CONVERGENCE OF OPERATOR SEMIGROUPS GENERATED BY ELLIPTIC OPERATORS

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ABSTRACT. Consider a sequence of operator semigroups $\left(T_t^{(n)}\right)_{t>0}$, $n \in \mathbb{N}$, whose generators are elliptic and are (informally) of type $L^{(n)} = \sum_{i,j=1}^d \partial_i \left(a_{ij}^{(n)} \partial_j + d_i^{(n)}\right) - \sum_{i=1}^d b_i^{(n)} \partial_i - c^{(n)}$ on a possibly unbounded open set $U \subset \mathbb{R}^d$ with measurable coefficients. Under weak assumptions on the coefficients we prove strong convergence of $\left(T_t^{(n)}\right)_{t>0}$, $n \in \mathbb{N}$, on $L^2(U;dx)$ resp. the Sobolev space $H_0^{1,2}(U;dx)$. In particular, this is done without assuming that $b_i^{(n)}, d_i^{(n)}, c^{(n)}$, are bounded in $L^\infty(U;dx)$ uniformly in n.

1. INTRODUCTION AND MAIN RESULTS

Let $U \subset \mathbb{R}^d$, $d \geq 3$, U open (not necessarily bounded), and let dx denote Lebesgue measure on U. Below all functions are supposed to be real–valued. Let $a_{ij}^{(n)}$, $b_i^{(n)}$, $d_i^{(n)}$, $c^{(n)} \in L^1_{\text{loc}}(U;dx)$, $1 \leq i,j \leq d$, $n \in \mathbb{N} \cup \{\infty\}$ satisfying the following conditions:

(1.1) There exists $\delta \in]0,\infty[$ such that for all $n \in \mathbb{N} \cup \{\infty\}$ and dx–a.e. $x \in U$

$$\sum_{i,j=1}^d a_{ij}^{(n)}(x)\xi_i\xi_j \geq \delta \sum_{i=1}^d \xi_i^2 \text{ for all } \xi_1,\dots,\xi_d \in \mathbb{R} .$$

(1.2) There exists $M \in [0,\infty[$ such that for all $n \in \mathbb{N}$ and dx–a.e. $x \in U$

$$\left|a_{ij}^{(n)}(x)\right| \leq M \ , \ 1 \leq i,j \leq d .$$

(1.3) There exist $p_{b,i}$, $p_{d,i}$, $p_c \in [d,\infty]$, $1 \leq i \leq d$, such that for all $n \in \mathbb{N} \cup \{\infty\}$

$$b_i^{(n)} \in L^{p_{b,i}}(U;dx) \ , \ d_i^{(n)} \in L^{p_{d,i}}(U;dx) \ , \ c^{(n)} \in L^{p_c/2}(U;dx) .$$

Note that (1.1) is a condition only on the symmetric part of $(a_{ij})_{1\leq i,j\leq d}$. Conditions (1.1) – (1.3) allow to construct the corresponding coercive closed forms (cf. e.g. [MR

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92, Chap. I, Sect. 2]) as follows. Let $C_0^\infty(U)$ denote the set of all infinitely differentiable functions with compact support in U. Fix $n \in \mathbb{N} \cup \{\infty\}$ and set $\partial_i := \frac{\partial}{\partial x_i}$, $1 \le i \le d$. Define

(1.4)

$$\mathcal{E}^{(n)}(u,v) := \sum_{i,j=1}^d \int \partial_i u \ \partial_j v \ a_{ij}^{(n)} \, dx + \sum_{i=1}^d \int u \ \partial_i v \ d_i^{(n)} \, dx + \sum_{i=1}^d \int \partial_i u \ v \ b_i^{(n)} \, dx + \int u \ v \ c^{(n)} \, dx \ ; \ u, v \in C_0^\infty(U) \ .$$

For $\alpha \in]0,\infty[$ set

$$\mathcal{E}_\alpha^{(n)}(u,v) := \mathcal{E}^{(n)}(u,v) + \alpha(u,v)_{L^2(U;dx)}; u, v \in C_0^\infty(U) \ .$$

E.g. by [MR 93, Theorem 2.2] we know that there exists $\alpha_n \in]0,\infty[$ such that $\left(\mathcal{E}_{\alpha_n}^{(n)}, C_0^\infty(U)\right)$ is closable on $L^2(U;dx)$ and its closure $\left(\mathcal{E}_{\alpha_n}^{(n)}, D(\mathcal{E}_{\alpha_n}^{(n)})\right)$ is a coercive closed form on $L^2(U;dx)$ in the sence of [MR 92, Chap. I, Definition 2.4]. It is well–known (and can e.g. easily be extracted from the proof of [MR 93, Theorem 2.2], or more precisely from the proof of the underlying [RSch 95, Theorem 1.7]) that there exist $\gamma_n \in]0,\infty[$ such that for all $u, v \in C_0^\infty(U)$

$$\left|\mathcal{E}_{\alpha_n}^{(n)}(u,v)\right| \le \gamma_n \ \mathcal{E}_{\alpha_n}^{(n)}(u,u)^{1/2} \mathcal{E}_{\alpha_n}^{(n)}(v,v)^{1/2} \tag{1.5}$$

$$\gamma_n^{-1} \ |u|_{1,2} \le \mathcal{E}_{\alpha_n}^{(n)}(u,u)^{1/2} \le \gamma_n \ |u|_{1,2} \ . \tag{1.6}$$

Here $| \ |_{1,2}$ is the norm on the classical Sobolev space $H_0^{1,2}(U;dx)$ of order 1 in $L^2(U;dx)$, defined as the completion of $C_0^\infty(U)$ w.r.t. $| \ |_{1,2}$ which is given by

$$|u|_{1,2}^2 := \sum_{i=1}^d \int (\partial_i u)^2 \, dx + \int u^2 \, dx \ ; \ u \in C_0^\infty(U) \ .$$

In particular, $D\left(\mathcal{E}_{\alpha_n}^{(n)}\right) = H_0^{1,2}(U,dx)$ and (1.5), (1.6) hold for all $u \in H_0^{1,2}(U;dx)$.

Remark 1.1. γ_n in (1.5), (1.6) only depends on α_n, δ, M and the L^p–norms of $b_i^{(n)}$, $d_i^{(n)}$, $c^{(n)}$, $1 \le i \le d$, (cf. condition (1.3)). This can also be seen e.g. from the respective proofs in [MR 93], [RSch 95] mentioned above. In particular, α_n and γ_n can be chosen to be independent of n, if all the L^p–norms in condition (1.3) are bounded uniformly in n.

Let $(L_{\alpha_n}, D(L_{\alpha_n}))$, $\left(T_{\alpha_n,t}\right)_{t>0}$ be the generator resp. the strongly continuous contraction semigroup associated with $\left(\mathcal{E}_{\alpha_n}^{(n)}, D(\mathcal{E}_{\alpha_n}^{(n)})\right)$ (cf. e.g. [MR 92, Chap. I., Sect. 2]). Define

$$T_t^{(n)} := e^{\alpha_n t} T_{\alpha_n,t} \ , \ t > 0 \ , \tag{1.7}$$

$$L^{(n)} := L_{\alpha_n} + \alpha_n \ , \ D(L^{(n)}) := D(L_{\alpha_n}) \ . \tag{1.8}$$

Then $(L^{(n)}, D(L^{(n)})$ generates $(T_t^{(n)})_{t>0}$ (on $L^2(U;dx)$.

Remark 1.2.

i) Obviously, $\left(L^{(n)}, D\left(L^{(n)}\right)\right)$ and $\left(T_t^{(n)}\right)_{t>0}$ are independent of the special choice of α_n.

ii) Informally, we have for $u \in C_0^\infty(U)$ that

$$L^{(n)}u = \sum_{i,j=1}^{d} \partial_i \left(a_{ij}^{(n)}\partial_j + d_i^{(n)}\right) u - \sum_{i=1}^{d} b_i^{(n)}\partial_i u - c^{(n)}u \ . \tag{1.9}$$

Though (1.9) is very suggestive, it is, of course, informal since $C_0^\infty(U)$ will in general not be a subset of $D\left(L^{(n)}\right)$.

iii) Note that e.g. by [MR 92, Chap. I, Theorem 2.20] $T_t^{(n)}f \in D\left(\mathcal{E}_{\alpha_n}^{(n)}\right) = H_0^{1,2}(U;dx)$ for all $f \in L^2(U;dx)$, $t > 0$.

Let $n \in \mathbb{N}\cup\{\infty\}$ and let $\left(G_\alpha^{(n)}\right)_{\alpha>\alpha_n}$ be the strongly continuous resolvent associated with $\left(T_t^{(n)}\right)_{t>0}$ on $L^2(U;dx)$, i.e., for $\alpha > \alpha_n$

$$G_\alpha^{(n)}f := \int_0^\infty e^{-\alpha t}\, T_t^{(n)}f\, dt \ , \ f \in L^2(U;dx) \ , \tag{1.10}$$

(where the integral is a Bochner integral in $L^2(U;dx)$). Note that for $\alpha > \alpha_n$ and $f \in L^2(U;dx)$

(1.11) $G_\alpha^{(n)}f \in D\left(\mathcal{E}_{\alpha_n}^{(n)}\right) = H_0^{1,2}(U;dx)$

and $\mathcal{E}_\alpha^{(n)}\left(G_\alpha^{(n)}f, v\right) = (f,v) = \mathcal{E}_\alpha^{(n)}(v, \widehat{G}_\alpha^{(n)}f)$ for all $v \in H_0^{1,2}(U;dx)$

(cf. e.g. [MR 92, Chap. I., Theorem 2.8] and recall (1.7)). Here for a densely defined operator $(T, D(T))$ on $L^2(U;dx)$ we denote its adjoint by $(\widehat{T}, D(\widehat{T}))$.
Consider for $1 \le i,j \le d$ the following conditions:

$$a_{ij}^{(n)} \underset{n\to\infty}{\longrightarrow} a_{ij}^{(\infty)} =: \ a_{ij} \qquad dx\text{–a.e. on U} \ . \tag{1.12}$$

$$b_i^{(n)} \underset{n\to\infty}{\longrightarrow} b_i^{(\infty)} =: \ b_i \text{ weakly}^* \text{ in } L^{p_{b,i}}(U;dx) \ . \tag{1.13}$$

$$d_i^{(n)} \underset{n\to\infty}{\longrightarrow} d_i^{(\infty)} =: \ d_i \text{ weakly}^* \text{ in } L^{p_{d,i}}(U;dx) \ . \tag{1.14}$$

$$c^{(n)} \underset{n\to\infty}{\longrightarrow} c^{(\infty)} =: \ c \text{ weakly}^* \text{ in } L^{p_c/2}(U;dx) \ . \tag{1.15}$$

Now we can formulate the main results of this paper.

Theorem 1.3. *Suppose that for $1 \le i,j \le d$ conditions (1.12), (1.13), and (1.15) are satisfied and that*

$$\left|d_i^{(n)} - d_i\right| \underset{n\to\infty}{\longrightarrow} 0 \ \textit{weakly}^* \ \textit{in} \ L^{p_{d,i}}(U;dx) \ , \ \textit{for all} \ 1 \le i \le d \ . \tag{1.16}$$

Then there exists $\alpha_0 \in]0, \infty[$ *such that for all* $\alpha > \alpha_0$ *and all* $f \in L^2(U; dx)$*:*

(i) $$G_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} G_\alpha^{(\infty)} f =: G_\alpha f \text{ and } \widehat{G}_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} \widehat{G}_\alpha^{(\infty)} f =: \widehat{G}_\alpha f$$

weakly in $H_0^{1,2}(U; dx)$*;*

(ii) $$G_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} G_\alpha f \text{ in } L^2(U; dx) \ ,$$

and hence for all $t > 0$

$$T_f^{(n)} f \underset{n\to\infty}{\longrightarrow} T_t f \text{ in } L^2(U; dx) \ .$$

Remark 1.4. (i) We use the notion "weakly*" rather than "weakly" since $p_{b,i}$, $p_{d,i}$, p_c can be equal to $+\infty$. Clearly, if we assume (1.14) then (1.16) holds if $d_i^{(n)} \underset{n\to\infty}{\longrightarrow} d_i$ in dx–measure for all $1 \le i \le d$. Note that (1.16), of course, implies (1.14).

(ii) Note that the last part of Theorem 1.3 (ii) is trivial, since (as is well–known and quite easy to prove) that strong convergence of strongly continuous contraction semigroups, (such as $e^{-\alpha t} T_t^{(n)} \underset{n\to\infty}{\longrightarrow} e^{-\alpha t} T_t, t > 0$, in our case) is equivalent to the strong convergence of their associated resolvents. (cf. e.g. [Me 94, Satz 1.7]).

(iii) If conditions (1.12), (1.14), and (1.15) hold and if, in addition,

$$\left| b_i^{(n)} - b_i \right| \underset{n\to\infty}{\longrightarrow} 0 \text{ weakly}^* \text{ in } L^{p_{b,i}}(U; dx) \text{ for all } 1 \le i \le d \ , \tag{1.17}$$

then by duality the assertion in part (i) of Theorem 1.3 still holds while part (ii) holds with all operators replaced by their adjoints on $L^2(U; dx)$.

By Rellich's compact embedding theorem we get the following as an immediate consequence of Theorem 1.3 (i).

Corollary 1.5. Suppose the U is bounded and that conditions (1.12) – (1.15) and (1.16) or (1.17) hold. Then there exists $\alpha_0 \in]0, \infty[$ such that for all $\alpha > \alpha_0$, $t > 0$, both $T_t^{(n)} \underset{n\to\infty}{\longrightarrow} T_t$ and $G_\alpha^{(n)} \underset{n\to\infty}{\longrightarrow} G_\alpha$ strongly on $L^2(U; dx)$. The same holds for their adjoints on $L^2(U; dx)$.

As another consequence we obtain:

Corollary 1.6. Assume that (1.12) holds and that for all $1 \le i \le d$, $b_i^{(n)} \underset{n\to\infty}{\longrightarrow} b_i$ in $L^{p_{b,i}}(U; dx)$, $d_i^{(n)} \underset{n\to\infty}{\longrightarrow} d_i$ in $L^{p_{d,i}}(U; dx)$, and $c^{(n)} \longrightarrow c$ in $L^{p_c/2}(U; dx)$. Then:

i) There exists $\alpha_0 \in]0, \infty[$ such that for all $f \in L^2(U; dx)$ and $\alpha > \alpha_0$,

$$G_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} G_\alpha f \text{ and } \widehat{G}_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} \widehat{G}_\alpha f \text{ in } H_0^{1,2}(U; dx) \ .$$

ii) For all $t > 0$ and all $f \in L^2(U; dx)$

$$T_t^{(n)} f \underset{n\to\infty}{\longrightarrow} T_t f \text{ and } \widehat{T}_t^{(n)} f \underset{n\to\infty}{\longrightarrow} \widehat{T}_t f \text{ in } H_0^{1,2}(U; dx) .$$

Our proofs of all results above are purely analytic. They are presented in the next section. Theorem 1.3 extends a result by D.W. Stroock (cf. [St 88, Theorem II.3.13], where the case where $U = \mathbb{R}^d$, $c \equiv 0$, $d_i^{(n)} \equiv 0$, $p_{b,i} = \infty$ for all $1 \le i \le d$, $n \in \mathbb{N}$, was treated and the $b_i^{(n)}$, $1 \le i \le d$, $n \in \mathbb{N}$, were assumed to be uniformly bounded. In contrast to Stroock's our proofs are not based on heat kernel estimates.
Finally, we note that we expect that by virtue of [S 95a], [S 95b] the results in this paper extend to the case of time–dependent coefficients (again without any uniform boundedness assumptions).

2. Proofs

Proof of Theorem 1.3. (i): For $q \in [1, \infty]$ let $\| \ \|_q$ denote the usual norm in $L^q(U; dx)$. By the conditions and Remark 1.1, α_n and γ_n can be chosen to be independent of n, i.e., $\gamma_n =: \gamma_0 > 0$ and $\alpha := \alpha_0 > 0$ for all $n \in \mathbb{N} \cup \{\infty\}$, say. In particular, for all $\alpha > \alpha_0$

$$\sup_n \left\| \alpha \widehat{G}_\alpha^{(n)} \right\| =: C_\alpha < \infty$$

where $\| \ \|$ denotes operator norm on $L^2(U; dx)$. Hence by (1.11)

$$\mathcal{E}_\alpha^{(n)} \left(\widehat{G}_\alpha^{(n)} f, \widehat{G}_\alpha^{(n)} f \right) = \left(f, \widehat{G}_\alpha^{(n)} f \right) \le \alpha^{-1} C_\alpha \|f\|_2^2 . \tag{2.1}$$

Fix $f \in L^2(U; dx)$, $\alpha > \alpha_0$. Since $\gamma_n = \gamma_0$ for all $n \in \mathbb{N}$, (1.6) and (2.1) imply that

$$\sup_n \left| \widehat{G}_\alpha^{(n)} f \right|_{1,2} =: C < \infty . \tag{2.2}$$

Then by the Banach–Alaoglu theorem there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and $\widehat{G} f \in H_0^{1,2}(U; dx)$ such that

$$\widehat{G}_\alpha^{(n_k)} f \underset{k\to\infty}{\longrightarrow} \widehat{G} f \quad \text{weakly in} \quad H_0^{1,2}(U; dx) .$$

So, it remains to show that $\widehat{G} f = \widehat{G}_\alpha f$. For simplicity of notation we replace $(n_k)_{k\in\mathbb{N}}$ again by $(n)_{n\in\mathbb{N}}$ and, since $\left(\widehat{G}_\alpha^{(n)} f \right)_{n\in\mathbb{N}}$ converges (strongly) in $L^2(V; dx)$ for every open ball V in U by Rellich's theorem, we may also assume that

$$\widehat{G}_\alpha^{(n)} f \longrightarrow \widehat{G} f \quad dx\text{–a.e. .} \tag{2.3}$$

Claim 1. Let $v \in C_0^\infty(U)$. Then

$$\lim_{n\to\infty}\left[\mathcal{E}_\alpha\left(v,\widehat{G}_\alpha^{(n)}f\right)-\mathcal{E}_\alpha^{(n)}\left(v,\widehat{G}_\alpha^{(n)}f\right)\right]=0\ .$$

Suppose Claim 1 has been proven. Then by the weak convergence of $\left(\widehat{G}_\alpha^{(n)}f\right)_{n\in\mathbb{N}}$ in $H_0^{1,2}(U;dx)$ and (1.5), (1.6) it follows that

$$\begin{aligned}\mathcal{E}_\alpha(v,\widehat{G}f)&=\lim_{n\to\infty}\mathcal{E}_\alpha\left(v,\widehat{G}_\alpha^{(n)}f\right)=\lim_{n\to\infty}\mathcal{E}_\alpha^{(n)}\left(v,\widehat{G}_\alpha^{(n)}f\right)=(v,f)\\&=\mathcal{E}_\alpha\left(v,\widehat{G}_\alpha f\right)\end{aligned}$$

for all $v\in C_0^\infty(U)$, hence $\widehat{G}_\alpha f=\widehat{G}f$ and the proof is complete.
To prove Claim 1 note that for $n\in\mathbb{N}$

$$\begin{aligned}&\mathcal{E}_\alpha\left(v,\widehat{G}_\alpha^{(n)}f\right)-\mathcal{E}_\alpha^{(n)}\left(v,\widehat{G}_\alpha^{(n)}f\right)\\&=\sum_{i,j=1}^d\int\left(a_{ij}-a_{ij}^{(n)}\right)\partial_i v\ \partial_j\widehat{G}_\alpha^{(n)}f\ dx+\sum_{i=1}^d\int\left(d_i-d_i^{(n)}\right)\ v\ \partial_i\widehat{G}_\alpha^{(n)}f\ dx\\&+\sum_{i=1}^d\int\left(b_i-b_i^{(n)}\right)\partial_i v\ \widehat{G}_\alpha^{(n)}f\ dx+\int\left(c-c^{(n)}\right)\ v\ \widehat{G}_\alpha^{(n)}f\ dx\ .\end{aligned}\tag{2.4}$$

By the Cauchy–Schwarz inequality and (2.2) the first summand converges to zero as $n\to\infty$ because of Lebesgue's dominated convergence theorem.

Let us recall that by Sobolev's Lemma if $\lambda:=\frac{2^{2/3}(d-1)}{(d-2)d^{1/2}}$, then for all $u\in C_0^\infty(U)$

$$\|u\|_{\frac{2d}{d-2}}\le\lambda\left(\int|\nabla u|_{\mathbb{R}^d}^2\ dx\right)^{1/2}\tag{2.5}$$

(cf. e.g. [Da 89, Theorem 1.7.1]). For $K:=\ \mathrm{supp}\ v$ (2.2) and (2.5) imply that $\{\widehat{G}_\alpha^{(n)}f\mid n\in\mathbb{N}\}$ is uniformly $\frac{d}{d-2}$–integrable on K w.r.t. dx. Hence by (2.3)

$$\widehat{G}_\alpha^{(n)}f\underset{n\to\infty}{\longrightarrow}\widehat{G}f\ \text{in}\ L^{\frac{d}{d-2}}(K;dx)\ .\tag{2.6}$$

Since for all $i\in\{1,\dots,d\}$

$$\sup_n\left\|b_i^{(n)}\right\|_{L^{d/2}(K;dx)}\le\infty$$

and

$$\sup_n\left\|c^{(n)}\right\|_{L^{d/2}(K;dx)}\le\infty$$

(because $p_{d,i} \geq d > d/2$ and $p_c \geq d/2$), it follows that both $b_i^{(n)} \underset{n\to\infty}{\longrightarrow} b_i$ and $c^{(n)} \underset{n\to\infty}{\longrightarrow} c$ weakly* in $L^{d/2}(K;dx)$. Hence (2.6) implies that both the third and fourth summand on the right hand side of (2.4) converge to zero.
To prove that the same holds for the second, fix $i \in \{1,\dots,d\}$ and note that

$$\begin{aligned} &\left| \int \left(d_i - d_i^{(n)}\right) v \; \partial_i \widehat{G}_\alpha^{(n)} f \; dx \right| \\ &\leq \left(\int \left(d_i - d_i^{(n)}\right)^2 v^2 \; dx \right)^{1/2} \left(\int \left(\partial_i \widehat{G}_\alpha^{(n)} f\right)^2 \; dx \right)^{1/2} . \end{aligned}$$

Hence by (2.2) it is sufficient to realize that by the Cauchy–Schwarz inequality (applied to the measure $|d_i - d_i^{(n)}| v^2 \; dx$)

$$\int \left(d_i - d_i^{(n)}\right)^2 v^2 \; dx \leq \left(\int \left|d_i - d_i^{(n)}\right| v^2 \; dx \right)^{1/2} \left(\int \left|d_i - d_i^{(n)}\right|^3 v^2 \; dx \right)^{1/2} ,$$

and to recall that by (1.16) $\left|d_i - d_i^{(n)}\right| \underset{n\to\infty}{\longrightarrow} 0$ weakly* in $L^{p_{d,i}}(U;dx)$ and thus, because $p_{d,i} \geq d \geq 3$, and supp v is compact,

$$\sup_n \int \left|d_i - d_i^{(n)}\right|^3 v^2 \; dx < \infty .$$

Now Claim 1 is proved.
To show that also $G_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} G_\alpha f$ weakly in $H_0^{1,2}(U;dx)$ we note that by (1.11) for all $n \in \mathbb{N}$

$$\mathcal{E}_\alpha^{(n)} \left(G_\alpha^{(n)} f, G_\alpha^{(n)} f \right) = \left(f, G_\alpha^{(n)} f \right) \leq \alpha^{-1} \; C_\alpha \; \|f\|_2^2 .$$

So, as above

$$\sup_n \left| G_\alpha^{(n)} f \right|_{1,2} =: C < \infty$$

and

$$G_\alpha^{(n_k)} f \underset{k\to\infty}{\longrightarrow} Gf \text{ weakly in } H_0^{1,2}(U;dx), \text{ hence weakly in } L^2(U;dx)$$

for some subsequence $(n_k)_{k\in\mathbb{N}}$ and some $Gf \in H_0^{1,2}(U;dx)$. Again we only have to show that $Gf = G_\alpha f$. But we know that $\widehat{G}_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} \widehat{G}_\alpha f$ weakly in $L^2(U;dx)$, hence $G_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} G_\alpha f$ weakly in $L^2(U;dx)$, so

$$G_\alpha f = Gf ,$$

and the proof of assertion (i) is complete.
(ii): By Remark 1.4 (i) it suffices to prove the first statement.

Claim 2. Let $f_n \in L^2(U;dx)$, $n \in \mathbb{N}$, such that $f_n \underset{n\to\infty}{\longrightarrow} 0$ weakly in $L^2(U;dx)$.
Then

$$\widehat{G}_\alpha^{(n)} f_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{weakly in } H_0^{1,2}(U;dx)\ .$$

Suppose Claim 2 has been proven, then for $\alpha > \alpha_0$ (where α_0 is as in assertion (i)), $f \in L^2(U;dx)$ and all $n \in \mathbb{N}$

$$\left\| G_\alpha^{(n)} f \right\|_2^2 = \int G_\alpha f \; G_\alpha^{(n)} f \; dx + \int \widehat{G}_\alpha^{(n)} \left(G_\alpha^{(n)} f - G_\alpha f \right) f \; dx\ .$$

By part (i) the first summand converges to $\|G_\alpha f\|_2^2$ while by Claim 2 the second summand converges to zero. Using part (i) again we conclude that

$$G_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} G_\alpha f \quad \text{in } L^2(U;dx)\ .$$

To prove the claim by (1.6) and Remark 1.1 as well as (2.2) it suffices to show that

$$\lim_{n\to\infty} \mathcal{E}_\alpha(v, \widehat{G}_\alpha^{(n)} f_n) = 0 \text{ for all } v \in C_0^\infty(U)\ .$$

So, let $v \in C_0^\infty(U)$, then by (1.11)

$$\mathcal{E}_\alpha\left(v, \widehat{G}_\alpha^{(n)} f_n\right) = (v, f_n) + \mathcal{E}_\alpha\left(v, \widehat{G}_\alpha^{(n)} f_n\right) - \mathcal{E}_\alpha^{(n)}\left(v, \widehat{G}_\alpha^{(n)} f_n\right)\ .$$

So, it remains to be shown that

$$\lim_{n\to\infty} \left(\mathcal{E}_\alpha\left(v, \widehat{G}_\alpha^{(n)} f_n\right) - \mathcal{E}_\alpha^{(n)}\left(v, \widehat{G}_\alpha^{(n)} f_n\right) \right) = 0\ .$$

But

$$\begin{aligned}
&\mathcal{E}_\alpha\left(v, \widehat{G}_\alpha^{(n)} f_n\right) - \mathcal{E}_\alpha^{(n)}\left(v, \widehat{G}_\alpha^{(n)} f_n\right) \\
&= \sum_{i,j=1}^d \int \left(a_{ij} - a_{ij}^{(n)}\right) \partial_i v \; \partial_j \widehat{G}_\alpha^{(n)} f_n \; dx + \sum_{i=1}^d \int \left(d_i - d_i^{(n)}\right) v \; \partial_i \widehat{G}_\alpha^{(n)} f_n \; dx \\
&+ \sum_{i=1}^d \int \left(b_i - b_i^{(n)}\right) \partial_i v \; \widehat{G}_\alpha^{(n)} f_n \; dx + \int \left(c - c^{(n)}\right) v \; \widehat{G}_\alpha^{(n)} f_n \; dx\ .
\end{aligned}$$

By (2.1) $\left(\widehat{G}_\alpha^{(n)} f_n\right)_{n\in\mathbb{N}}$ is bounded in $H_0^{1,2}(U;dx)$, hence by exactly the same arguments as in the proof of Claim 1 (with $K :=$ supp v) we obtain that

$$\widehat{G}_\alpha^{(n)} f_n \underset{n\to\infty}{\longrightarrow} h \quad \text{in} \quad L^{\frac{d}{d-2}}(K;dx)$$

for some $h \in H_0^{1,2}(U;dx)$. Now also the rest of the proof of Claim 2 is entirely analogous to that of Claim 1.

Thus the proof of assertion (ii) is complete. □

Proof of Corollary 1.6. (i): Let $\alpha_0 \in]0,\infty[$ be as in Theorem 1.3. Fix $\alpha > \alpha_0$ and $f \in L^2(U;dx)$. Then by Remark 1.1 (cf. the beginning of the proof for Theorem 1.3) it suffices to prove

$$\lim_{n\to\infty} \mathcal{E}_\alpha^{(n)} \left(G_\alpha^{(n)} f - G_\alpha f, G_\alpha^{(n)} f - G_\alpha f \right) = 0 \ , \tag{2.7}$$

since by duality the same then holds for $\widehat{G}_\alpha f$, $\widehat{G}_\alpha^{(n)} f$, $n \in \mathbb{N}$. But by applying (1.11) twice we have for all $n \in \mathbb{N}$

$$\begin{aligned}
& \mathcal{E}_\alpha^{(n)} \left(G_\alpha^{(n)} f - G_\alpha f, G_\alpha^{(n)} f - G_\alpha f \right) \\
= \ & \mathcal{E}_\alpha \left(G_\alpha f, G_\alpha^{(n)} f - G_\alpha f \right) - \mathcal{E}_\alpha^{(n)} \left(G_\alpha f, G_\alpha^{(n)} f - G_\alpha f \right) \\
= \ & \sum_{i,j=1}^{d} \int \left(a_{ij} - a_{ij}^{(n)} \right) \partial_i G_\alpha f \ \partial_j \left(G_\alpha^{(n)} f - G_\alpha f \right) \ dx \\
+ \ & \sum_{i=1}^{d} \int \left(d_i - d_i^{(n)} \right) G_\alpha f \ \partial_i \left(G_\alpha^{(n)} f - G_\alpha f \right) \ dx \\
+ \ & \sum_{i=1}^{d} \int \left(b_i - b_i^{(n)} \right) \partial_i G_\alpha f \left(G_\alpha^{(n)} f - G_\alpha f \right) \ dx \\
+ \ & \int \left(c - c^{(n)} \right) G_\alpha f \left(G_\alpha^{(n)} f - G_\alpha f \right) \ dx \ .
\end{aligned}$$

Since by Theorem 1.3, $G_\alpha^{(n)} f \underset{n\to\infty}{\longrightarrow} G_\alpha f$ weakly in $H_0^{1,2}(U;dx)$, it is clear that the first summand converges to zero as $n \to \infty$. To see that the same is true for the others we only have to realize that after applying Hölder's inequality we have to deal with integrals of type

$$I_n := \int g_n^2 \ u_n^2 \ dx \ , \quad n \in \mathbb{N} \ ,$$

where $g_n \to 0$ in $L^p(U;dx)$, $p \in [d,\infty[$, $u_n \in H_0^{1,2}(U;dx)$ such that $\sup_n |u_n|_{1,2} < \infty$. But using Hölder's inequality and (2.5) we obtain that

$$\begin{aligned}
I_n &\le \left(\int u_n^2 \, dx \right)^{\frac{p-d}{p}} \left(\int g_n^{2p/d} \ u_n^2 \, dx \right)^{d/p} \\
&\le \|u_n\|_2^{\frac{2(p-d)}{p}} \ \|g_n\|_p^2 \ \lambda^{2d/p} \ |u_n|_{1,2}^{2d/p} \ ,
\end{aligned}$$

hence $I_n \underset{n\to\infty}{\longrightarrow} 0$ and the proof of assertion (i) is complete.

(ii): E.g. by [AFRS 95, Theorem 3.4 (iii)], (1.6) and Remark 1.1 it follows that $\left(T_t^{(n)} \right)_{n\in\mathbb{N}}$ is a strongly continuous semigroup on $H_0^{1,2}(U;dx)$ and that $\left(G_\alpha^{(n)} \right)_{\alpha>\alpha_0}$ is the associated resolvent. Hence assertion (ii) follows by Remark 1.4 (ii). □

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