x) d\mu \tag{3.5}$$

holds for any $f \in \mathcal{S}(X)$, this (necessarily unique) random variable is said to be A's divergence with respect to μ (or, alternatively, μ's logarithmic derivative along A). In this case we say that $A \in \mathrm{dom}(\delta_\mu)$. Note that if $\delta_\mu A \in L^2(\mu)$, then whenever (3.5) holds $\forall f \in \mathcal{S}(X)$, it follows by a simple density argument that it also holds $\forall f \in W^{2,1}(H,\mu)$.

Clearly, it is possible to define divergence for more general vector fields $A : X \to X$. Namely, let $\partial_A f :=_{X^*} \langle f', A\rangle_X$ be μ-integrable for any $f \in \mathcal{S}(X)$. Then $\delta_\mu A$ is defined as above by means of (3.5).

Some examples in which the divergence can be identified are

(i) It can be seen from (3.1) that for $h \in H$, $\delta_\mu A^h = \beta_h$ whenever the right hand side exists in $L^1(\mu)$;

(ii) Let $\{e_i\}$ be an orthonormal basis in H, such that β_{e_i} exist and let $A \in L^1(\mu,H)$, $A(x) = \sum_i A^i(x)e_i$, where $A^i \in L^1(\mu)$ has the stochastic partial derivative $\partial_{e_i}A_i$. Then

$$\delta_\mu A = \sum_{i=1}^{\infty}\left(\partial_{e_i}A^i + A^i\beta_{e_i}\right), \tag{3.6}$$

provided the series converges in $L^1(\mu)$. In particular, this is true for finite sums if $A^i\beta_{e_i} \in L^1(\mu)$ for all i.

(iii) Assume that A takes values in $j_H(X^*)$ and is Gâteaux differentiable along H such that $\nabla_H A : H \to H$ is a trace class opeartor. If β exists and $trace\nabla_H A$, $\langle j_H^{-1}A, \beta\rangle \in L^1(\mu)$, then $\delta_\mu A = trace\nabla_H A + \langle j_H^{-1}A, \beta\rangle$. In particular, let X be a Hilbert space, $H = T(X)$, where T is a Hilbert–Schmidt operator, $A(x) = T^2B(x)$, where $B : X \to X$ is Gâteaux differentiable with $\|B\|_X$, $\|D_X B\|_{L(X)} \in L^2(\mu)$, and let β exist with $\|\beta\|_X \in L^2(\mu)$. Then integration by parts easily yields (cf. [24, Chapter VIII] or [7, Proposition 8.4.3])

$$\delta_\mu A(x) = trace(T^2 D_X B(x)) + (B(x), \beta(x))_X. \tag{3.7}$$

This is a case in which the sum in (3.6) can be split into two converging series, which are represented by the two terms in (3.7).

(iv) If μ is a centered Gaussian measure on a locally convex space X with the Cameron–Martin space H, then every field $A \in W^{2,1}(H, \mu, H)$ possesses the divergence $\delta_\mu A \in L^2(\mu)$ (see [24] or [7, Theorem 7.2.5]) which is in fact a stochastic integral (often called the Ramer–Itô integral). This section owes its name to the fact that it can be considered to deal with a "nongaussian stochastic integral".

Note that the definition of the divergence is consistent with the one given in finite dimensions (compare (3.5) and (3.7) with (2.1) and (2.2) respectively).

4 Main Results

In all that follows we shall assume the following separability property for the locally convex measure space (X, μ) introduced in Section 3: there is a sequence $\{l_n\}$ of continuous linear functionals on X such that the completion of the σ-field generated by $\{l_n\}$ coincides with $\mathcal{B}_\mu(X)$.

For example, this is the case if there is a sequence $\{l_n\}_{n=1}^\infty \subset X^*$ which separates the points in X. It is also worth mentioning that this separability property holds for every Radon Gaussian measure on an arbitrary locally convex space (see, e.g., [9] or [13]).

Note that we can always choose $\{l_n\}$ in such a way that $\{j_H(l_n)\}$ is dense in H, adding a countable sequence of continuous linear functionals on X which separates the points in H. Therefore, without loss of generality (by means of the Gramm–Schmidt procedure) we may and shall assume that $(e_n := j_H(l_n))_{n=1}^\infty$ is a orthonormal basis of H which generates the following projections (cf. the notation introduced at the end of subsection 3.2):

Definitions 4.1 *For each $n \in \mathbb{N}$ let L_n be the linear span of $\{l_1, \dots, l_n\} \subset X^*$ and H_n the linear span of $\{e_1, \dots, e_n\} \subset H$ ($H_n = j(L_n)$), denote $\mathcal{F}_n = \mathcal{F}_{L_n}$ and define $E_n : L^1(\mu) \longrightarrow L^1_{\mathcal{F}_n}(\mu)$ to be the conditional expectation $E_n\varphi = E(\varphi|\mathcal{F}_n)$ as well as $P_n = P_{H_n}$ the orthogonal projection of H onto H_n which can be extended to the continuous linear operator (denoted by the same symbol) $P_n : X \to H_n \subset X$ by setting*

$$P_n x = \sum_{i=1}^n l_i(x) e_i \qquad x \in X.$$

Finally $\mu_n = \mu \circ P_n^{-1}$ is μ's marginal measure on H_n and $\beta^{(n)} = \sum_{k=1}^n \beta_{e_k} e_k \in L^1(\mu, H)$.

The role of our separability assumption is to ensure that for every μ–integrable map F with values in H, the sequence $\{E_n F\}$ is an H-valued martingale convergent to F in $L^1(\mu, H)$.

We shall now state our main results, preceded by the assumptions we impose on the probability measure μ and on the vector field $A : X \to H$.

A1) μ is differentiable along all directions in $j_H(X^*)$. In addition, for every $h \in j_H(X^*)$ there exists a $c = c(h) > 0$ such that

$$Ee^{c|\beta_h|} = \int_X e^{c|\beta_h(x)|}\,\mu(dx) < \infty. \tag{4.1}$$

(equivalently, (4.1) could have been written without absolute values).

A2) $A \in G_{\alpha,1}(\mu, H)$ for some $\alpha > 1$ and for every $\lambda > 0$

$$\int_X e^{\lambda \|\nabla_H A\|_{L(H)}}\,\mu(dx) < \infty. \tag{4.2}$$

A3) For every $\lambda > 0$

$$\sup_n Ee^{\lambda|\delta_\mu(P_n A)|} = \sup_n \int_X e^{\lambda|\delta_\mu(P_n A(x))|}\, \mu(dx) < \infty. \tag{4.3}$$

Remarks 4.2 *(i) Assumption* **A3)** *implicitly uses the fact that $\delta_\mu(P_n A)$ exist for each $n \in \mathbb{N}$ (this follows from equality (3.6)). Moreover, (4.3) implies that $\delta_\mu A$ exists and, by Fatou's lemma, for every $\lambda > 0$*

$$Ee^{\lambda|\delta_\mu A|} = \int_X e^{\lambda|\delta_\mu A(x)|}\, \mu(dx) < \infty. \tag{4.4}$$

We have been unable to show that it is enough to assume the more elegant (4.4) in **A3)** *instead of (4.3) (which might be strictly stronger). However it seems that in many cases the verification of (4.4) will in any case be achieved by checking that (4.3) holds.*

(ii) Note that (4.4) follows also from the condition

A3') *there exist two increasing sequences of integers $\{k_n\}$ and $\{m_n\}$ with $k_n \geq m_n$ such that for every $\lambda > 0$*

$$\sup_n \int_X e^{\lambda|\delta_\mu(E_{k_n} P_{m_n} A)|}\, \mu(dx) < \infty \tag{4.5}$$

$$\sup_n \int_X e^{\lambda\|\nabla_H(E_{k_n} P_{m_n} A)\|_{L(H)}}\, \mu(dx) < \infty, \tag{4.6}$$

which is somewhat less constructive but in some special cases works better than **A3)**. *An obvious example when this happens is the case where $E_n\beta^{(n)} = \beta^{(n)}$, which takes place, e.g., for product-measures (then $\beta^{(n)}$ is a function of $P_n x$ for a suitable choice of $\{l_n\}$, so that with $k_n = m_n = n$ (4.4) is equivalent to (4.5) and (4.2) is equivalent to (4.6)). Another important example where (4.4) is equivalent to (4.5) is connected with Gibbs measures in the so called models with a finite radius of interaction (see [2]). In our notation, this corresponds to the situation where for every n there is $r(n)$ such that $\beta^n = E_k\beta^n$ for $k = n + r(n)$ (in this case, for $k_n = n + r(n)$ and $m_n = n$, the equivalence of (4.4) and (4.5) follows from Proposition 5.9 below).*

(iii) Note that if H is dense in X, then so is $j_H(X^*)$. By virtue of condition **A3**, *μ is quasi-invariant along $j_H(X^*)$ (this well-known result from [37] is also a corollary of Proposition 2.18 applied to conditional measures), hence, it has full support, which enables us to define Sobolev classes $W^{p,k}(H, \mu, H)$ as explained in Section 3.2.*

Theorem 4.3 *Let μ be a probability measure on a locally convex space X described above, and $A : X \longrightarrow H$ a measurable vector field. Assume that* **A1)**, **A2)** *and* **A3)** *(or* **A3')**) *are satisfied. Then (1.1) has a flow solution and the Radon–Nikodym process r_t has a measurable modification all whose sample paths are continuous; in addition, for every $t \in \mathbb{R}^n$ and $p > 1$*

$$\|r_t\|_{L^p(\mu)} \leq 4e^{\Lambda(p,t)|t|}, \tag{4.7}$$

where $\Lambda(p,t) = \left(\int e^{(|t|\vee 1)pq|\delta_\mu A(x)|}\, \mu(dx)\right)^{1/q}$, $\frac{1}{p} + \frac{1}{q} = 1$.
Moreover, $V_t := U_t - I \in G_{\theta,1}(H, \mu, H)$ for each $\theta < \alpha$, and for every $p \geq 1$, letting $\nabla_H U_t := \nabla_H V_t + I_H$, one has

$$\|\nabla_H U_t\|_{L^p(\mu)} \leq 4\left(\int_X e^{2(|t|+1)p\|\nabla_H A(x)\|}\, \mu(dx)\right)^{\frac{1}{2p}} e^{(\Lambda(2,t)+1)|t|/p}; \tag{4.8}$$

Finally, $\psi \circ U_t \in G_{\alpha,1}(\mu)$ for every $\psi \in S(X)$ and

$$\frac{d}{dt}\int_X \psi \circ U_t(x)\,\mu(dx) = -\int_X \psi(U_t(x))\delta_\mu A(x)\,\mu(dx). \tag{4.9}$$

(If we assume $A \in W^{\alpha,1}(H,\mu,H)$, then $\psi \circ U_t \in W^{\alpha,1}(H,\mu))$ as well for every $\psi \in \mathcal{S}(X)$).

Theorem 4.4 *Let X, H, μ and A be as in Theorem 4.3. For a flow solution $\{U_t, t \in \mathbb{R}\}$ of (1.1) which satisfies the properties stated therein, the following are true as well:*

(i) The continuous version of r_t is given by

$$r_t(x) = \exp\left\{-\int_0^t \delta_\mu A(U_{-s}x)\,ds\right\}. \tag{4.10}$$

(ii) U_t is unique in the sense that if $W_t(x)$ is another solution of (1.1) satisfying (4.7) then $U_\bullet x = W_\bullet x$ in $C(\mathbb{R}, X)$ for μ–almost every $x \in X$.

Finally, if **A3)** *is satisfied only for some $\lambda > 0$, then there exist $T > 0$ and $p_T > 1$ such that all the statements above are valid on the time interval $[-T, T]$ with $p = p_T$.*

Example 4.5 *Let μ be a centered Radon Gaussian measure on a locally convex space X and let H be its Cameron–Martin space. Then our conditions on A read as follows: $A \in G_{\alpha,1}(H,\mu,H)$ for some $\alpha > 1$, $\exp(c\|\nabla_H A\|_{L(H)})$, $\exp(c|\delta_\mu A|) \in L^1(\mu)$ for all $c > 0$. For the validity of the results above on a bounded interval it suffices to have $\exp(c|\delta_\mu A|) \in L^1(\mu)$ for some $c > 0$. In this case condition* **A3')** *applies, since for any orthonormal basis $\{e_n\}$ in H formed by elements from $j_H(X^*)$ one has $\beta^{(n)}(x) = -\sum_{i=1}^n l_i(x)e_i$, whence $E_n\beta^{(n)} = \beta^{(n)}$.*

Remark 4.6 *Note that if in Example 4.5 one defines $\nabla_H A$ using the directional description as in Lemma 5.4 below, then this can be done also for $p = 1$. In this case, the inclusion $\|\nabla_H A\|_{L(H)} \in L^r(\gamma)$ implies the inlusion $\|A\|_H \in L^r(\gamma)$. This is a well-known corollary of a Poincaré-type inequality for real functions or for H-valued maps with the Hilbert-Schmidt norms on derivatives (see, e.g., [35] or [9], Chapter 3, Theorem 2.10). However, as pointed out by A. S. Ustunel, in our case the same is true for the operator norm, since the scalar case can be applied to the function $\psi : x \mapsto \|A(x)\|_H$, taking into account that $\|\nabla_H \psi(x)\|_H \le \|A(x)\|_{L(H)}$.*

Remark 4.7 *We emphasize that the nontrivial part of these results is the existence of the solution. The absolute continuity of the corresponding flow turns out to be a by-product of our method of the proof. The explicit expressions for $\frac{d(\mu\circ U_t^{-1})}{d\mu}$ are fairly well-known ("nonlinear Cameron–Martin formulas"). In the finite dimensional case they appear in [29] and [16]. In infinite dimensions formulas of this type were obtained in [37] (constant vector fields) and in [17] (the Gaussian case). A simple derivation of such expressions based on the integration by parts was proposed in [4] – [5] and generalized significantly in [23] and [24, Chapter VIII]. The same idea was used also in [38]. Clearly, for $A(x) \equiv h$ one gets Skorohod's formula for the density of the shifted measure*

$$\frac{d\mu_{th}}{d\mu}(x) = e^{\int_0^t \beta_h(x-sh)\,ds},$$

since $U_t(x) = x + th$. Note that μ if is not quasi-invariant along the vector h (such a vector always exists for any nonzero measure on an infinite dimensional space), then the flow $U_t(x) = x + th$ is well-defined and unique, but is not absolutely continuous.

The proofs of these theorems involve projections onto finitely many coordinates and thus rely on our finite dimensional results. However, unlike [16], the procedure of constructing finite dimensional approximations is more involved.

5 Proofs: Infinite dimensional case

In this section we prove Theorem 4.3. Following some preliminary technical results, the flow solution is constructed in subsection 5.2. It is formally defined in (5.19), and its various claimed properties will result from Lemma 5.11 and Lemma 5.17. Both items of Theorem 4.4 will then be proved in subsections 5.3 and 5.4 respectively. The last statement of Theorem 4.4 will be clear from the proofs.

5.1 Smoothness of conditional expectations

The following lemma is straightforward and is included for future reference. In it we use the notation $\eta^T = \eta \circ T^{-1}$ for the the image of a measure η under the measure transformation induced by a mapping T from η's underlying space into a second space.

Lemma 5.1 *Assume μ is a probability measure on a locally convex space X and for a measurable linear mapping T from X into another locally convex space Y let $\mathcal{B}_X^T$ be the subsigma algebra $\{T^{-1}(A),\ A \in \mathcal{B}_Y\}$. If μ is differentiable along $h \in X$ then μ^T is differentiable along Th and*

$$E\left(\beta_h^\mu \middle| \mathcal{X}^T\right) = \beta_{Th}^{\mu^T} \circ T. \tag{5.1}$$

Proof: For any $\psi \in \mathcal{S}(Y)$

$$\begin{aligned}\int_X \beta_{Th}^{\mu^T}(Tx)\psi(Tx)\,\mu(dx) &= \int_Y \beta_{Th}^{\mu^T}(y)\psi(y)\,\mu(dy) = -\int_Y \left(\partial_{Th}\psi\right)(y)\,\mu^T(dy)\\ &= -\int_X \partial_{Th}\psi(Tx)\,\mu(dx) = -\int_X \partial_h(\psi\circ T)(x)\,\mu(dx) = \int_X \beta_h^\mu(x)\psi(Tx)\,\mu(dx),\end{aligned}$$

from which (5.1) follows since $\psi \circ T$ generate $\mathcal{B}_X^T$. □

Corollary 5.2 *Assume that a probability measure μ on a locally convex space X is differentiable in the directions $h_1, \ldots, h_n$ and for some $c > 0$*

$$\int \exp(c|\beta_{h_i}(x)|)\,\mu(dx) < \infty.$$

Let $T : X \to \mathbb{R}^n$ be a measurable linear map such that $\{y_i = Th_i\}_{i=1}^n$ spans $\mathbb{R}^n$. Then $\mu \circ T^{-1}$ has a continuous strictly positive density p.

Proof: By Lemma 5.1 $\mu^T = \mu \circ T^{-1}$ is differentiable along $\{y_i\}$ and

$$E\left(\beta_{h_i}\middle|\mathcal{X}^T\right) = \beta_{y_i}^{\mu^T} \circ T.$$

Moreover, changing variables and applying Jensen's inequality we obtain

$$\int_{\mathbb{R}^n} e^{c|\beta_{y_i}^{\mu^T}(y)|}\,\mu^T(dy) = \int_X e^{c|\beta_{y_i}^{\mu^T}(Tx)|}\,\mu(dx) = Ee^{c\left|E\left(\beta_{h_i}|\mathcal{X}^T\right)\right|} \le Ee^{c|\beta_{h_i}|} < \infty.$$

The conclusion then follows from Proposition 2.18. □

In the next subsection we shall prove Theorem 4.3 by first projecting the vector field onto finite dimensional subspaces and then establishing the existence of a limit of the resulting flows as the dimensionality tends to infinity. In the course of the first of these two stages, it will turn out that in contrast with the Gaussian case treated in [17], the gradient ∇_H and divergence δ_μ do not commute with the operation of conditioning on linear subsigma algebras (that is, subsigma algebras generated by a family of linear functionals). The remainder of this subsection is devoted to these issues.

Lemma 5.3 *Let $F_n \to F$ in $L^p(\mu, G)$. Assume that the partial stochastic derivatives $\partial_h F_n$ exist and converge weakly in $L^p(\mu, G)$ to some mapping Φ_h. If $\beta_h \in L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$, then Φ_h is the partial stochastic derivative of F. If, in addition, there is an $L(H, G)$-valued mapping Φ, measurable in the sense above, such that $\|\Phi\|_{L(H)} \in L^1(\mu)$ and for each h from a dense subset $H_0 \subset H$ one has $\nabla_H F_n(\cdot)(h) \to \Phi(\cdot)(h)$ weakly in $L^1(\mu, G)$, then Φ is the stochastic gradient of F.*

Proof: It suffices to note that the conditions above enable us to pass to the limit in equalities (3.3) for F_n. □

Lemma 5.4 *Given $h \in H$ and a Hilbert space G assume $\exp|\beta_h| \in L^1(\mu)$ and let $F \in L^p(\mu, G)$, where $p > 1$. The following are equivalent:*

(i) There exists the stochastic derivative $\partial_h F \in L^p(\mu, G)$;

(ii) There is a modification $\widetilde{F}$ of F such that $t \mapsto \widetilde{F}(x + th)$ is locally absolutely continuous for all x and $\frac{\partial \widetilde{F}}{\partial h} := \frac{d}{dt}\widetilde{F}(\,\cdot\, + th)|_{t=0}$ is in $L^p(\mu, G)$ (note that $\frac{\partial \widetilde{F}}{\partial h}$ automatically exists μ-a.e.),

in which case $\partial_h F = \frac{\partial \widetilde{F}}{\partial h}$ μ–a.e.
(If G is a general Banach space then the same is true provided in (ii) the existence of $\frac{\partial \widetilde{F}}{\partial h}$ a.e. is assumed — this holds automatically for spaces with the Radon–Nikodym property).

Proof: To simplify notation we consider the scalar case $G = \mathbb{R}$ (in fact, below we apply this result only to finite dimensional mappings). Let Y be the complementary hyperplane to $\mathbb{R}^1 h$ and μ_Y the projection of μ to Y as explained above in Subsection 3.1. Then X is identified with $\mathbb{R}^1 \oplus Y$. By our condition, $\beta_h^\mu F$ is μ-integrable. It follows then from the definition of $\partial_h F$ and the integration by parts formula characterization (3.1) of the differentiability that the measure $\nu := F\mu$ is differentiable along h with $\beta_h^\nu = \beta_h^\mu + \partial_h F/F$, where by definition $\partial_h F/F = 0$ on $\{F = 0\}$. As noted above, both measures μ and ν admit differentiable conditional measures $\mu(\,\cdot\,|y)$ and $\nu(\,\cdot\,|y)$, respectively, on the lines $L_y = y + \mathbb{R}^1 h$, $y \in Y$. In addition, $\exp(|\beta_h^{\mu(\,\cdot\,|y)}|) \in L^1(\mu(\,\cdot\,|y)$ for μ_Y-a.e. y. Hence, the measures $\mu(\,\cdot\,|y)$ admit absolutely continuous strictly positive densities f^y with respect to the natural Lebesgue measures λ^y on L_y (induced by the maps $t \mapsto y + th$). The measures $\nu(\,\cdot\,|y)$ admit absolutely continuous densities g^y. Hence, $\Phi(\,\cdot\,, y) : t \mapsto g^y(t)/f^y(t)$ is locally absolutely continuous for μ_Y-almost all $y \in Y$. Note that $\Phi(t, y) = F(y + th)$ for μ-almost all $(t, y) \in X$. Differentiating $g^y(t)/f^y(t)$ in t and taking into account that $g^y(t)' = 0$ a.e. on $\{g^y = 0\}$, we get the expression $\frac{g^y(t)}{f^y(t)}\left(\frac{g^y(t)'}{g^y(t)} - \frac{f^y(t)'}{f^y(t)}\right)$. Since $g^y(t)/f^y(t) = F(y + th)$, $f^y(t)'/f^y(t) = \beta_h^\mu(y + th)$ and $g^y(t)'/g^y(t) = \beta_h^\nu(y + th)$ for μ-almost all (t, y), we arrive at the equality $\partial\Phi(t, y)/\partial t = \partial_h F(y + th)$ μ-a.e. Thus, we get a modification with the desired property.

The converse follows directly from the integration by parts formula (3.1) and the remark following it. □

Remark 5.5 *It is clear from the proof that Lemma 5.4 remains valid if instead of the exponential integrability of β_h one has that $F\beta_h$ is integrable and the local absolute continuity of $t \mapsto \widetilde{F}(y + th)$ takes place on the set $\{t : f^y(t) > 0\}$.*

Corollary 5.6 *Let $F \in G_{\alpha,1}(H, \mu, H)$. Then $\psi \circ (I + F) \in G_{\alpha,1}(H, \mu)$ for every bounded Borel function ψ on X such that $|\psi(x + th) - \psi(x)| \le C\|h\|_H$ for all $x \in X$ and $h \in H$. In addition,*

$$\nabla_H(\psi \circ (I + F)) = (I_H + \nabla_H F)^*(\nabla_H \psi \circ (I + F)).$$

In particular, this assertion holds true for $\psi \in \mathcal{S}(X)$.

Proof: Clearly, for a suitable version $\widetilde{F}$ of F the functions $t \mapsto \psi(x + th + \widetilde{F}(x + th))$ are locally absolutely continuous (which can be easily seen from the definition) and their derivatives at zero exist μ-a.e. and are given by $(\nabla\psi(x + \widetilde{F}(x)), h + \frac{\partial \widetilde{F}}{\partial h}(x))_H$. It remains to note that $(I_H + \nabla_H F)^*(\nabla_H \psi \circ (I + F)) \in L^\alpha(\mu, H)$. $\square$

Corollary 5.7 *Let μ and F be as in Lemma 5.4 such that the stochastic derivative $\partial_h F \in L^p(\mu, G)$ exists. Then, for every $n \in \mathbb{N}$, $E_n F$ has the stochastic derivative $\partial_h E_n F \in L^r(\mu, G)$ for every $r < p$. In addition,*

$$\partial_h E_n F = E_n \partial_h F + E_n(F\beta_h) - E_n F E_n \beta_h \tag{5.2}$$

Proof: Let $a = P_n h$. We consider again the scalar case. Let $\nu = F\mu$, $\nu_n = \nu \circ P_n^{-1}$. Then $E_n F = \Phi \circ P_n$, where $\Phi = d\nu_n/d\mu_n$. As noted in the proof of Lemma 5.4, $\nu_h' = \partial_h F\mu + F\beta_h\mu$. There exist Borel functions Q_1, Q_2 and Q_3 on H_n such that

$$E_n(\partial_h F) = Q_1 \circ P_n, \quad E_n(F\beta_h) = Q_2 \circ P_n \quad E_n(\beta_h) = Q_3 \circ P_n.$$

Then we have

$$(\nu_n)_a' = (\nu_h') \circ P_n^{-1} = Q_1\mu_n + Q_2\mu_n. \tag{5.3}$$

Let E be the orthogonal complement to a in H_n. Since μ_n and ν_n are differentiable along a, they admit absolutely continuous conditional densities φ^y and ψ^y on the lines $y + \mathbb{R}a$ for all $y \in E$. In addition, $\varphi^y > 0$. Therefore, the function $t \mapsto \psi^y(t)/\varphi^y(t) = \Phi(y + ta)$ is locally absolutely continuous for all $y \in E$. Passing to conditional measures in relationship (5.3) we get the equality

$$\frac{d}{dt}\psi^y(t) = Q_1(y + ta)\varphi^y(t) + Q_2(y + ta)\varphi^y(t) \quad \text{for } \mu_n\text{-a.e. } (t, y).$$

Note that $\psi^y(t) = \Phi(y + th)\varphi^y(t)$ for μ_n-a.e. (t, y). This gives

$$\frac{\partial}{\partial t}\Phi(y + ta) = -\Phi\frac{d}{dt}\varphi^y(t) + Q_1\varphi^y(t) + Q_2\varphi^y(t).$$

Since $Q_3(y+ta) = \frac{d\varphi^y(t)}{dt}/\varphi^y(t)$ for μ_n-almost all (t, y), we arrive at the equality $\partial_a\Phi = Q_1 + Q_2 - Q_3$ μ_n-a.e. In particular, we get that for μ-almost all x the function $t \mapsto E_n F(x + th)$ is locally absolutely continuous and its usual directional derivative $\frac{\partial E_n F}{\partial h} \equiv \partial_a \Phi \circ P_n$ coincides with the right hand side of (5.2) μ-a.e. Since by a basic property of conditional expectations, $E_n(\partial_h F) \in L^p(\mu)$, $E_n(F\beta_h) \in L^r(\mu)$, $E_n F\, E_n \beta_h \in L^r(\mu)$ for any $r < p$, we get the inclusion $\partial_a \Phi \circ P_n \in L^r(\mu)$. The integration by parts formula then implies that $\frac{\partial E_n F}{\partial h} \equiv \partial_a \Phi \circ P_n$ serves as the stochastic derivative $\partial_h F$. $\square$

Corollary 5.8 *Let $L_1 \subset X^*$ be a linear subspace and let H_2 be a closed subspace of H orthogonal to $H_1 := j_H(L_1)$.*

a) For any $p \in (1, \infty)$ and $f \in W^{p,1}(X, H, \mu)$ which is $\mathcal{F}_{L_1}$–measurable, $\nabla_{H_2} f = 0$ a.s.

b) If $A \in \mathrm{dom}(\delta_\mu)$ is H_2–valued then $E(\delta_\mu A \,|\, \mathcal{F}_{L_1}) = 0$ a.s.

In particular if L is a linear subspace of X^* with $H_0 = j_H(L)$ closed in H, and $P_0 A \in \mathrm{dom}(\delta_\mu)$, then

$$E(\delta_\mu(P_{H_0} A \,|\, \mathcal{F}_L)) = E(\delta_\mu A \,|\, \mathcal{F}_L). \tag{5.4}$$

Proof: a) follows directly from Lemma 5.4, since $l(x+h) \equiv l(x)$ for every $l \in L_1$, $h \in H_2$ by virtue of the equality $l(h) = (j_H(l), h)_H = 0$.

b) Let $Z \in L^\infty(\mathcal{F}_{L_1}, \mu) \cap \mathcal{S}(X)$. Then $EZ\delta_\mu A = -E(\nabla_H Z, A)_H = -E(\nabla_{H_2} Z, A)_H = 0$, the last two equalities following respectively from the fact that A is H_2–valued and from part a). Since

Z was arbitrary in $Z \in L^\infty(\mathcal{F}_{L_1},\mu) \cap \mathcal{S}(X)$, and the latter is dense in $L^\infty(\mathcal{F}_{L_1},\mu)$, it follows that $E\left(\delta_\mu A \,|\mathcal{F}_{L_1}\right) = 0$ a.s. By choosing $L_1 = L$ and $H_2 = H_0^\perp$ we obtain

$$E\left(\delta_\mu(P_{H_0}A)\,|\mathcal{F}_L\right) = E\left(\delta_\mu A\,|\mathcal{F}_L\right) - E\left(\delta_\mu(P_{H_0^\perp}A)\,|\mathcal{F}_L\right) = E\left(\delta_\mu A\,|\mathcal{F}_L\right).$$

□

We now turn to the divergence operator (see Definitions 4.1 for the meaning of H_n, P_n, $\mathcal{F}_n$, E_n, μ_n and $\beta^{(n)}$). The projections P_n act on a given vector field $A \in L^1(\mu, H)$ producing the finite dimensional vector fields $A_{k,n} = E_k P_n A$, $n \le k$. Since $A_{k,n}$ is $\mathcal{F}_k$–measurable, and its range is contained in H_n, there is a unique vector field $\widetilde{A}_{k,n} \in L^1(\mu_k, H_n)$ such that

$$A_{k,n} = \widetilde{A}_{k,n} \circ P_k.$$

Although $A_{k,n}$ and $\widetilde{A}_{k,n}$ represent in many ways the same object, the divergence operator distinguishes between them:

Proposition 5.9 *Let $A \in L^1(\mu, H)$ and, given $n, k \in \mathbb{N}$, $n \le k$, assume that μ is differentiable along $e_1, \ldots, e_n$ and $\delta_\mu P_n A$ exists. Then $\delta_{\mu_k}\widetilde{A}_{k,n}$ and $\delta_\mu A_{k,n}$ exist as well and*

$$\left(\delta_{\mu_k}\widetilde{A}_{k,n}\right) \circ P_k = E_k \delta_\mu(P_n A), \tag{5.5}$$

$$\delta_\mu A_{k,n} = E_k\left(\delta_\mu(P_n A)\right) + (\beta^{(n)} - E_k\beta^{(n)}, A_{k,n})_H. \tag{5.6}$$

If $n = k$ and $\delta_\mu A$ exists, then (5.5) and (5.6) remain true with A replacing $P_n A$ on the right.

Proof: The last claim is a direct consequence of (5.4) in Corollary 5.8. Let φ be a $C_b^1(H_k)$ function. Then

$$\begin{aligned}\int_{H_k}\left(\nabla_{H_k}\varphi, \widetilde{A}_{k,n}\right)_{H_k} d\mu_k &= \int_X \left((\nabla_{H_k}\varphi)\circ P_k, E_k P_n A\right)_{H_k} d\mu \\ &= \int_X \left(\nabla_H(\varphi \circ P_k), P_n A\right)_H d\mu = -\int_X (\varphi\circ P_k)\delta_\mu(P_n A)\, d\mu. \end{aligned}\tag{5.7}$$

Hence, $\delta_{\mu_k}\widetilde{A}_{k,n}$ exists and $-\int_X(\delta_{\mu_k}\widetilde{A}_{k,n})\circ P_k(\varphi\circ P_k)\,d\mu$ coincides with the left hand side of (5.7). Since any bounded $\mathcal{F}_n$–measurable function can be approximated by functions $\varphi \circ P_n$ of the above form, (5.5) follows from (5.7).

As for (5.6), first note that applying (5.2) to $h = e_1, \cdots, e_n$ we obtain the equality

$$\nabla_{H_n}E_k f = E_k\nabla_{H_n} f + E_k\left(f(\beta^{(n)} - E_k\beta^{(n)})\right). \tag{5.8}$$

Then, for any $f \in \mathcal{S}(X)$

$$\begin{aligned}\int_X(\nabla_H f, A_{k,n})_H d\mu &= \int_X (\nabla_{H_n} f, A_{k,n})_H d\mu = \int_X \left(E_k\nabla_{H_n} f, A_{k,n}\right)_H d\mu \\ &= \int_X \left(\nabla_{H_n}E_k f, A_{k,n}\right)_H d\mu - \int_X \left(E_k[f(\beta^{(n)} - E_k\beta^{(n)})], A_{k,n}\right)_H d\mu.\end{aligned}$$

Since $(\nabla_{H_n}E_k f, P_n A)_H = (\nabla_H E_k f, P_n A)_H$, the first term on the right hand side above equals

$$\int_X (\nabla_{H_n}E_k f, P_n A)_H d\mu = -\int_X E_k f \delta_\mu(P_n A) d\mu = -\int_X f E_k\delta_\mu(P_n A) d\mu$$

while the second one is given by $-\int_X f\left(\beta^{(n)} - E_k\beta^{(n)}, A_{k,n}\right)_H d\mu$. Collecting the terms we obtain the existence of $\delta_\mu A_{k,n}$ and the equality

$$\int_X f\delta_\mu A_{k,n} d\mu = \int_X f\left\{E_k\delta_\mu(P_n A) + \left(\beta^{(n)} - E_k\beta^{(n)}, A_{k,n}\right)_H\right\} d\mu$$

from which (5.6) follows since f was an arbitrary smooth function. □

Corollary 5.10 *Under the assumptions of the previous proposition, we have for all* $\lambda > 0$

$$\int_{H_k} \exp\{\lambda|\delta_{\mu_k}\widetilde{A}_{k,n}(x)|\}\,\mu_k(dx) \le \int_X \exp\{\lambda|\delta_\mu P_n A(x)|\}\,\mu(dx).$$

Proof: By Jensen's inequality

$$\int_{H_k} \exp\{\lambda|\delta_{\mu_k}\widetilde{A}_{k,n}(x)|\}\,\mu_k(dx) = \int_X \exp\{\lambda|E_k\delta_\mu P_n A(y)|\}\,\mu(dy) \le \int_X \exp\{\lambda|\delta_\mu P_n A(y)|\}\,\mu(dy).$$

□

5.2 Construction of the Flow

Lemma 5.11 *Let* μ *a Radon measure on a locally convex space* X, H *a separable Hilbert (or Banach space) embedded continuously into* X *and let* $A : X \to H$ *be a vector field such that there exists a sequences of vector fields* $A_n : X \to H$ *generating the flows* $\{U_t^n\}$ *with the following properties:*

(i) $\|A - A_n\|_{L^\alpha(\mu,H)} \to 0$ *for some* $\alpha > 1$;

(ii) There is a measurable mapping $(t,x) \in \mathbb{R}^1 \times X \to V_t(x) \in H$ *such that* $\int\limits_{-T}^{T} \int_X \|V_t(x) - U_t^n(x) + x\|_H\,\mu(dx)\,dt \to 0$ *for every* $T > 0$;

(iii) For all n, t *and* $p > 1$ *one has* $\mu \circ (U_t^n)^{-1} = r_t^n \mu$ *and*

$$\sup_n \sup_{-t\le s\le t} \|r_s^n\|_{L^p(\mu)} \le C(p,t).$$

Then, the mapping $U_t(x) = x + V_t(x)$ *on* $(\mathbb{R}^1 \times X, dt\otimes\mu)$ *admits an equivalent modification, denoted by the same letter, which is a flow solution of (1.1) with* $\sup\limits_{-t\le s\le t} \|r_s\|_{L^p(\mu)} \le C(p,t)$. *Moreover, for any sequence* $\{F_n \in L^d(\mu,H)\}$ *convergent to some map* F *in* $L^d(\mu,H)$, $d > 1$, *one has* $\|F_n \circ U_t^n - F \circ U_t\|_{L^p(dt\otimes\mu,H)} \to 0 \quad \forall p < d$.

Proof: Put $V_t^n(x) = U_t^n(x) - x$. Passing to a subsequence we may assume that $\|V_t(x) - V_t^n(x)\|_H \to 0$ for $\mu\otimes dt$-a.e. (t,x). Hence $U_t(x) - U_t^n(x) \to 0$ in H (hence, also in X) a.e. Let $\varphi \in C_b(X)$. We have for almost all t

$$\int\limits_X \varphi(U_t(x))\,\mu(dx) = \lim_{n\to\infty} \int\limits_X \varphi(U_t^n(x))\,\mu(dx) = \lim_{n\to\infty} \int\limits_X \varphi(x) r_t^n(x)\,\mu(dx).$$

The right hand side is majorized by $C(p,t)\|\varphi\|_q$. Hence, there is a function $r_t \in L^p(\mu)$ with $\|r_t\|_p \le C(p,t)$ such that

$$\int \varphi(U_t(x))\,\mu(dx) = \int \varphi(x) r_t(x)\,\mu(dx)$$

for all $\varphi \in L^q(\mu)$. It follows from the relationship above that $r_t^n \to r_t$ in the weak topology of $L^p(\mu)$ (for almost all t).

Let us prove that for any $p < \alpha$

$$\int\limits_0^t \int\limits_X \|A \circ U_s(x) - A_n \circ U_s^n(x)\|_H^p\,\mu(dx)\,ds \to 0 \tag{5.9}$$

Note that, letting $p_1 = \alpha/p$, $\frac{1}{p_1} + \frac{1}{q} = 1$, one has

$$\begin{aligned}
&\int_X \{\|A \circ U_s(x)\|_H^p + \|A_n \circ U_s^n(x)\|_H^p\}\, \mu(dx) \\
&\qquad \le \int_X \|A(x)\|_H^p r_s(x)\, \mu(dx) + \int_X \|A_n(x)\|_H^p r_s^n(x)\, \mu(dx) \\
&\qquad \le \|A\|_\alpha^p \|r_s\|_q + \|A_n\|_\alpha^p \|r_s^n\|_q \le \|A\|_p (\|r_s\|_q + \|r_s^n\|_q),
\end{aligned} \tag{5.10}$$

which is uniformly bounded on $[0,t]$. Thus, by the Lebesgue theorem it suffices to prove the convergence to zero of the inner integrals in (5.9) for every $s \in [0,t]$.

Fix s. Since μ is Radon and $\|A(x) - A_n(x)\|_H \to 0$ μ-a.e., for every $\varepsilon > 0$ there exists a compact set X_ε with $\mu(X \backslash X_\varepsilon) < \varepsilon$ such that A is uniformly continuous on this set and $\sup_{x\in X_\varepsilon} \|A(x) - A_n(x)\|_H \to 0$. There is a neighborhood of zero $W \subset X$ such that $\|A(x) - A(y)\|_H \le \varepsilon$ if $x, y \in X_\varepsilon$ and $x - y \in W$. Let us pick N_1 such that

$$\sup_{x\in X_\varepsilon} \|A(x) - A_n(x)\|_H \le \varepsilon \quad \forall n \ge N_1.$$

Since $\|U_s(x) - U_s^n(x)\|_H \to 0$ a.e., there exist a compact set Z_ε and an integer $N_2 \ge N_1$ such that $\mu(X \backslash Z_\varepsilon) \le \varepsilon$ and

$$U_s(x) - U_s^n(x) \in W \quad \forall x \in Z_\varepsilon\ \forall n \ge N_2.$$

Put $\Omega(n) = U_s^{-1}(X_\varepsilon) \cap (U_s^n)^{-1}(X_\varepsilon) \cap Z_\varepsilon$. Let $n \ge N_2$. Then

$$\int_{\Omega(n)} \|A(U_s(x)) - A_n(U_s^n(x))\|_H^p\, \mu(dx) \le \varepsilon,$$

since $U_s(x) \in X_\varepsilon$, $U_s^n(x) \in X_\varepsilon$ and $U_s(x) - U_s^n(x) \in W$ for $x \in \Omega(n)$. Let us estimate the integral over the complement of $\Omega(n)$. It is bounded above by the integrals over the sets $S_1 = X \backslash U_s^{-1}(X_\varepsilon)$, $S_2 = X \backslash (U_s^n)^{-1}(X_\varepsilon)$, $S_3 = X \backslash Z_\varepsilon$. Note that $\mu(S_3) \le \varepsilon$,

$$\mu(S_1) = \int_{X\backslash X_\varepsilon} r_s(x)\, \mu(dx) \le \sqrt{\varepsilon} \|r_s\|_2, \qquad \mu(S_2) = \int_{X\backslash X_\varepsilon} r_s^n(x)\, \mu(dx) \le \sqrt{\varepsilon} \|r_s^n\|_2.$$

It remains to note that the integrals of $\|A \circ U_s(x)\|_H^p$ and $\|A_n \circ U_s^n(x)\|_H^p$ over S_i, $i = 1, 2, 3$, can be estimated by $C(s)\varepsilon^a$, $a = \frac{p+\alpha-2}{p+\alpha}$. This follows from Hölder's inequality with powers $k = \frac{p+\alpha}{2}$ and $k' = \frac{k}{k-1}$ combined with the estimates (5.10).

Thus, for all $n \ge N_2$ the integral above is bounded by $\varepsilon + C(s)\varepsilon^a$, which gives the desired conclusion. Clearly, the same arguments apply to F_n and F as well, whence the last assertion of this lemma.

Therefore, for $\mu \otimes dt$-a.e. (t, x) we have $U_t(x) = x + \int_0^t A(U_s(x))\, ds$.

Denote the right hand side of the equality above by $\widetilde{U}$. We shall prove that for every t, $\widetilde{U}_t(x) = \lim_n \widetilde{U}_t^n(x)$ μ–a.e., which is enough to establish the flow property (1.2). To see this note that our modification $\widetilde{U}_t(x)$ is continuous in t for every x while on the other hand, there is a subsequence of $\{f_n(t) = V_t^n(x)\}$ that converges locally uniformly to a continuous limit for μ-a.e. x. Indeed, it converges for almost all t. In addition, for every time interval J and for any $p \in (1, \alpha)$ $\liminf_n \int_J \|f_n'(t)\|_H^p\, dt < \infty$ μ-a.e., by Fatou's lemma, since

$$\int_J \int_X \|A \circ U_t^n(x)\|_H^p\, \mu(dx)\, dt = \int_J \int_X \|A(x)\|_H^p r_t^n(x)\, \mu(dx)\, dt,$$

which, by virtue of Hölder's inequality and the uniform bound for $\int_J \|r_t^n\|_\gamma dt$, is itself estimated by $const\|A\|_\alpha$. Thus, for a suitable subsequence (denoted for simplicity by the same indices)

$$\|f_n(t) - f_n(s)\|_H \leq \int_s^t \|f_n'(\tau)\|_H \, d\tau \leq (t-s)^{\frac{1}{q}} \|f_n'\|_{L^p(J)} \leq const(t-s)^{\frac{1}{q}},$$

which implies the locally uniform convergence of such a subsequence.

Finally, it follows from the arguments above, that all the transformations U_t are absolutely continuous with the required estimates for $\|r_t\|_p$. □

We now intend to construct approximating fields A_n which satisfy the conditions of Lemma 5.11. We shall consider the more difficult case where the condition **A3)** is fulfilled. It would be nice to take for A_n the fields $E_n P_n A$ (this works perfectly in the case where condition **A3')** is fulfilled with $k_n = m_n = n$), but the extra term which appears in the expression for $\delta_\mu(E_n P_n A)$ in Proposition 5.9 may destroy the uniform exponential integrability. For this reason we shall use a modified sequence $\{A_n\}$ of the form $A_n = E_{k(n)}(G_{j(n)} \circ P_n A)$ with suitably chosen $j(n)$ and $k(n)$ and G_j of the same type as in the proof of Theorem 2.6. It follows from our assumptions that $\nabla(P_n A) = P_n \nabla A$,

$$\delta_\mu(P_n A) = trace(\nabla(P_n A)) + (P_n A, \beta^{(n)})_H \quad \text{where} \quad \beta^{(n)} = \sum_{i=1}^n \beta_{e_i} e_i,$$

and that for every $c > 0$

$$\int_X \exp\Big(c|trace\nabla(P_n A)(x)|\Big)\,\mu(dx) \leq \int_X \exp\Big(cn\|\nabla(P_n A)(x)\|\Big)\,\mu(dx)$$
$$\leq \int_X \exp\Big(cn\|\nabla A(x)\|\Big)\,\mu(dx)$$

whence by virtue of **A3)**

$$\int_X \exp\Big(c\Big|\Big(P_n A(x), \beta^{(n)}(x)\Big)_H\Big|\Big)\,\mu(dx) < \infty.$$

Therefore, for every fixed n, taking the sequence of mappings G_j on H_n defined as in the proof of Theorem 2.6 ($G_j(x) = g_j(x)x/\|x\|$) and putting $B_j(x) = G_j(P_n A(x))$, we get for sufficiently big j

$$\int_X \exp\Big(n\|\nabla B_j(x)\|\Big)\,\mu(dx) \leq \int_X \exp\Big((n+1)\|\nabla P_n A(x)\|\Big)\,\mu(dx), \tag{5.11}$$

$$\int_X \exp\Big(n|\delta_\mu B_j(x)|\Big)\,\mu(dx) \leq \int_X \exp\Big((n+1)|\delta_\mu P_n A(x)|\Big)\,\mu(dx). \tag{5.12}$$

Indeed, $B_j(x) \to P_n A(x)$ and $\nabla B_j(x) \to \nabla(P_n A)(x)$ pointwise. In addition,

$$\delta_\mu B_j = trace\nabla(G_j \circ P_n A) + (G_j \circ P_n A, \beta^{(n)})_H,$$

$$|trace\nabla(G_j \circ P_n A)| \leq n\|\nabla(G_j \circ P_n A)\| \leq n\|\nabla(P_n A)\|,$$

$$|(G_j \circ P_n A, \beta^{(n)})_H| \leq |(P_n A, \beta^{(n)})_H|.$$

Hence, we arrive at (5.11), (5.12) by virtue of the exponential estimates above and Lebesgue's theorem.

Our next step is to fix such j and approximate $B := G_j \circ (P_n A)$ by a cylindrical field $A_n = E_k B$ with some $k = k(n) \geq n$ so that (5.11) and (5.12) are preserved with B_j replaced by A_n and $n+1$ replaced by $n+2$. This is indeed possible as one can see from (5.6) (note that $P_k B = B$). As $k \to \infty$ the terms $(\beta^{(n)} - E_k\beta^{(n)}, B)_H$ converge to zero by the martingale convergence. In addition, letting $M = \sup \|B(x)\|_H$, one has

$$\int_X \exp\Big\{ c\Big|\Big(\beta^{(n)}(x) - E_k\beta^{(n)}(x), B(x)\Big)_H\Big|\Big\}\,\mu(dx)$$

$$\leq \int_X \exp\Big\{ cM\|\beta^{(n)}(x)\|_H + cM\|E_k\beta^{(n)}(x)\|_H \Big\}\,\mu(dx)$$

$$\leq \sqrt{\int_X \exp\Big\{ 2cM\|\beta^{(n)}(x)\|_H \Big\}\,\mu(dx)}\sqrt{\int_X \exp\Big\{ 2cM\|E_k\beta^{(n)}(x)\|_H \Big\}\,\mu(dx)}$$

$$\leq \int_X \exp\Big\{ 2cM\|\beta^{(n)}(x)\|_H \Big\}\,\mu(dx),$$

which is finite by virtue of (4.1). Hence,

$$\int_X \exp\Big\{ c\Big|\Big(\beta^{(n)}(x) - E_k\beta^{(n)}(x), B(x)\Big)_H\Big|\Big\}\,\mu(dx) \to 1.$$

$\square$

Lemma 5.12 *The fields A_k constructed above generate global flows $\{U_t^k\}$ under which measure μ is quasi-invariant with the Radon–Nikodym densities*

$$r_t^k(x) := \frac{d(\mu \circ (U_t^k)^{-1})}{d\mu}(x) = \exp\Big\{ -\int_0^t \delta A_k(U_{-s}^k(x))\,ds \Big\}. \tag{5.13}$$

Proof: To simplify notation we shall assume that n is the same as $k(n)$ above in $E_k(G_j(P_n A))$, which is possible, since $k(n) \geq n$ (otherwise we could consider a suitable subsequence of indices). Let Y_k be a closed subspace in X complementary to H_k. Denote by ν the image of μ under the natural projection on Y_k. Recall that μ has differentiable conditional measures $\mu(\,\cdot\,|y)$ on the subspaces $y + H_k$, $y \in Y_k$. Moreover, these conditional measures a.s. satisfy the conditions of Theorem 2.6 (as well as the restrictions of the maps A_k to $H_k + x$). Indeed, for every μ-integrable function F its restrictions on $y + H_k$ are $\mu(\,\cdot\,|y)$-integrable for ν-a.e. y, and

$$\int_X F(x)\,\mu(dx) = \int_{Y_k}\int_{y+H_k} F(x)\,\mu(dx|y)\,\nu(dy).$$

On the other hand, for ν-a.e. y the function $\delta_{\mu(\cdot|y)}A_k$ coincides with the restriction of $\delta_\mu A_k$ to $H_k + y$. This follows from the explicit formula

$$\delta_\mu A_k = trace \nabla A_k + (A_k, \beta^{(k)})_H$$

combined with the already mentioned fact that for ν-a.e. y the logarithmic derivative $\beta_h^{\mu(\cdot|y)}$ is given by the restriction of β_h to $H_k + y$.

Note that according to Corollary 5.10 and the exponential estimate of ∇A_k, the field A_k satisfies the conditions of Theorem 2.6 also on H_k with the measures μ_k (μ's projection on H_k).

It may be thus concluded by Theorem 2.6 that for every $k \in \mathbb{N}$ there exists a μ_k–quasi-invariant flow $\widetilde{U}_t^k : H_k \longrightarrow H_k$ which satisfies for μ_k–a.e. $x \in H_k$ the equation

$$\widetilde{U}_t^k(x) = x + \int_0^t A_k(\widetilde{U}_s^k(x))\, ds \qquad \forall t \in \mathbb{R}. \tag{5.14}$$

Now extend these flows to all of X by setting

$$U_t^k x = \widetilde{U}_t^k(P_k x) + (x - P_k x) \qquad x \in X, t \in \mathbb{R}. \tag{5.15}$$

In fact, U^k is repeating the action of $\widetilde{U}^k$ in every k–dimensional plane parallel to H_k. Moreover

$$\begin{aligned} U_t^k(x) &= P_k x + \int_0^t A_k\left(\widetilde{U}_s^k(P_k x)\right) ds + (x - P_k x) \\ &= x + \int_0^t A_k\left(\widetilde{U}_s^k(P_k x) + x - P_k x\right) ds = x + \int_0^t A_k\left(U_s^k(x)\right) ds, \end{aligned} \tag{5.16}$$

so that $\{U_t^k\}$ is the flow generated by A_k.

It is important that for ν-a.e. y this is exactly the flow $\{U_t^{k,y}\}$ on $H_k + y$ which arises when Theorem 2.6 is applied to the measure $\mu(\,\cdot\,|y)$. It has already been explained that $\mu(\,\cdot\,|y)$ and the restriction of A_k to $H_k + y$ satisfy the conditions of Theorem 2.6 (strictly speaking, since $H_k + y$ is not a linear space, we should talk about the flow lifted to $H_k + y$ from H_k equipped with the translate of the measure $\mu(\,\cdot\,|y)$). The coincidence of these two flows (one associated with μ_k, another with $\mu(\,\cdot\,|y)$) is obvious from the construction in Theorem 2.6 which gives both flows by one and the same procedure (this can be also deduced from the uniqueness results).

It is clear from the above that ν-a.e. conditional measures $\mu(\,\cdot\,|y)$ are quasi-invariant under $\{U_t^k\}$. Hence, μ is quasi-invariant under these flows. The expression (5.13) follows from the finite dimensional result applied to the conditional measures.

□

Our aim is now to show that U^k converges as $k \to \infty$ to a limiting flow.

Lemma 5.13 *With a suitable choice of $j = j(n)$ and $k = k(n)$ above one has*

$$\lim_{n\to\infty} \|A - A_n\|_\alpha = 0,$$

and for any $p \in [1, \alpha)$

$$\lim_{n\to\infty} \int \|\nabla A(x)(h) - \nabla A_n(x)(h)\|_H^p\, \mu(dx) = 0 \qquad \forall h \in H.$$

Below by $\{A_n\}$ we mean the sequence constructed in this lemma.

Proof: Clearly, $\|A - P_n A\|_\alpha \to 0$. By construction, for a fixed n we have $\|B_j - P_n A\|_\alpha \to 0$ as $j \to \infty$, where $B_j = G_j \circ P_n A$. By the vector martingale convergence (see [43, Section II.4.2]) we have:

$$\lim_{n\to\infty} \|E_k B - B\|_\alpha = 0.$$

This provides the desired convergence for A_n. A similar argument works for ∇A_n. Indeed, $\|\partial_h(P_n A) - \partial_h A\|_p \to 0$ by the Lebesgue theorem. Clearly, $\|\partial_h(G_j \circ P_n A) - \partial_h(P_n A)\|_p \to 0$ as $j \to \infty$. For any fixed j and h, we get from (5.2)

$$\|\partial_h(E_k B_j) - \partial_h B_j\|_p \leq \|E_k(\partial_h B_j) - \partial_h B_j\|_p + \|E_k(\beta_h)B_j - E_k\beta_h E_k B_j\|_p,$$

which tends to zero as $k \to \infty$ by the martingale convergence combined with the estimate $\|\beta_h B_j - \beta_h E_k B_j\|_p \leq \|E_k B_j - B_j\|_\alpha \|\beta_h\|_q$, $\quad (q = \frac{\alpha p}{\alpha - p})$. □

Lemma 5.14 *Let $B \in G_{\alpha,1}(\mu, H_k)$ be a vector field of the form $B(x) = B_0(P_k x)$, where $B_0 : H_k \to H_k$. Assume that for all $c > 0$*

$$\exp(c\|\nabla B\|) + \exp(c|\delta_\mu B|) \in L^1(\mu).$$

Then all the assertions and estimates of Theorem 4.3 hold true. If, in addition, C is another field of the same type, then for the corresponding flows one has for each $T > 0$ and $p < \alpha$

$$\sup_{-T\le t\le T} \|U_t^B - U_t^C\|_{L^p(\mu,H)} \le \widetilde{C}(p,T)\|B - C\|_\alpha,$$

where the constant $C(p,T)$ depends only on p, T, α, and the exponential moments of $\|\nabla B\|$, $\|\nabla C\|$, $|\delta_\mu B|$, $|\delta_\mu C|$ up to some order d, depending only on α, p, and T. Finally, if $\exp(c|\delta_\mu B|) + \exp(c|\delta_\mu C|) \in L^1(\mu)$ for some $c > 0$, then the assertions above hold for some $T > 0$ as in Theorem 4.3.

Proof: Let us fix a time interval $[-T, T]$. Put $L := H_k$. Let Y be a closed subspace in X complementary to L. Let us, as above, denote by ν the projection of μ on Y and choose differentiable conditional measures $\mu(\,\cdot\,|y)$ on $L + y$, $y \in Y$. As we have already explained, there is a flow $\{U_t\}$ acting in $L + y$, $y \in Y$, such that the images of these conditional measures under $\{U_t\}$ have the densities r_t^y with respect to the measures themselves. In addition, the finite dimensional result implies

$$\|r_t^y\|_{L^p(\mu(\cdot|y))} \le 4\exp\Big\{\int_{L+y} \exp(c|\delta_\mu A(x)|)\,\mu(dx|y)\Big\}. \tag{5.17}$$

However, the right hand side of (5.17) need not be ν-integrable. For this reason, in order to get an estimate of $\|r_t\|_p$, we shall use the following trick. For every N let us define the measure μ^N on X by

$$\mu^N(E) = \int_{Y_N} \mu(E|y)\,\nu(dy),$$

where

$$Y_N = \Big\{y \in Y : \int_{L+y} \exp\Big(c|\delta_\mu A(x)|\Big)\,\mu(dx|y) \le N\Big\}.$$

For $y \in Y_N$ the measure μ^N has the same conditional measures on $L + y$ as μ; for other y it has zero conditional measures. Therefore, it is differentiable along L (see [12], Theorem 2.5, or [7], Chapter 2) and for $y \in Y_N$ the logarithmic derivatives of $\mu^N(\,\cdot\,|y)$ coincide with those of $\mu(\,\cdot\,|y)$. Obviously, μ^N is quasi-invariant under $\{U_t\}$. Put $\psi_t^N = d(\mu^N \circ U_t^{-1})/d\mu^N$. Clearly, $\psi_t^N = r_t$ on $L + y$ if $y \in Y_N$. Since the right hand side of (5.17) is integrable over Y_N we get $\|\psi_t^N\|_{L^p(\mu^N)} \le 4e^N$. By Lemma 2.2 this leads to the uniform (in N) estimate of $\|r_t^N\|_{L^p(\mu^N)}$, $t \in [-T, T]$. Indeed,

$$\|\exp(c|\delta_{\mu^N} B|)\|_{L^p(\mu^N)} \le \|\exp(c|\delta_\mu B|)\|_{L^p(\mu)},$$

since $\delta_{\mu^N} B$ coincides with $\delta_\mu B$ on $L + y$ for $y \in Y_N$. Noting that

$$\|\psi_t^N\|_{L^p(\mu^N)} = \int_{Y_N}\int_{L+y} r_t(x)^p\,\mu(dx|y)\,\nu(dy),$$

we conclude by the monotone convergence theorem that $\|r_t\|_p$ are uniformly bounded on $[-T, T]$. Applying Lemma 2.2 once again we obtain an upper bound. By the finite dimensional case, ∇U_t exists and

$$\nabla U_t(x) = I + \int_0^t \nabla B(U_s(x))\nabla U_s(x)\,ds.$$

Hence, the same reasoning as in Lemma 2.17 leads to the desired estimates of $\|\nabla U_t\|_p$. This concerns the estimates of $\|U_t^B - U_t^C\|_p$ as well. The last claim of this lemma is clear from the proof. $\square$

Lemma 5.15 *For any $p < \alpha$ and $T > 0$ the sequence $V_t^n(x) := U_t^n(x) - x$ converges in $L^p(\mu, H)$ uniformly in $t \in [-T, T]$. In addition,*

$$\int \|U_t^n(x) - U_t^m(x)\|_H^p \, \mu(dx) \leq \eta(p,T) \|A_n - A_m\|_\alpha^p. \tag{5.18}$$

Proof: This follows from Lemma 5.14 and Lemma 5.13 combined with (5.11) and (5.12). $\square$

Denote the limit of $\{V_t^k\}$ by V_t and define

$$U_t(x) = x + V_t(x). \tag{5.19}$$

¿From Lemma 5.11 we get

Corollary 5.16 *For any $t > 0$ one has:*

$$\lim_{n\to\infty} \int_{-t}^{t} \int_X \|A_n \circ U_s^n(x) - A \circ U_s(x)\|_H \, \mu(dx) \, ds = 0.$$

Lemma 5.17 *(i) $U_t - I = V_t \in G_{\theta,1}(H, \mu, H)$ for every $\theta < \alpha$ and (4.8) is satisfied;*

(ii) For every $\psi \in S(X)$, letting $\nabla U_t := I + \nabla V_t$, one has

$$\nabla(\psi \circ U_t)(x) = \nabla U_t(x)^* \nabla \psi(U_t(x)); \tag{5.20}$$

(iii) For all t and μ-a.e. x the operators $\nabla U_t^n(x)$ converge to $\nabla U_t(x)$ in the strong operator topology and

$$\nabla U_t(x) = I + \int_0^t \nabla A(U_s(x)) \nabla U_s(x) \, ds. \tag{5.21}$$

(iv) Equality (4.9) holds true.

Proof: According to the finite dimensional case the operator-valued mappings $\nabla U_t^n(x)$ exist, satisfy the uniform estimates of the type (4.8) and solve the equations

$$\nabla U_t^n(x) = I + \int_0^t \nabla A_n(U_s^n(x)) \nabla U_s^n(x) \, ds.$$

By Lemma 5.11 and Lemma 5.13 for every $h \in H$, we get the convergence

$$\nabla A_k(U_s^k(\cdot))(h) \to \nabla A(U_s(\cdot))(h) \qquad \text{in } L^p(dt \otimes \mu, H) \quad \forall p < \alpha.$$

For a subsequence (denoted by the same indices) one has a.s. convergence. Lemma 2.3 implies that for μ–a.e. x the sequence $\{\nabla U_t^n(x)(h)\}$ converges uniformly on bounded intervals to the unique solution $\{\nabla U_t(x)(h)\}$ of the equation obtained from (5.21) by applying both sides to h. Since $\sup_n \|\nabla U_t^n(\cdot)(h)\|_{L^p(\mu,H)} < \infty$ for every t, we have also the convergence $\nabla U_t^n(\cdot)(h) \to \nabla U_t(\cdot)(h)$ in all $L^p(\mu, H)$. Hence, Lemma 5.3 applies. The case of $W^{\alpha,1}(H, \mu, H)$ is similar.

Now (5.20) follows from Corollary 5.6 and the estimate (4.8) results by passing to the limit in the corresponding estimates for A_k (it also follows from the integral equation (5.21) in the same manner as in Lemma 2.17).

To prove (4.9) first note that

$$\int (\nabla\psi(x), A(x))_H\, \mu(dx) = \lim_{n\to\infty} \int (\nabla\psi(x), A_n(x))_H\, \mu(dx)$$
$$= -\lim_{n\to\infty} \int \psi(x)\delta_\mu A_n(x)\, \mu(dx) \qquad \forall\psi \in \mathcal{S}(X),$$

which together with the uniform exponential integrability of $\delta_\mu A_n$ implies the weak convergence of $\delta_\mu A_n$ to $\delta_\mu A$ in all $L^p(\mu)$. Since $\{\psi \circ U_t^n\}$ converges to $\psi \circ U_t$ in all $L^p(\mu)$ (as it follows from the results above) one gets

$$\int \psi(U_t(x))\delta_\mu A(x)\, \mu(dx) = \lim_{n\to\infty} \int \psi(U_t^n(x))\delta_\mu A_n(x)\, \mu(dx).$$

The right hand side can be rewritten as

$$-\lim_{n\to\infty} \int \Big(\nabla(\psi \circ U_t^n)(x), A_n(x)\Big)_H \mu(dx) = -\lim_{n\to\infty} \int \Big(\nabla\psi(U_t^n(x)), A_n \circ U_t^n(x)\Big)_H \mu(dx)$$
$$= -\int \Big(\nabla\psi(U_t(x)), A \circ U_t(x)\Big)_H \mu(dx) = -\int \frac{d}{dt}\psi \circ U_t(x)\, \mu(dx) \qquad .$$

It is easy to see that one can interchange integration and differentiation on the right hand side of the equality above (note that $\psi(U_t(x))$ is absolutely continuous in t for almost all x). □

Remark 5.18 *The influence of Cruzeiro's work [17] for Wiener space has already been mentioned in the Introduction. However, at this point, we have chosen a different method of constructing the flow (not only because in the Gaussian case it yields a considerably shorter proof): the reader might have noticed that the approximations above actually converge in H and that the topology of X is in fact irrelevant (other than being weaker than H's). In particular, note that in the Gaussian case the linear functionals l_n need not be continuous in X's topology.*

At first glance this feature seems to be present in Cruzeiro's scheme (c.f. [17, Corollary 3.2.3]). However, due to a small oversight at the beginning of the proof of [17, Lemma 3.2.2], the last term of the equality

$$\widetilde{U}_t^{A_{n+p}}(x) - \widetilde{U}_t^{A_n}(x) = U_t^{A_{n+p}}(P_{n+p}x) - U_t^{A_n}(P_n x) + (P_n - P_{n+p})x$$

is missing, and it is precisely this term which converges in X (so that while the validity of the Corollary is indeed affected, that of the Lemma as stated — and of the remainder of the construction in [17] — is not) but not in H.

5.3 The density evolution

The expression (4.10) can actually be deduced from [5], [24] or [38]. However, it is easier to derive it directly using Bell's method used in those papers than to verify the validity of the corresponding conditions.

Proof of Theorem 4.4(i)

Fix $\varphi \in \mathcal{S}(X)$. According to equality (4.9)

$$\begin{aligned}
\int_X \varphi(x)r_t(x)\, \mu(dx) &= \int_X \varphi(U_t(x))\, \mu(dx) = \int_X \left(\varphi(x) - \int_0^t \varphi(U_s(x))\delta_\mu A(x)\, ds\right) \mu(dx) \\
&= \int_X \varphi(x)\, \mu(dx) - \int_0^t \int_X \varphi(x)\delta_\mu A(U_{-s}(x)) r_s(x)\, \mu(dx)\, ds \\
&= \int_X \varphi(x) \left(1 - \int_0^t \delta_\mu A(U_{-s}(x)) r_s(x)\, ds\right) \mu(dx). \qquad (5.22)
\end{aligned}$$

Since φ was arbitrary in $\mathcal{S}(X)$, it may be concluded that $\forall t \in \mathbb{R}$

$$r_t(x) = 1 - \int_0^t \delta_\mu A(U_{-s}(x)) r_s(x)\, ds \tag{5.23}$$

holds μ–a.e. Let $\tilde{r}_t$ be the modification of r_t given by the right hand side of (5.23). It follows from (5.22) that this modification also satisfies the integral equation (5.23), from which (4.10) follows.

□

5.4 Uniqueness

Proof of Theorem 4.4(ii)

Let us observe that similarly to the finite dimensional case we have the following equality which holds for almost all t and x:

$$A(U_t(x)) = \nabla U_t(x) A(x). \tag{5.24}$$

Indeed, by the finite dimensional case $A_n(U_t^n(x)) = \nabla U_t^n(x) A_n(x)$. As we have seen above (see Corollary 5.16), for a suitably chosen subsequence, the left hand side of the latter converges a.s. to $A(U_t(x))$, while the right hand side converges to $\nabla U_t(x) A(x)$. Indeed, $\nabla U_t^n(x) \to \nabla U_t(x)$ in the strong operator topology and $A_n(x) \to A(x)$ μ-a.e. Thus, we can use the following elementary fact: if operators T_i converge to T pointwise, and vectors v_i converge to v in norm, then $T_i v_i \to Tv$, as can be readily seen from the Banach–Steinhaus theorem.

Let $\{W_t(x)\}$ be another family of solutions such that $\mu \circ W_t^{-1} = \psi_t \mu$ and $\|\psi_t\|_r$ is locally integrable for all $r > 1$. Again it suffices to check that for every t

$$F(t,x) := U_t \circ W_{-t}(x) - x = 0 \quad \mu\text{-a.e.}$$

Clearly, by the continuity in t, it suffices to check that for μ-a.e. x this equality holds for all t. Using that W_{-t} preserves measure equivalence we get from (5.24)

$$A(U_t \circ W_{-t}(x)) = \nabla U_t(W_{-t}(x)) A(W_{-t}(x)) \qquad \mu\text{-a.e.}$$

Since $F(t,x) = \lim_n (U_t^n \circ W_{-t}(x) - x)$ μ-a.e. for all t, it remains to note that for every time interval J and $p = \frac{1+\alpha}{2}$, $\gamma \in (1, \frac{2\alpha}{1+\alpha})$, one has

$$\begin{aligned}
&\int_J \int_X \left\| \frac{\partial}{\partial t} U_t^n \circ W_{-t}(x) \right\|_H^p \mu(dx)\, dt \\
&\quad = \int_J \int_X \left\| A_n \circ U_t^n \circ W_{-t}(x) - \nabla U_t^n(W_{-t}(x)) A_n(W_{-t}(x)) \right\|_H^p \mu(dx)\, dt \\
&\quad \le \int_J \int_X \left\| A_n \circ U_t^n \circ W_{-t}(x) - A \circ U_t \circ W_{-t}(x) \right\|_H^p \mu(dx)\, dt \\
&\qquad\quad + \int_J \int_X \left\| \nabla U_t^n(W_{-t}(x)) A_n(W_{-t}(x)) - \nabla U_t(W_{-t}(x)) A(W_{-t}(x)) \right\|_H^p \mu(dx)\, dt \\
&\quad = \int_J \int_X \left\| A_n \circ U_t^n(x) - A \circ U_t(x) \right\|_H^p \psi_{-t}(x)\, \mu(dx)\, dt \\
&\qquad\quad + \int_J \int_X \left\| \nabla U_t^n(x) A_n(x) - \nabla U_t(x) A(x) \right\|_H^p \psi_{-t}(x)\, \mu(dx)\, dt
\end{aligned}$$

$$\leq \|A_n\circ U_t^n - A\circ U_t\|_{\gamma p}^p \sup_{t\in J}\|\psi_{-t}\|_{\gamma'}$$
$$+\sup_{t\in J}\|\psi_{-t}\|_{\gamma'}\int_J\Big(\int_X\Big\|\nabla U_t^n(x)A_n(x)-\nabla U_t(x)A(x)\Big\|_H^{\gamma p}\mu(dx)\,dt\Big)^{1/\gamma},$$

which tends to zero since the sequence $\|\nabla U_t^n(x)A_n(x)-\nabla U_t(x)A(x)\|_H^{\gamma p}$ converges to zero a.e. and is uniformly integrable. □

6 Flows generated by non-Cameron–Martin space valued fields

We shall now discuss a more general class of vector fields with values not necessarily in the Cameron–Martin space. Such a situation has already been studied in several works on Wiener processes on loop-spaces (see [21], [22], [26], [28]). Related problems arise when general absolutely continuous transformations of Gaussian measures are investigated (see [40]–[42]).

Let X be a locally convex space equipped with a centered Gaussian measure γ. Such a space will be called below "a Gaussian space". Denote by $H := H(\gamma)$ the Cameron–Martin space of X associated with γ and by X^*_γ the closure of X^* in $L^2(\gamma)$ (which is known to coincide with the space of all γ-measurable linear functionals on X, see [9], [13]). We say that two Gaussian spaces (X_1,γ_1) and (X_2,γ_2) are linearly isomorphic if there is a γ_1-measurable linear mapping $J: X_1\to X_2$ such that $\gamma_1\circ J^{-1}=\gamma_2$ and $J: H(\gamma_1)\to H(\gamma_2)$ is an isometry of Hilbert spaces. In this case there is an analogous map from X_2 to X_1 (see [9], Chapter 1). It is also known that any two Gaussian spaces with infinite dimensional Cameron–Martin spaces are linearly isomorphic in this sense (see [9], Chapter 1, Theorem 5.3 and Theorem 5.6). Recall that each operator $T\in L(H)$ admits a unique (up to an equivalent modification) extension to measurable linear (in the usual algebraic sense) mapping $\widetilde{T}: X\to X$ (see, e.g., [9], Chapter 1, Theorem 5.6). Such an extension satisfies the condition $\widetilde{T}x\in H$ a.e. if and only if $T\in \mathcal{HS}(H)$.

As above, for any orthonormal basis $\{e_n\}$ in H we denote by l_n the measurable linear functional associated with e_n and put $P_n x=\sum_{i=1}^n l_i(x)e_i$, just as in Definition 4.1.

Let $A: X\to X$ be a Borel (or just γ-measurable) vector field on X. We say that a flow of γ-measurable transformations $\{U_t\}$ of X solves the equation

$$U_t(x)=x+\int_0^t A(U_s(x))\,ds, \tag{6.1}$$

if there exist a Gassian space (Y,ν) linearly isomorhic to (X,γ) by means of an isomorphism J and a linear subspace $\Lambda\subset Y^*$ dense with respect to the weak-* topology such that, letting $A^Y := JAJ^{-1}$, $U_t^Y := JU_tJ^{-1}$, for ν-a.e. x one has

$$\langle l, U_t^Y(x)\rangle = \langle l,x\rangle + \int_0^t \langle l, A^Y(U_s^Y(x))\rangle\,ds \quad \forall l\in\Lambda,\quad \forall t,$$

where the integral in the right hand side is to be understood in the Lebesgue sense and $\langle\cdot,\cdot\rangle$ is the duality pairing between Y^* and Y.

Proposition 6.1 *(i) Let $L\in L(H)$ be such that $L+L^*\in \mathcal{HS}(H)$. Let $T_t:=\exp(tL)$. Then the flow $\{\widetilde{T}_t\}$ is a solution to (6.1) with $A=\widetilde{L}$ such that γ is quasiinvariant under this flow.*

(ii) Let $\{T_t\}$ be a strongly continuous semigroup on H commuting with the semigroup $\{T_t^*\}$. Assume that the generator G of the semigroup $\{T_tT_t^*\}$ is an element of $\mathcal{HS}(H)$. Then the previous assertion holds true with $A=\widetilde{G}$. In addition, in this case the inclusion $G\in \mathcal{HS}(H)$ is also a necessary condition for the quasiinvariance of γ under $\{\widetilde{T}_t\}$.

In particular, this assertion holds true if L is antisymmetric, that is, $L^*=-L$.

Proof: According to Theorem 1.6 in [11], there exists a Hilbert space E such that: (a) H is embedded densely into E by a Hilbert–Schmidt map; (b) the semigroup $\{T_t\}$ extends to a strongly continuous semigroup $\{T_t^E\}$ on E; (c) the domain D_E of generator of $\{T_t^E\}$ on E has full measure with respect to the image of the standard cylindrical Gaussian measure on H (this image, denoted by γ_0, is a Radon Gaussian measure on E). Since the Gaussian space (X,γ) is linearly isomorphic to (E,γ_0), we may assume from the very beginning that $X = E$ and $\gamma = \gamma_0$. In this case, equation (6.1) is trivially satisfied for all $l \in E^*$ and all $x \in D_E$. Thus, it remains to check the absolute continuity of the flow. According to a well-known result (see, e.g., [9], Chapter IV, Theorem 1.2), this is equivalent to the statement that $T_tT_t^* - I \in \mathcal{HS}(H)$. In case (i), the derivative of $T_tT_t^* - I$ in t equals $LT_tT_t^* + T_tL^*T_t^* = T_t(L+L^*)T_t^*$, whence $T_tT_t^* - I = \int_0^t T_s(L+L^*)T_s^*\,ds$, which is a Hilbert–Schmidt operator, since the Hilbert–Schmidt norm of $T_s(L+L^*)T_s^*$ is majorized by $\|L+L^*\|_{\mathcal{HS}}\exp(2s\|L\|_{\mathcal{L}(H)})$. In case (ii), we have $T_tT_t^* = \exp(tG)$, where $G \in \mathcal{HS}(H)$. Hence $\exp(tG) - I \in \mathcal{HS}(H)$. Note that if $\{T_t\}$ and $\{T_t^*\}$ commute, then $\{T_tT_t^*\}$ is a strongly continuous semigroup consisting of self-adjoint operators. Hence, its generator G is self-adjoint and nonpositive. Then by the spectral theorem, the condition $\exp(tG) - I \in \mathcal{HS}(H)$ implies that $G \in \mathcal{HS}(H)$. Indeed, we conclude that G is bounded and has the spectrum consisting of zero and a sequence $\lambda_n \to 0$. Then $\sum_n \lambda_n^2 < \infty$. $\square$

In [34] the following situation was investigated. Let (X, H, μ) be an abstract Wiener space. Then the Sobolev classes $\widetilde{W}^{p,r}(H,\mu,H)$ are defined in [34] by means of the operator norms on the gradients (if H in our setup is the Cameron–Martin space of μ, then $\widetilde{W}^{p,r}(H,\mu,H) = G_{p,r}(H,\mu,H)$). Let $\{Q_t\}_{t\in\mathbb{R}}$ be a strongly continuous group of bounded operators on X such that $Q_t|_H$ are orthogonal operators.

Assuming that $A \in \bigcap_{p>1}\widetilde{W}^{p,2}(H,\mu,H)$ has exponentially integrable $\|A\|_H$, $\|\nabla A\|_{\mathcal{L}(H)}$ and $|\delta_\mu A|$, it is proved in [34] that there is an absolutely continuous flow $\{U_t\}$ solving (μ-a.e.) the equation

$$U_t(x) = Q_t(x) + \int_0^t Q_{t-s}A(U_s(x))\,ds.$$

If $\{Q_t\}$ has generator L defined μ-a.e., then this equation corresponds to the ordinary differential equation $x'(t) = A(x(t)) + Lx(t)$, $x(0) = x$. Our approach enables us to get the following more general statement, where Q_t are not supposed to be continuous on X, only the one-fold differentiability of A is required and the exponential integrability of A is dropped.

Theorem 6.2 *Let μ be a centered Radon Gaussian measure on a locally convex space X, H its Cameron–Martin space, $\{Q_t\}_{t\in\mathbb{R}}$ a strongly continuous group of orthogonal operators on H. Assume that $A \in G_{\alpha,1}(H,\mu,H)$, where $\alpha > 1$, is such that $\exp(\|\nabla A\|_{\mathcal{L}(H)}) \in \bigcap_{p>1} L^p(\mu)$ and that $\delta_\mu A$ exists and $\exp(c|\delta_\mu A|) \in L^1(\mu)$ for some $c > 0$. Then there is $T > 0$ and a measurable family of transformations $\{U_t\}_{t\in\mathbb{R}}$ such that μ-a.e.*

(i) $U_t(x) = \widetilde{Q}_t(x) + \int_0^t Q_{t-s}A(U_s(x))\,ds$, where $\widetilde{Q}_t$ stands for the measurable linear extension of Q_t to X (which always exists and preserves measure μ);

(ii) $U_t \circ U_s = U_{t+s}$ for all $t, s \in \mathbb{R}$;

(iii) For all $t \in [-T, T]$ one has $\mu \circ U_t^{-1} = r_t\mu$, where

$$r_t(x) = \exp\Bigl\{-\int_0^t \delta_\mu A(U_{-s}(x))\,ds\Bigr\},$$

and $\sup_{-T\le t\le T}\|r_t\|_{L^p(\mu)} \le C(p,T)$ for some $p = p_c > 1$.

Proof: The proof (the details will be published elsewhere) uses the same method as above and Theorem 1.6 from [11] (already cited) which implies that there exist a separable Hilbert space E with a centered Gaussian measure γ whose Cameron–Martin space H_0 is dense and a measurable linear isomorphism $J : X \to E$ such that $\mu \circ J^{-1} = \gamma$, $J : H \to H_0$ is a unitary isomorphism and the strongly continuous group $\{Q_t^E = JQ_tJ^{-1}\}$ on H_0 can be extended to a strongly continuous semigroup $\{\widetilde{Q}_t^E\}$ on E whose generator has the domain of full γ-measure. Since our claim is invariant under linear isomorphisms, we can deal with E instead of X. Taking as above the finite dimensional fields $A_n = P_nE_nA$, we can solve the equations

$$W_t^n(x) = x + \int_0^t \widetilde{Q}_{-s}^E A_n(\widetilde{Q}_s^E W_s^n(x))\, ds.$$

Indeed, each of these nonautonomous equations is equivalent to a system which satisfies our standard assumptions (due to our conditions on the extended semigroup). Then the family $U_t^n(x) = \widetilde{Q}_t^E W_t^n(x)$ gives a flow solving the desired equation with A replaced by A_n. Now it is trivial to see that the flows $\{U_t^n\}$ converge (a.e.) to a flow $\{U_t\}$ with the announced properties. □

Remark 6.3 *The previous theorem also holds under weaker assumption that the semigroups $\{T_t\}$ and $\{T_t^*\}$ commute and the generator L of $\{T_tT_t^*\}$ on H is a Hilbert–Schmidt operator.*

The last equation to be considered in this section is given by vector fields of the form

$$A(x) = \widetilde{L}(x)x, \tag{6.2}$$

where $x \mapsto L(x)$ is a map with values in the space of antisymmetric operators on H. Fields of this kind on the Wiener space $C([0,1], \mathbb{R}^d)$ appear in [21], [22], [26] in the form $A(w)(t) = \int_0^t C(w(s))\, dw(s)$, where $C(\,\cdot\,)$ is a map with values in antisymmetric operators on $\mathbb{R}^d$. In this case, $L(w)$ acts on $H = W_0^{2,1}([0,1], \mathbb{R}^d)$ by the formula

$$L(w)h(t) = \int_0^t C(w(s))h'(s)\, ds,$$

hence, it is antisymmetric, provided H is equipped with its natural inner product $(h,k)_H = \langle h', k' \rangle_{L^2([0,1],\mathbb{R}^d)}$.

It would be reasonable to look for a solution of the form

$$U_t(x) = \widetilde{V}_t(x)x, \tag{6.3}$$

where $V_t(x)$ are orthogonal operators on H obtained from the equation

$$\frac{d}{dt}V_t(x) = L(\widetilde{V}_t(x)x)V_t(x), \quad V_0(x) = I. \tag{6.4}$$

Individual transformations (i.e. not embedded in flows) of the type $x \mapsto \widetilde{V}(x)x$, where $V(x)$ are unitary operators on H, have been investigated in [41], [42]. The main problem which arises in [41], [42], is that (unlike the case where V is a constant unitary operator which can be extended to a measurable linear map preserving the measure), one has to "extend" all the operators $V(x)$ simultaneously, so as to get a well-defined map $\widetilde{V}(x)x$ preserving the measure. For this, one needs some extra conditions, which seem to be difficult to verify in our situation (where $V_t(x)$ are not given in advance).

Consider thus a mapping $x \in X \mapsto L(x) \in L(H)$ such that $L(x)$ is antisymmetric for all x, and to simplify technical details, assume that $\widetilde{L} : X \to L(X)$ is uniformly bounded and uniformly Lipschitz.

Proposition 6.4 *If for some orthonormal basis $\{e_n\}$ in H the following limit exists in $L^1(\gamma)$:*

$$(\operatorname{div} L(x), x)_H := -\lim_{n\to\infty}\left(\sum_{i=1}^{n} \partial_{e_i} L(x)e_i, P_n x\right)_H. \tag{6.5}$$

then the field $A(x) = \widetilde{L}(x)x$ generates a flow $\{U_t\}$. Moreover, if $\exp|(\mathrm{div}L(x), x)_H| \in L^1(\gamma)$, then γ is quasiinvariant along $\{U_t\}$ and, formally denoting $\delta_\gamma A = (\mathrm{div}L(x), x)_H$, the Radon-Nikodym density r_t is given by the same expression as in our main theorem, namely (4.10).
In particular, if $(\mathrm{div}L(x), x)_H = 0$ then γ is invariant under the flow U_t.

Proof: Clearly, A is locally Lipschitzian on X and $\|A(x)\|_X \le C\|x\|_X$, hence equation (6.1) is uniquely solvable in the usual sense. Moreover, equation (6.4) is uniquely solvable, hence the solution $\{U_t(x)\}$ of the initial equation can be written as (6.3). Note that this solution is locally Lipschitzian. Hence, for the quasiinvariance of γ under U_t it suffices to check that for every bounded Lipschitz function f on X one has

$$\int \langle f'(x), A(x)\rangle\ \gamma(dx) = -\int f(x)(\mathrm{div}L(x), x)_H\ \gamma(dx), \tag{6.6}$$

where f' stands for the Gâteaux derivative of f (which is known to exist γ-a.e., see [9], Theorem 9.3.10 or [12]). Indeed, one has by the semigroup property $U_{t+\tau} = U_t \circ U_\tau$ that $\frac{d}{dt}(f\circ U_t) = \langle (f\circ U_t)', A\rangle$.

Assume first that f is a bounded Lipschitz function of $l_1, \ldots, l_n$ and $m \ge n$. In this case $f \in W^{2,1}(H,\mu)$ and $f'(x)$ can be identified with $\nabla_{H_m} f(x)$. Defining thus

$$A_m(x) = L(x)P_m(x)$$

it follows from (3.5) and the remark following it that

$$\int \langle f'(x), A_m(x)\rangle\, \gamma(dx) = \int (\nabla_H f(x), P_m A_m(x))_H\ \gamma(dx) = -\int f(x)\delta_\gamma(P_m A_m)(x)\,\gamma(dx). \tag{6.7}$$

Inserting the equality $\beta(x) = -x$ (Example (ii) in Section 3.1) in (3.6) (or (3.7)) we obtain

$$\begin{aligned}
\delta_\gamma(P_m A_m)(x) &= \mathrm{trace}\nabla_H P_m A_m(x) - (A_m(x), P_m x)_H \\
&= \sum_{i=1}^{m} \partial_{e_i}\,(L(x)P_m x, e_i)_H - (L(x)P_m x, P_m x)_H = -\sum_{i=1}^{m} \partial_{e_i}\,(P_m(x), L(x)e_i)_H \\
&= -\sum_{i=1}^{m} (e_i, L(x)e_i)_H - \sum_{i=1}^{m} (P_m x, \partial_{e_i} L(x)e_i)_H = -\sum_{i=1}^{m} (P_m x, \partial_{e_i} L(x)e_i)_H
\end{aligned} \tag{6.8}$$

(note that by antisymmetry $(Lh, h) = 0\ \ \forall h \in H$). The combination of (6.7) and (6.8) yields

$$\int \langle f'(x), A_m(x)\rangle\gamma(dx) = \int f(x)\left(\sum_{i=1}^{m} (P_m x, \partial_{e_i} L(x)e_i)_H\right)\ \gamma(dx). \tag{6.9}$$

Letting $m \to \infty$, (6.9) converges to (6.6). Indeed, the left hand side converges since

$$\|A_m - A\|_{L^1(\gamma,X)} \le (\sup_{x\in X}\|L(x)\|)\|x - P_m x\|_{L^1(\gamma,X)} \longrightarrow 0 \quad \text{as} \quad m\to\infty,$$

while the convergence of the right hand side follows from that of $(\mathrm{div}L(x), x)_H$ in (6.5).

Finally, let f be an arbitrary bounded Lipschitz function and recall from Definition 4.1 the sigma–algebra $\mathcal{F}_n$ generated by $l_1, \ldots, l_n$. It is straighforward to check that $f_n := E(f|\mathcal{F}_n)$ is also bounded and Lipschitzian with the same constant — actually $f_n' = E(f'|\mathcal{F}_n)$ γ-a.s., — and thus (6.6) holds replacing f by f_n. When $n\to\infty$ one obtains (6.6) by applying the martingale and dominated convergence theorems, noting that $\|A\|_X$ obviously belongs to $L^1(\gamma)$. □

Remark 6.5 *Combining Proposition 6.4 with the methods above one gets an analogous assertion for fields of the form* $A(x) = C(x)x + A_0(x)$, *where* A_0 *takes values in* H. *It is also possible, but technically more involved, to extend results of this kind to the case where the operators* $C(x)$ *are not assumed to be continuous on* X.

7 Additional Remarks

Even if $\delta_\mu A = 0$ one cannot guarantee the existence of the flow without extra integrability conditions on A or ∇A. Consider the following example from [16].

Example 7.1 *Let* μ *be the standard Gaussian measure on* $\mathbb{R}^2$ *and let* A *be the field on* $\mathbb{R}^2$ *defined by*

$$A : x = (x_1, x_2) \mapsto \Big(x_1^2, (2x_1 - x_1^3) e^{x_2^2/2} \int_{x_2}^{\infty} e^{-s^2/2} ds \Big).$$

Then $A \in C^\infty$ *and* $\|A\|,\ \|\nabla A\| \in L^{1-\varepsilon}(\mu)$ *for each* $\varepsilon \in (0,1)$. *In addition,*

$$\delta_\mu A(x) = \mathrm{div} A(x) - (A(x), x) = 0,$$

since in this case $\beta(x) = -x$. *However, for every starting point* x *the corresponding solution is explosive.*

Let us give some examples where conditions **A3)** or **A3')** are fulfilled.

Example 7.2 *(i) Let* $X = \prod_n E_n$, *where* $E_n = \mathbb{R}^{d_n}$ *are equipped with probability measures* μ_n *possessing densities* $\varphi_n \in W^{1,1}(E_n)$. *Let* $\mu = \otimes_n \mu_n$ *and*

$$H = \Big\{ (h_n) \in X : \sum_{n=1}^{\infty} \|\partial_{h_n} \varphi_n\|^2_{L^2(E_n)} < \infty \Big\}.$$

Assume that conditions (4.1), (4.2), and (4.4) are satisfied. Then conditions **A1)**, **A2)** *and* **A3')** *are fulfilled. In particular, this example covers the Gaussian case (with* H *equal to the Cameron–Martin space).*

(ii) Let X *be a Hilbert space. Assume that* $A(x) = j_H \circ B(x)$, *where* $B : X \to X^*$ *is Gâteaux differentiable with*

$$\exp \|D_X B\|,\ \exp\{\|B(x)\|_{X^*} \|\beta(x)\|_X\} \in L^p(\mu) \quad \forall p \geq 1.$$

Then condition **A3)** *is fulfilled.*

It is possible to formulate sufficient conditions for the existence of a flow which do not involve ∇A explicitly. The proof of the following reproduces some of the steps above.

Theorem 7.3 *Let* μ *be a measure on* X *differentiable along all directions in* $j_H(X^*)$. *Assume that a vector field* $A : X \to H$ *is such that there exists a sequence of vector fields* A_n *of the form* $A_n(x) = B_n(P_n(x))$, $B_n \in \mathrm{Lip}_{loc}(H_n, H_n)$, *with the following properties:*
(i) $\int_X \|A(x) - A_n(x)\|_H^\alpha \, \mu(dx) \to 0$ *for some* $\alpha > 1$;
(ii) $\sup_n \int_X \exp\Big\{ \lambda |\delta_\mu A_n(x)| + \lambda \|\nabla A_n(x)\|_{L(H)} \Big\} \mu(dx) < \infty \quad \forall \lambda$;
Then (1.1) has a flow solution $U_t(x)$ *for which* $\sup_{-t \leq s \leq t} \|r_s\|_{L^p(\mu)} \leq C(p,t)$. *In addition,* $\delta_\mu A$ *exists and (4.10) holds.*

Remark 7.4 *It is worth mentioning that if ν is a measure equivalent to μ with $\nu = g\mu$, then* $\frac{d(\nu\circ U_t^{-1})}{d\nu}(x) = r_t(x)\frac{g(U_{-t}(x))}{g(x)}$.

Remark 7.5 *As it has been shown above, the mapping $U_t(x)$ on $\mathbb{R}^1 \times X$ solves the so called transport equation*

$$\frac{\partial U_t(x)}{\partial t} = \nabla_H U_t(x)(A(x)),$$

which played a crucial role in [25] in the finite dimensional case. In particular, for every $\psi \in \mathcal{S}(X)$ the function $u(t,x) = \psi(U_t(x))$ solves the Cauchy problem $\frac{\partial u}{\partial t} = (\nabla_H u, A)_H$, $u(0,x) = \psi(x)$. In our approach the transport equation has been involved implicitly in the proof of uniqueness. In some more special situations (e.g., in the Gaussian case) $u(t,x)$ can be shown to be the limit of the solutions of the parabolic equations

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{1}{2}\varepsilon^2\Delta_H u^\varepsilon(t,x) + (\nabla_H u^\varepsilon(t,x), A(x))_H, \quad \Delta_H u = trace\nabla_H^2 u,$$

as $\varepsilon \to 0$ (this is related with viscosity solutions). Such a parabolic equation corresponds to the transition probabilities of the diffusion ξ_t^ε governed by the stochastic differential equation

$$d\xi_t^\varepsilon = \varepsilon dW_t + A(\xi_t^\varepsilon)dt,\ \xi_0^\varepsilon = x,$$

where W_t is the Wiener process in X generated by H (see, e.g., [8]). It should be noted, however, that in our non-Gaussian case, the conditions above do not guarantee in general the solvability of that stochastic equation.

Remark 7.6 *Let $A : X \to H$ be a Borel mapping with $\sup_x \|A(x)\|_H < \infty$. Assume that A is continuous along H. Then the integral equation*

$$x(t) = x + \int_0^t A(x(s))\, ds$$

is solvable for every x on every segment $J = [-M, M]$.

Indeed, let K be the set of all continuous functions $\phi : J \to X$ such that $\phi(t) = x + v(t)$ where v takes values in the centered closed ball U of radius $M \sup_x \|A(x)\|_H$ in H. We endow K with the topology of uniform convergence (one can check that K is metrizable for this topology). Clearly, K is convex and complete subset of the locally convex space of all continuous functions on J with values in X equipped with the topology of uniform convergence. The map

$$F(\phi)(t) = x + \int_0^t A(\phi(s))\, ds$$

is continuous by virtue of the continuity of A along H. One can check that $F(K) \subset K$ and that $F(K)$ is relatively compact in K. Therefore, by Tychonoff's fixed point theorem, there is a function $\phi \in K$ such that $F(\phi) = \phi$. Note that one cannot refer here to standard existence theorems for infinite dimensional ordinary differential equations (e.g., [8]), since A need not be continuous on X, and, on the other hand, its image need not be compact in H.

Certainly, such a solution in general is neither unique nor a flow.

Remark 7.7 *(i) In a similar way one can investigate a nonautonomous case where A may depend on t (so that the family of solutions need not be a flow). Clearly, if dependence on t is sufficiently* *regular, this case can be reduced to an autonomous system which satisfies all the conditions required above.*

(ii) It is possible to show higher regularity of the flow provided A has more stochastic derivatives. Due to the length of this paper, this question will be considered separately.

(iii) The methods above apply also to equations on manifolds. Related problems have been discussed in [21], [22], [26]. This will be the subject of a separate paper.

References

[1] S. Albeverio, Yu. G. Kondratiev, and M. Röckner, Dirichlet operators via stochastic analysis, J. Funct. Anal. **128** (1995), 102–138.

[2] A. V. Antoniouk and A. V. Antoniouk, Smoothing properties of semigroups for Dirichlet operators of Gibbs measures, J. Funct. Anal. **127** (1995), 390–430.

[3] D. Bell, A quasi-invariance theorem for measures on Banach spaces, Trans. Amer. Math. Soc. **290** (1985), n 2, 841–845.

[4] D. Bell, "The Malliavin calculus," Pitman Monogr. Surv. Pure Appl. Math., vol. 34, Wiley, New York, 1987.

[5] D. Bell, Transformations of measure on an infinite-dimensional vector space, Seminar on Stochastic Processes, 1990 (Vancouver, BC, 1990), 15–25, Progr. Probab., **24**, Birkhäuser Boston, Boston, MA, 1991.

[6] V. I. Bogachev, Locally convex spaces with the CLT property and supports of measures, Moscow Univ. Math. Bull. **41**:6 (1986), 19–23.

[7] V. I. Bogachev, Differentiable measures and the Malliavin calculus, Preprint n 16 (1995), Scuola Normale Superiore, Pisa, 197 pp. (to appear in: Itogi Nauki i Tehn. VINITI, Probability Theory, Vol. 27; English transl.: J. Math. Sci.).

[8] V. I. Bogachev, Deterministic and stochastic differential equations in infinite dimensional spaces, Acta Appl. Math. **40** (1995), 25–93.

[9] V. I. Bogachev, Gaussian measures on linear spaces, Preprint n 10 (1995), Warwick Univ., 161 pp. (to appear in: Itogi Nauki i Tehn. VINITI, Math. Anal., Vol. 16; English transl.: J. Math. Sci.).

[10] V. I. Bogachev and M. Röckner, Regularity of invariant measures on finite and infinite dimensional spaces and applications, J. Funct. Anal. **133** (1995), 168–223.

[11] V. I. Bogachev, M. Röckner, and B. Schmuland, Generalized Mehler semigroups and applications, Preprint SFB 343 N 94–088, Bielefeld Univ. (1994), 35 pp.; to appear in Probab. Theor. Relat. Fields.

[12] V. I. Bogachev and O. G. Smolyanov, Analytic properties of infinite dimensional distributions, Russian Math. Surv. **45** (1990), n 3, 3–104.

[13] Ch. Borell, Gaussian Radon measures on locally convex spaces, Math. Scand. **38** (1976), n 2, 265–284.

[14] A. Brandao, Global finite dimensional flows, Exposit. Math. **13** (1995), 377–384.

[15] R. Buckdahn, Anticipative Girsanov transformations, Probab. Theor. Relat. Fields **89** (1991), 211–238.

[16] A.-B. Cruzeiro, Équations différentielles ordinaires: Non explosion et mesures quasi–invariantes, J. Funct. Anal. **54** (1983), 193–205.

[17] A.-B. Cruzeiro, Équations différentielles sur l'espace de Wiener et formules de Cameron-Martin non linéaires, J. Funct. Anal. **54** (1983), 206–227.

[18] A.-B. Cruzeiro, Unicité de solutions d'équations différentielles sur l'espace de Wiener, J. Funct. Anal. **58** (1984), 335–347.

[19] A.-B. Cruzeiro, Estimations capacitaires sur l'espace de Wiener, I, Bull. Sci. Math. (2e ser.) **110** (1986), 139–147.

[20] A.-B. Cruzeiro, Flows in infinite dimensions and associated transformations of Gaussian measures, in: "Stochastic methods in mathematics and physics" (Karpacz, 1988), pp. 290–301, World Sci. Publishing, Teaneck, NJ, 1989.

[21] A.-B. Cruzeiro and P. Malliavin, Repère mobile et géometrie riemannienne sur les espaces des chemins, C. R. Acad. Sci. Paris, sér. 1, **319** (1994), 859–864.

[22] A.-B. Cruzeiro and P. Malliavin, Curvatures of path spaces and stochastic analysis, Report n 16 (1995), Institut Mittag–Leffler, 35 pp.

[23] Yu. L. Daletskii and G. Sohadze, Absolute continuity of smooth measures, Funct. Anal. and Appl. **22** (1988), n 2, 77–78.

[24] Yu. L. Daletskii and S. V. Fomin, "Measures and differential equations in infinite-dimensional spaces," Kluwer Acad. Publ., 1991.

[25] R. J. DiPerna and P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. **98** (1989), n 3, 511–547.

[26] B. Driver, A Cameron–Martin type quasi-invariance theorem for the Brownian motion on a compact manifold, J. Funct. Anal. **110** (1992), 272–376.

[27] S. Kusuoka, The nonlinear transformation of Gaussian measure on Banach space and its absolute continuity, J. Fac. Sci. Univ. Tokyo, Sect. IA **29** (1982), 567–597.

[28] M. Malliavin and P. Malliavin, Integration on loop groups I. Quasi-invariant measures, J. Funct. Anal. **93** (1990), 207–237.

[29] P. Malliavin, "Géometrie différentielle stochastique," Univ. de Montreal Presses, 1978.

[30] P. Malliavin, Smooth σ-fields, in: "Stochastic Analysis" (Mayer-Wolf E., Merzbach E., Schwartz A., Eds.), pp. 371–382, Academic Press, New York, Boston, MA, 1991.

[31] P. Malliavin, Infinite dimensional analysis, Bull. Sc. Math. (sér. 2) **117** (1993), 63–90.

[32] E. Mayer-Wolf, Preservation of measure continuity under conditioning, J. Funct. Anal. **115** (1993), 227–246.

[33] D. Nualart, A. S. Ustunel, and M. Zakai, Some relations among classes of σ-fields, Probab. Theor. Relat. Fields **85** (1990), 119–129.

[34] G. Peters, Flows on the Wiener space generated by vector fields with low regularity, C. R. Acad. Sci. Paris, sér. 1, **320** (1995), 1003–1008.

[35] G. Pisier, Probability methods in the geometry of Banach spaces, Lect. Notes Math. **1206** (1985), 167–241.

[36] R. Ramer, On nonlinear transformations of Gaussian measures, J. Funct. Anal. **15** (1974), 166–187.

[37] A. V. Skorohod, "Integration in Hilbert space," Springer, 1974.

[38] O. G. Smolyanov and H. Weizsäcker, Differentiable families of measures, J. Funct. Anal. **118** (1993), 454–476.

[39] A. S. Üstünel and M. Zakai, On the structure of independence on Wiener space, J. Funct. Anal. **90** (1990), n 1, 113–137.

[40] A. S. Üstünel and M. Zakai, Transformation of Wiener measure under anticipative flows, Probab. Theor. Relat. Fields **93** (1992), 91–136.

[41] A. S. Üstünel and M. Zakai, Analyse de rotations aléatoires sur l'espace de Wiener, C. R. Acad. Sci. Paris, sér. 1, **319** (1994), 1069–1073.

[42] A. S. Üstünel and M. Zakai, Random rotations of the Wiener path, Probab. Theor. Relat. Fields **103** (1995), n 3, 409–430.

[43] N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan, "Probability distributions in Banach spaces," Nauka, Moscow, 1985 [in Russian]; English translation: Kluwer, Dordrecht, 1990.

[44] M. Zakai and O. Zeitouni, When does the Ramer formula look like Girsanov formula ?, Ann. Probab. **20** (1992), n 3, 1436–1440.

[45] W. Ziemer, "Weakly differentiable functions," Springer, 1989.